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Construction of non-constant lower and upper functions for impulsive periodic problems

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Summary. We present conditions ensuring the existence of piecewise linear lower and upper functions for the nonlinear impulsive periodic boundary value problem $u''=f(t,u,u')$, $u(t_i+)=\mathrm{J}_i(u(t_i))$, $u'(t_i+)=\mathrm{M}_i(u'(t_i))$, $i=1,2,\dots,m$, $u(0)=u(T)$, $u'(0)=u'(T)$. This together with the existence principles which we proved in [5]–[7] allows us to prove new existence criteria, see Theorems 3.1 and 3.2.
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1. Introduction

This paper deals with the impulsive periodic boundary value problem

$$u''=f(t,u,u'), \tag{1.1}$$

$$u(t_i+)=\mathrm{J}_i(u(t_i)),\quad u'(t_i+)=\mathrm{M}_i(u'(t_i)),\quad i=1,2,\dots,m, \tag{1.2}$$

$$u(0)=u(T),\quad u'(0)=u'(T), \tag{1.3}$$

where

$$\begin{cases} 0<t_1<\dots<t_m<T<\infty,\\ f \text{ satisfies the Carathéodory conditions on } [0,T]\times\mathbb{R}^2,\\ \mathrm{J}_i \text{ and } \mathrm{M}_i,\ i=1,2,\dots,m, \text{ are continuous functions on } \mathbb{R}. \end{cases} \tag{1.4}$$

There are several papers providing the existence results for such problems in terms of lower and upper functions, see e.g. [1]–[4], [8] and our papers [5]–[7]. However, up to now, only Proposition 1.3 in [6] gives conditions ensuring the existence of nonconstant (in particular, piecewise constant) lower and upper functions. The main

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goal of this paper is to find conditions for f, J_i, M_i giving piecewise linear lower and upper functions for (1.1)–(1.3). This together with the existence principles which we proved in [5]–[7] allows us to present new existence criteria.

Throughout the paper we keep the following notation and conventions: For $J \subset \mathbb{R}$, $\mathbb{C}(J)$ is the set of real valued functions which are continuous on J, $\mathbb{C}^1(J)$ is the set of functions having continuous first derivatives on J, $\mathbb{L}(J)$ is the set of functions Lebesgue integrable on J and $\mathbb{L}_\infty(J)$ is the set of functions essentially bounded on J. For $u \in \mathbb{L}_\infty[0,T]$, we denote $\|u\|_\infty = \sup \operatorname{ess}_{t\in[0,T]} |u(t)|$. Further, $\mathrm{D} = \{t_1, t_2, \dots, t_m\}$, $t_0{=}0$, $t_{m+1}{=}T$ and $\mathbb{C}^1_{\mathrm{D}}[0,T]$ is the set of functions $u\colon [0,T] \mapsto \mathbb{R}$ of the form

$$u(t) = \begin{cases} u_{[0]}(t) & \text{if } t \in [0, t_1], \\ u_{[1]}(t) & \text{if } t \in (t_1, t_2], \\ \dots & \quad \dots \\ u_{[m]}(t) & \text{if } t \in (t_m, T], \end{cases}$$

where $u_{[i]} \in \mathbb{C}^1[t_i, t_{i+1}]$ for $i = 0, 1, \dots, m$. Moreover, $\mathbb{AC}^1_{\mathrm{D}}[0,T]$ stands for the set of functions $u \in \mathbb{C}^1_{\mathrm{D}}[0,T]$ having first derivatives absolutely continuous on each subinterval (t_i, t_{i+1}), $i = 0, 1, \dots, m$. For $u \in \mathbb{C}^1_{\mathrm{D}}[0,T]$ and $i = 1, 2, \dots, m+1$ we define $u'(t_i) = u'(t_i-) = \lim_{t\to t_i-} u'(t)$ and $u'(0) = u'(0+) = \lim_{t\to 0+} u'(t)$.

1.1 Definition. A *solution of the problem* (1.1)–(1.3) is a function $u \in \mathbb{AC}^1_{\mathrm{D}}[0,T]$ which satisfies the conditions (1.2) and (1.3) and for a.e. $t \in [0,T]$ fulfils the equation (1.1).

1.2 Definition. A function $\sigma_1 \in \mathbb{AC}^1_{\mathrm{D}}[0,T]$ is called a *lower function of the problem* (1.1)–(1.3) if

$$\sigma_1''(t) \ge f(t, \sigma_1(t), \sigma_1'(t)) \quad \text{for a.e.} \quad t \in [0,T], \tag{1.5}$$

$$\sigma_1(t_i+) = \mathrm{J}_i(\sigma_1(t_i)), \quad \sigma_1'(t_i+) \ge \mathrm{M}_i(\sigma_1'(t_i)), \quad i = 1, 2, \dots, m, \tag{1.6}$$

$$\sigma_1(0) = \sigma_1(T), \quad \sigma_1'(0) \ge \sigma_1'(T). \tag{1.7}$$

A function $\sigma_2 \in \mathbb{AC}^1_{\mathrm{D}}[0,T]$ is an *upper function of* (1.1)–(1.3) if it satisfies

$$\sigma_2''(t) \le f(t, \sigma_2(t), \sigma_2'(t)) \quad \text{for a.e.} \quad t \in [0,T], \tag{1.8}$$

$$\sigma_2(t_i+) = \mathrm{J}_i(\sigma_2(t_i)), \quad \sigma_2'(t_i+) \le \mathrm{M}_i(\sigma_2'(t_i)), \quad i = 1, 2, \dots, m, \tag{1.9}$$

$$\sigma_2(0) = \sigma_2(T), \quad \sigma_2'(0) \le \sigma_2'(T). \tag{1.10}$$

2. Construction of nonconstant lower and upper functions

2.1 Theorem. *Assume* (1.4) *and*

$$\lim_{x\to\infty}(\mathrm{J}_i(x)-x)=c_i\in\mathbb{R}\quad for\quad i=1,2,\dots,m. \tag{2.1}$$

Denote

$$c=-\frac{1}{T}\sum_{i=1}^{m}c_i \tag{2.2}$$

and suppose that there are $A\in\mathbb{R}$, $\delta>0$ *and* $\nu\in\{1,2\}$ *such that*

$$(-1)^{\nu}f(t,x,y)\ge 0\quad for\ a.e.\ t\in[0,T]\quad and\ all\ x\ge A, y\in[c-\delta,c+\delta] \tag{2.3}$$

and

$$(-1)^{\nu}(M_i(y)-y)\ge 0\quad for\ y\in[c-\delta,c+\delta],\ i=1,2,\dots,m. \tag{2.4}$$

Then for each $\widetilde{A}\in[A,\infty)$ *there exist* $\widetilde{k}\in(c-\delta,c+\delta)$ *and* $\sigma_\nu\in\mathbb{AC}^1_{\mathrm{D}}[0,T]$ *such that* $\sigma_\nu(t)\ge\widetilde{A}$, $\sigma'_\nu(t)=\widetilde{k}$ *for* $t\in[0,T]$ *and, for* $\nu=1$ $(\nu=2)$, σ_ν *is a lower* (*upper*) *function of* (1.1)–(1.3).

Proof. Let $\nu=2$ and $\widetilde{A}\in[A,\infty)$.

- Step 1. For a, $k\in\mathbb{R}$, $x\in\mathbb{R}$ and $i=1,2,\dots,m$, define

$$\varphi(t,a,k)=\begin{cases} a+k\,t & \text{if } t\in[0,t_1],\\ \mathrm{J}_i(\varphi(t_i,a,k))+k\,(t-t_i) & \text{if } t\in(t_i,t_{i+1}],\ i=1,2,\dots,m,\end{cases}$$

and

$$\varepsilon_i(x)=\mathrm{J}_i(x)-x-c_i.$$

By virtue of (2.1), there are constants $A_i>\widetilde{A}$, $i=1,2,\dots,m$, such that

$$|\varepsilon_i(x)|<\frac{\delta T}{m}\quad\text{for all}\quad x\ge A_i\quad\text{and}\quad i=1,2,\dots,m. \tag{2.5}$$

Furthermore, we have

$$\varphi(t,a,k)=\begin{cases} a+k\,t & \text{if } t\in[0,t_1],\\ a+k\,t+\sum_{j=1}^{i}\big(c_j+\varepsilon_j(\varphi(t_j,a,k))\big) & \\ & \text{if } t\in(t_i,t_{i+1}],\ i=1,2,\dots,m.\end{cases} \tag{2.6}$$

Put

$$\widetilde{a}=\max_{j=1,\dots,m}A_j+2\Big(\sum_{j=1}^{m}|c_j|+\delta\,T\Big)$$

and suppose that $k\in[c-\delta,c+\delta]$. Then

$$\varphi(t_1,\widetilde{a},k)=\widetilde{a}+k\,t_1\ge A_1+\big(1-\frac{t_1}{T}\big)\sum_{j=1}^{m}|c_j|\ge A_1.$$

Furthermore, by (2.5), we have $|\varepsilon_1(\varphi(t_1,\widetilde{a},k))|\le\dfrac{\delta T}{m}$. Consequently, in view of (2.6), we get

$$\begin{aligned}\varphi(t_1+,\widetilde{a},k)&=\widetilde{a}+k\,t_1+c_1+\varepsilon_1(\varphi(t_1,\widetilde{a},k))\\&\ge A_2+|c_1|+\delta\,T+(1-\frac{t_1}{T})\sum_{j=1}^{m}|c_j|-|c_1|-\frac{\delta T}{m}\ge A_2.\end{aligned}$$

Now, let $1<i\le m$ and let

$$\varphi(t_j+,\widetilde{a},k)\ge A_{j+1}\quad\text{and}\quad\varphi(t_j,\widetilde{a},k)\ge A_j \tag{2.7}$$

for each $j=1,2,\dots,i-1$. Then, by (2.5), we have

$$|\varepsilon_j(\varphi(t_j,\widetilde{a},k))|\le\frac{\delta T}{m}\quad\text{for each}\;\; j=1,2,\dots,i-1.$$

Hence, using (2.6) and (2.5), we get

$$\varphi(t_i,\widetilde{a},k)\ge A_i+\sum_{j=1}^{m}\big(1-\frac{t_i}{T}\big)\,|c_j|+\sum_{j=i}^{m}|c_j|+\big(1-\frac{i-1}{m}\big)\,\delta\,T\ge A_i$$

and $|\varepsilon_i(\varphi(t_i,\widetilde{a},k))|\le\dfrac{\delta T}{m}$. In view of (2.6), we have

$$\begin{aligned}\varphi(t_i+,\widetilde{a},k)&=\widetilde{a}+k\,t_i+\sum_{j=1}^{i}(c_j+\varepsilon_j(\varphi(t_j,\widetilde{a},k)))\\&\ge A_{i+1}+\sum_{j=1}^{m}\big(1-\frac{t_i}{T}\big)\,|c_j|+\big(1-\frac{i}{m}\big)\,\delta\,T\ge A_{i+1}\end{aligned}$$

which means that (2.7) is true for any $j\in\{1,2,\dots,m\}$. Similarly we can show that $\varphi(T,\widetilde{a},k)\ge A_m$. Thus

$$\begin{cases}\varphi(t,\widetilde{a},k)\ge\widetilde{A}\quad\text{and}\quad|\varepsilon_i(\varphi(t_i,\widetilde{a},k))|<\dfrac{\delta T}{m}\\\qquad\text{for all}\;\; t\in[0,T],\;\; k\in[c-\delta,c+\delta]\quad\text{and}\quad i=1,2,\dots,m.\end{cases}\tag{2.8}$$

• STEP 2. We will prove that there is $\widetilde{k}\in(c-\delta,c+\delta)$ such that

$$\varphi(0,\widetilde{a},\widetilde{k})=\varphi(T,\widetilde{a},\widetilde{k}).\tag{2.9}$$

By virtue of (2.6) and (2.8), $\sum_{i=1}^m |\varepsilon_i(\varphi(t_i,\widetilde{a},k)| < \delta\, T$ and

$$\varphi(T,\widetilde{a},k) - \varphi(0,\widetilde{a},k) = T\left(k - c + \frac{1}{T}\sum_{i=1}^{m} \varepsilon_i(\varphi(t_i,\widetilde{a},k))\right)$$

for each $k \in [c-\delta, c+\delta]$. In particular, $\varphi(T,\widetilde{a},c-\delta) - \varphi(0,\widetilde{a},c-\delta) < 0$ and $\varphi(T,\widetilde{a},c+\delta) - \varphi(0,\widetilde{a},c+\delta) > 0$. Since φ is continuous, the existence of $\widetilde{k} \in (c-\delta, c+\delta)$ satisfying (2.9) follows.

- STEP 3. Define $\sigma_2(t) = \varphi(t,\widetilde{a},\widetilde{k})$ for $t \in [0,T]$. Then $\sigma_2'(t) = \widetilde{k}$ for $t \in [0,T]$ and $\sigma_2(t_i+) = \mathrm{J}_i(\sigma_2(t_i))$ for $i = 1,2,\dots,m$. By (2.4) we have $\widetilde{k} \le \mathrm{M}_i(\widetilde{k})$, i.e. σ_2 satisfies (1.9). Moreover, by (2.8) and (2.9), we have $\sigma_2(t) \ge \widetilde{A}$ on $[0,T]$ and $\sigma_2(0) = \sigma_2(T)$ and so (1.10) is true. Finally, by (2.3), σ_2 fulfils (1.8), i.e. σ_2 is an upper function for (1.1)–(1.3).

The case $\nu = 1$ can be treated analogously. □

Theorem 2.1 gives piecewise linear lower and upper functions which are bounded below. Now we will show conditions guaranteeing the existence of lower or upper functions bounded above. This is the contents of the next theorem. Its proof is similar to that of Theorem 2.1.

2.2 Theorem. *Assume* (1.4). *Further, let* $d_i \in \mathbb{R}$, $i = 1,2,\dots,m$, d, $B \in \mathbb{R}$, $\eta > 0$ *and* $\nu \in \{1,2\}$ *be such that*

$$\lim_{x\to-\infty} (\mathrm{J}_i(x) - x) = d_i \in \mathbb{R} \quad \textit{for } i = 1,2,\dots,m, \qquad d = -\frac{1}{T}\sum_{i=1}^{m} d_i,$$

$$(-1)^\nu f(t,x,y) \ge 0 \quad \textit{for a.e. } t \in [0,T] \;\; \textit{and all } x \le B,\; y \in [d-\eta, d+\eta],$$

$$(-1)^\nu (M_i(y) - y) \ge 0 \quad \textit{for } y \in [d-\eta, d+\eta],\; i = 1,2,\dots,m,$$

Then for each $\widetilde{B} \le B$ *there exist* $\widetilde{k} \in (d-\eta, d+\eta)$ *and* $\sigma_\nu \in \mathbb{AC}^1_{\mathrm{D}}[0,T]$ *such that* $\sigma_\nu(t) \le \widetilde{B}$, $\sigma_\nu'(t) = \widetilde{k}$ *for* $t \in [0,T]$ *and, for* $\nu = 1$ ($\nu = 2$), σ_ν *is a lower* (*upper*) *function of* (1.1)–(1.3).

2.3 Remark. Let (1.4) hold. Assume that $c_i, d_i \in \mathbb{R}$ and $A \in (0,\infty)$ are such that

$$\mathrm{J}_i(x) = \begin{cases} x + c_i & \text{for } x \ge A \\ x + d_i & \text{for } x \le -A, \; i = 1,2,\dots,m, \end{cases}$$

$$\text{(2.10)} \qquad \begin{cases} f(t,x,c) \ge 0 & \text{for a.e. } t \in [0,T] \text{ and all } x \ge A, \\ f(t,x,d) \le 0 & \text{for a.e. } t \in [0,T] \text{ and all } x \le -A, \end{cases}$$

and

$$(2.11) \qquad \mathrm{M}_i(c) \ge c, \quad \mathrm{M}_i(d) \le d, \quad i = 1,2,\dots,m.$$

Let $\widetilde{A} \ge A$. Then, according to the proof of Theorem 2.1, we can see that

$$\sigma_2(t) = \begin{cases} \widetilde{A} + \sum_{j=1}^m |c_j| + c\,t & \text{for } t \in [0,t_1), \\ \widetilde{A} + \sum_{j=1}^m |c_j| + c\,t + \sum_{j=1}^i c_j & \text{for } t \in (t_i, t_{i+1}], \quad i = 1,2,\dots,m \end{cases}$$

and

$$\sigma_1(t) = \begin{cases} -\widetilde{A} - \sum_{j=1}^m |d_j| + d\,t & \text{for } t \in [0,t_1), \\ -\widetilde{A} - \sum_{j=1}^m |d_j| + d\,t + \sum_{j=1}^i d_j & \text{for } t \in (t_i, t_{i+1}], \quad i = 1,2,\dots,m \end{cases}$$

are respectively upper and lower functions of (1.1)–(1.3) satisfying

$$\widetilde{A} \le \sigma_2(t) \le \widetilde{A} + 2\sum_{j=1}^m |c_j| \quad \text{and} \quad -\widetilde{A} - 2\sum_{j=1}^m |d_j| \le \sigma_1(t) \le -\widetilde{A} \quad \text{for } t \in [0,T].$$

If all inequalities in (2.10) and (2.11) are reversed, then σ_2 becomes a lower function and σ_1 an upper function.

3. New existence criteria

Our main results are Theorems 3.1 and 3.2 which provides new existence criteria for the problem (1.1)–(1.3).

3.1 Theorem. *Let the assumptions of Theorem* 2.1 *be satisfied for* $\nu = 2$ *and let the assumptions of Theorem* 2.2 *be satisfied for* $\nu = 1$. *Assume that* J_i *are increasing on* $\mathbb{R}$ *and* M_i *are nondecreasing on* $\mathbb{R}$ *for* $i = 1,2,\dots,m$. *Finally, let for each compact interval* $K \subset \mathbb{R}$ *there exist* $h_K \in \mathbb{L}[0,T]$ *and* $\omega_K \in \mathbb{C}([1,\infty))$ *such that* $h_K \ge 0$ *on* $[0,T]$, $\omega_K > 0$ *on* $[1,\infty)$, $\int_1^\infty ds/\omega_K(s) = \infty$ *and* $|f(t,x,y)| \le \omega_K(|y|)\,(|y| + h_K(t))$ *for a.e.* $t \in [0,T]$ *and all* $x \in K$, $|y| > 1$. *Then the problem* (1.1)−(1.3) *has a solution.*

Proof. By Theorem 2.1, for each $\widetilde{A} \ge A$, there is an upper function σ_2 of (1.1)–(1.3) such that $\sigma_2 \ge \widetilde{A}$ on $[0,T]$. By Theorem 2.2, for each $\widetilde{B} \le B$, there is a lower function σ_1 of (1.1)–(1.3) such that $\sigma_1 \le \widetilde{B}$ on $[0,T]$. Choose $\widetilde{A}$, $\widetilde{B}$ in such a way that $\widetilde{B} \le \widetilde{A}$. Hence $\sigma_1 \le \sigma_2$ on $[0,T]$ and all the assumptions of [5, Theorem 3.1] are satisfied. Therefore (1.1)–(1.3) has a solution. □

3.2 Theorem. *Let the assumptions of Theorem* 2.1 *be satisfied for* $\nu = 1$ *and let the assumptions of Theorem* 2.2 *be satisfied for* $\nu = 2$. *Assume that* J_i *are increasing on* $\mathbb{R}$ *and* M_i *are nondecreasing on* $\mathbb{R}$ *for* $i = 1,2,\dots,m$. *Finally, let there exist* $h \in \mathbb{L}[0,T]$ *such that* $|f(t,x,y)| \le h(t)$ *for a.e.* $t \in [0,T]$ *and all* $x,y \in \mathbb{R}$. *Then the problem* (1.1)−(1.3) *has a solution.*

Proof. By Theorems 2.1 and 2.2 there are a lower function σ_1 and an upper function σ_2 of (1.1)–(1.3). The existence of a solution to (1.1)–(1.3) follows by [7, Theorem 3.1]. ☐

3.3 Example. Let $k \in \mathbb{N} \cup \{0\}$, $\gamma \in (0,\infty)$, $p_j \in \mathbb{L}_\infty[0,T]$, $j = 0,1,\dots,2k$, $p_{2k+1} \in \mathbb{L}[0,T]$, $p_{2k+1} \ge \gamma$ a.e. on $[0,T]$, $q_1, q_2 \in \mathbb{L}_\infty[0,T]$, $c_i \in (0,\infty)$, $\alpha_i \in [0,1)$ and $\beta_i \in \mathbb{R}$, $i = 1,2,\dots,m$. Consider the problem (1.1)–(1.3), where

$$f(t,x,y) = \sum_{j=0}^{2k+1} p_j(t)\,x^j + q_1(t)\,y + q_2(t)\,y^2,\ \mathrm{J}_i(x) = x + \frac{2c_i}{\pi}\arctan x,\ \mathrm{M}_i(y) = \alpha_i\,y + \beta_i$$

for a.e. $t \in [0,T]$, all $x, y \in \mathbb{R}$ and $i = 1,2,\dots,m$. Let c be given by (2.2) and let $\beta_i \in (c\,(1-\alpha_i), -c\,(1-\alpha_i))$, $i = 1,2,\dots,m$. Then the conditions of Theorem 2.1 are satisfied for $\nu = 2$ and the conditions of Theorem 2.2 are satisfied for $\nu = 1$ and $d_i = -c_i$, $i = 1,2,\dots,m$. Since $c_i > 0$ and $\alpha_i \ge 0$, the functions J_i are increasing and M_i are nondecreasing on $\mathbb{R}$ for $i = 1,2,\dots,m$. Choose an arbitrary compact interval $K \subset \mathbb{R}$ and denote $\varkappa_1 = \max_{x\in K}\left(|x|^{2k+1}\right)$ and $\varkappa_2 = \sum_{j=1}^{2k}\left(\|p_j\|_\infty \max_{x\in K}|x|^j\right)$. Then $|f(t,x,y)| \le \omega(|y|)\,(|y| + h(t))$ for a.e. $t \in [0,T]$ and all $x \in K$, $|y| > 1$, where $\omega(s) = 1 + \|q_1\|_\infty + \|q_2\|_\infty\, s$ and $h(t) = \varkappa_1\,|p_{2k+1}(t)| + \varkappa_2$. Thus, the existence of a solution to (1.1)–(1.3) follows by means of Theorem 3.1.

3.4 Example. Let $\gamma \in (0,\infty)$, $p \in \mathbb{L}_\infty[0,T]$, $q_1 \in \mathbb{L}[0,T]$, $q_2 \in \mathbb{L}_\infty[0,T]$, $q_1 \ge 0$, $q_2 \ge 0$ and $q_1 + q_2 \ge \gamma$ a.e. on $[0,T]$, $\varphi \in \mathbb{C}(\mathbb{R})$, $\lim_{|x|\to\infty}\varphi(x) = 0$. Consider the problem (1.1)–(1.3), where

$$f(t,x,y) = p(t)\,\varphi(x) + q_1(t)\,y + q_2(t)\,y^2\,\operatorname{sgn} y \quad \text{for a.e. } t \in [0,T] \text{ and all } x, y \in \mathbb{R}$$

and $\mathrm{J}_i(x)$ and $\mathrm{M}_i(y)$, $i = 1,2,\dots,m$, are given as in Example 3.3, but with $c_i \in (-\pi/2, 0)$ and $\alpha_i \in (1,\infty)$, $i = 1,2,\dots,m$. Let c be given by (2.2) and let $\beta_i \in (c\,(1-\alpha_i), c\,(\alpha_i - 1))$ for $i = 1,2,\dots,m$. We have $c > 0$, $M_i(c) > c$, $M_i(-c) < -c$, $f(t,x,c) > 0$ and $f(t,x,-c) < 0$ for a.e. $t \in [0,T]$ and all $x \in \mathbb{R}$ with $|x|$ sufficiently large. Thus, the assumptions of Theorem 2.1 are satisfied for $\nu = 2$ and the assumptions of Theorem 2.2 are satisfied for $\nu = 1$ and $d_i = -c_i$, $i = 1,2,\dots,m$. Furthermore, J_i are increasing and M_i are nondecreasing on $\mathbb{R}$ for $i = 1,2,\dots,m$. Since φ is bounded on $\mathbb{R}$, we can find $\omega \in \mathbb{C}([1,\infty))$ such that $\omega > 0$ on $[1,\infty)$, $\int_1^\infty \mathrm{d}s/\omega(s) = \infty$ and $|f(t,x,y)| \le \omega(|y|)|y|$ for a.e. $t \in [0,T]$ and all $x \in \mathbb{R}$, $|y| > 1$. Thus, by Theorem 3.1, the given problem has a solution.

3.5 Example. Let $\gamma \in (0,\infty)$, $p \in \mathbb{L}_\infty[0,T]$, $q \in \mathbb{L}[0,T]$, $q \ge \gamma$ a.e. on $[0,T]$, $\varphi \in \mathbb{C}(\mathbb{R})$, $\lim_{|x|\to\infty}\varphi(x) = 0$. Consider the problem (1.1)–(1.3), where

$$f(t,x,y) = p(t)\,\varphi(x) + \frac{q(t)\,y}{1+y^2} \quad \text{for a.e. } t \in [0,T] \text{ and all } x, y \in \mathbb{R}$$

and $\mathrm{J}_i(x)$ and $\mathrm{M}_i(y)$, $i = 1, 2, \ldots, m$, are given as in Example 3.4, but with $c_i \in (0, \infty)$. We can see that the assumptions of Theorem 2.1 are satisfied for $\nu = 1$ and the assumptions of Theorem 2.2 are satisfied for $\nu = 2$ and $d_i = -c_i$, $i = 1, 2, \ldots, m$. As in the previous examples, J_i are increasing and M_i are nondecreasing on $\mathbb{R}$ for $i = 1, 2, \ldots, m$. Moreover, since the functions φ and $\frac{y}{y^2+1}$ are bounded on $\mathbb{R}$, we can find $h \in \mathbb{L}[0, T]$ such that $|f(t, x, y)| \le h(t)$ for a.e. $t \in [0, T]$ and all $x, y \in \mathbb{R}$. Thus, by Theorem 3.2, our problem has a solution.

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