Local projection stabilization in the finite element method

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Outline

- stabilization for incompressible flow problems
- local projection stabilization for convection–diffusion–reaction equations
- generalized formulation with overlapping projection domains
- stability and error analysis with respect to the SUPG norm
- numerical results
- local projection stabilization for the Stokes problem
- local projection stabilization for the Oseen problem

Motivation: numerical solution of the incompressible Navier–Stokes equations

$$\mathbf{u}_t - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \tilde{\mathbf{f}}, \quad \operatorname{div} \mathbf{u} = 0 \qquad \text{in } \Omega \times (0, T]$$

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Oseen problems

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Two sources of instabilities:

– dominant convection

– violation of the inf–sup condition

$$\sup_{\mathbf{v}_h\in\mathbf{V}_h}\frac{(q_h,\operatorname{div}\mathbf{v}_h)}{|\mathbf{v}_h|_{1,\Omega}}\geq\beta\,\|q_h\|_{0,\Omega}\quad\forall\,q_h\in Q_h$$

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Residual-based stabilization (SUPG/PSPG/div-div)

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Drawback: strong coupling between velocity and pressure

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Projection-based stabilization

$$\kappa_h = id - \pi_h$$

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$$\mathscr{A}([\mathbf{u}_h,p_h],[\mathbf{v}_h,q_h]) + (\kappa_h((\mathbf{b}\cdot\nabla)\mathbf{u}_h), \delta^u\,\kappa_h((\mathbf{b}\cdot\nabla)\mathbf{v}_h))$$
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Codina (2000), Kaya, Layton (2003), Braack, Burman (2006)

Steady convection–diffusion–reaction equation

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Integration by parts $\quad\Rightarrow\quad$ coercivity:

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inappropriate if $\varepsilon \ll |\mathbf{b}|$!!!

solution globally polluted by spurious oscillations

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$$\delta|_T = \frac{h_T}{2\, l\, |\mathbf{b}|} \left(\coth \mathrm{Pe}_T - \frac{1}{\mathrm{Pe}_T} \right) \quad \text{with} \quad \mathrm{Pe}_T = \frac{|\mathbf{b}|\, h_T}{2\, l\, \varepsilon}$$

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If $0 \le \delta|_T \le \min\left\{ \dfrac{\sigma_0}{2\,\|c\|^2_{0,\infty,T}}, \dfrac{h_T^2}{2\,\varepsilon\,\mu^2} \right\} \quad \forall\, T \in \mathscr{T}_h$,

then coercivity on V_h w.r.t.

$$|||v|||_{SUPG} = \left(|||v|||_G^2 + \|\delta^{1/2}\,\mathbf{b} \cdot \nabla v\|^2_{0,\Omega} \right)^{1/2}$$

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Drawbacks: non–symmetric, second–order derivatives, difficulties for non–steady problems

Local projection stabilizations

Becker, Braack (2001) Stokes

Becker, Braack (2004) transport, Navier–Stokes

Braack, Burman (2006) Oseen

Braack, Richter (2006, 2007) Stokes; Navier–Stokes; react. flows

Becker, Vexler (2007) conv.–diff.–react., optimal control

Lube, Rapin, Löwe (2007) Oseen

Ganesan, Tobiska (2007) conv.–diff.–react., Stokes, Oseen

Matthies, Skrzypacz, Tobiska (2007) Oseen, enrichment

Matthies, Skrzypacz, Tobiska (2008) conv.–diff.–react.

Knobloch, Lube (2009) conv.–diff.–react.

Knobloch, Tobiska (2009) conv.–diff.–react.

Braack (2008, 2009) Navier–Stokes; Oseen, optimal control

Braack, Lube (2009) review on LPS for incompressible flows

Assumptions

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sufficient condition: $b_M \cdot D_M \subset B_M$ with $b_M \in H_0^1(M) \cap C(\overline{M})$,

$b_M \geq 0, \quad b_M \neq 0, \quad b_M$ defined via the reference element

One–level approach

Matthies, Skrzypacz, Tobiska (2007)

$\mathscr{M}_h = \mathscr{T}_h$

examples of spaces: $D_M = P_{l-1}(M) \qquad \forall\, M \in \mathscr{M}_h\,,$

$$V_h = P_{l,\mathscr{T}_h} \;+\; \bigoplus_{M\in\mathscr{M}_h} b_M \cdot P_{l-1}(M)$$

or

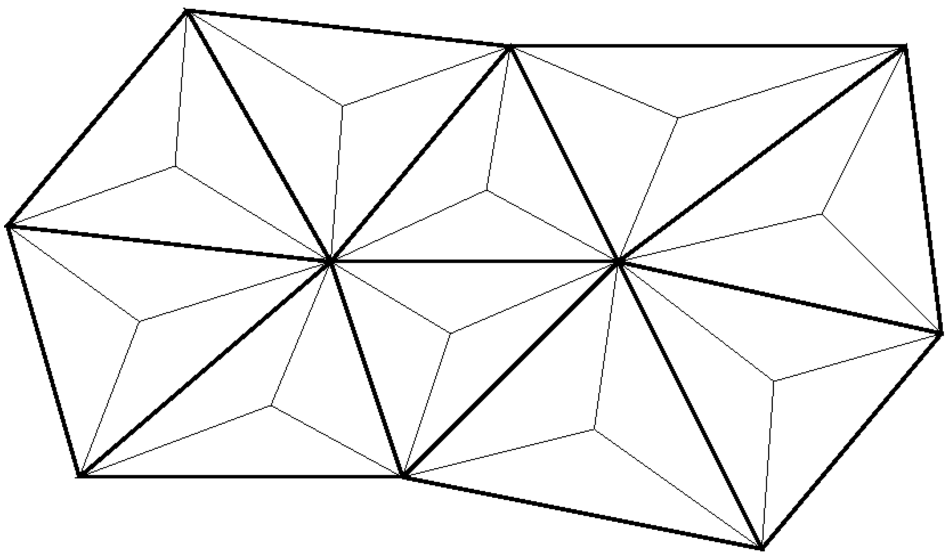
$$V_h = Q_{l,\mathscr{T}_h} \;+\; \bigoplus_{M\in\mathscr{M}_h} b_M \cdot Q_{l-1}(M)$$

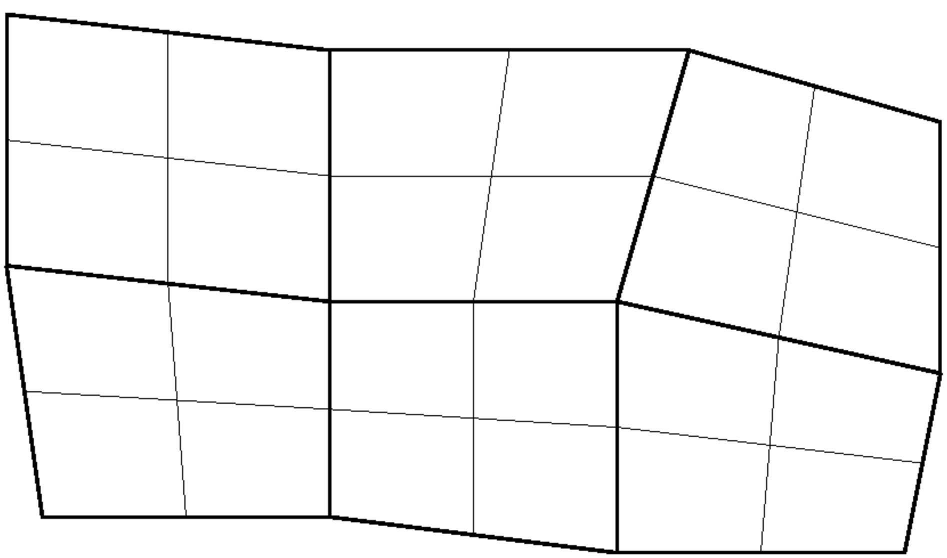
(mapped)

Two–level approach

Becker, Braack (2001)

$\mathscr{T}_h$ is obtained by a refinement of $\mathscr{M}_h$





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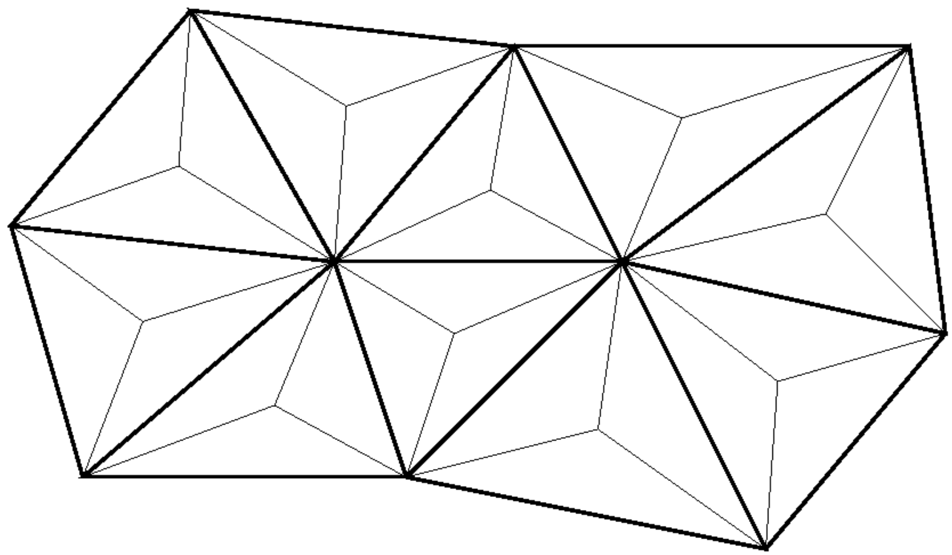

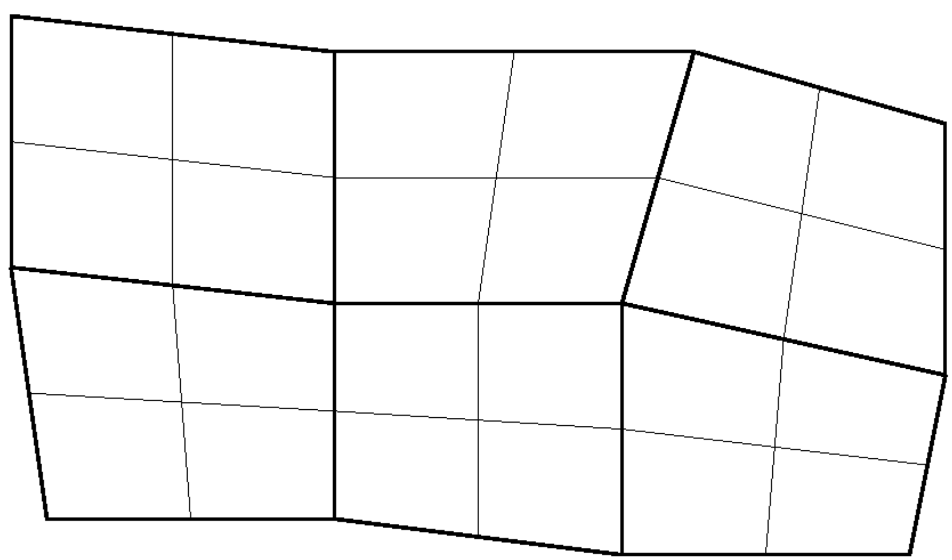

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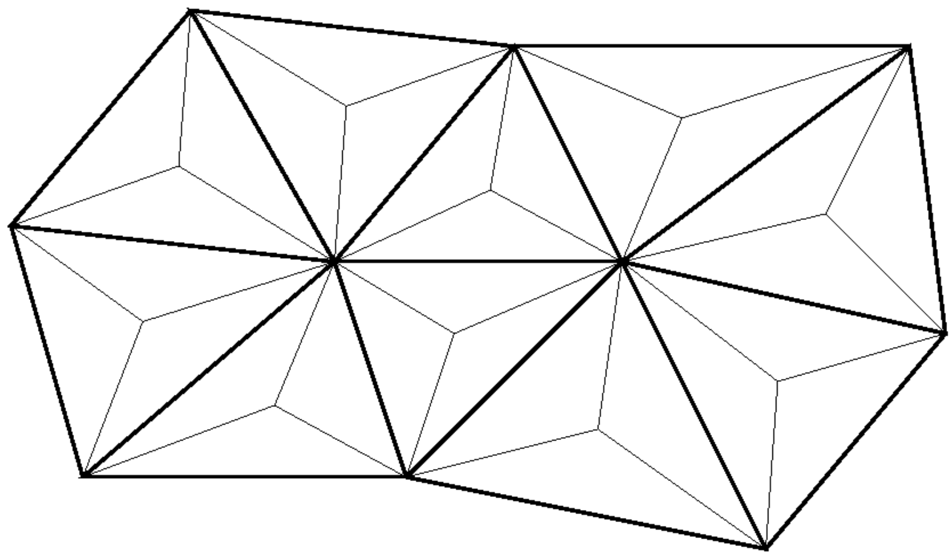

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can be viewed as one–level approach for simplicial meshes

Overlapping sets $M \in \mathscr{M}_h$

K. (2009)

Let any element of $\mathscr{T}_h$ have a vertex in Ω.

Let $x_1, \dots, x_{N_h}$ be the vertices of $\mathscr{T}_h$ lying in Ω.

Set
$$M_i = \operatorname{int} \bigcup_{T \in \mathscr{T}_h,\, x_i \in \overline{T}} \overline{T}\,, \qquad i = 1, \dots, N_h\,,$$

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cheaper and more robust than the previous approaches

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Find $u_h \in V_h$ such that

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K. (2009)

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$$\sup_{v_h \in V_h} \frac{a_h^{LP}(u_h, v_h)}{|||v_h|||_{LPSD}} \geq \beta\, |||u_h|||_{LPSD} \qquad \forall\, u_h \in V_h\,,$$

where β is a positive constant independent of h and ε.

Proof: Consider any $u_h \in V_h$ and $M \in \mathscr{M}_h$ $\Rightarrow$ $\exists\, z_M \in B_M$:

$$(z_M, q)_M = \tau_M\,(\mathbf{b}\cdot\nabla u_h, q)_M \qquad \forall\, q \in D_M\,,$$

$$\|z_M\|_{0,M} \leq \beta_{LP}^{-1}\,\tau_M\,\|\mathbf{b}\cdot\nabla u_h\|_{0,M}$$

$$\Rightarrow \quad (z_M, \mathbf{b}\cdot\nabla u_h)_M = \tau_M\,\|\mathbf{b}\cdot\nabla u_h\|_{0,M}^2 - \tau_M\,(\mathbf{b}\cdot\nabla u_h, \kappa_M(\mathbf{b}\cdot\nabla u_h))_M + (z_M, \kappa_M(\mathbf{b}\cdot\nabla u_h))_M$$

$$z_h = \sum_{M\in\mathscr{M}_h} z_M\,, \qquad |||z_h|||_{LPSD} \leq \hat{C}\,|||u_h|||_{LPSD}$$

$$a_h^{LP}(u_h, z_h) = \varepsilon\,(\nabla u_h, \nabla z_h) + (\mathbf{b}\cdot\nabla u_h, z_h) + (c\,u_h, z_h) + s_h(u_h, z_h)$$
$$\geq \frac{1}{2}\sum_{M\in\mathscr{M}_h} \tau_M\,\|\mathbf{b}\cdot\nabla u_h\|_{0,M}^2 - \bar{C}\,|||u_h|||_{LP}^2$$

Inf–sup condition

K. (2009)

$$\sup_{v_h \in V_h} \frac{a_h^{LP}(u_h, v_h)}{|||v_h|||_{LPSD}} \geq \beta\, |||u_h|||_{LPSD} \qquad \forall\, u_h \in V_h\,,$$

where β is a positive constant independent of h and ε.

Proof: Consider any $u_h \in V_h \quad \Rightarrow \quad \exists\, z_h \in V_h :$

$$|||z_h|||_{LPSD} \leq \hat{C}\, |||u_h|||_{LPSD},$$

$$a_h^{LP}(u_h, z_h) \geq \tfrac{1}{2} \sum_{M \in \mathscr{M}_h} \tau_M\, \|\mathbf{b} \cdot \nabla u_h\|_{0,M}^2 - \bar{C}\, |||u_h|||_{LP}^2$$

Inf–sup condition

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$$\sup_{v_h \in V_h} \frac{a_h^{LP}(u_h, v_h)}{|||v_h|||_{LPSD}} \geq \beta\, |||u_h|||_{LPSD} \qquad \forall\, u_h \in V_h\,,$$

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Proof: Consider any $u_h \in V_h \quad \Rightarrow \quad \exists\, z_h \in V_h :$

$$|||z_h|||_{LPSD} \leq \hat{C}\, |||u_h|||_{LPSD},$$

$$a_h^{LP}(u_h, z_h) \geq \frac{1}{2} \sum_{M \in \mathscr{M}_h} \tau_M \, \|\mathbf{b} \cdot \nabla u_h\|_{0,M}^2 - \bar{C}\, |||u_h|||_{LP}^2$$

$$a_h^{LP}(u_h, u_h) = |||u_h|||_{LP}^2$$

Inf–sup condition

K. (2009)

$$\sup_{v_h \in V_h} \frac{a_h^{LP}(u_h, v_h)}{|||v_h|||_{LPSD}} \geq \beta \, |||u_h|||_{LPSD} \qquad \forall \, u_h \in V_h \, ,$$

where β is a positive constant independent of h and ε.

Proof: Consider any $u_h \in V_h \quad \Rightarrow \quad \exists \, z_h \in V_h :$

$$|||z_h|||_{LPSD} \leq \hat{C} \, |||u_h|||_{LPSD},$$

$$a_h^{LP}(u_h, z_h) \geq \frac{1}{2} \sum_{M \in \mathscr{M}_h} \tau_M \, \|\mathbf{b} \cdot \nabla u_h\|_{0,M}^2 - \bar{C} \, |||u_h|||_{LP}^2$$

$$a_h^{LP}(u_h, u_h) = |||u_h|||_{LP}^2$$

$\Rightarrow \quad v_h := 2\, z_h + (1 + 2\,\bar{C})\, u_h$ satisfies

$a_h^{LP}(u_h, v_h) \geq |||u_h|||_{LPSD}^2$ and $|||u_h|||_{LPSD} \geq \beta \, |||v_h|||_{LPSD}$

Inf–sup condition

K. (2009)

$$\sup_{v_h \in V_h} \frac{a_h^{LP}(u_h, v_h)}{|||v_h|||_{LPSD}} \geq \beta\, |||u_h|||_{LPSD} \qquad \forall\, u_h \in V_h\,,$$

where β is a positive constant independent of h and ε.

General error estimate

$$\beta\, |||u - u_h|||_{LPSD} \leq \inf_{w_h \in V_h} \left\{ \beta\, |||u - w_h|||_{LPSD} + \sup_{v_h \in V_h} \frac{a_h^{LP}(u - w_h, v_h)}{|||v_h|||_{LPSD}} \right\} + \sup_{v_h \in V_h} \frac{s_h(u, v_h)}{|||v_h|||_{LPSD}}$$

Inf–sup condition

K. (2009)

$$\sup_{v_h \in V_h} \frac{a_h^{LP}(u_h, v_h)}{|||v_h|||_{LPSD}} \geq \beta\, |||u_h|||_{LPSD} \qquad \forall\, u_h \in V_h\,,$$

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General error estimate

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$$+ \sup_{v_h \in V_h} \frac{s_h(u, v_h)}{|||v_h|||_{LPSD}}$$

For an optimal estimate of the consistency error, it is essential that we use $\mathbf{b}_M$ instead of $\mathbf{b}$ in s_h.

Estimate of the consistency error

Recall that $$s_h(u,v) = \sum_{M \in \mathscr{M}_h} \tau_M \, s_M(u,v)$$

with $$s_M(u,v) = (\kappa_M(\mathbf{b}_M \cdot \nabla u), \kappa_M(\mathbf{b}_M \cdot \nabla v))_M$$

Estimate of the consistency error

Recall that $$s_h(u,v) = \sum_{M\in\mathscr{M}_h} \tau_M\, s_M(u,v)$$

with $$s_M(u,v) = (\kappa_M(\mathbf{b}_M\cdot\nabla u), \kappa_M(\mathbf{b}_M\cdot\nabla v))_M$$

$\Rightarrow$ $$s_h(u,v) \le \sqrt{s_h(u,u)}\,\sqrt{s_h(v,v)} \le \sqrt{s_h(u,u)}\,|||v|||_{LPSD}$$

Estimate of the consistency error

Recall that $s_h(u,v) = \sum_{M \in \mathcal{M}_h} \tau_M \, s_M(u,v)$

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$\Rightarrow \quad s_h(u,v) \leq \sqrt{s_h(u,u)}\,\sqrt{s_h(v,v)} \leq \sqrt{s_h(u,u)}\,|||v|||_{LPSD}$

$\Rightarrow \quad s_M(u,u) = \|\kappa_M(\mathbf{b}_M \cdot \nabla u)\|_M^2 = \|\mathbf{b}_M \cdot \kappa_M(\nabla u \qquad)\|_M^2$

Estimate of the consistency error

Recall that $$s_h(u,v) = \sum_{M\in\mathscr{M}_h} \tau_M\, s_M(u,v)$$

with $$s_M(u,v) = (\kappa_M(\mathbf{b}_M\cdot\nabla u), \kappa_M(\mathbf{b}_M\cdot\nabla v))_M$$

$$\Rightarrow \quad s_h(u,v) \le \sqrt{s_h(u,u)}\,\sqrt{s_h(v,v)} \le \sqrt{s_h(u,u)}\,|||v|||_{LPSD}$$

$$\Rightarrow \quad s_M(u,u) = \|\kappa_M(\mathbf{b}_M\cdot\nabla u)\|_M^2 = \|\mathbf{b}_M\cdot\kappa_M(\nabla u - \mathbf{q}_M)\|_M^2$$

$$\forall\ \mathbf{q}_M \in [D_M]^d$$

Estimate of the consistency error

Recall that $$s_h(u,v) = \sum_{M\in\mathcal{M}_h} \tau_M\, s_M(u,v)$$

with $$s_M(u,v) = (\kappa_M(\mathbf{b}_M\cdot\nabla u), \kappa_M(\mathbf{b}_M\cdot\nabla v))_M$$

$$\Rightarrow \quad s_h(u,v) \le \sqrt{s_h(u,u)}\,\sqrt{s_h(v,v)} \le \sqrt{s_h(u,u)}\,|||v|||_{LPSD}$$

$$\Rightarrow \quad s_M(u,u) = \|\kappa_M(\mathbf{b}_M\cdot\nabla u)\|_M^2 = \|\mathbf{b}_M\cdot\kappa_M(\nabla u - \mathbf{q}_M)\|_M^2$$

$$\forall\, \mathbf{q}_M \in [D_M]^d$$

$$\Rightarrow \quad \tau_M\, s_M(u,u) \le \tau_M\, \|\mathbf{b}\|_{0,\infty,M}^2\, \|\nabla u - \mathbf{q}_M\|_{0,M}^2$$

Estimate of the consistency error

Recall that $$s_h(u,v) = \sum_{M \in \mathcal{M}_h} \tau_M \, s_M(u,v)$$

with $$s_M(u,v) = (\kappa_M(\mathbf{b}_M \cdot \nabla u), \kappa_M(\mathbf{b}_M \cdot \nabla v))_M$$

$$\Rightarrow \quad s_h(u,v) \leq \sqrt{s_h(u,u)}\,\sqrt{s_h(v,v)} \leq \sqrt{s_h(u,u)}\,|||v|||_{LPSD}$$

$$\Rightarrow \quad s_M(u,u) = \|\kappa_M(\mathbf{b}_M \cdot \nabla u)\|_M^2 = \|\mathbf{b}_M \cdot \kappa_M(\nabla u - \mathbf{q}_M)\|_M^2$$

$$\forall\, \mathbf{q}_M \in [D_M]^d$$

$$\Rightarrow \quad \tau_M \, s_M(u,u) \leq \tau_M \, \|\mathbf{b}\|_{0,\infty,M}^2 \, \|\nabla u - \mathbf{q}_M\|_{0,M}^2$$

$$\tau_M \, \|\mathbf{b}\|_{0,\infty,M} \leq \tau_0 \, h_M$$

Estimate of the consistency error

Recall that $$s_h(u,v) = \sum_{M\in\mathscr{M}_h} \tau_M\, s_M(u,v)$$

with $$s_M(u,v) = (\kappa_M(\mathbf{b}_M\cdot\nabla u), \kappa_M(\mathbf{b}_M\cdot\nabla v))_M$$

$$\Rightarrow \quad s_h(u,v) \le \sqrt{s_h(u,u)}\,\sqrt{s_h(v,v)} \le \sqrt{s_h(u,u)}\;|||v|||_{LPSD}$$

$$\Rightarrow \quad s_M(u,u) = \|\kappa_M(\mathbf{b}_M\cdot\nabla u)\|_M^2 = \|\mathbf{b}_M\cdot\kappa_M(\nabla u - \mathbf{q}_M)\|_M^2$$

$$\forall\ \mathbf{q}_M \in [D_M]^d$$

$$\Rightarrow \quad \tau_M\, s_M(u,u) \le \tau_M\, \|\mathbf{b}\|_{0,\infty,M}^2\, \|\nabla u - \mathbf{q}_M\|_{0,M}^2$$

$$\tau_M\, \|\mathbf{b}\|_{0,\infty,M} \le \tau_0\, h_M$$

Consequently,

$$\sup_{v\in V_h} \frac{s_h(u,v)}{|||v|||_{LPSD}} \le C\, h^{1/2}\, \|\mathbf{b}\|_{0,\infty,\Omega}^{1/2} \left(\sum_{M\in\mathscr{M}_h} \inf_{\mathbf{q}_M\in[D_M]^d} \|\nabla u - \mathbf{q}_M\|_{0,M}^2 \right)^{1/2}$$

Error estimate

Let $\exists\ i_h \in \mathscr{L}(H^2(\Omega)\cap H_0^1(\Omega), V_h)$ and

$j_M \in \mathscr{L}(H^1(M), D_M)$, $M \in \mathscr{M}_h$ such that

$$\left(\sum_{M\in\mathscr{M}_h} \{|v-i_h v|_{1,M}^2 + h_M^{-2}\,\|v-i_h v\|_{0,M}^2\}\right)^{1/2} \leq C\,h^k\,|v|_{k+1,\Omega}$$

$$\forall\, v \in H^{k+1}(\Omega),\ k=1,\dots,l,$$

$$\|q-j_M q\|_{0,M} \leq C\,h_M^k\,|q|_{k,M} \qquad \forall\, q \in H^k(M),\ M\in\mathscr{M}_h,\ k=1,\dots,l\,.$$

Error estimate

Let $\exists\ i_h \in \mathscr{L}(H^2(\Omega) \cap H^1_0(\Omega), V_h)$ and

$j_M \in \mathscr{L}(H^1(M), D_M)$, $M \in \mathscr{M}_h$ such that

$$\left(\sum_{M \in \mathscr{M}_h} \{ |v - i_h v|^2_{1,M} + h_M^{-2} \, \| v - i_h v \|^2_{0,M} \} \right)^{1/2} \leq C\, h^k \, |v|_{k+1,\Omega}$$

$$\forall\ v \in H^{k+1}(\Omega),\ k = 1, \dots, l,$$

$$\| q - j_M q \|_{0,M} \leq C\, h_M^k \, |q|_{k,M} \qquad \forall\ q \in H^k(M),\ M \in \mathscr{M}_h,\ k = 1, \dots, l\,.$$

Let $u \in H^{k+1}(\Omega)$ for some $k \in \{1, \dots, l\}$.

Error estimate

Let $\exists\ i_h \in \mathscr{L}(H^2(\Omega)\cap H_0^1(\Omega), V_h)$ and

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$$\forall\, v \in H^{k+1}(\Omega),\ k = 1,\dots,l,$$

$$\|q - j_M q\|_{0,M} \le C\,h_M^k\,|q|_{k,M} \quad \forall\, q \in H^k(M),\ M \in \mathscr{M}_h,\ k = 1,\dots,l\,.$$

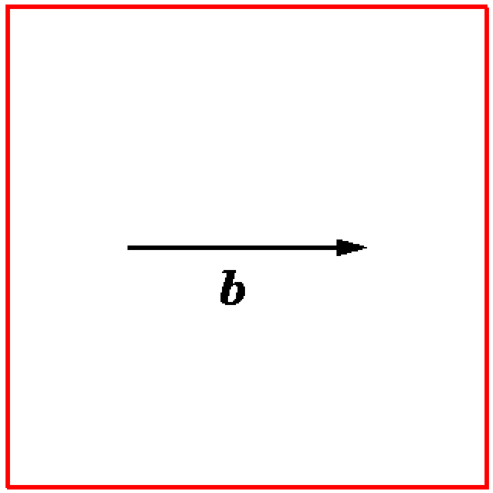
Let $u \in H^{k+1}(\Omega)$ for some $k \in \{1,\dots,l\}$. Then

$$|||u - u_h|||_{LPSD} \le C\,(\varepsilon + h\,\|\mathbf{b}\|_{0,\infty,\Omega} + h^2\,\|\sigma\|_{0,\infty,\Omega})^{1/2}\,h^k\,|u|_{k+1,\Omega}\,,$$

where the constant C is independent of h and ε.

Example (convection with a constant nonzero source term)

$u = 0$

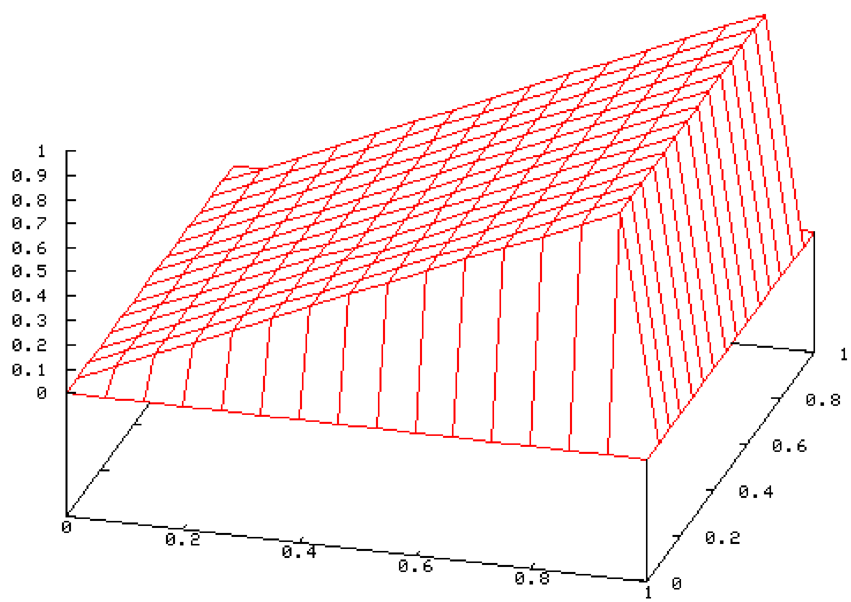


$$\varepsilon = 10^{-8}$$

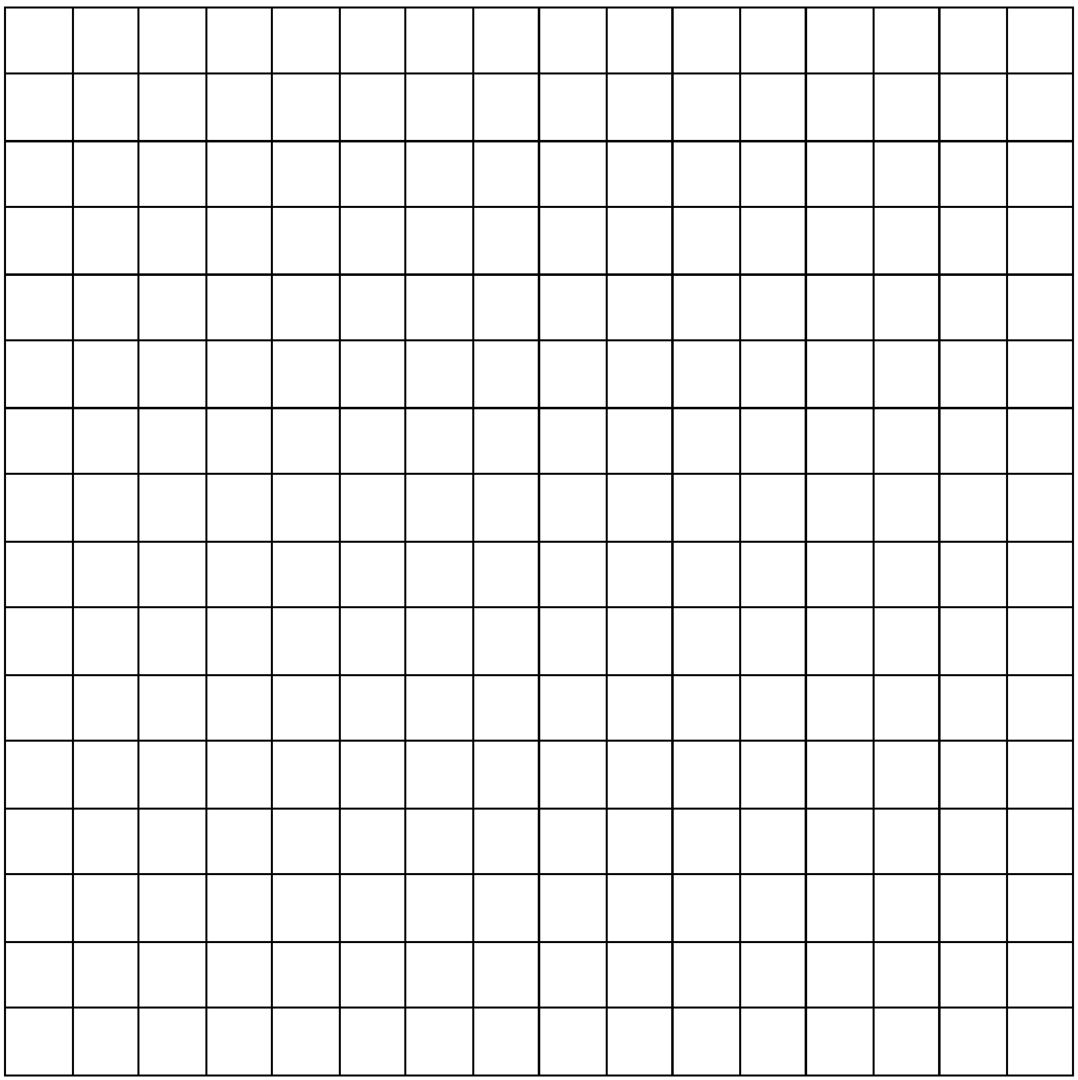
$$|\mathbf{b}| = 1$$

$$c = 0$$

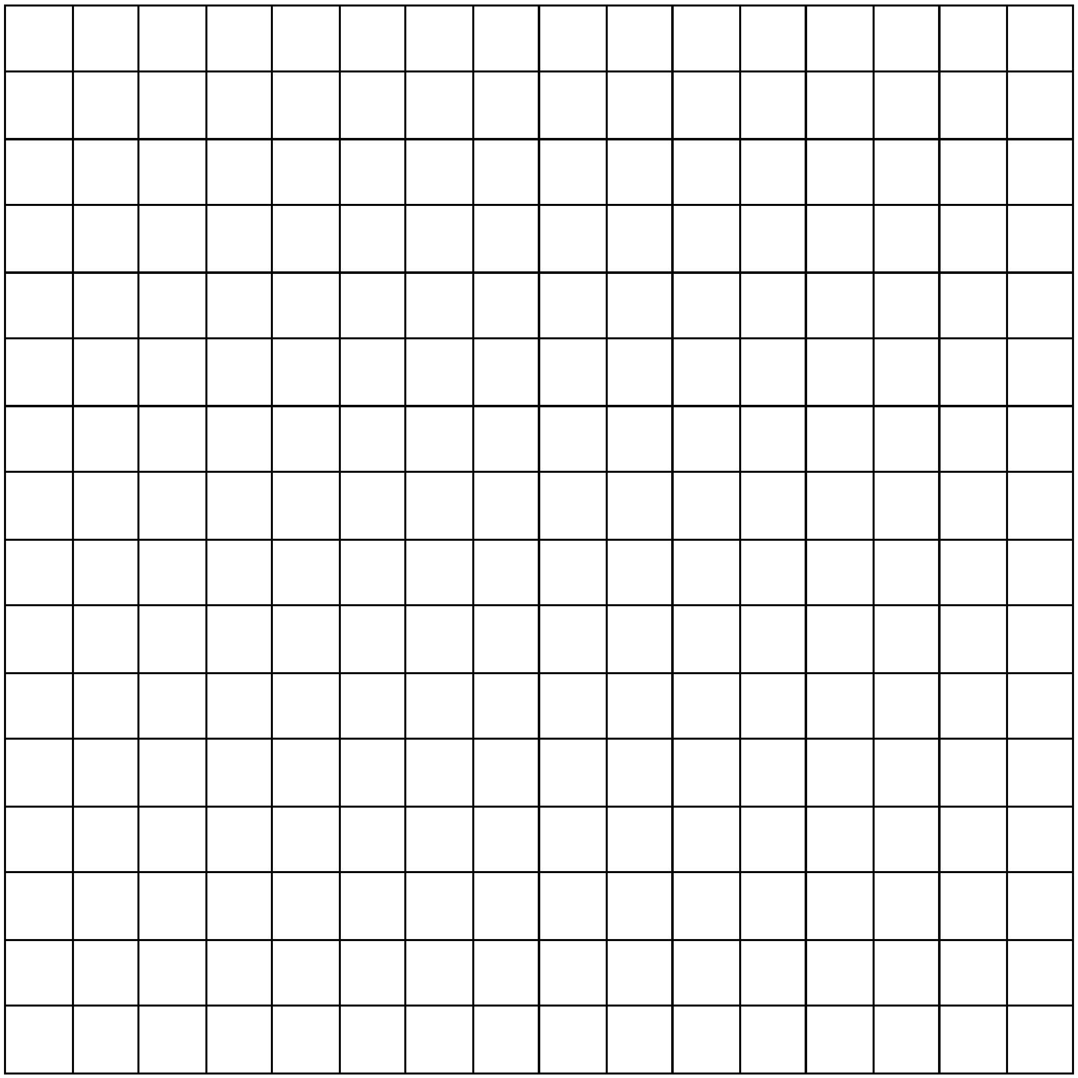
$$f = 1$$



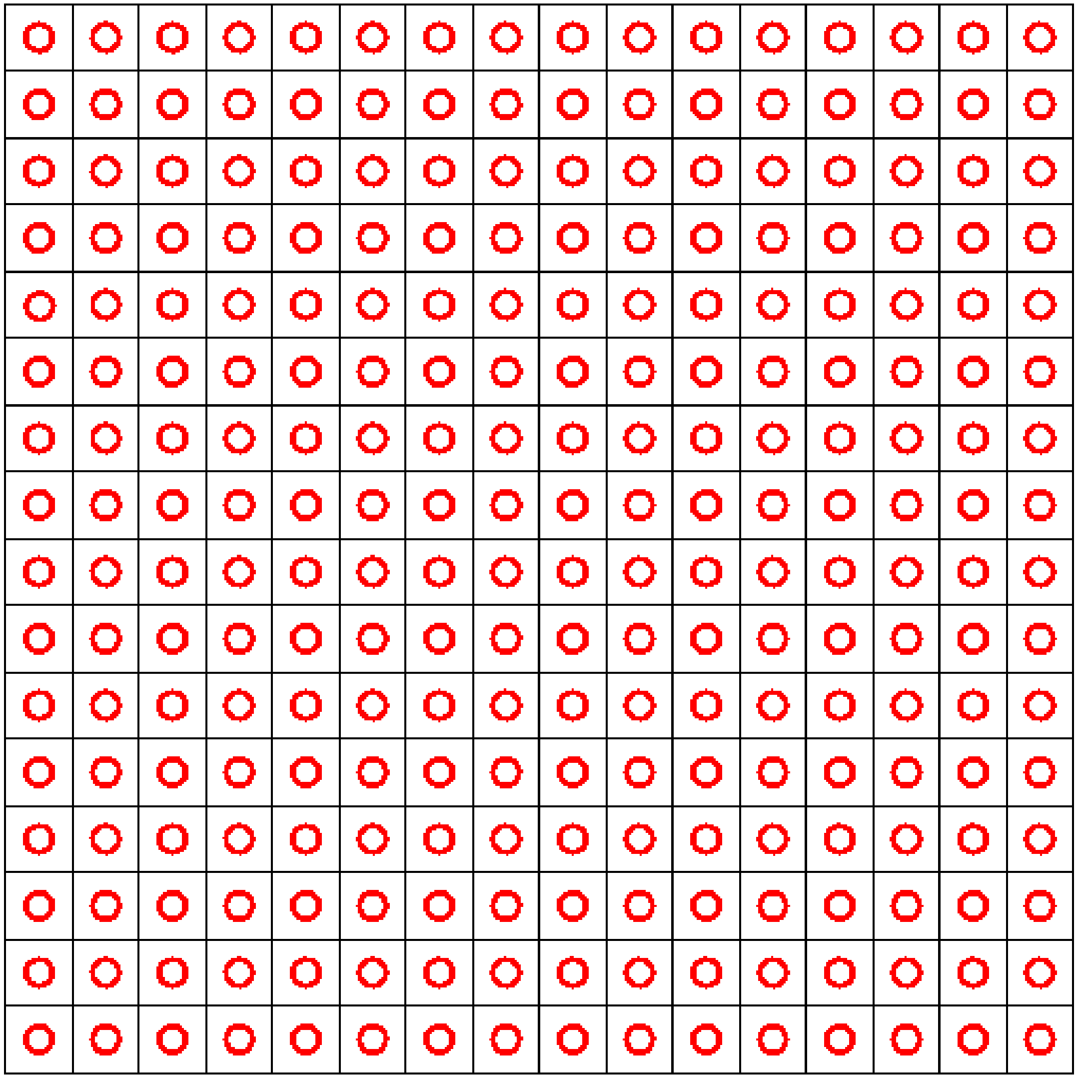
Uniform triangulation (16×16 squares)



LPS discretizations: $V_h \ \ldots \ Q_2, \quad D_M = P_1(M)$

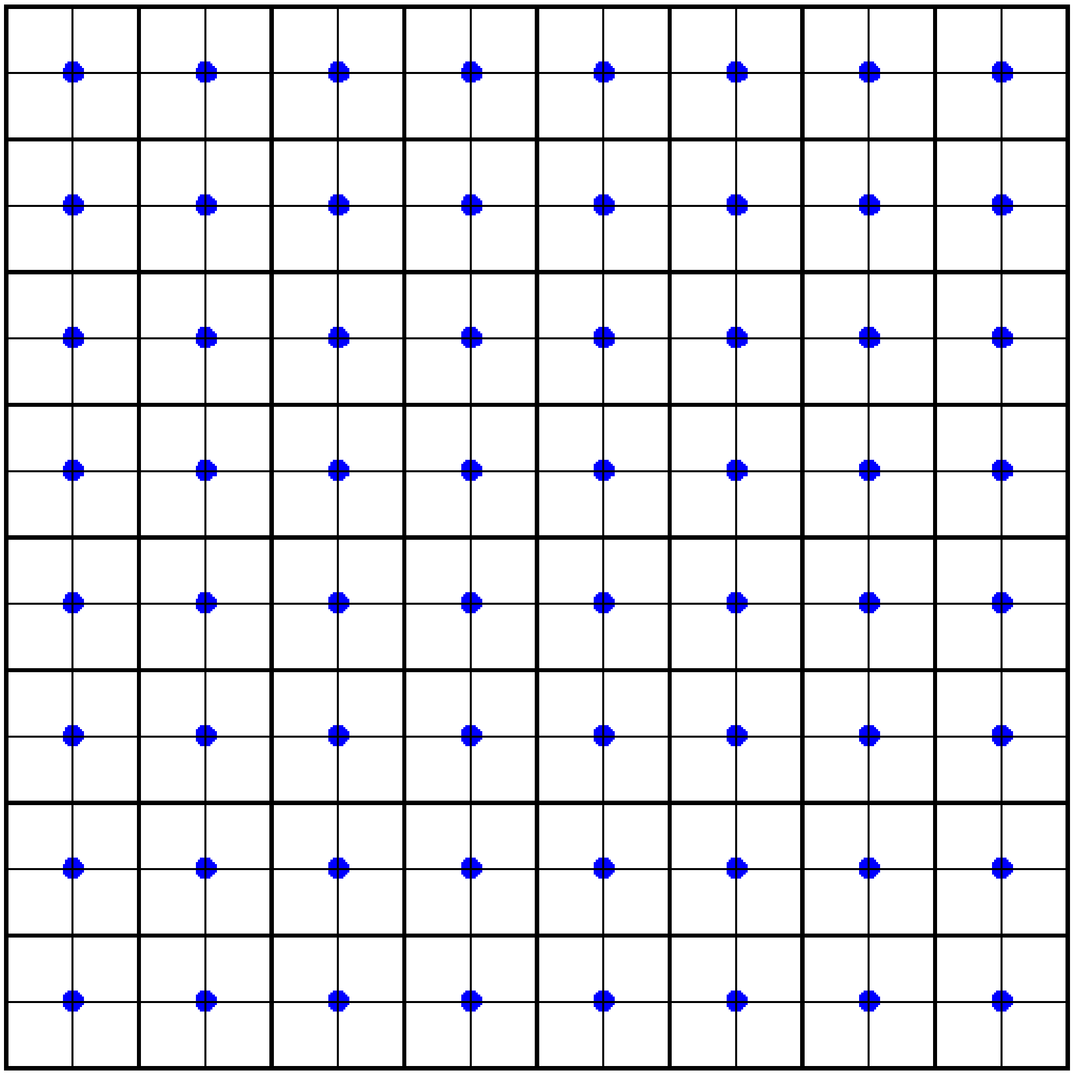


LPS discretizations: $V_h \ldots Q_2, \quad D_M = P_1(M)$



one–level LPS

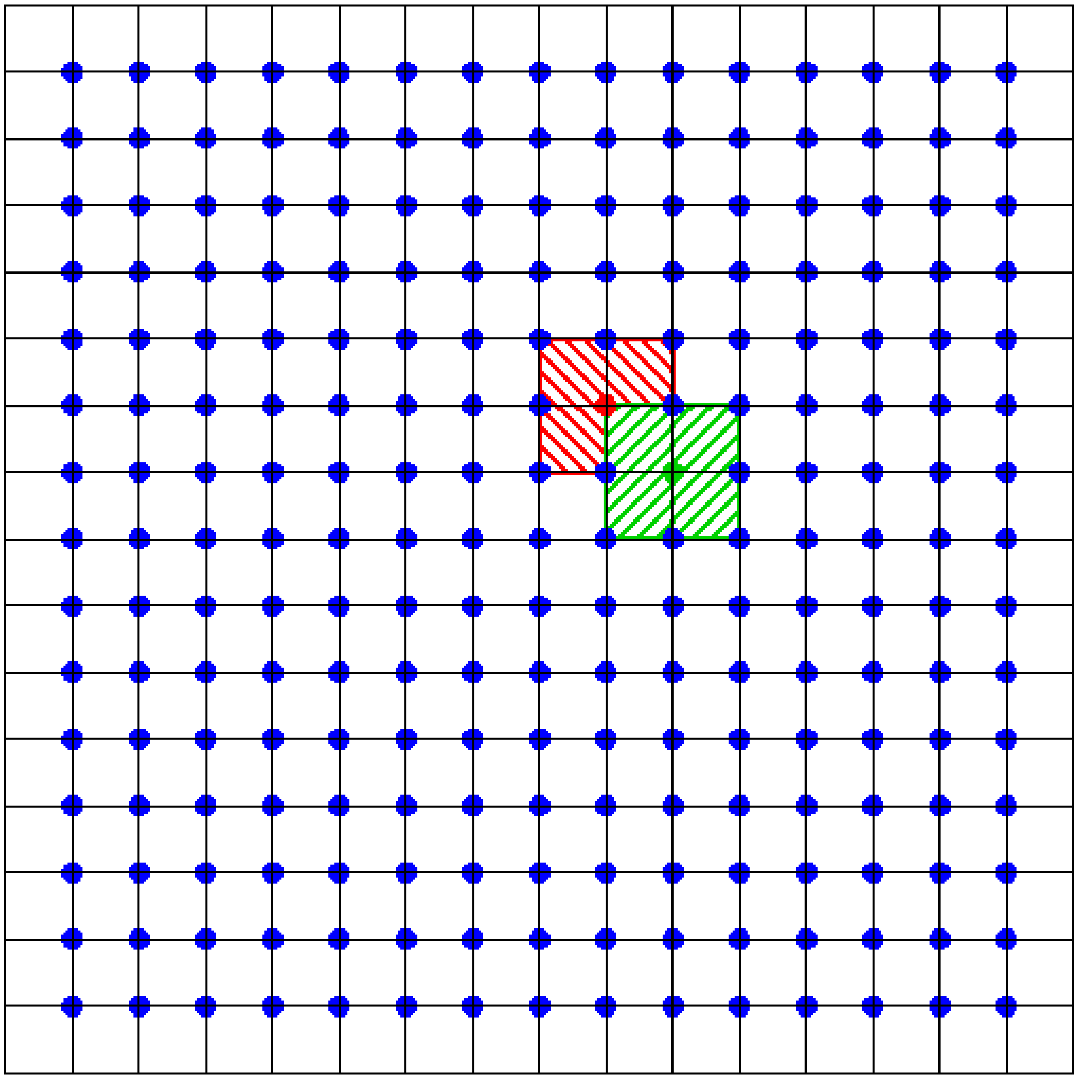
LPS discretizations: $V_h \ \ldots \ Q_2, \quad D_M = P_1(M)$



one–level LPS

two–level LPS

LPS discretizations: $V_h \ldots Q_2, \quad D_M = P_1(M)$

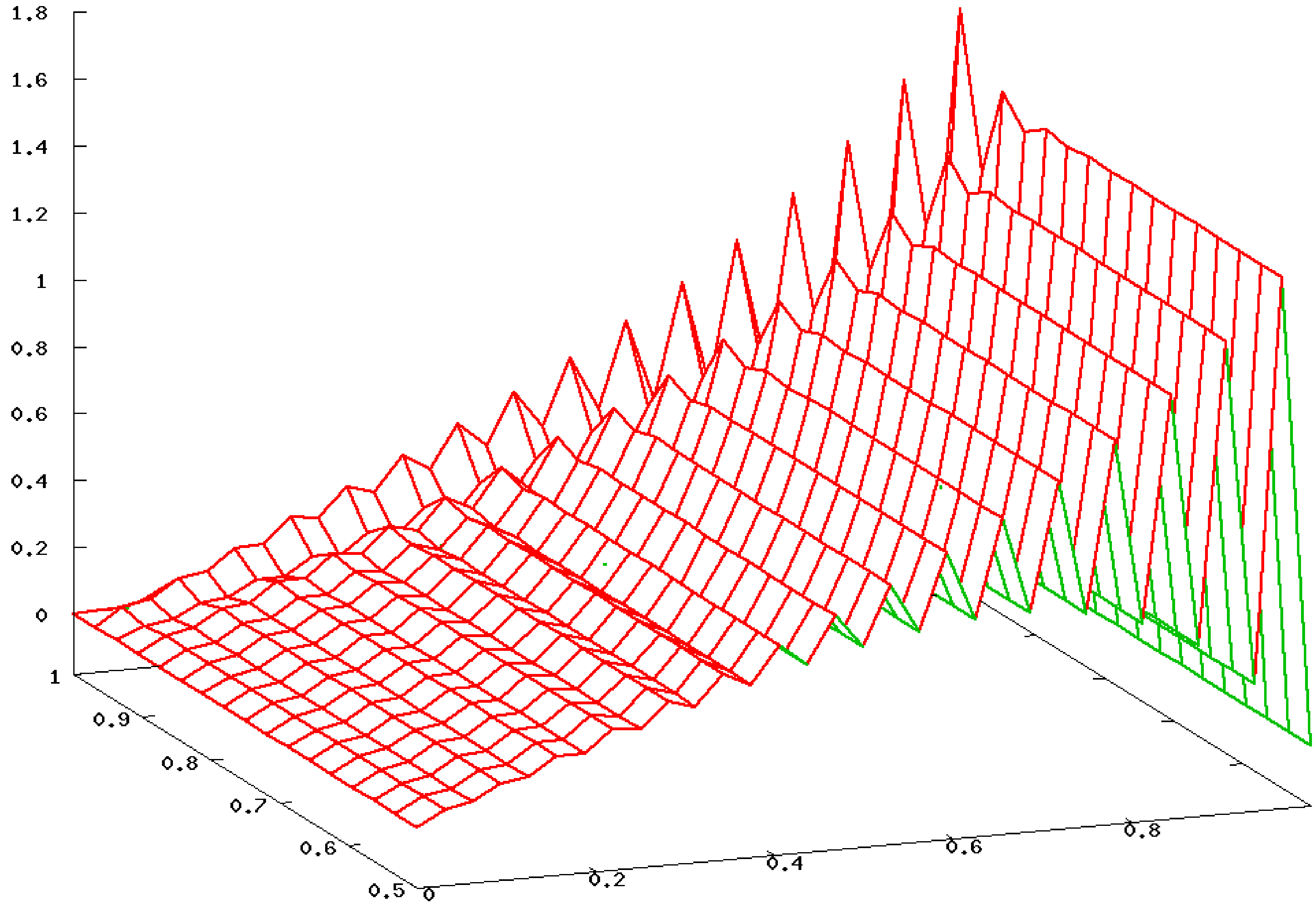


one–level LPS

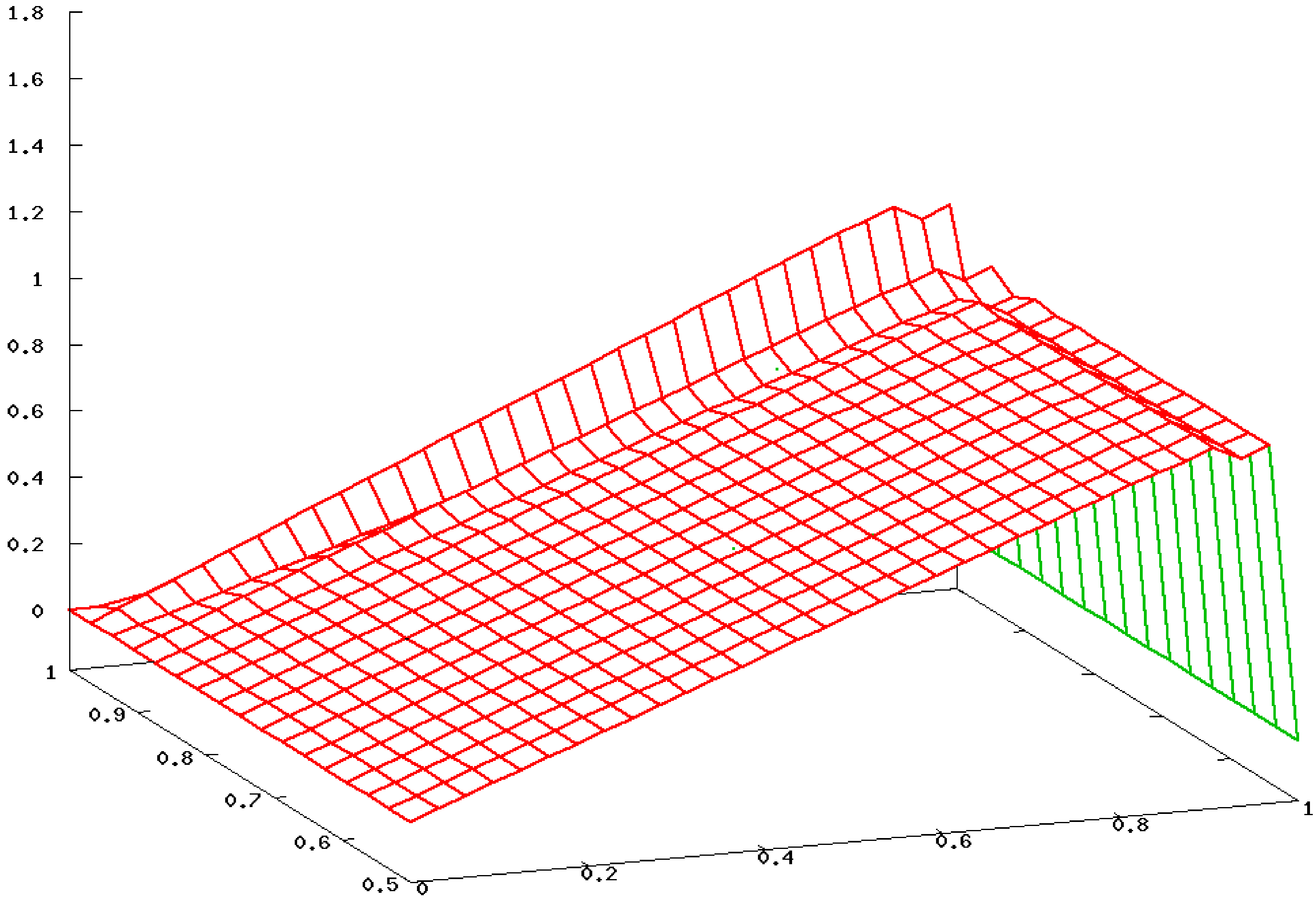
two–level LPS

overlapping LPS

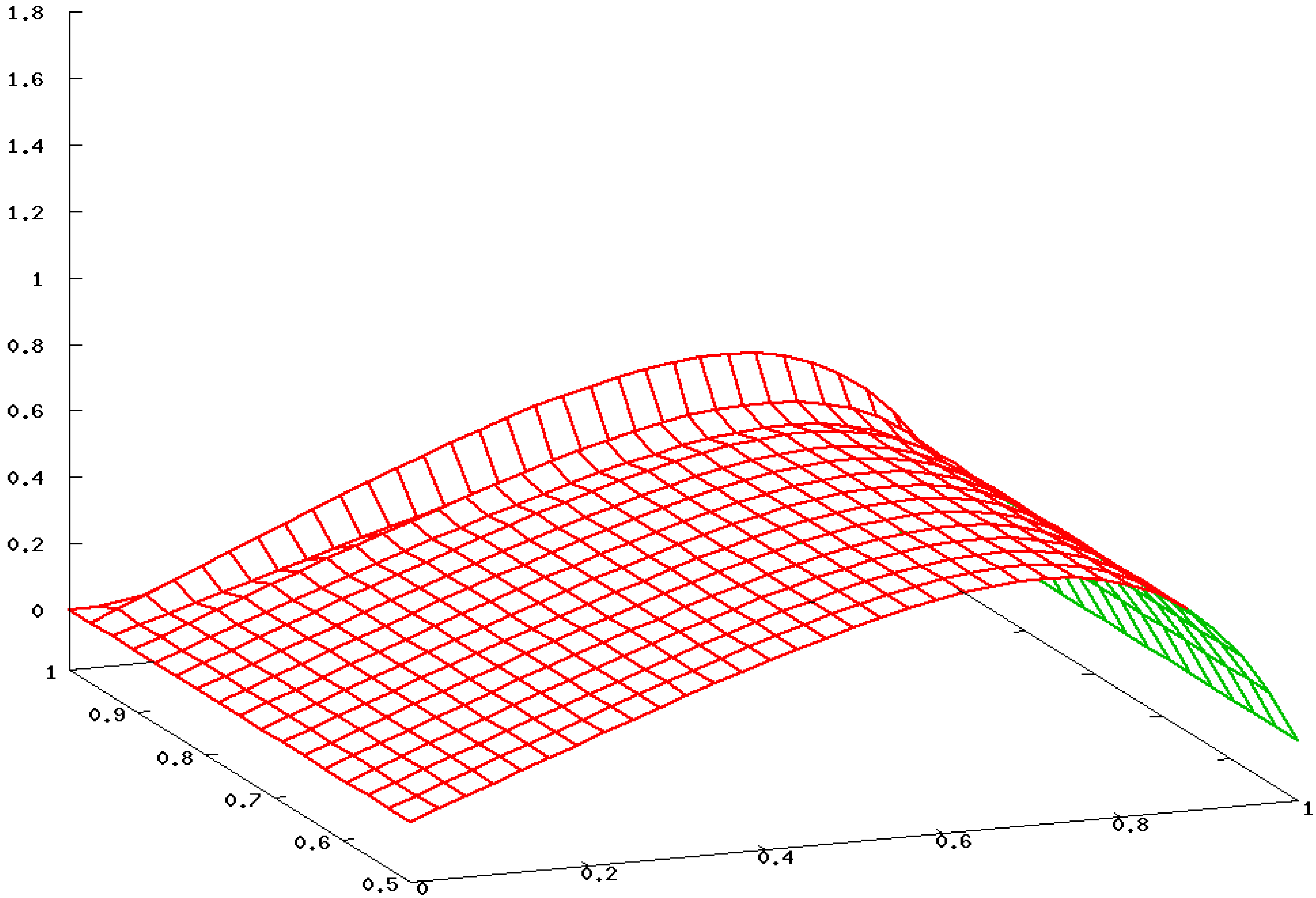
SUPG method, $\delta \times 0.1$



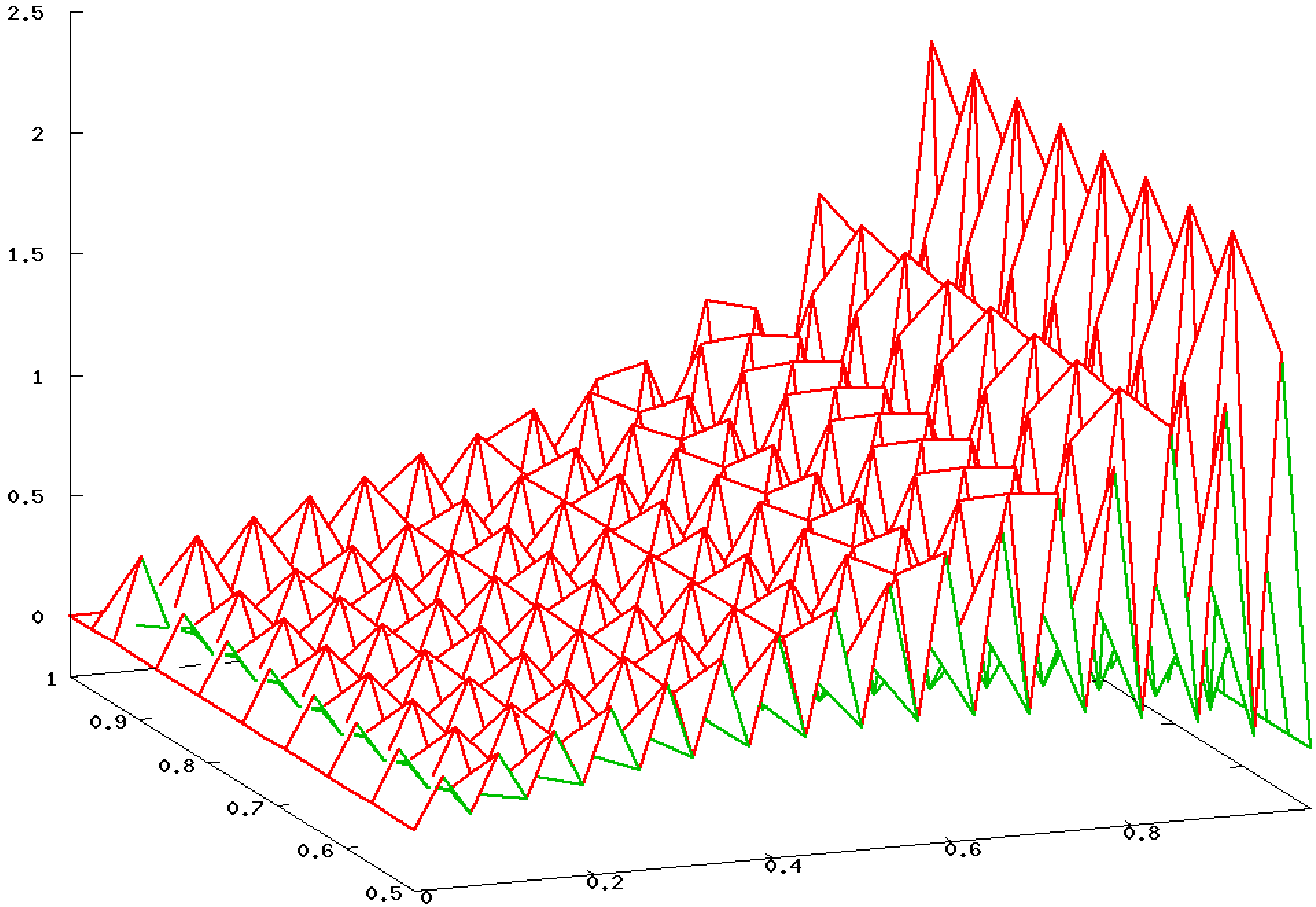
SUPG method, $\delta \times 1$



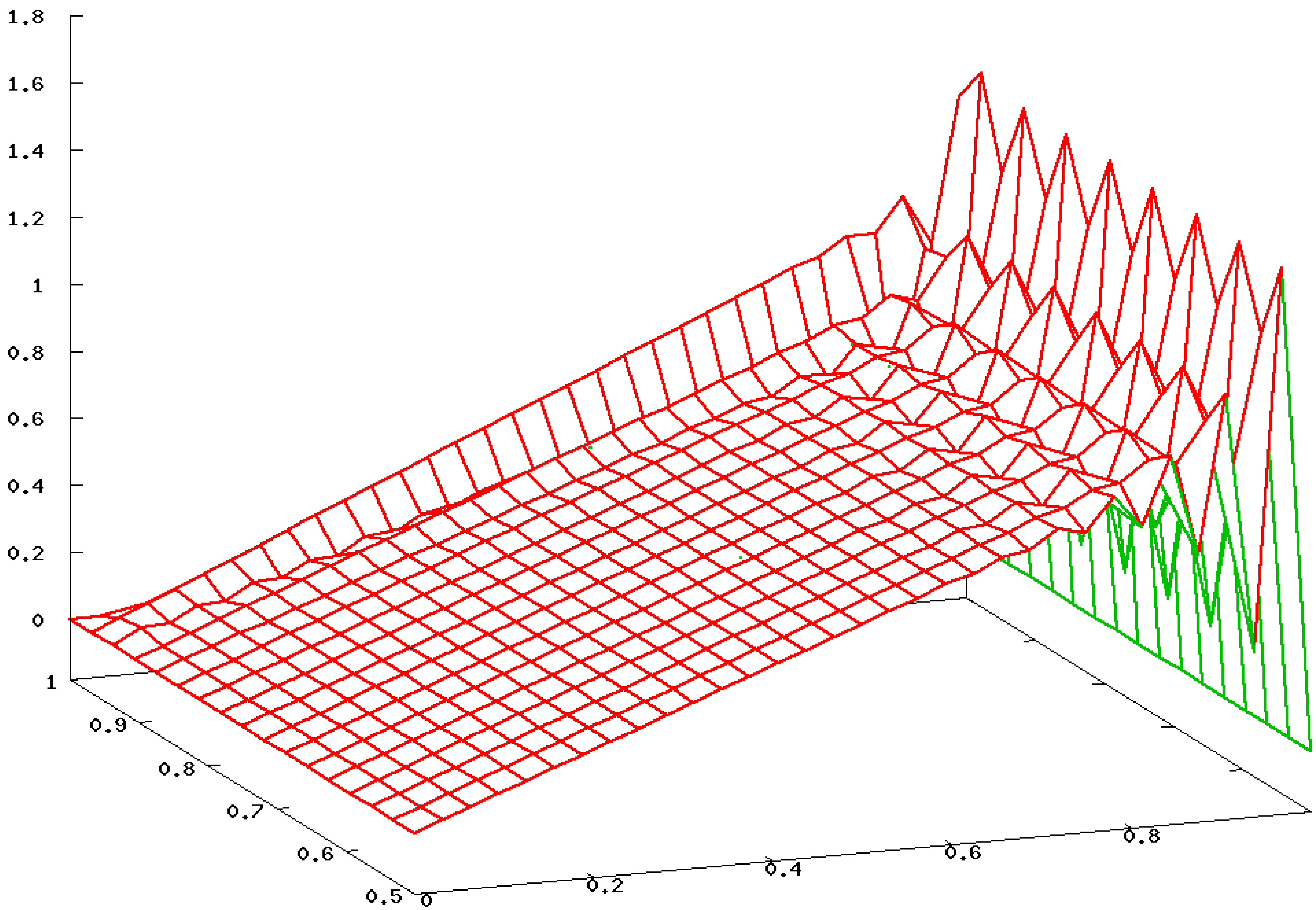
SUPG method, $\delta \times 10$



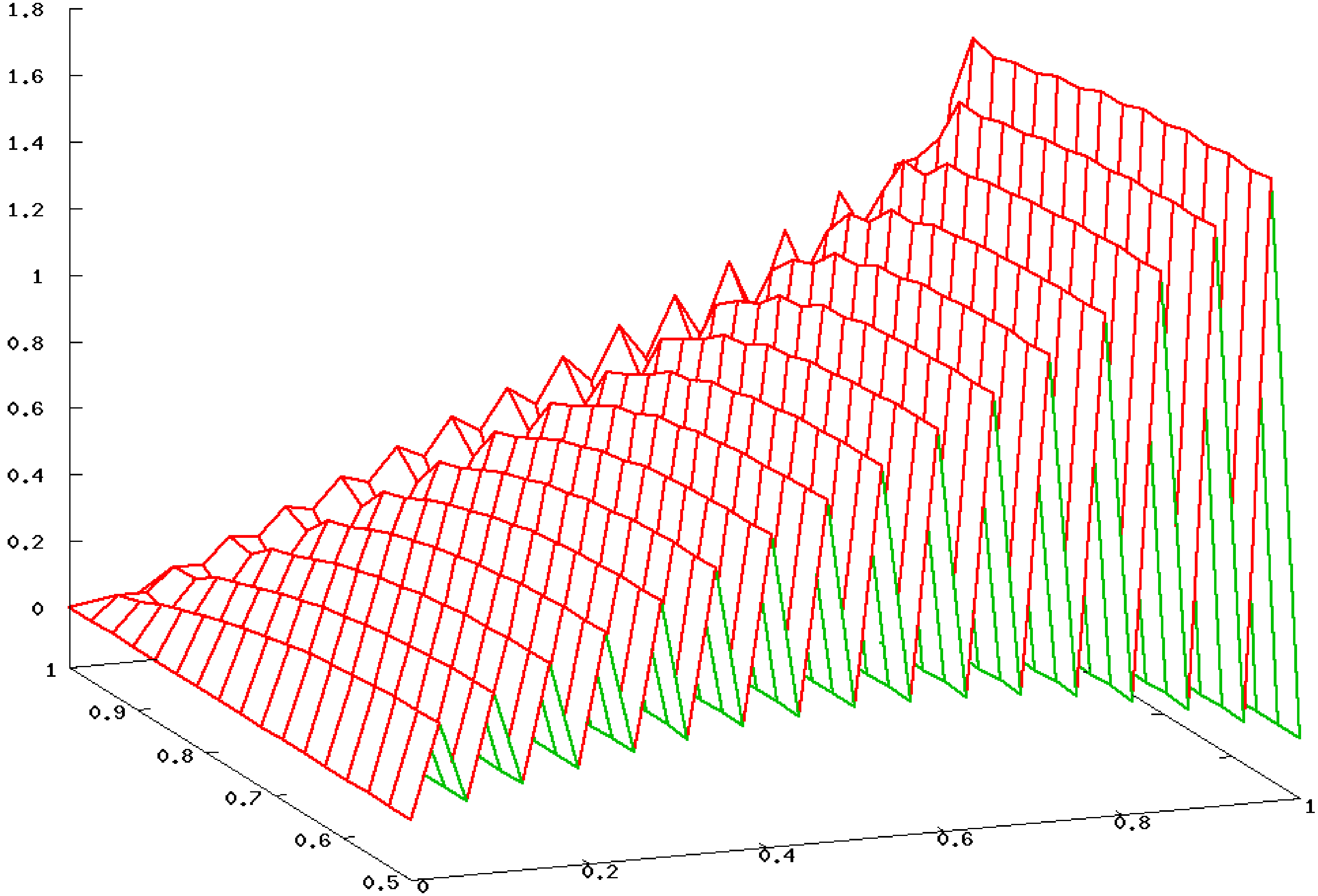
One–level LPS, $\tau_0 = 0.01$



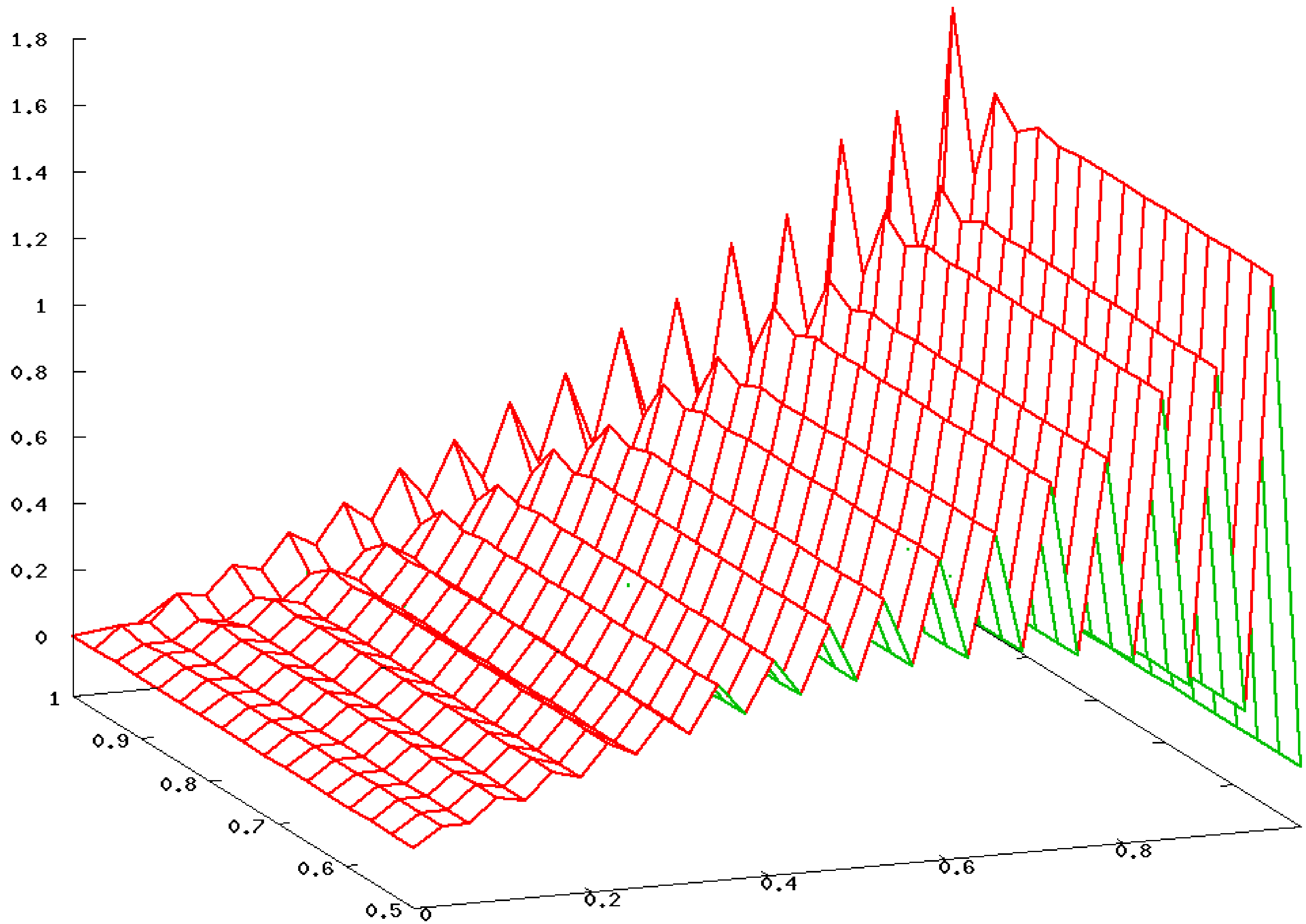
One–level LPS, $\tau_0 = 0.1$



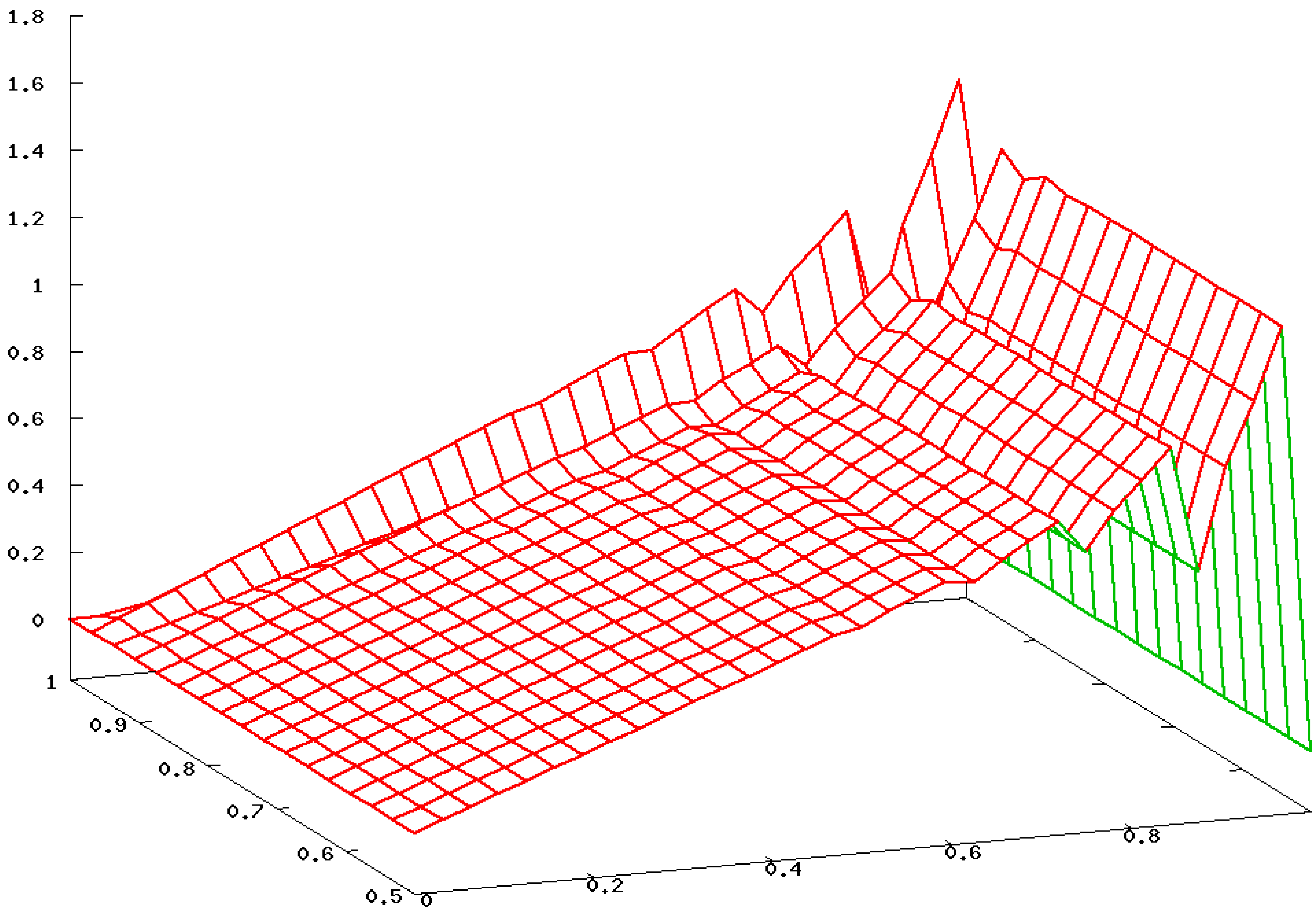
One–level LPS, $\tau_0 = 1$



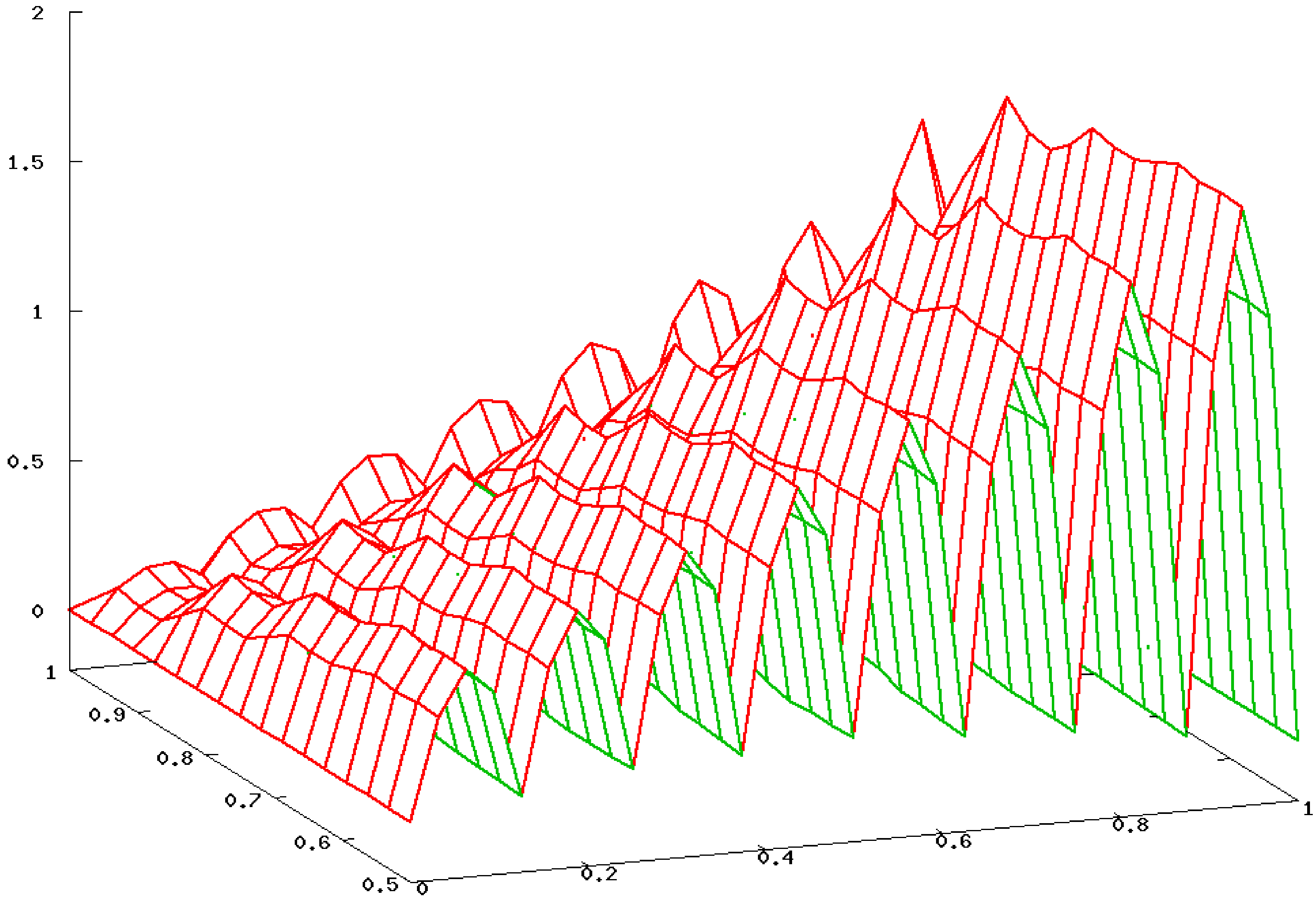
Two–level LPS, $\tau_0 = 0.01$



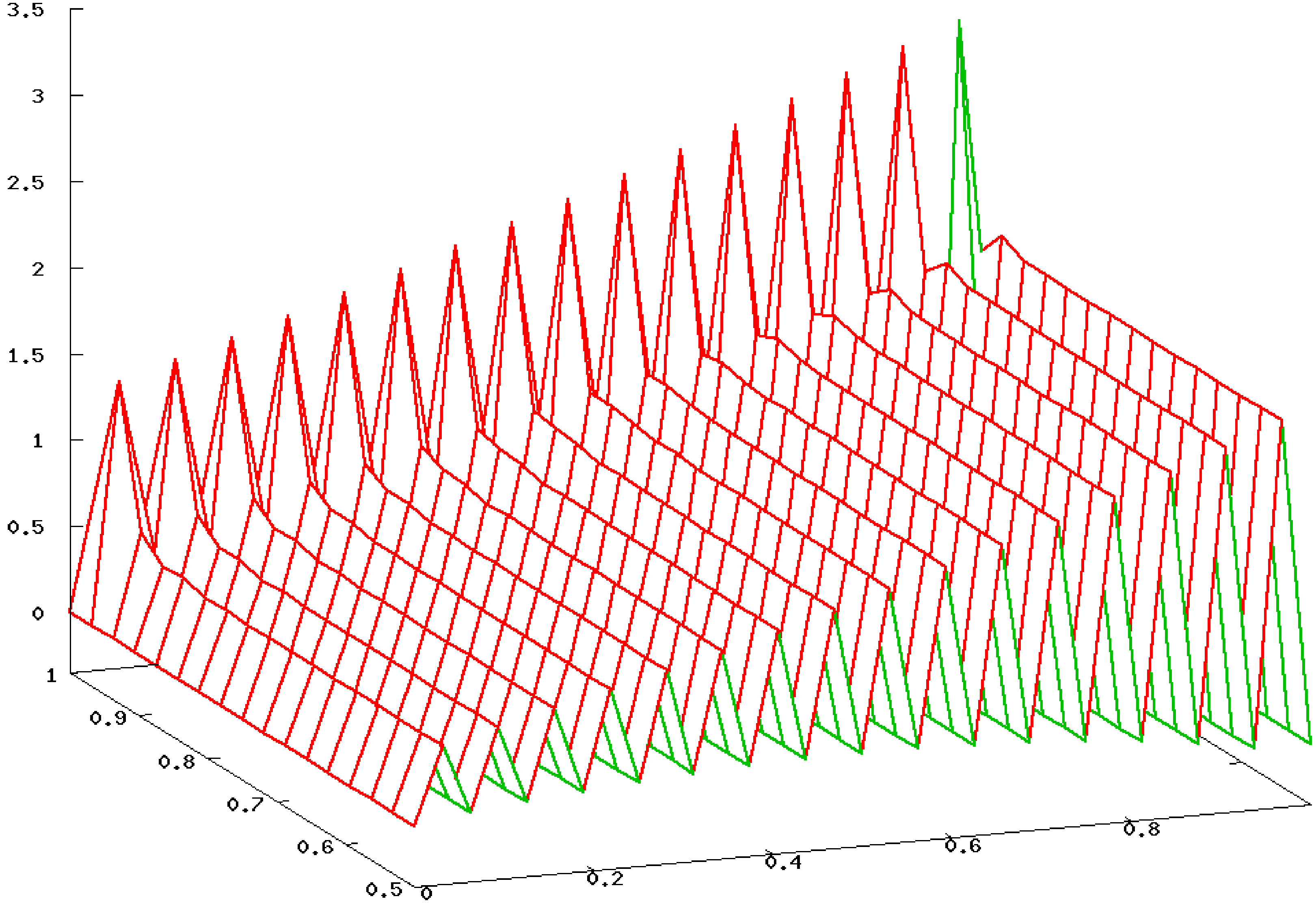
Two–level LPS, $\tau_0 = 0.1$



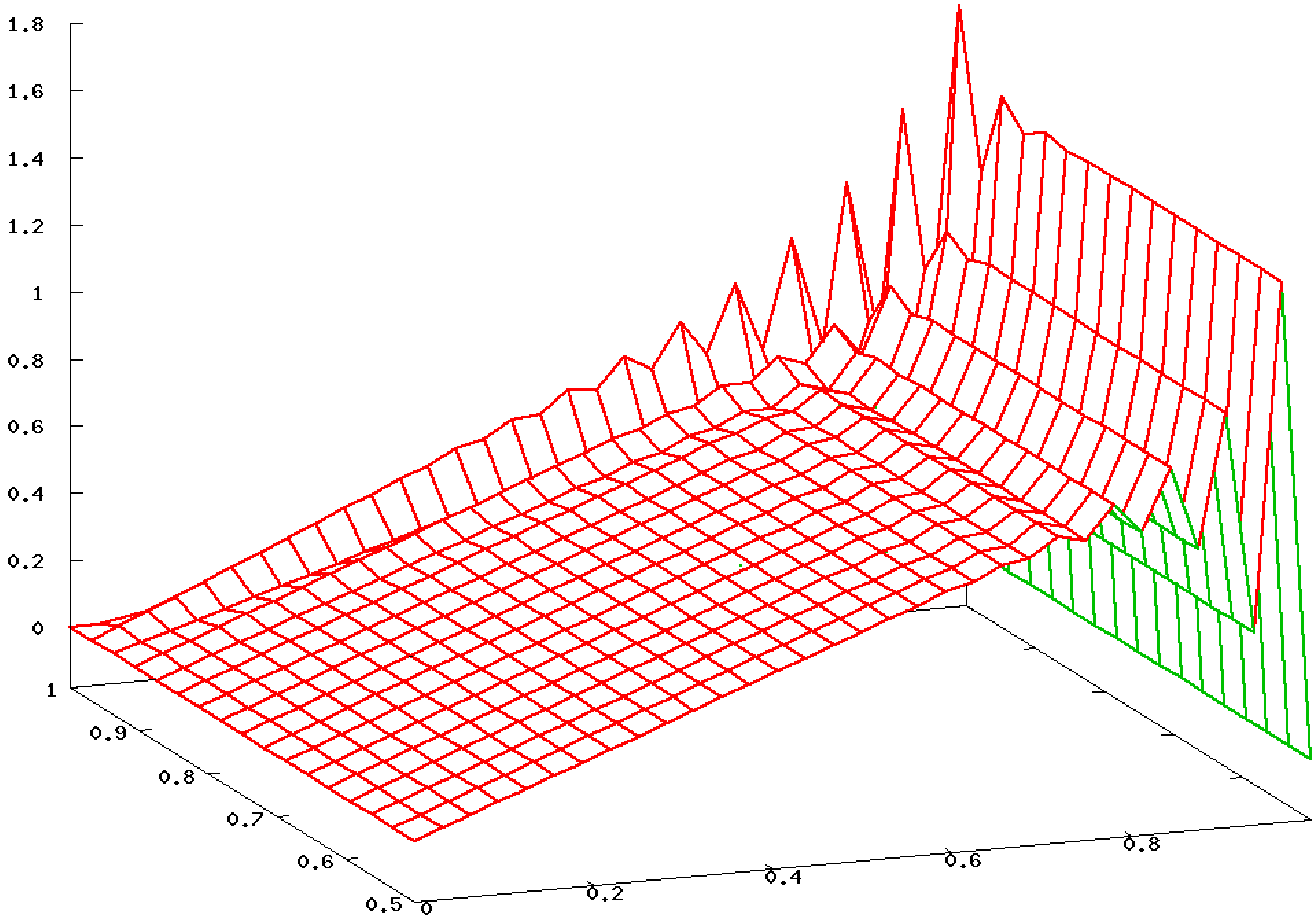
Two–level LPS, $\tau_0 = 1$



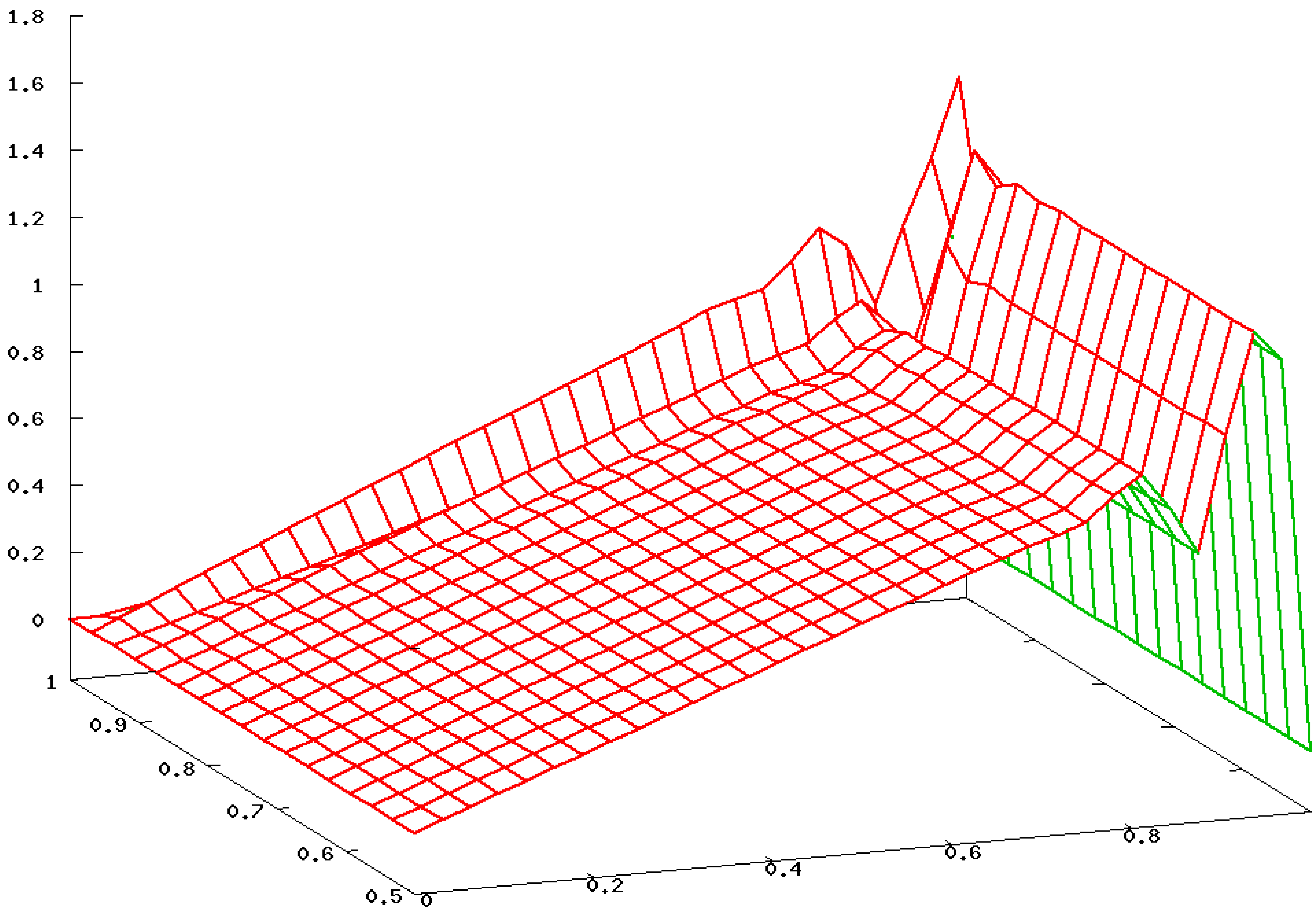
Overlapping LPS, $\tau_0 = 0.001$



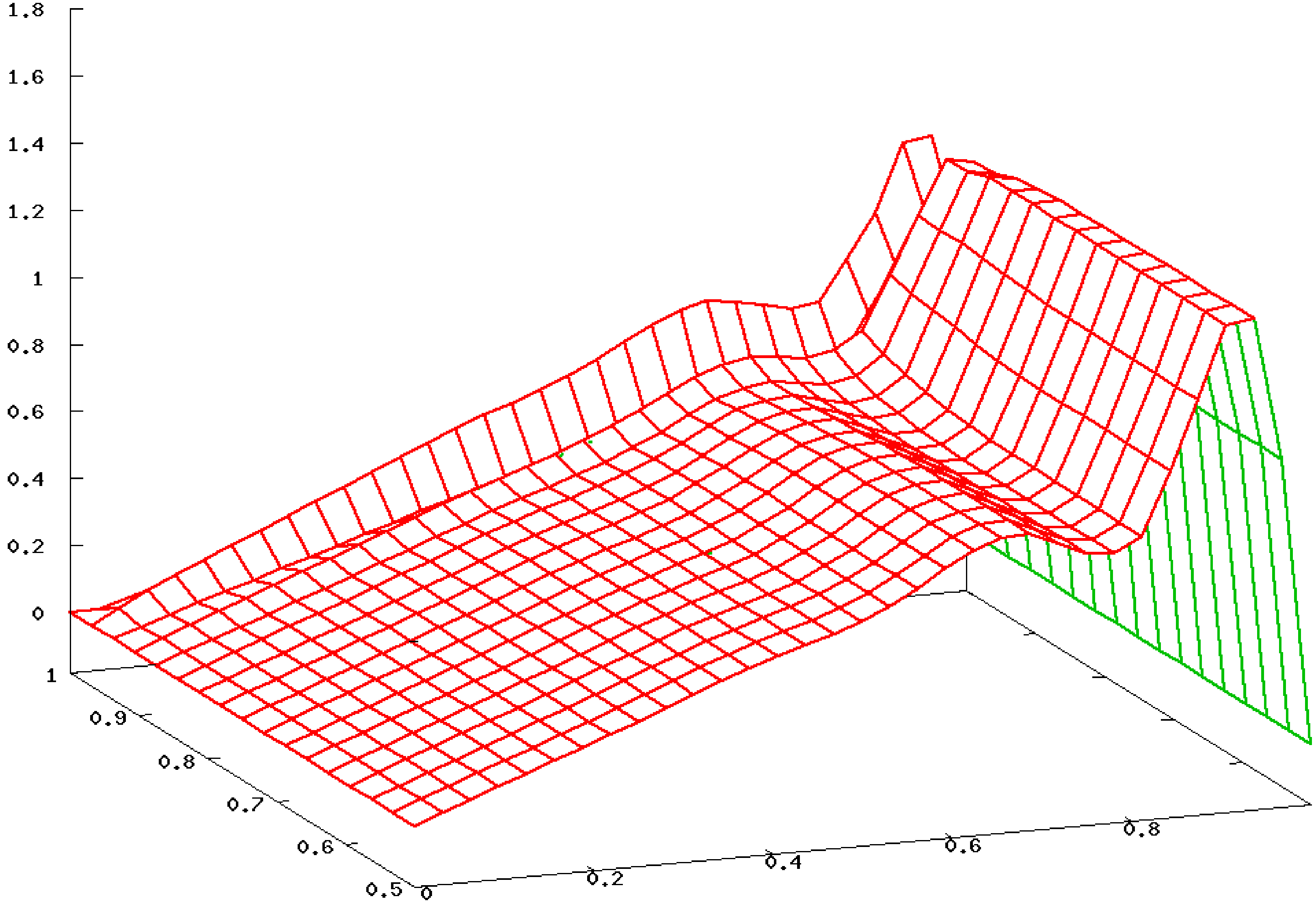
Overlapping LPS, $\tau_0 = 0.01$



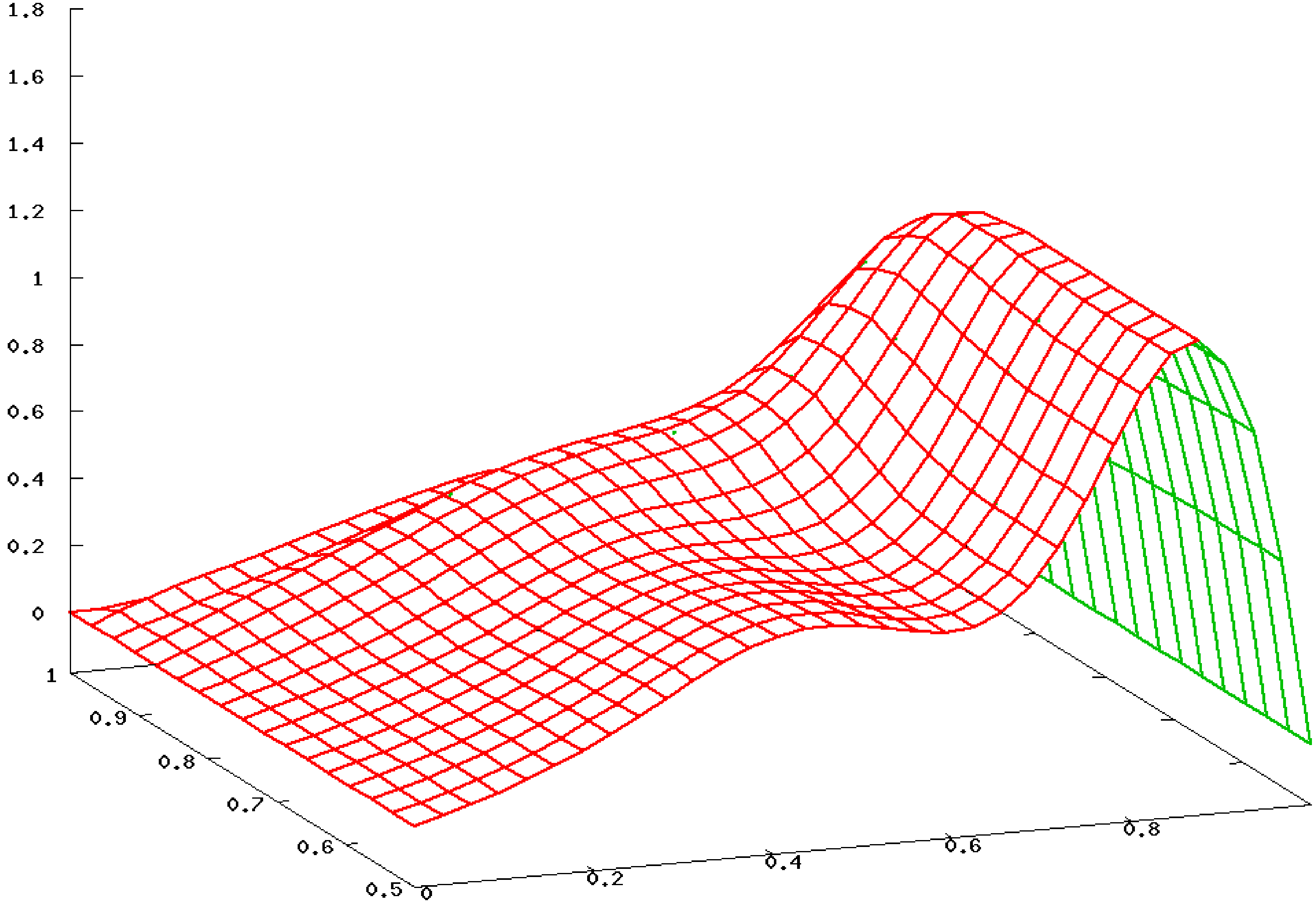
Overlapping LPS, $\tau_0 = 0.1$



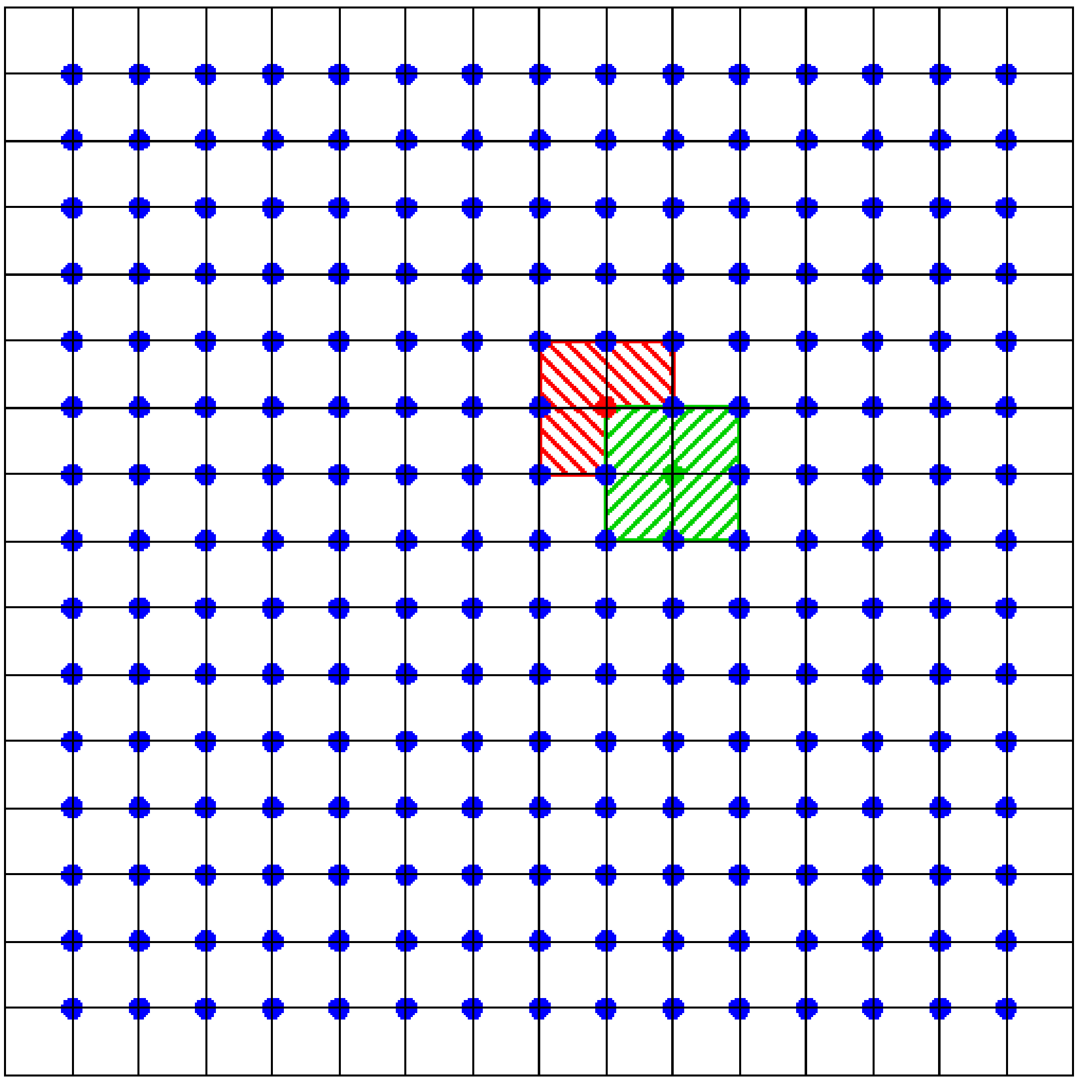
Overlapping LPS, $\tau_0 = 1$



Overlapping LPS, $\tau_0 = 10$



LPS discretizations: $V_h \ldots Q_2, \quad D_M = P_1(M)$

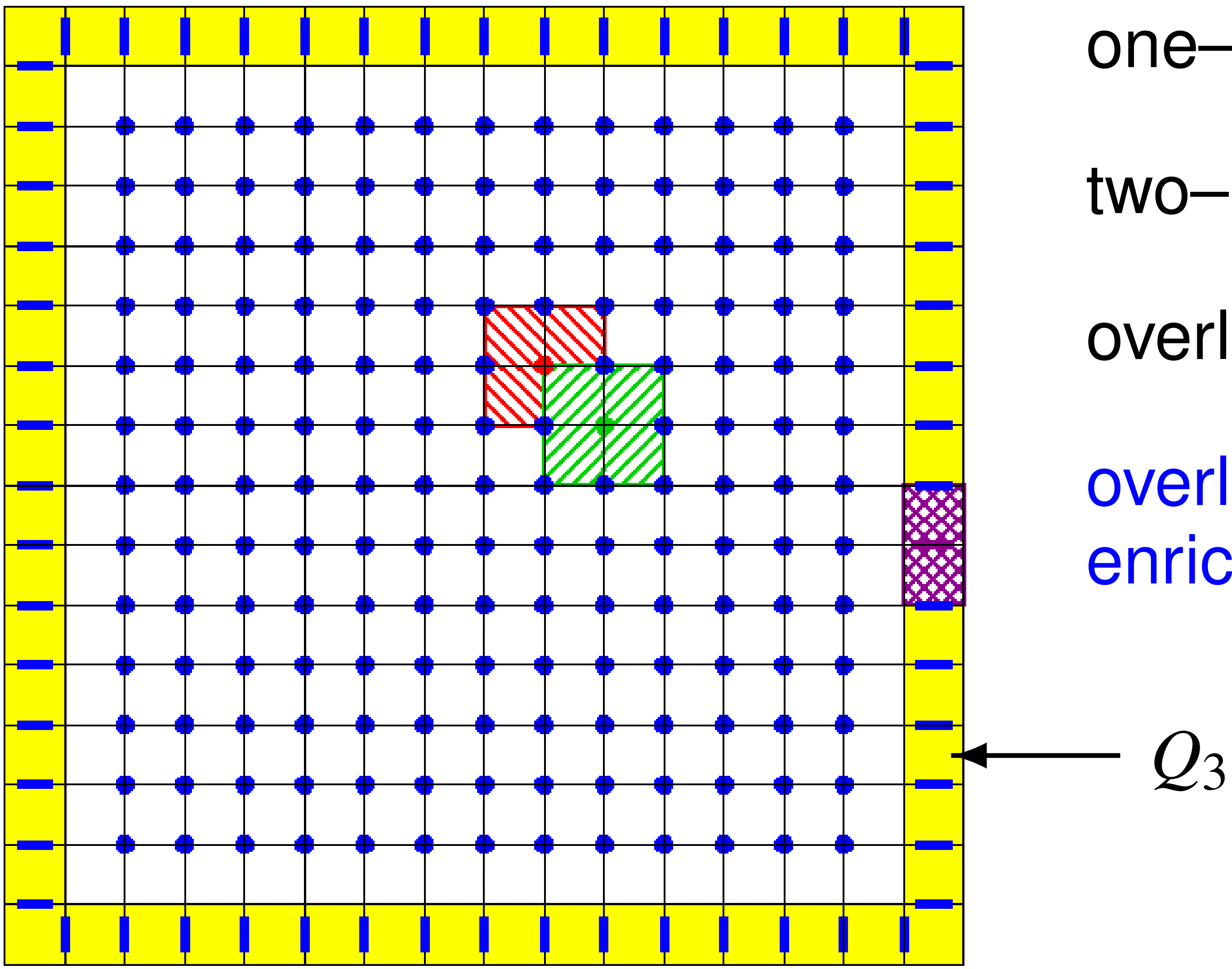


one–level LPS

two–level LPS

overlapping LPS

LPS discretizations: $V_h \ldots Q_2, \quad D_M = P_1(M)$



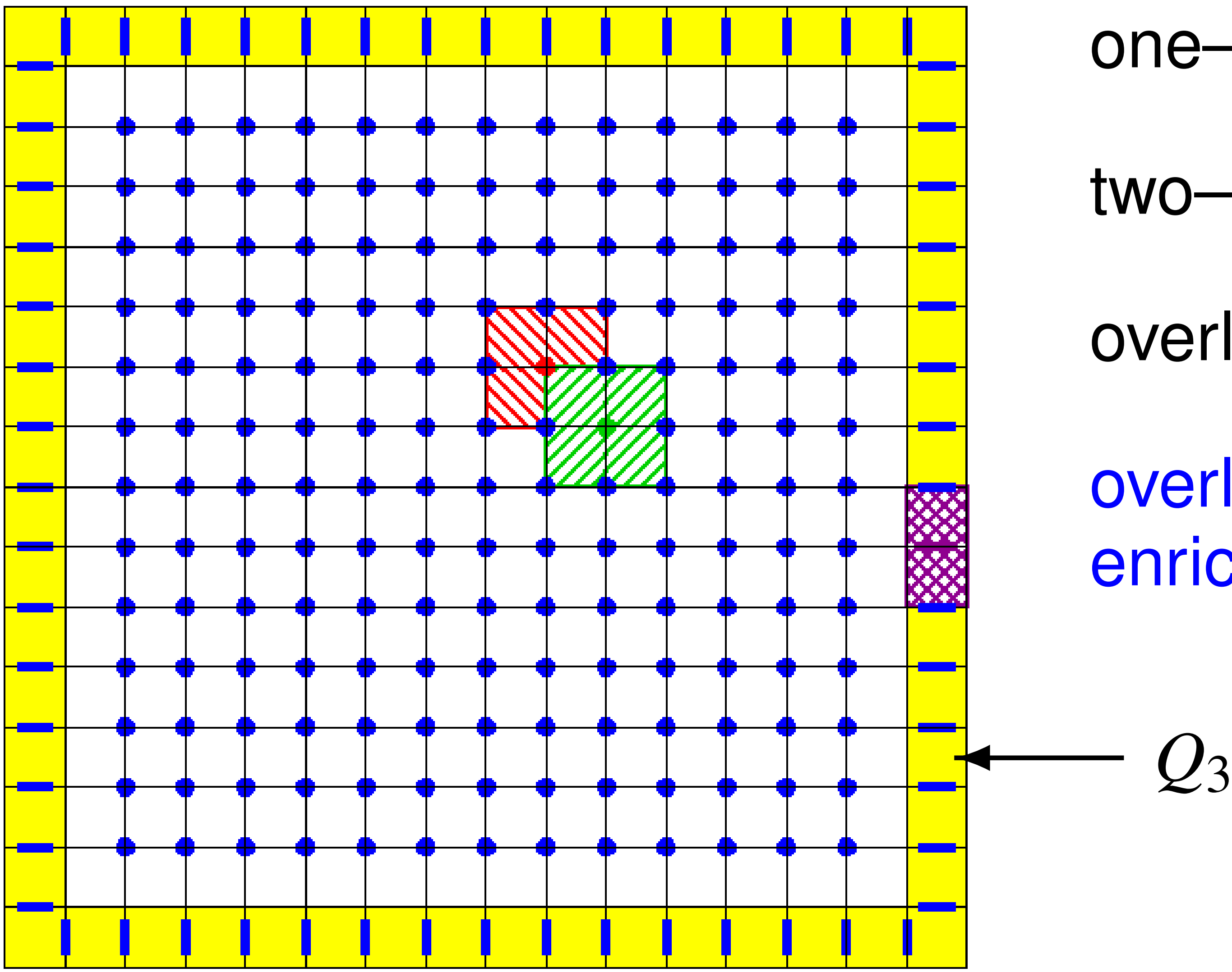
one–level LPS

two–level LPS

overlapping LPS

overlapping LPS enriched at $\partial\Omega$

LPS discretizations: $V_h \ldots Q_2, \quad D_M = P_1(M)$



one–level LPS

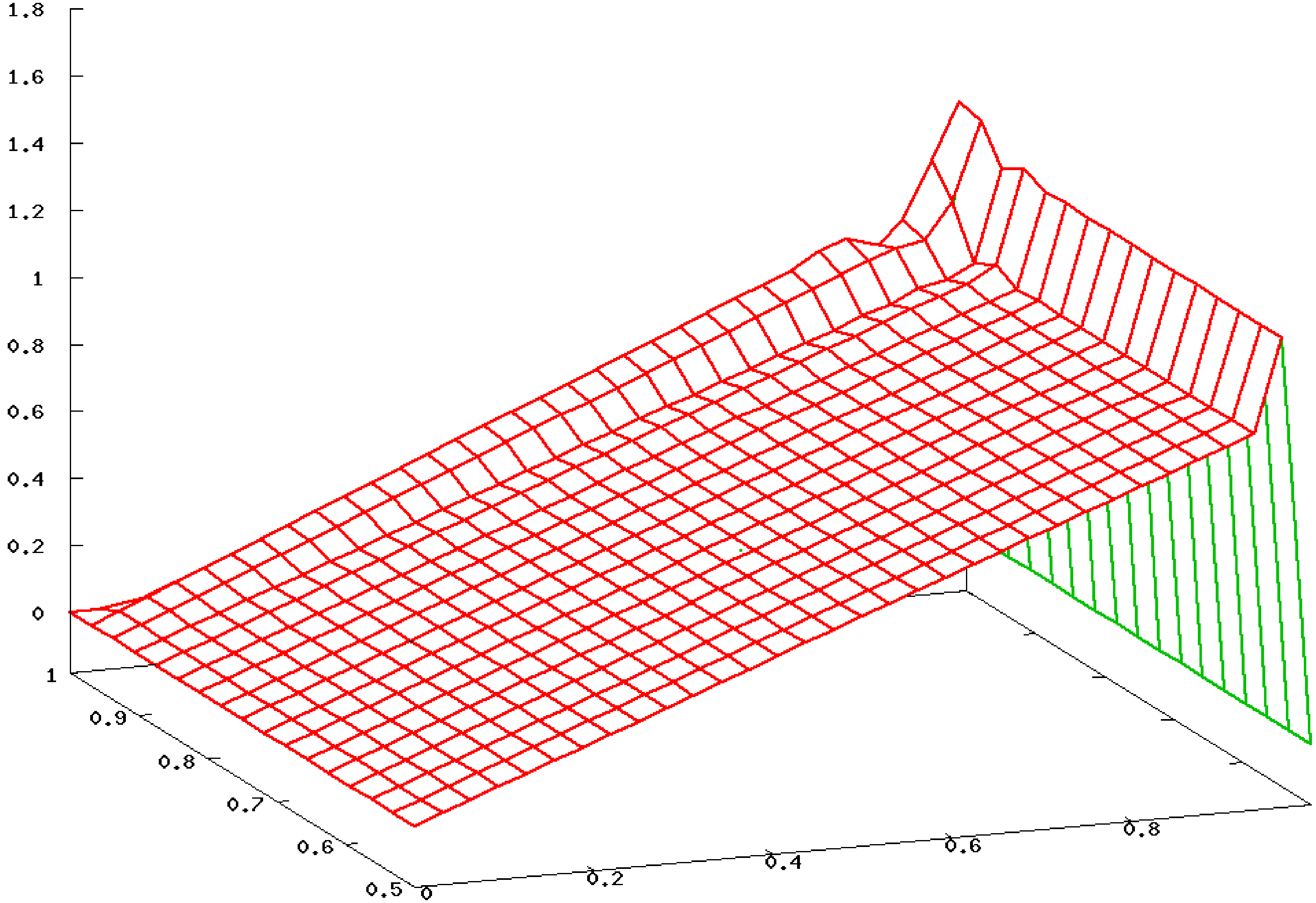
two–level LPS

overlapping LPS

overlapping LPS enriched at $\partial\Omega$

Motivated by a one–dimensional analysis of Tobiska (2009).

Overlapping LPS enriched at $\partial\Omega$, $\tau = 0.05\,h_1/b_1$



Oseen problem

$$-\nu\,\Delta\mathbf{u}+(\mathbf{b}\cdot\nabla)\mathbf{u}+c\,\mathbf{u}+\nabla p=\mathbf{f},\quad \operatorname{div}\mathbf{u}=0\quad \text{in }\Omega,$$
$$\mathbf{u}=\mathbf{0}\quad \text{on }\partial\Omega$$

Galerkin discretization

Find $\mathbf{u}_h\in\mathbf{V}_h,\ p_h\in Q_h$ such that

$$\mathscr{A}([\mathbf{u}_h,p_h],[\mathbf{v}_h,q_h])=(\mathbf{f},\mathbf{v}_h)\qquad \forall\,\mathbf{v}_h\in\mathbf{V}_h,q_h\in Q_h.$$

$\mathbf{V}_h\subset H_0^1(\Omega)^d, Q_h\subset L_0^2(\Omega)$ ... finite–dimensional spaces

Two sources of instabilities:

– dominant convection

– violation of the inf–sup condition

$$\sup_{\mathbf{v}_h\in\mathbf{V}_h}\frac{(q_h,\operatorname{div}\mathbf{v}_h)}{|\mathbf{v}_h|_{1,\Omega}}\geq\beta\,\|q_h\|_{0,\Omega}\quad \forall\,q_h\in Q_h$$

Stokes problem

$$-\Delta\mathbf{u}+\nabla p=\mathbf{f},\quad \operatorname{div}\mathbf{u}=0\quad \text{in } \Omega,\qquad \mathbf{u}=\mathbf{0}\quad \text{on } \partial\Omega$$

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Galerkin discretization

Find $\mathbf{u}_h \in \mathbf{V}_h$, $p_h \in Q_h$ such that

$$\mathscr{A}([\mathbf{u}_h, p_h], [\mathbf{v}_h, q_h]) = (\mathbf{f}, \mathbf{v}_h) \qquad \forall\, \mathbf{v}_h \in \mathbf{V}_h, q_h \in Q_h,$$

where

$$\mathscr{A}([\mathbf{u}, p], [\mathbf{v}, q]) = (\nabla \mathbf{u}, \nabla \mathbf{v}) - (p, \operatorname{div} \mathbf{v}) + (q, \operatorname{div} \mathbf{u})$$

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$\mathbf{V}_h \subset H_0^1(\Omega)^d,\ Q_h \subset H^1(\Omega) \cap L_0^2(\Omega)$ ... finite–dim. spaces

Assume violation of the inf–sup condition

$$\sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{(q_h, \operatorname{div} \mathbf{v}_h)}{|\mathbf{v}_h|_{1,\Omega}} \geq \beta\, \|q_h\|_{0,\Omega} \quad \forall\, q_h \in Q_h$$

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$$\mathscr{A}_h([\mathbf{u}, p], [\mathbf{v}, q]) = \mathscr{A}([\mathbf{u}, p], [\mathbf{v}, q]) + s_h^p(p, q),$$

$$s_h^p(p, q) = \sum_{M \in \mathscr{M}_h} \alpha_M \,(\kappa_M \nabla p, \kappa_M \nabla q)_M\,, \qquad \alpha_M \geq C\, h_M^2$$

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Then

$$\sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{(q, \operatorname{div} \mathbf{v}_h)}{|\mathbf{v}_h|_{1,\Omega}} + \sqrt{s_h^p(q, q)} \geq \gamma \|q\|_{0,\Omega} \quad \forall\, q \in H^1(\Omega) \cap L_0^2(\Omega)$$

Proof: For simplicity, let $\mathscr{M}_h$ consist of non-overlapping sets.

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where $i_h : H_0^1(\Omega) \to V_h$ satisfies

$$|i_h v|_{1,\Omega} + \left(\sum_{M \in \mathscr{M}_h} h_M^{-2}\, \|v - i_h v\|_{0,M}^2 \right)^{1/2} \le C_i\, |v|_{1,\Omega} \qquad \forall\, v \in H_0^1(\Omega)\,.$$

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$$\begin{aligned}(q, \operatorname{div}(\mathbf{v}-\mathbf{v}_h)) &= -(\nabla q, \mathbf{v}-\mathbf{v}_h) = (\nabla q, \mathbf{z}_h) - (\nabla q, \mathbf{v} - i_h\mathbf{v}) \\ &= \sum_{M\in\mathscr{M}_h} [(\nabla q, \mathbf{z}_M)_M - (\nabla q, \mathbf{v} - i_h\mathbf{v})_M]\end{aligned}$$

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&= \sum_{M \in \mathscr{M}_h} [(\kappa_M \nabla q, \mathbf{z}_M)_M - (\kappa_M \nabla q, \mathbf{v} - i_h \mathbf{v})_M] \\
&\le \sum_{M \in \mathscr{M}_h} \|\kappa_M \nabla q\|_{0,M} \left(\|\mathbf{z}_M\|_{0,M} + \|\mathbf{v} - i_h \mathbf{v}\|_{0,M} \right)
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$$\sup_{\mathbf{v} \in H_0^1(\Omega)^d} \frac{(q, \operatorname{div} \mathbf{v})}{|\mathbf{v}|_{1,\Omega}} \le C_1 \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{(q, \operatorname{div} \mathbf{v}_h)}{|\mathbf{v}_h|_{1,\Omega}} + C_2 \sqrt{s_h^p(q,q)}$$

Proof: For simplicity, let $\mathscr{M}_h$ consist of non-overlapping sets.

Consider any $\mathbf{v} \in H_0^1(\Omega)^d \quad \Rightarrow \quad \forall\, M \in \mathscr{M}_h \quad \exists\, \mathbf{z}_M \in B_M^d$:

$$(\mathbf{z}_M, \mathbf{r})_M = (\mathbf{v} - i_h \mathbf{v}, \mathbf{r})_M \qquad \forall\, \mathbf{r} \in D_M^d\,,$$

$$\|\mathbf{z}_M\|_{0,M} \leq \beta_{LP}^{-1}\, \|\mathbf{v} - i_h \mathbf{v}\|_{0,M}$$

$$\mathbf{z}_h = \sum_{M \in \mathscr{M}_h} \mathbf{z}_M\,, \qquad \mathbf{v}_h = i_h \mathbf{v} + \mathbf{z}_h \quad \Rightarrow \quad |\mathbf{v}_h|_{1,\Omega} \leq C_1\, |\mathbf{v}|_{1,\Omega}$$

For any $q \in H^1(\Omega) \cap L_0^2(\Omega)$:

$$(q, \operatorname{div}(\mathbf{v} - \mathbf{v}_h)) \leq C_2 \sqrt{s_h^p(q,q)}\, |\mathbf{v}|_{1,\Omega}$$

Then

$$\gamma_0\, \|q\|_{0,\Omega} \leq \sup_{\mathbf{v} \in H_0^1(\Omega)^d} \frac{(q, \operatorname{div} \mathbf{v})}{|\mathbf{v}|_{1,\Omega}} \leq C_1 \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{(q, \operatorname{div} \mathbf{v}_h)}{|\mathbf{v}_h|_{1,\Omega}} + C_2 \sqrt{s_h^p(q,q)}$$

Inf–sup condition for $\mathscr{A}_h$

$$\inf_{[\mathbf{u}_h,p_h]\in\mathbf{V}_h\times Q_h}\ \sup_{[\mathbf{v}_h,q_h]\in\mathbf{V}_h\times Q_h}\frac{\mathscr{A}_h([\mathbf{u}_h,p_h],[\mathbf{v}_h,q_h])}{|||[\mathbf{u}_h,p_h]|||\ |||[\mathbf{v}_h,q_h]|||}\geq C\,,$$

where $|||[\mathbf{v},q]||| = \left(|\mathbf{v}|_{1,\Omega}^2 + \|q\|_{0,\Omega}^2 + s_h^p(q,q)\right)^{1/2}$

and $C > 0$ depends only on γ.

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where $|||[\mathbf{v},q]|||=\left(|\mathbf{v}|^2_{1,\Omega}+\|q\|^2_{0,\Omega}+s^p_h(q,q)\right)^{1/2}$

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$$\sup_{\mathbf{v}_h\in\mathbf{V}_h}\frac{(p_h,\operatorname{div}\mathbf{v}_h)}{|\mathbf{v}_h|_{1,\Omega}}+\sqrt{s^p_h(p_h,p_h)}\geq\gamma\|p_h\|_{0,\Omega}.$$

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$\Rightarrow\quad \exists\,\overline{\mathbf{v}}_h\in V^d_h\qquad\qquad :$

$$-(p_h,\operatorname{div}\overline{\mathbf{v}}_h)\geq\frac{\gamma}{2}\,\|p_h\|_{0,\Omega}\,|\overline{\mathbf{v}}_h|_{1,\Omega}-\sqrt{s^p_h(p_h,p_h)}\,|\overline{\mathbf{v}}_h|_{1,\Omega}$$

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$$-(p_h, \operatorname{div}\overline{\mathbf{v}}_h) \geq \frac{\gamma}{2}\,\|p_h\|_{0,\Omega}\,|\overline{\mathbf{v}}_h|_{1,\Omega} - \sqrt{s_h^p(p_h,p_h)}\,|\overline{\mathbf{v}}_h|_{1,\Omega}$$

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$$\mathscr{A}_h([\mathbf{u}_h,p_h],[\overline{\mathbf{v}}_h,0])=-(p_h,\operatorname{div}\overline{\mathbf{v}}_h)+(\nabla\mathbf{u}_h,\nabla\overline{\mathbf{v}}_h)$$

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$\Rightarrow \quad \exists\, \overline{\mathbf{v}}_h \in V_h^d,\ |\overline{\mathbf{v}}_h|_{1,\Omega} = \|p_h\|_{0,\Omega}$:

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$$\begin{aligned}\mathscr{A}_h([\mathbf{u}_h,p_h],[\overline{\mathbf{v}}_h,0]) &= -(p_h,\operatorname{div}\overline{\mathbf{v}}_h) + (\nabla\mathbf{u}_h,\nabla\overline{\mathbf{v}}_h)\\ &\geq \frac{\gamma}{2}\,\|p_h\|^2_{0,\Omega} - \sqrt{s_h^p(p_h,p_h)}\,\|p_h\|_{0,\Omega} - |\mathbf{u}_h|_{1,\Omega}\,\|p_h\|_{0,\Omega}\end{aligned}$$

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$$\mathscr{A}_h([\mathbf{u}_h,p_h],[\overline{\mathbf{v}}_h,0]) \geq \frac{\gamma}{4}\,\|p_h\|^2_{0,\Omega} - \frac{2}{\gamma}\,s_h^p(p_h,p_h) - \frac{2}{\gamma}\,|\mathbf{u}_h|^2_{1,\Omega}$$

Inf–sup condition for $\mathscr{A}_h$

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$$\mathscr{A}_h([\mathbf{u}_h,p_h],[\overline{\mathbf{v}}_h,0])\ge\frac{\gamma}{4}\|p_h\|^2_{0,\Omega}-\frac{2}{\gamma}s_h^p(p_h,p_h)-\frac{2}{\gamma}|\mathbf{u}_h|^2_{1,\Omega}$$

$$\mathscr{A}_h([\mathbf{u}_h,p_h],[\mathbf{u}_h,p_h])=\qquad s_h^p(p_h,p_h)\quad+|\mathbf{u}_h|^2_{1,\Omega}$$

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$$\mathscr{A}_h([\mathbf{u}_h,p_h],[\mathbf{u}_h,p_h]) = s^p_h(p_h,p_h) + |\mathbf{u}_h|^2_{1,\Omega}$$

Set $[\mathbf{v}_h,q_h] = \frac{4}{\gamma}\,[\overline{\mathbf{v}}_h,0] + \left(1+\frac{8}{\gamma^2}\right)[\mathbf{u}_h,p_h]$.

Inf–sup condition for $\mathscr{A}_h$

$$\inf_{[\mathbf{u}_h,p_h]\in\mathbf{V}_h\times Q_h}\ \sup_{[\mathbf{v}_h,q_h]\in\mathbf{V}_h\times Q_h}\frac{\mathscr{A}_h([\mathbf{u}_h,p_h],[\mathbf{v}_h,q_h])}{|||[\mathbf{u}_h,p_h]|||\;|||[\mathbf{v}_h,q_h]|||}\geq C\,,$$

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$$\mathscr{A}_h([\mathbf{u}_h,p_h],[\overline{\mathbf{v}}_h,0])\geq\frac{\gamma}{4}\|p_h\|^2_{0,\Omega}-\frac{2}{\gamma}s_h^p(p_h,p_h)-\frac{2}{\gamma}|\mathbf{u}_h|^2_{1,\Omega}$$

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Set $[\mathbf{v}_h,q_h]=\dfrac{4}{\gamma}[\overline{\mathbf{v}}_h,0]+\left(1+\dfrac{8}{\gamma^2}\right)[\mathbf{u}_h,p_h]$. Then

$$\mathscr{A}_h([\mathbf{u}_h,p_h],[\mathbf{v}_h,q_h])\geq|||[\mathbf{u}_h,p_h]|||^2,\qquad |||[\mathbf{u}_h,p_h]|||\geq C\,|||[\mathbf{v}_h,q_h]|||\,.$$

Inf–sup condition for $\mathscr{A}_h$

$$\inf_{[\mathbf{u}_h,p_h]\in\mathbf{V}_h\times Q_h}\ \sup_{[\mathbf{v}_h,q_h]\in\mathbf{V}_h\times Q_h} \frac{\mathscr{A}_h([\mathbf{u}_h,p_h],[\mathbf{v}_h,q_h])}{|||[\mathbf{u}_h,p_h]|||\ |||[\mathbf{v}_h,q_h]|||} \geq C\,,$$

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and $C > 0$ depends only on γ.

An optimal estimate of $|\mathbf{u}-\mathbf{u}_h|_{1,\Omega}$ and $\|p-p_h\|_{0,\Omega}$ follows analogously as for the convection–diffusion–reaction equation.

Oseen problem

$$-\nu\Delta\mathbf{u}+(\mathbf{b}\cdot\nabla)\mathbf{u}+c\,\mathbf{u}+\nabla p=\mathbf{f},\quad \operatorname{div}\mathbf{u}=0\quad \text{in } \Omega,$$

$$\mathbf{u}=\mathbf{0}\quad \text{on } \partial\Omega$$

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$$\mathbf{u}=\mathbf{0}\quad \text{on }\partial\Omega$$

Galerkin discretization

Find $\mathbf{u}_h\in\mathbf{V}_h,\ p_h\in Q_h$ such that

$$\mathscr{A}([\mathbf{u}_h,p_h],[\mathbf{v}_h,q_h])=(\mathbf{f},\mathbf{v}_h)\qquad \forall\,\mathbf{v}_h\in\mathbf{V}_h,q_h\in Q_h,$$

where

$$\mathscr{A}([\mathbf{u},p],[\mathbf{v},q])=\nu\,(\nabla\mathbf{u},\nabla\mathbf{v})+((\mathbf{b}\cdot\nabla)\mathbf{u}+c\,\mathbf{u},\mathbf{v})$$
$$-(p,\operatorname{div}\mathbf{v})+(q,\operatorname{div}\mathbf{u})\,,$$

$\mathbf{V}_h\subset H^1_0(\Omega)^d,\ Q_h\subset H^1(\Omega)\cap L^2_0(\Omega)$ ... finite–dim. spaces

Local projection stabilization

Find $\mathbf{u}_h \in \mathbf{V}_h,\ p_h \in Q_h$ such that

$$\mathscr{A}_h([\mathbf{u}_h, p_h], [\mathbf{v}_h, q_h]) = (\mathbf{f}, \mathbf{v}_h) \qquad \forall\, \mathbf{v}_h \in \mathbf{V}_h, q_h \in Q_h,$$

where

$$\mathscr{A}_h([\mathbf{u}, p], [\mathbf{v}, q]) = \mathscr{A}([\mathbf{u}, p], [\mathbf{v}, q]) + s_h^u(\mathbf{u}, \mathbf{v}) + s_h^p(p, q) + s_h^{\mathrm{div}}(\mathbf{u}, \mathbf{v}),$$

$$s_h^u(\mathbf{u}, \mathbf{v}) = \sum_{M \in \mathscr{M}_h} \tau_M \left(\kappa_M((\mathbf{b}_M \cdot \nabla)\mathbf{u}), \kappa_M((\mathbf{b}_M \cdot \nabla)\mathbf{v})\right)_M,$$

$$s_h^p(p, q) = \sum_{M \in \mathscr{M}_h} \alpha_M \left(\kappa_M \nabla p, \kappa_M \nabla q\right)_M,$$

$$s_h^{\mathrm{div}}(\mathbf{u}, \mathbf{v}) = \sum_{M \in \mathscr{M}_h} \gamma_M \left(\kappa_M \operatorname{div} \mathbf{u}, \kappa_M \operatorname{div} \mathbf{v}\right)_M,$$

$$\tau_M \sim \min\left\{ \frac{h_M}{\|\mathbf{b}\|_{0,\infty,M}}, \frac{h_M^2}{\varepsilon} \right\}, \qquad \alpha_M \sim h_M, \qquad \gamma_M \sim h_M$$

for equal order spaces

Inf–sup condition for $\mathscr{A}_h$

$$\inf_{[\mathbf{u}_h,p_h]\in\mathbf{V}_h\times Q_h}\ \sup_{[\mathbf{v}_h,q_h]\in\mathbf{V}_h\times Q_h}\frac{\mathscr{A}_h([\mathbf{u}_h,p_h],[\mathbf{v}_h,q_h])}{|||[\mathbf{u}_h,p_h]|||\ |||[\mathbf{v}_h,q_h]|||}\geq C\,,$$

where $|||[\mathbf{v},q]||| = \Big(\nu\,|\mathbf{v}|_{1,\Omega}^2+\|\sigma^{1/2}\,\mathbf{v}\|_{0,\Omega}^2+(\nu+\sigma_0)\,\|q\|_{0,\Omega}^2$
$$+s_h^u(\mathbf{v},\mathbf{v})+s_h^p(q,q)+s_h^{\mathrm{div}}(\mathbf{v},\mathbf{v})\Big)^{1/2}.$$

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$$\inf_{[\mathbf{u}_h,p_h]\in\mathbf{V}_h\times Q_h}\ \sup_{[\mathbf{v}_h,q_h]\in\mathbf{V}_h\times Q_h}\frac{\mathscr{A}_h([\mathbf{u}_h,p_h],[\mathbf{v}_h,q_h])}{|||[\mathbf{u}_h,p_h]|||\ |||[\mathbf{v}_h,q_h]|||}\geq C\,,$$

where $|||[\mathbf{v},q]||| = \Big(\nu\,|\mathbf{v}|^2_{1,\Omega}+\|\sigma^{1/2}\,\mathbf{v}\|^2_{0,\Omega}+(\nu+\sigma_0)\,\|q\|^2_{0,\Omega}$

$$+s_h^u(\mathbf{v},\mathbf{v})+s_h^p(q,q)+s_h^{\mathrm{div}}(\mathbf{v},\mathbf{v})\Big)^{1/2}.$$

Error estimate

$$|||[\mathbf{u}-\mathbf{u}_h,p-p_h]|||\leq C\,(\nu+h)^{1/2}\,h^k\,(|\mathbf{u}|_{k+1,\Omega}+|p|_{k+1,\Omega})$$

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– operations *discretization* and *optimization* commute