Sunflowers and Fibonacci series

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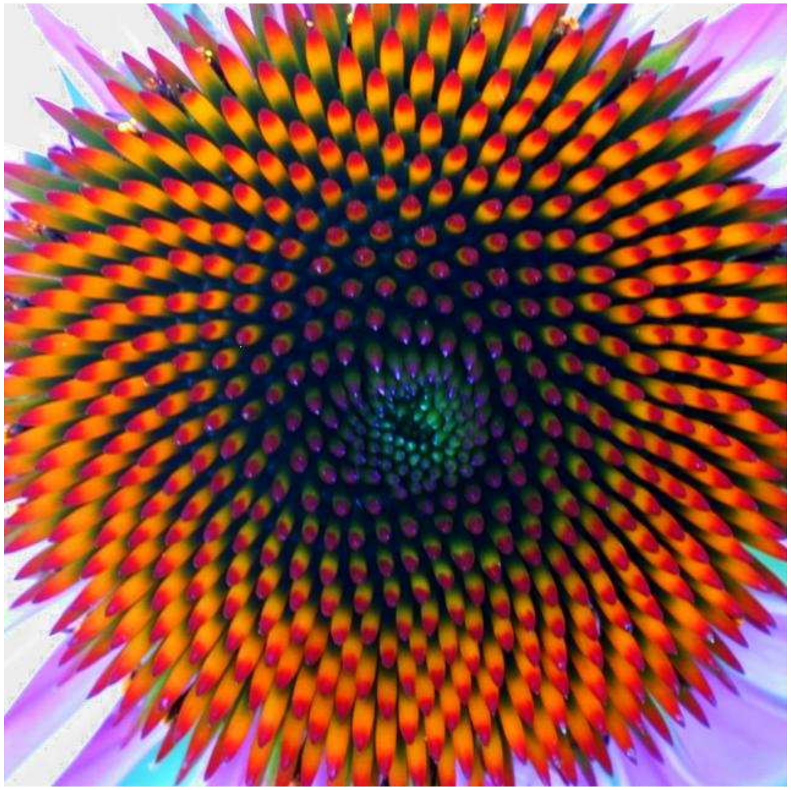
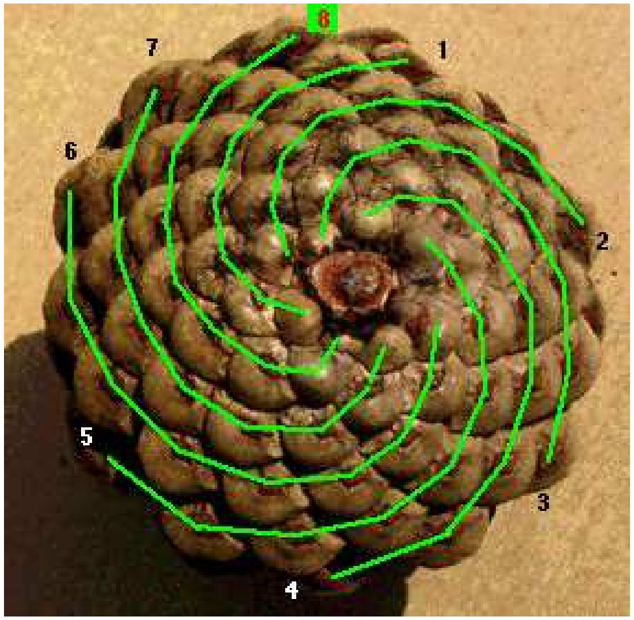
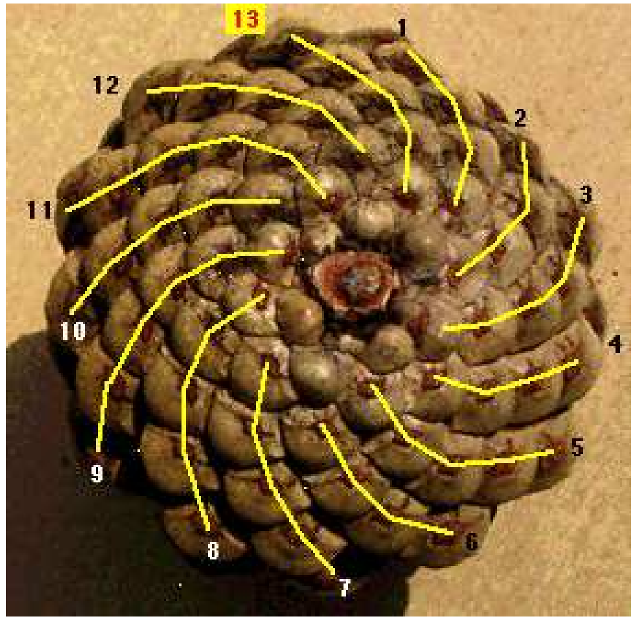


Look over the pictures carefully ...





... spot the numbers of spirals, do they have something in common?

Observations

Phyllotaxis (Greek phyllon = leaf, taxis = ordering) is ordering of the elements (seeds, leaves, scales, florets ...) of a plant.

The eye is attracted to conspicuous spirals, the parastichies, linking each element to its nearest neighbors.

The whole surface is covered with a number i of parallel spirals running in one direction, and j in the other.

Main observation: the most striking feature is that (i, j) are nearly always two consecutive numbers of the Fibonacci series (1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144,...)

A bit of history

- Theopharastus (370-285 B.C.) says about plants that *those that have flat leaves have them in a regular series*

- Leonardo Fibonacci of Pisa (1175-1240). Solved a practical problem of his father, merchant, concerning the monthly growth of a population of rabbits.

- around 1830 Karl Shimper and Alexander Braun - first spotted connection between number of spirals and Fibonacci sequence. They pointed out the importance of divergence angle. They measured this angle and what they got was the Golden angle $\gamma = 137°30'28''$

$$\frac{360° - \gamma}{\gamma} = \tau = \frac{1 + \sqrt(5)}{2}.$$

Why Golden angle?

Simple model:

Each seed is going straight on from the middle to the border with velocity $v \approx \sqrt{t}$. The next one grows in time unit. The angle between directions of two following seeds is ϕ.

Question:

What should be ϕ, or the ratio $r = \phi/2\pi$, to reproduce the observed facts?

We simulate this model behavior with Maple.

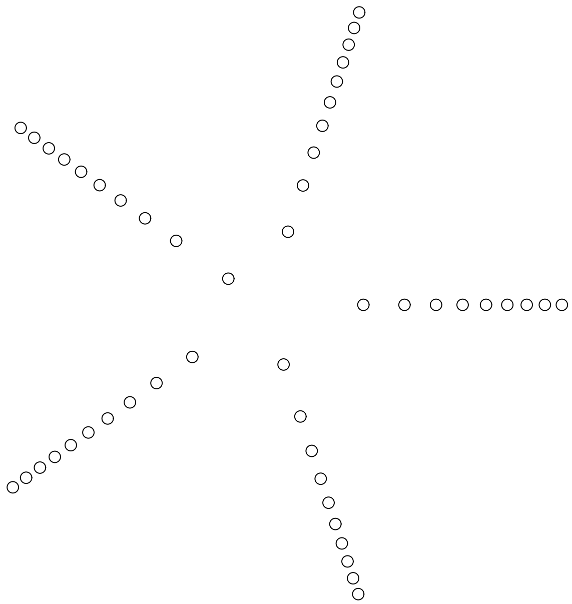


Figure 1: $r = 0.6 = \frac{3}{5}$.

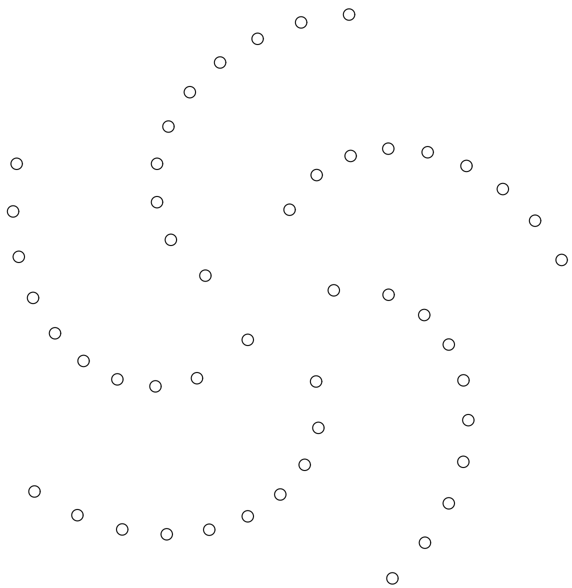


Figure 2: $r = 0.605$.

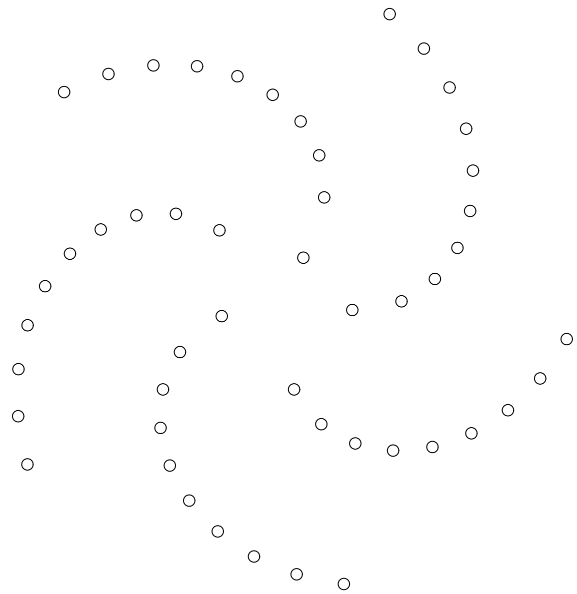


Figure 3: $r = 0.595$.

What we see: For r rational $r = p/q$ we observe q branches. If r is near to a rational number with a small denominator we see q spirals.

Suggestion: Use irrational number r. For example $r = \pi$.

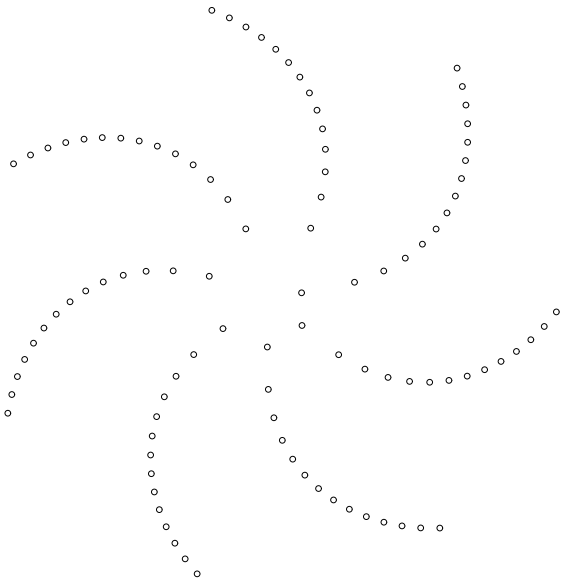


Figure 4: $t = 100$.

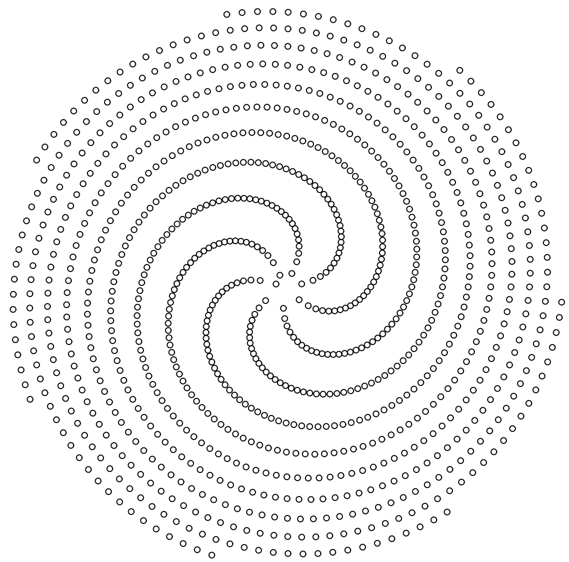


Figure 5: $t = 1000$.

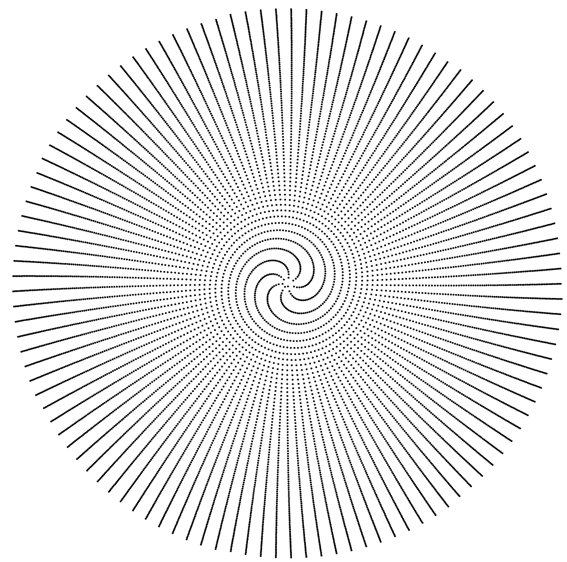


Figure 6: $t = 10000$.

Good rational approximation for π are $3, \frac{22}{7}, \frac{355}{113}, \ldots$. Each number can be uniquely written in the form of continued fraction from which we can see the rational approximations

$$\pi = 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1 + \frac{1}{292 + \ldots}}}}$$

Good candidate: Number with very bad approximations! The most irrational!

$$z = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \dots}}}} = \frac{1}{1+z} \quad \to z = \frac{1+\sqrt{5}}{2}$$

Golden section! Its rational approximations $1, 2, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \dots$

Moreover an inequality holds:

$$1, \frac{3}{2}, \frac{8}{5}, \frac{21}{13}, \dots < z < 2, \frac{5}{3}, \frac{13}{8}, \frac{34}{21}, \dots$$

That explains the direction of spirals.

With divergence of golden section we can see spirals in both directions. The number of them is two consecutive Fibonacci numbers.

At the same time the use of space is good.

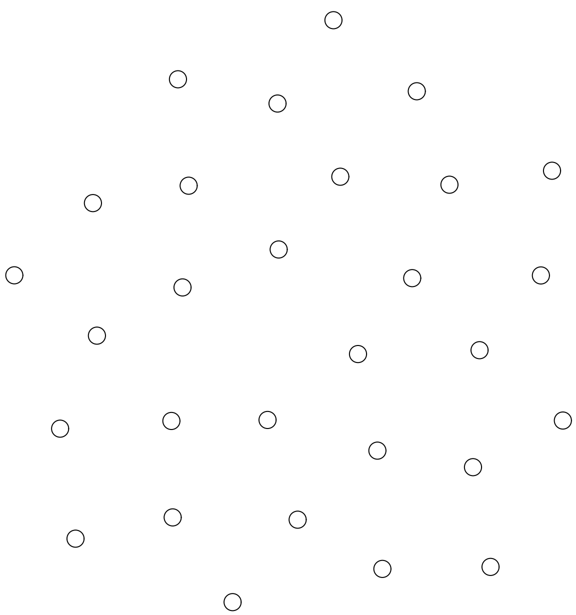


Figure 7: $t = 30$.

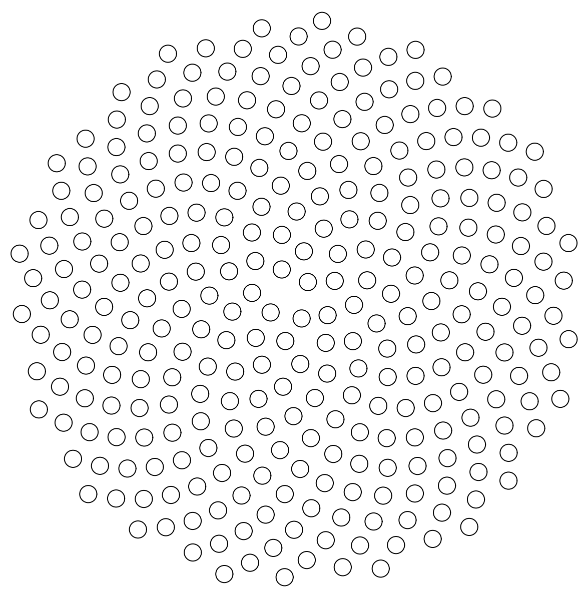


Figure 8: $t = 300$.

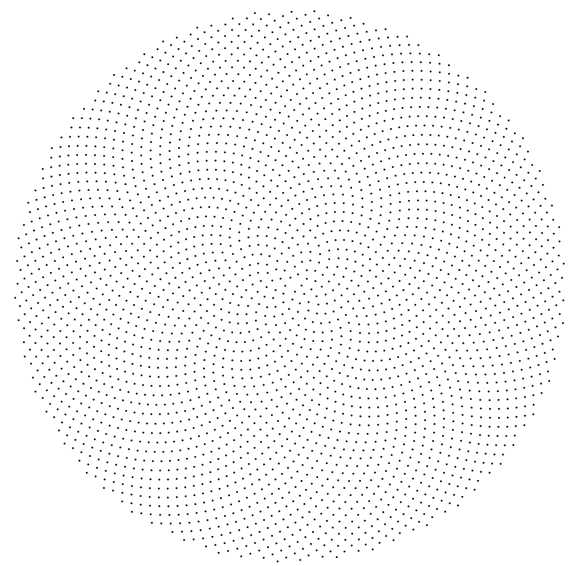
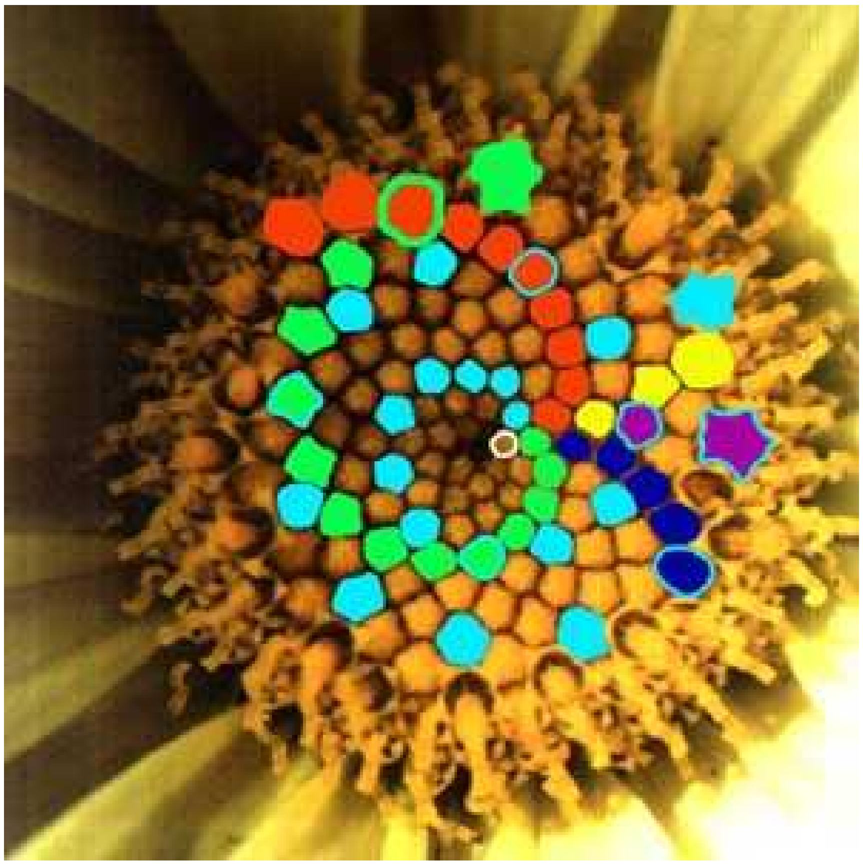
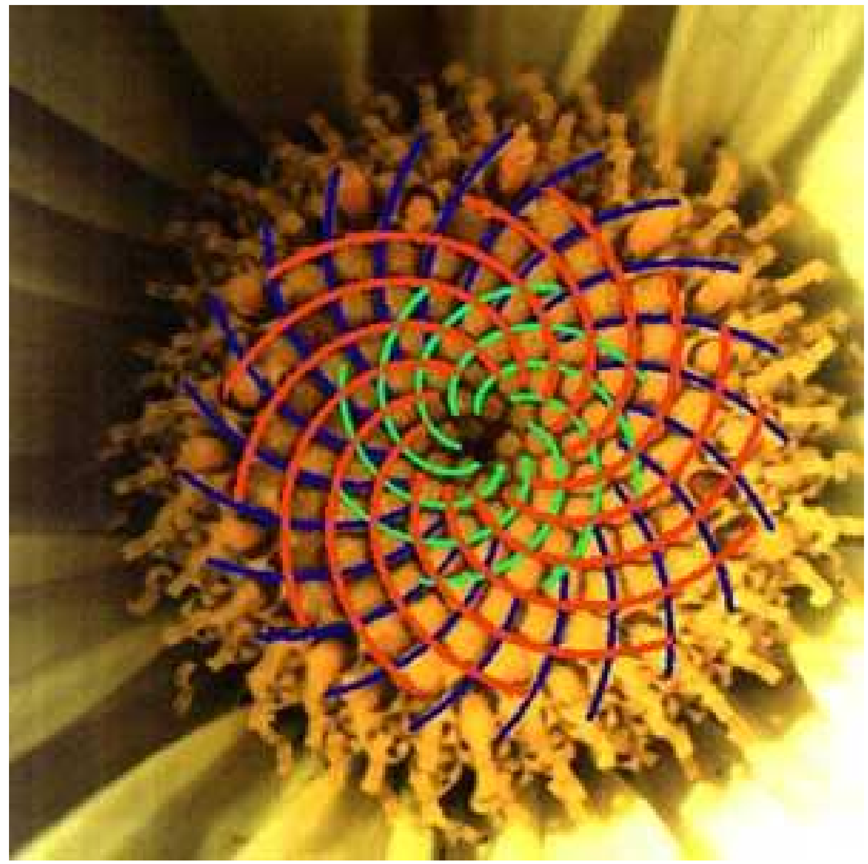


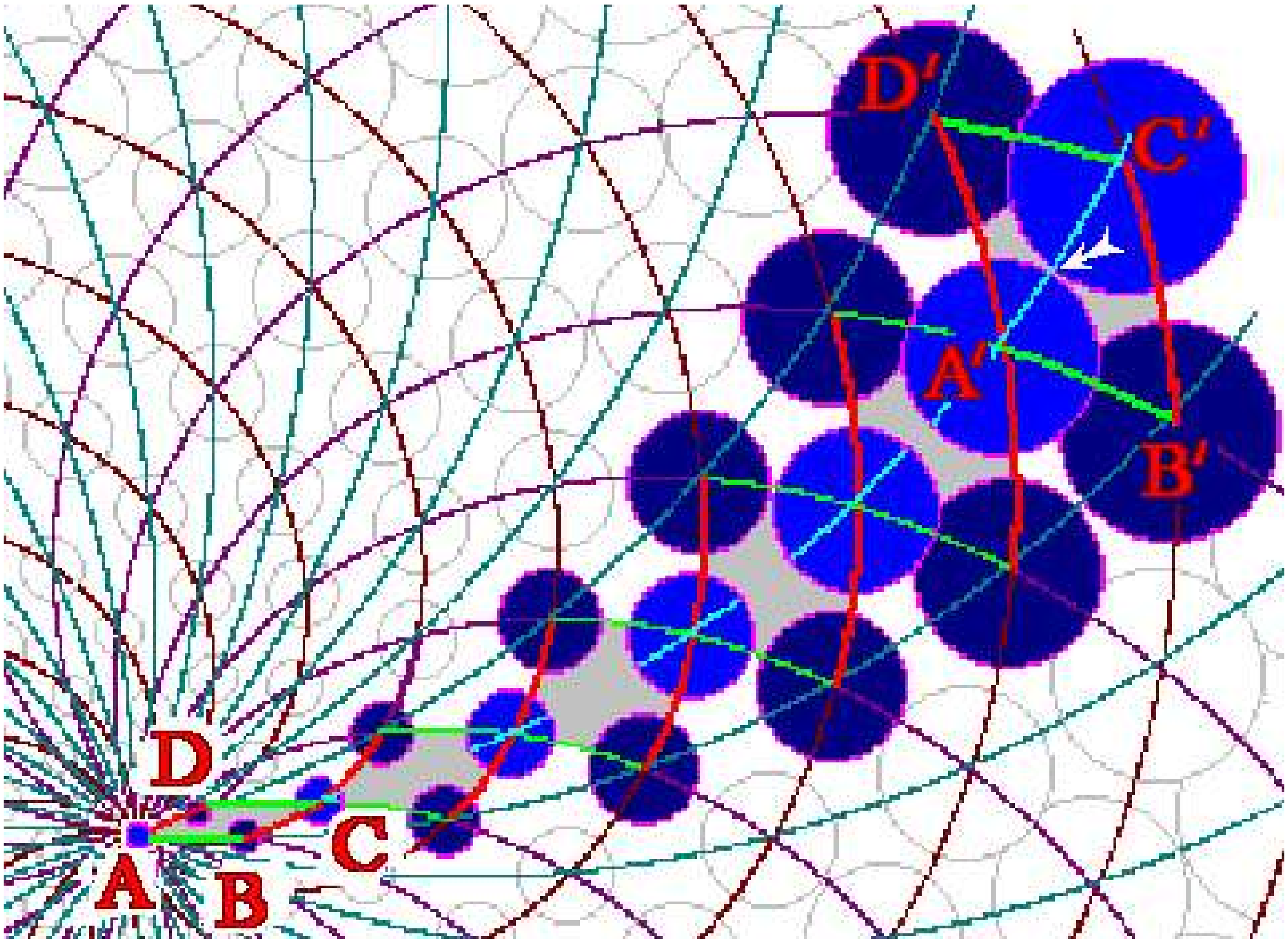
Figure 9: $t = 3000$.

Which two Fibonacci numbers do we see, or which spirals are the most visible, depends on the number of seeds (size of the flower).





How the seeds change its neighbors?



BUT!!!

There are other number with bad rational approximation, for example:

$$z_1 = \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \dots}}}} \quad \text{or} \quad z_2 = \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \dots}}}} \quad \text{or}$$

$$z_3 = \sqrt{2} - 1 = \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \dots}}}}.$$

- z_1: $1, 3, 4, 7, 11, 18, 29, \dots$ spirals,
- z_2: $1, 4, 5, 9, 14, 23, 37, \dots$ spirals,
- z_3: $2, 5, 12, 29, 70, \dots$ spirals.

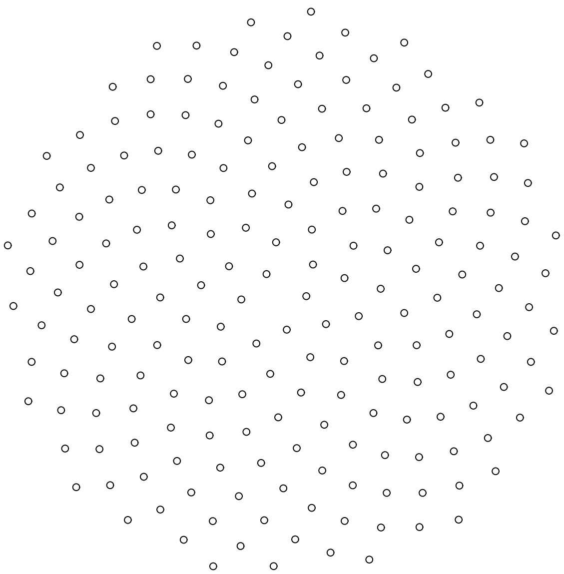


Figure 10: $r = z_1$.

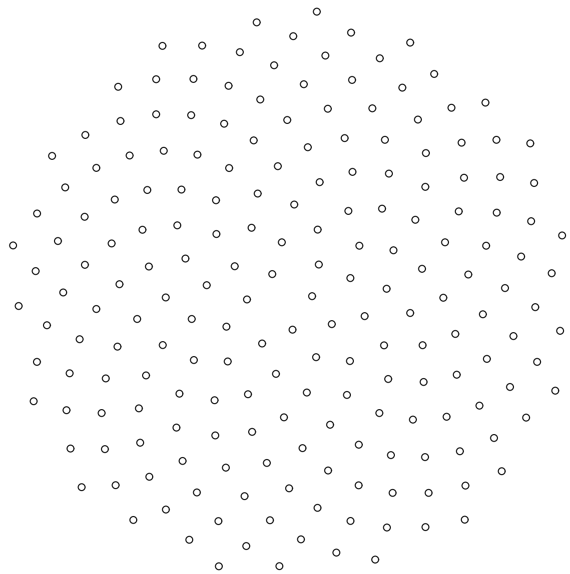


Figure 11: $r = z_2$.

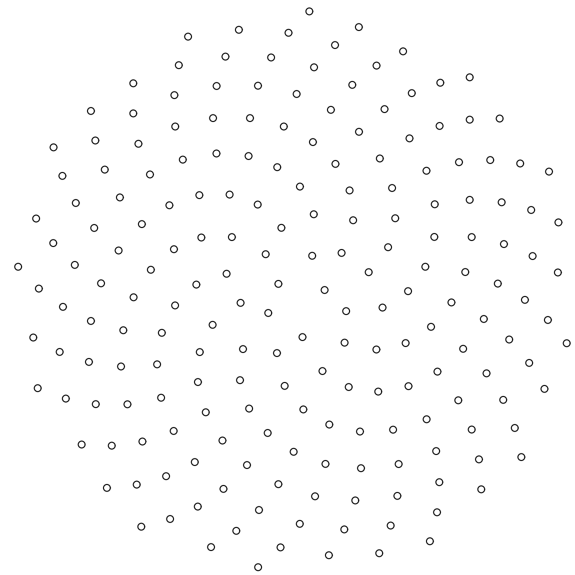


Figure 12: $r = z_3$.

Not very different from the previous patterns ...

The second a bit of history

- Around 1835 Louis and August Bravais: point lattice on a cylinder, then Auguste crystallography.
- 1868 Hofmeister: the new seed appears in the largest available gap.
- 1875 Wiesner: the least shading of the lower leaves by the upper ones.
- 1878 Simon Schwendener: result of contact pressure, constructed mechanical machine, exact measurements.
- 1913 Schoute: chemical inhibitor, not found up to now.
- 1907 G. van Iterson: geometrical model of close packing, got the right result together with many "wrong", did not know how to explain.

Big question

In 20th century two groups of scientists:

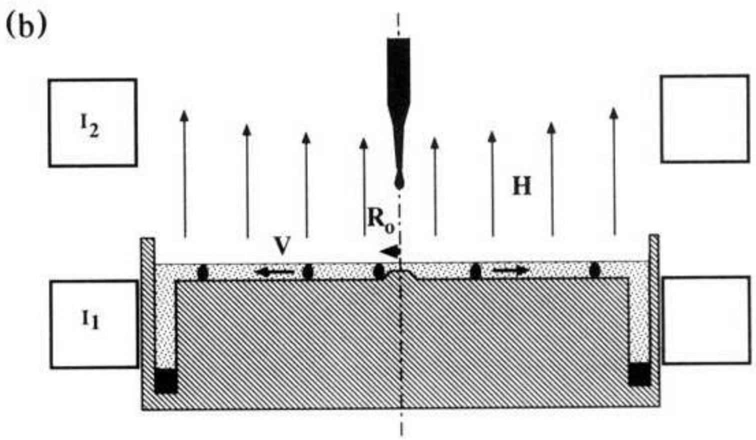
- Evolutionary explanation: The "golden" packing is the best one. Current plants were chosen by the natural selection.
- Natural selection is not excuse for everything. There should be different explanation.

Who is right?

Energy model

Douady S., Couder Y., Physical Review Letters 68 (13), 2098 (1992)

Experiment with ferro-fluid. Relevant parameter: $G = \frac{VT}{R_0}$.



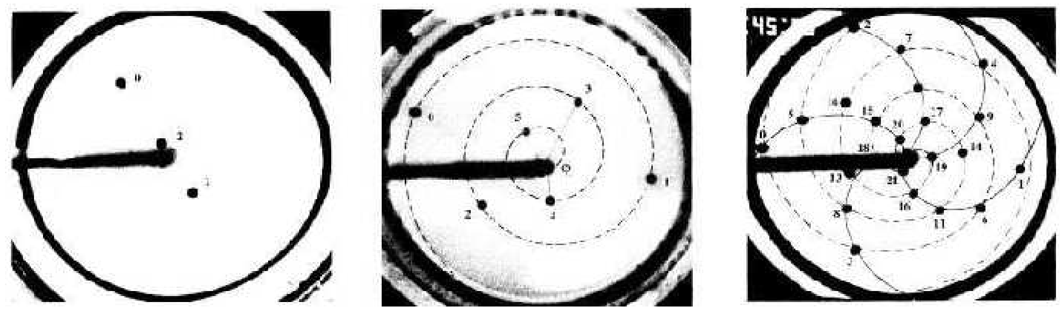
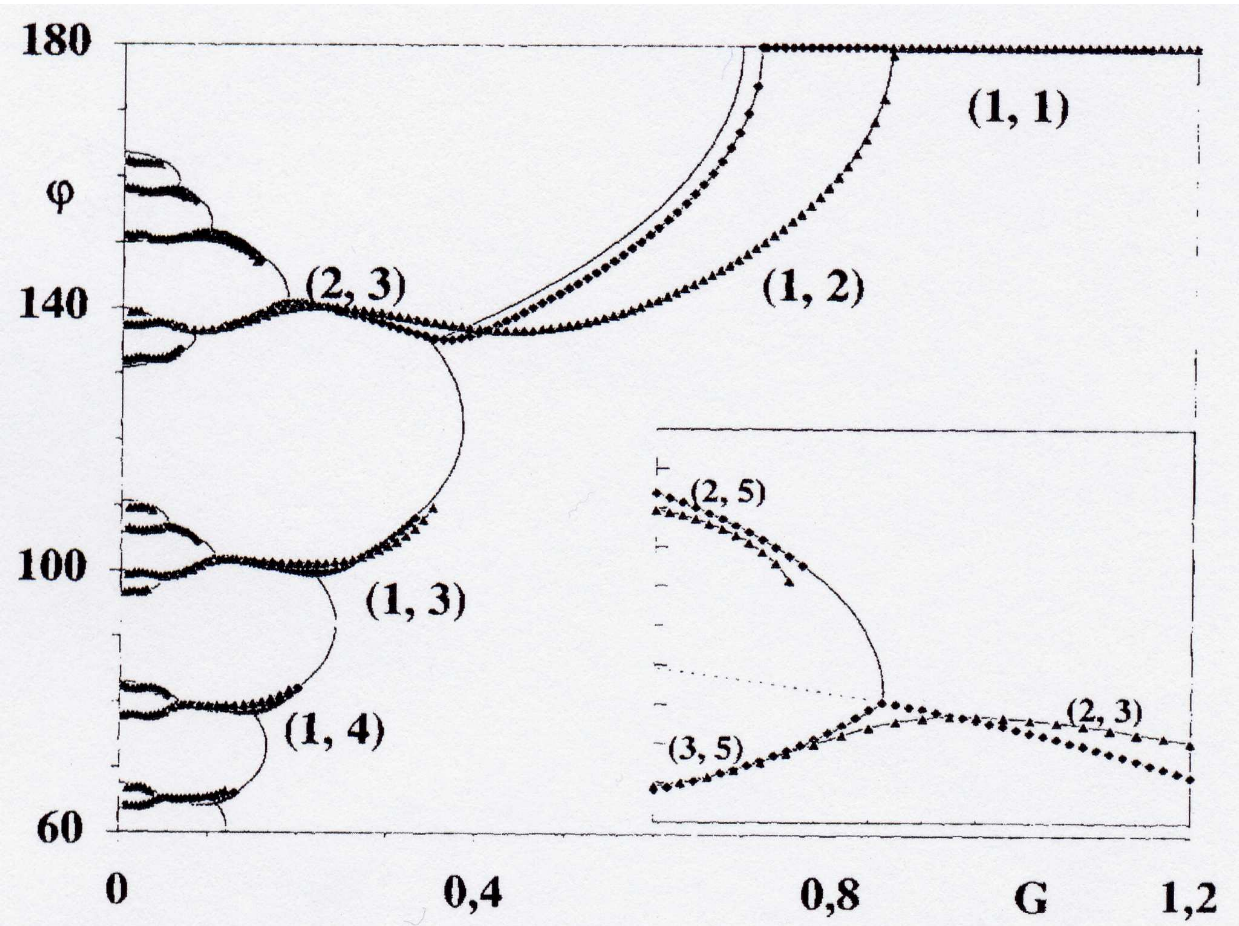


Figure 13: a) $G \approx 1$, $\phi \approx 180°$, b) $G \approx 0.7$, $\phi \approx 150°$, c) $G \approx 0.15$, $\phi \approx 139°$ $(i = 3,\ j = 5)$.

Iterson-like bifurcation diagram. But!! Bifurcation point are disconnected, the only connected line is going to the Golden angle!



Summary

- The patterns are given just by the growth process. The sunflower would not know what is its connection with Fibonacci (it is not written in its DNA).

- The phyllotaxis goes beyond botany. Levitov found ”phyllotactic patterns” in a flux lattice of a superconductor.