Thermal fluctuations and order parameters in the SK model of a spin glass

Asymptotic solution near the critical point

V. Janiš

FZÚ AV ČR, 13/12/2005

Grant: GAAV IAA-1010307 (2003-5)
Collaborators: L. Zdeborová & A. Klíč

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Outline of Part I

1. *Mean-field theory for spin glasses*
   - Sherrington-Kirkpatrick model
2. *Averaging over randomness*
   - Replica trick
   - Parisi RSB solution
3. *Summation over spin configurations*
   - TAP free energy
   - TAP & RSB

Outline of Part II

4 *Hierarchical TAP theory*
- Thermodynamic homogeneity and multiple TAP states

5 *One-level hierarchical solution*
- 1-TAP free energy and order parameters
- Stability conditions

6 *Asymptotic solution near the critical point*
- Fixed internal magnetic field
- Equilibrium value of the local field expanded

Outline

Sherrington-Kirkpatrick model

- Ising Hamiltonian (classical spins) $S_I = \pm 1$

$$H[J,S] = \sum_{i<j} J_{ij} S_i S_j + h \sum_i S_i$$

- Long-range random spin couplings

$$N \langle J_{ij} \rangle_{av} = \sum_{j=1}^{N} J_{ij} = 0, \quad N \left\langle J_{ij}^2 \right\rangle_{av} = \sum_{j=1}^{N} J_{ij}^2 = J^2$$

Spin couplings J_{ij} : Gaussian random variables

- Free energy (self-averaging) – summation over lattice sites ⇔ averaging over spin couplings (ergodic theorem)

$$F = -\frac{1}{\beta} \lim_{N\to\infty} \ln \mathrm{Tr}_S \left[\exp\left\{-\beta H[J,S]\right\}\right] = -\frac{1}{\beta} \lim_{N\to\infty} \left\langle \ln \mathrm{Tr}_S \left[\exp\left\{-\beta H[J,S]\right\}\right]\right\rangle_{av}$$

Averaging the logarithm not straightforward

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Averaging the logarithm not straightforward

Outline

Replica trick – basic idea

Replica trick: averaging of logarithm (quenched) converted to averaging of a partition function (annealed)

- Logarithm: limit of the replication factor to zero (derived perturbatively)

$$\ln Z = \lim_{n \to 0} \frac{1}{n}(Z^n - 1)$$

with the replicated partition function (n integer)

$$Z^n = \prod_{i<j} \int dJ_{ij} \mu(J_{ij}) \prod_{\alpha=1}^{n} \prod_{i=1}^{N} \int dS_i^\alpha \rho(S_i^\alpha) \exp\{-\beta H[J, S^\alpha]\}$$

- Integration over spin configurations and randomnes interchanged
- After averaging over randomness — partition sum diagonal in lattice indices & nondiagonal in replica indices

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Replica trick – averaging

- Saddle point ($N \to \infty$) evaluation of the integral over spin configurations – order parameters emerge –

matrices in the replica (integer) indices

$$Q^{\alpha\beta} = \frac{J^2}{N} \sum_i \langle S_i^\alpha S_i^\beta \rangle, \quad \alpha \neq \beta$$

- Free energy density averaged over spin couplings J_{ij}

$$f = \frac{1}{\beta} \max f_T(Q)$$

$$f_T(Q) = -\frac{\beta^2}{4} + \ln 2 + \lim_{n \to 0} \left\{ \frac{1}{4} \sum_{\alpha<\beta} \beta^2 Q_{\alpha\beta}^2 - \ln \left[\mathrm{Tr} \exp \left(\sum_{\alpha<\beta} \beta^2 Q_{\alpha\beta} S^\alpha S^\beta \right) \right] \right\}$$

Next: separation of summation over replica indices in order to perform explicitly summation over spin configurations

Problem: matrix (visual) representation for integer numbers of replicas — needed for real numbers $n \to 0$

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Replica trickk – analytic continuation

- Only specific matrices $n \times n$ allow for analytic continuation to real n
- The most general case – ultrametric structure

$$\begin{pmatrix} 0 & q_0 & q_1 & q_1 & q_2 & q_2 & q_2 & q_2 \\ q_0 & 0 & q_1 & q_1 & q_2 & q_2 & q_2 & q_2 \\ q_1 & q_1 & 0 & q_0 & q_2 & q_2 & q_2 & q_2 \\ q_1 & q_1 & q_0 & 0 & q_2 & q_2 & q_2 & q_2 \\ q_2 & q_2 & q_2 & q_2 & 0 & q_0 & q_1 & q_1 \\ q_2 & q_2 & q_2 & q_2 & q_0 & 0 & q_1 & q_1 \\ q_2 & q_2 & q_2 & q_2 & q_1 & q_1 & 0 & q_0 \\ q_2 & q_2 & q_2 & q_2 & q_1 & q_1 & q_0 & 0 \end{pmatrix}$$

Ultrametric structure — only bloc matrices of identical elements
— larger blocks multiples of smaller blocks
— hierarchy of embeddings around the diagonal

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Outline

Replica trick – decoupling of spin variables

- Mean-field approximation — effective separation of different spin replicas – makes summations over replica indices independent
- We convert $Q[S] = \sum_{\alpha<\beta} Q^{\alpha\beta} S^\alpha S^\beta$ to sums of squares
- K different values of $Q^{\alpha\beta}$: $q_1, q_2, \ldots, q_K$
- *Multilplicity* of individual values q_1 $(n_1 - 1)$-times, q_2 $(n_2 - n_1)$-times, …, q_K, $(n_K - n_{K-1})$-times
- Spin decouplings

$$2Q[S] = q_K \left(\sum_{\alpha=1}^{n_k=n} S^\alpha\right)^2 + (q_{K-1} - q_K) \sum_{i=1}^{n_k/n_{K-1}-1} \left(\sum_{\alpha=in_{K-1}+1}^{(i+1)n_{K-1}} S^\alpha\right)^2 \ldots - nq_1$$

Squares in the partition sum decoupled via Hubbard-Startonovich transformations

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Replica trick – continuous limit

- In RSB discrete hierarchies have no direct physical meaning — limit to infinite number of hierarchies $K \to \infty$
- Traces of functions of replica matrices

$$\lim_{n\to 0} \frac{1}{n} \mathrm{Tr} Q^m = -\sum_{l=1}^{K} (n_{l-1} - n_l) q_l^m$$

with $1 = n_0 > n_1 > \ldots > n_K \geq 0$

- Continuous limit: $K \to \infty, \quad n_{l-1} - n_l = dx, \quad n_l / n_{l+1} = 1 + g(x) dx$
- *Order parameters*: $q(x)$ for $x \in [0, 1]$

$$q(x) = q_l, \qquad 0 < n_l \leq x \leq n_{l-1} < 1$$

- Integral representation

$$\lim_{n\to 0} \frac{1}{n} \mathrm{Tr} Q^m = \int_0^1 d\mu(x) q(x)^m$$

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Replica trick – Parisi's solution

Parisi's RSB free energy

$$f_{av} = \max_{q(x)} k_B T \, f_T[q]$$

$$f_T[q] = -\frac{1}{4}\beta^2 \left(1 + \int_0^1 d\mu(x) q(x)^2 + 2q(1)\right) + \widetilde{f}_T[q]$$

$$\widetilde{f}_T[q] = -f(0,h)$$

$$\frac{\partial f(x,h)}{\partial x} = -\frac{1}{2}\frac{dq}{dx}\left[\frac{\partial^2 f(x,h)}{\partial h^2} + x\left(\frac{\partial f(x,h)}{\partial h}\right)^2\right]$$

$$f(1,h) = \ln\left[2\cosh(\beta h)\right]$$

Talagrand (04): RSB construction exact

$$f_{av} = f_{SK}$$

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Fixed configurations of spin couplings J_{ij}

MF ($d = \infty$) solution for SK model for a fixed configuration of spin couplings J_{ij}

- *Inhomogeneous free energy*: local magnetizations m_i and local internal magnetic fields η_i^0 – order parameters

$$F_{TAP} = \sum_i \left\{ m_i \eta_i^0 - \frac{1}{\beta} \ln 2\cosh[\beta(h + \eta_i^0)] \right\} - \frac{1}{2} \sum_{ij} \left[J_{ij} m_i m_j + \frac{1}{2} \beta J_{ij}^2 (1 - m_i^2)(1 - m_j^2) \right]$$

- Stationarity equations for the order parameters

$$m_i = \tanh[\beta(h + \eta_i^0)] ,$$

$$\eta_i^0 = \sum_j J_{ij} m_j - m_i \sum_j \beta J_{ij}^2 (1 - m_j^2)$$

- Numerical solution for finite volumes viable – many solutions (degenerate in free energy)

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Stability conditions

Linear susceptibility

Not all solutions of the TAP equations physically acceptable — only stable ones can represent equilibrium states

Positivity of *linear (nonlocal) susceptibility*

◂ nonlocal χ

$$\left(\chi^{-1}\right)_{ij} = \frac{\partial^2 \beta F_{TAP}}{\partial m_i \partial m_j} + \sum_l \left[\frac{\partial^2 \beta F_{TAP}}{\partial m_i \partial \eta_l^0}\frac{\partial \eta_l^0}{\partial m_j} + \frac{\partial^2 \beta F_{TAP}}{\partial m_j \partial \eta_l^0}\frac{\partial \eta_l^0}{\partial m_i}\right]$$

$$+ \sum_{kl} \frac{\partial^2 \beta F_{TAP}}{\partial \eta_k^0 \partial \eta_l^0}\frac{\partial \eta_k^0}{\partial m_i}\frac{\partial \eta_l^0}{\partial m_j} = -\beta J_{ij} + \delta_{ij}\left(\frac{1}{1 - m_i^2} + \sum_l \beta^2 J_{il}^2 (1 - m_l^2)\right)$$

Only *local minima* of F_{TAP} are physical

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Positivity of *linear (nonlocal) susceptibility*

nonlocal χ

$$\left(\chi^{-1}\right)_{ij} = \frac{\partial^2 \beta F_{TAP}}{\partial m_i \partial m_j} + \sum_l \left[\frac{\partial^2 \beta F_{TAP}}{\partial m_i \partial \eta_l^0}\frac{\partial \eta_l^0}{\partial m_j} + \frac{\partial^2 \beta F_{TAP}}{\partial m_j \partial \eta_l^0}\frac{\partial \eta_l^0}{\partial m_i}\right]$$

$$+ \sum_{kl} \frac{\partial^2 \beta F_{TAP}}{\partial \eta_k^0 \partial \eta_l^0}\frac{\partial \eta_k^0}{\partial m_i}\frac{\partial \eta_l^0}{\partial m_j} = -\beta J_{ij} + \delta_{ij}\left(\frac{1}{1-m_i^2} + \sum_l \beta^2 J_{il}^2 (1 - m_l^2)\right)$$

Only *local minima* of F_{TAP} are physical

Stability conditions
Spin-glass susceptibility

Uniqueness of the equilibrium state
– consistency demand from the derivation of the TAP free energy

Positivity of spin-glass susceptibility

$$\chi_{SG} \equiv \frac{1}{N}\sum_{ij}\chi_{ij}^2 = \frac{1}{N}\sum_i \frac{\chi_{ii}^2}{1-\sum_j \beta^2 J_{ij}^2 \chi_{jj}^2} \, .$$

Local susceptibility: $\chi_{ii} = 1 - m_i^2$

Consistency condition to be fulfilled (Plefka: convergence of LCE)

$$\lambda = 1 - \frac{\beta^2 J^2}{N}\sum_i (1-m_i^2)^2 \geq 0 \qquad (1)$$

$\lambda = 0$ defines the de Almeida-Thouless transition line to the SG phase

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Outline

1. Mean-field theory for spin glasses
   - Sherrington-Kirkpatrick model

2. Averaging over randomness
   - Replica trick
   - Parisi RSB solution

3. *Summation over spin configurations*
   - TAP free energy
   - TAP & RSB

Properties of TAP states

1. Multitude of solutions at low temperatures – local minima not separable in free energy from unstable saddle points — complexity
2. High degeneracy in free energy – complex free-energy landscape
3. Convergence rather rare – ubiquitous unstable states (do not obey Plefka's stability condition)
4. Majority of configurations do not possess a well defined equilibrium state (minimum of free energy) – non-self-averaging FE – direct averaging leads to the SK (replica symmetric) solution
5. Composite equilibrium state (De Dominicis-Young ansatz)

$$\mathrm{Tr}_S \exp\left[-\beta H\{S\}\right] = \sum_{\alpha}^{\mathcal{N}} \exp\left[-\beta F_{TAP}\{m_i^\alpha\}\right]$$

$\mathcal{N}$ – numer of TAP solutions,

6. TAP states independent – separated by infinite energy barriers – quasi-equilibrium states

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Relation of TAP to RSB

Questions without unambiguous (rigorous) answers

1. *How do we derive Parirsi's solution from TAP?* – Cavity method – interpretation of the order parameters
2. *Is the TAP free energy an exact solution of the SK model? Is the TAP phase space complete?* – YES
   – disconnected phase space & independent TAP solutions, non-self-averaging TAP free energy
3. *Does the TAP approach lead to a unique stable equilibrium state?* – weighted sum of quasiequilbrium TAP states (solutions of TAP equations)
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Outline

Homogeneity of free energy I

Thermodynamic lmit exists only if there is a unique thermodynamic equilibrium state – degeneracy in free energy must be lifted

- Thermodynamic homogeneity – thermodynamic potentials depend only on spatial densities of extensive variables (Gibbs paradox)
- Euler homogeneity condition

$$\alpha F(T,V,N,\ldots,X_i,\ldots) = F(T,\alpha V,\alpha N,\ldots,\alpha X_i,\ldots)$$

- Real spin replicas ($\alpha = \nu$ integer) – each TAP state – one spin replica (independence of TAP states)

$$[\mathrm{Tr}\ \exp\{-\beta H\}]^\nu = \mathrm{Tr}_\nu \exp\left\{\sum_{a=1}^{\nu}\sum_{<ij>} J_{ij} S_i^a S_j^a\right\}$$

Homogeneity of free energy II

- Breaking independence of spin replicas – softening of energy barriers between TAP states

$$\Delta H(\mu) = \sum_i \sum_{a<b} \mu^{ab} S_i^a S_i^b$$

- Replicated free energy with weakly coupled replicas

$$F_\nu(\mu) = -k_B T \frac{1}{\nu} \left\langle \ln \mathrm{Tr} \exp\left\{ -\beta \sum_\alpha H^\alpha - \beta \Delta H(\mu) \right\} \right\rangle_{av}$$

- *Stability of homogeneity*

$$\frac{d}{d\nu} \lim_{\mu \to 0} F_\nu(\mu) \equiv 0$$

- Necessary condition: Analytic continuation to $\nu \in \mathbb{R}$ (no neeed for the limit $\nu \to 0$)

Replicated TAP free energy

General solution for interger number of real replicas

VJ, L. Zdeborová, cond-mat/0504132

$$F_\nu = \frac{1}{\nu}\sum_{a=1}^{\nu}\left\{\sum_i M_i^a\left[\eta_i^a + \beta J^2\sum_{b=1}^{a-1}\chi^{ab}M_i^b\right] + \frac{\beta J^2 N}{2}\sum_{b=1}^{a-1}(\chi^{ab})^2 - \frac{1}{4}\sum_{i,j}\beta J_{ij}^2\left[1-(M_i^a)^2\right]\left[1-(M_j^a)^2\right] - \frac{1}{2}\sum_{i,j}J_{ij}M_i^a M_j^a\right\} - \frac{1}{\beta\nu}\sum_i \ln \mathrm{Tr}\exp\left\{\beta^2 J^2\sum_{a<b}^{\nu}\chi^{ab}S_i^a S_i^b + \beta\sum_{a=1}^{\nu}(h+\eta_i^a)\,S_i^a\right\}$$

- Order parameters: M_i, η_i, χ^{ab}
- TAP recovered for $\chi^{ab} = 0$
- Decoupling of spin replicas: integer ν – trivial solution (RS)
- Analytic continuation – maximally general form of $\nu \times \nu$ matrices χab

Uniqueness of the equilibrium state

- Equivalence of replicas

$$M_i^a \equiv \langle S_i^a \rangle_T = M_i, \quad \eta_i^a = \eta$$

- Symmetry

$$\chi^{ab} = \chi^{ba}, \quad \chi^{aa} = 0$$

- Indistinguishability of spin replicas

$$\{\chi^{a1}, \ldots, \chi^{a\nu}\} = \{\chi^{b1}, \ldots, \chi^{b\nu}\}$$

  permutation of elements within rows (columns)

- Hierarchical (ultrametric) structure of χ^{ab} as in the RSB trick
  – consequence of stability conditions hierarchically applied
  – most general structure allowing for analytic continuation

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Hierarchical TAP free energy

Hierarchical solution with K-levels — K different values of χ^{ab}
$(\nu_1 - 1)\chi_1,\ (\nu_2 - \nu_1)\chi_2, \ldots, (\nu_K - \nu_{K-1})\chi_K$ – homogeneous order parameters

K-TAP free energy – analytic representation

$$F_K(\chi_1, \nu_1, \ldots, \chi_K, \nu_K) = -\frac{1}{4}\sum_{i,j}\beta J_{ij}^2(1 - M_i^2)(1 - M_j^2) - \frac{1}{2}\sum_{i,j} J_{ij} M_i M_j$$
$$+ \sum_i M_i \left[\eta_i + \frac{1}{2}\beta J^2 M_i \sum_{l=1}^{K}(\nu_l - \nu_{l-1})\chi_l\right] + \frac{\beta J^2 N}{4}\sum_{l=1}^{K}(\nu_l - \nu_{l-1})\chi_l[\chi_l + 2]$$
$$- \frac{1}{\beta\nu_K}\sum_i \ln\left[\int_{-\infty}^{\infty}\mathcal{D}\lambda_K \left\{\ldots \int_{-\infty}^{\infty}\mathcal{D}\lambda_1 \left\{2\cosh\left[\beta\left(h + \eta_i + \sum_{l=1}^{K}\lambda_l\sqrt{\chi_l - \chi_{l+1}}\right)\right]\right\}^{\nu_1} \ldots\right\}^{\nu_K/\nu_{K-1}}\right]$$

$\mathcal{D}\lambda_l \equiv \mathrm{d}\lambda_l\ e^{-\lambda_l^2/2}/\sqrt{2\pi}, \quad \nu_0 = 1$

Outline

Free energy

$K = 1$ TAP theory – "replica symmetric" solution for replicated TAP
– apart from local inhomogeneous sets M_i, η_i
– two *homogeneous* order parameters χ and ν

$$F_1(\chi,\nu) = -\frac{1}{4}\sum_{i,j}\beta J_{ij}^2(1-M_i^2)(1-M_j^2) - \frac{1}{2}\sum_{i,j} J_{ij}M_iM_j + \frac{\beta J^2 N}{4}\chi[(\nu-1)\chi+2] + \sum_i M_i\left[\eta_i + \frac{1}{2}\beta J^2(\nu-1)\chi M_i\right] - \frac{1}{\beta\nu}\sum_i \ln\int \mathcal{D}\lambda_i\left[2\cosh[\beta(h+\lambda_i J\sqrt{\chi}+\eta_i)]\right]^\nu$$

$F_1(\chi,\nu)$ analytic function of all variables (ν)

Stationarity equations
Local variables

- Local magnetization

$$M_i = \left\langle \rho^{(\nu)}(h+\eta_i;\lambda,\chi)\tanh[\beta(h+\eta_i+\lambda J\sqrt{\chi})]\right\rangle_\lambda \equiv \langle \rho_i^{(\nu)} t_i\rangle_\lambda \ ,$$

where $\langle X(\lambda)\rangle_\lambda = \int \mathcal{D}\lambda X(\lambda)$ and

$$\rho_i^\nu \equiv \rho^{(\nu)}(h+\eta_i;\lambda,\chi) = \frac{\cosh^\nu[\beta(h+\eta_i+\lambda J\sqrt{\chi})]}{\left\langle\cosh^\nu[\beta(h+\eta_i+\lambda J\sqrt{\chi})]\right\rangle_\lambda}$$

is a density matrix (weight of the other TAP solutions affecting the chosen one)

- Internal magnetic field

$$\eta_i = \sum_j J_{ij}M_j - M_i\left[\beta J^2(\nu-1)\chi + \sum_j \beta J_{ij}^2(1-M_j^2)\right]$$

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Staionarity equations

Homogeneous variables

- Homogeneous overlap susceptibility

$$\chi = \frac{1}{N}\sum_i \left[\left\langle \rho_i^{(\nu)} t_i^2 \right\rangle_\lambda - \left\langle \rho_i^{(\nu)} t_i \right\rangle_\lambda^2\right]$$

- Multiplicity (geometric/replication) factor

$$\beta^2 J^2 \chi(2Q+\chi)\nu = \frac{4}{N}\sum_i \left[\langle \ln\cosh[\beta(h+\eta_i+\lambda J\sqrt{\chi})]\rangle_\lambda - \ln\left\langle \cosh^\nu[\beta(h+\eta_i+\lambda J\sqrt{\chi})]\right\rangle_\lambda^{1/\nu}\right]$$

$Q \equiv N^{-1}\sum_i M_i^2$

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$Q \equiv N^{-1}\sum_i M_i^2$

Dependence on the geometric parameter ν I

Reconstruction of TAP – $\chi = 0$ – high-temperature phase (Plefka's condition fulfilled)

When else do we recover TAP?

- Single spin replica: $\nu = 1, \quad F_1(\chi_1, 1) = F_{TAP}$
- Limit to infinite number of replicas: $\nu \to \infty, \quad \nu\chi = \Gamma^2$
  – saddle point evaluation of λ-integral

$$\bar{F}_1(\Gamma, \bar{\lambda}_i) = -\frac{1}{4}\sum_{i,j}\beta J_{ij}^2(1 - M_i^2)(1 - M_j^2) - \frac{1}{2}\sum_{i,j} J_{ij} M_i M_j + \sum_i M_i \left[\eta_i + \frac{1}{2}\beta J^2 \Gamma^2 M_i\right] + \frac{1}{\beta}\sum_i \left\{\frac{\bar{\lambda}_i^2}{2} - \ln\left[2\cosh[\beta(h + \eta_i + J\Gamma\bar{\lambda}_i)]\right]\right\} = F_{TAP}$$

$$\bar{\lambda}_i = \beta J \Gamma M_i$$

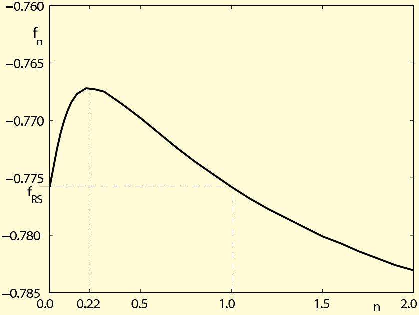
Dependence on the geometric parameter ν II

- Zero number of replicas: $\nu \to 0$
  – annealed averaging goes over to quenched

$$F_1(\chi, 0) = \frac{\beta J^2 N}{4}\chi(2-\chi) - \frac{1}{4}\sum_{i,j}\beta J_{ij}^2(1-M_i^2)(1-M_j^2)$$
$$-\frac{1}{2}\sum_{i,j} J_{ij} M_i M_j + \sum_i M_i\left[\eta_i - \frac{1}{2}\beta J^2 \chi M_i\right]$$
$$-\frac{1}{\beta}\sum_i \int \mathcal{D}\lambda_i \ln\left[2\cosh[\beta(h+\eta_i+\lambda_i J\sqrt{\chi})]\right] = F_{TAP}$$

- Substitution: $\xi_i = \eta_i + \lambda_i J\sqrt{\chi}\,, \chi = 1 - Q,\ Q = N^{-1}\sum_i M_i^2$
- Integration absorbed into lattice sum

ν-dependence of 1-TAP



Outline

4 Hierarchical TAP theory
- Thermodynamic homogeneity and multiple TAP states

5 *One-level hierarchical solution*
- 1-TAP free energy and order parameters
- Stability conditions

6 Asymptotic solution near the critical point
- Fixed internal magnetic field
- Equilibrium value of the local field expanded

Linear susceptibility

- Positivity of linear susceptibility – minimum of free energy w.rt. inhomogeneous parameters

$$\left(\chi^{-1}\right)_{ij} = -\beta J_{ij} + \delta_{ij}\left[\beta^2 J^2 \left(1 - Q - (1-\nu)\chi\right) + \frac{1}{\chi_{ii}}\right]$$

- Inhomogeneous local susceptibility

$$\chi_{ii} = 1 - M_i^2 - (1-\nu)\left[\left\langle \rho_i^{(\nu)} t_i^2 \right\rangle_\lambda - \left\langle \rho_i^{(\nu)} t_i \right\rangle_\lambda^2\right]$$

.

Stability criteria

- Positivity of spin-glass susceptibility – uniqueness of the equilibrium state

$$\Lambda_0 = 1 - \frac{\beta^2 J^2}{N} \sum_i \left[1 - (1-\nu) \left\langle \rho_i^{(\nu)} t_i^2 \right\rangle_\lambda - \nu \left\langle \rho_i^{(\nu)} t_i \right\rangle_\lambda^2 \right]^2 \geq 0 \qquad (2)$$

- Extremum of free energy w.r.t. variation of the homogeneous parameter

$$\Lambda_1 = 1 - \frac{\beta^2 J^2}{N} \sum_i \left\langle \rho_i^{(\nu)} (1 - t_i^2)^2 \right\rangle_\lambda \geq 0 \qquad (3)$$

- TAP solution: $\Lambda_0 = \Lambda_1 = \lambda = 1 - \beta^2 J^2 N^{-1} \sum_i (1 - m_i^2)^2$ instability

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Properties of 1-TAP theory

Nontriviality of the homogeneous order parameters

- Overlap susceptibility $\chi > 0$, if TAP free energy thermodynamically inhomogeneous
- Free energy F_1 depends on the geometric parameter ν
- Physical solution – $\nu < 1$
  – thermodynamic inhomogeneity *minimized*
  – free energy *maximized*

  ▸ TAP instability
- Instability parameters – Λ_0 decreasing in ν
  – Λ_1 increasing in ν

  ▸ figure
- 1-TAP stable – only if both stability conditions fulfilled for the equilibrium ν_{eq}
- If F_1 unstable $\Rightarrow$ 2-TAP solution etc.
- Free energy F_K either exact or an exact lower bound

Properties of 1-TAP theory

Nontriviality of the homogeneous order parameters

- Overlap susceptibility $\chi > 0$, if TAP free energy thermodynamically inhomogeneous
- Free energy F_1 depends on the geometric parameter ν
- Physical solution – $\nu < 1$
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Explicit χ-dependence

Expansion around TAP – χ - expansion parameter

Two-step expansion

1. Internal magnetic field η_i fixed (χ independent)
2. equilibrium value of η_i expanded in χ

- Local magnetization

$$M_i \doteq \mu_i - \beta^2 J^2 (1-\nu)\mu_i(1-\mu_i^2)\chi + \beta^4 J^4 (1-\nu)\mu_i(1-\mu_i^2)\left[2-\nu-(3-2\nu)\mu_i^2\right]\chi^2$$

$\mu_i = \tanh[\beta(h+\eta_i)]$, ($\eta_i$ depends on χ)

- Homogeneous parameter Q

$$Q \doteq \left\langle \mu_i^2 \right\rangle_{av} - 2\beta^2 J^2 (1-\nu)\left\langle \mu_i^2(1-\mu_i^2)\right\rangle_{av}\chi + \beta^4 J^4 (1-\nu)\left\langle \mu_i^2(1-\mu_i^2)\left[5-3\nu-(7-5\nu)\mu_i^2\right]\right\rangle_{av}\chi^2$$

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Global order parameters

Asymptotic equations for the homogeneous order parameters

- Asymptotic equation for χ

$$\chi \doteq \beta^2 J^2 (1-\mu_i^2)^2 \chi - \beta^4 J^4 (1-\mu_i^2)^2 [2-\nu-(8-5\nu)\mu_i^2]\chi^2$$

- Asymptotic equation for ν

$$0 \doteq \nu\chi^2 \left\{ 1 - \beta^2 J^2 \left\langle (1-\mu_i^2)^2 \right\rangle_{av} + \frac{2}{3}\beta^4 J^4 \chi \left\langle (1-\mu_i^2)^2 \left[3 - 2\nu - (11-8\nu)\mu_i^2 \right] \right\rangle_{av} \right\}$$

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Outline

Local magnetization μ

Only linear order in χ sufficient

- Local magnetization

$$\mu_i \doteq m_i + (1 - m_i^2)\chi\beta\dot{\eta}_i$$

where $\dot{\eta}_i = d\eta_i/d\chi$ and $m_i = \tanh[\beta(h + \eta_i^0)]$

- Derivative of the local field

$$\beta\dot{\eta}_i = \beta^2 J^2 \left[(1-\nu) + \dot{Q}\right] M_i + \sum_j \left[\beta J_{ij} - \delta_{ij}\beta^2 J^2 (1-Q)\right] \dot{M}_j$$

with $\dot{M}_i = dM_i/d\chi$

- Using the expansions for M_i and Q together with the definition of the TAP susceptibility

$$\beta\dot{\eta}_i \doteq \beta^2 J^2 (1-\nu)\left[m_i - 2\beta^2 J^2 \frac{\langle m_i^2(1-m_i^2)\rangle_{av}}{(1-m_i^2)} \sum_j \chi_{ij}^{TAP} m_j\right]$$

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Mean-field nonolocal susceptibility I

Mean-field approximation – separation of distinct lattice sites

- *Decoupling of sums with linear susceptibility*:

$$C[f,g] = \frac{1}{N}\sum_{ij}\chi_{ij} f(m_i) g(m_j)$$

- *Representation of the matrix inverse via self-avoiding random walks*

$$\chi_{ij} = \chi_{ii}\left[\delta_{ij} + {\sum_k}' \beta J_{ik}\chi_{kj}\right]$$

▸ TAP susceptibility

Mean-field nonolocal susceptibility II

- *Operator representation of the spin coupling*

$$\beta J_{ij} = \frac{\beta^2 J^2}{N} \left[\nabla_i m_j + m_i \nabla_j\right]$$

$$\nabla_i \equiv \chi_{ii} \frac{\partial}{\partial m_i}$$

.

- *Operator representation of the linear susceptibility* (along the AT line)

$$\chi_{ij} = \chi_{ii}\delta_{ij} + \frac{\beta^2 J^2}{2N} \left[2\nabla_i \chi_{ii} m_j \chi_{jj} + 2m_i \chi_{ii} \nabla_j \chi_{jj} + \nabla_i \chi_{ii} \nabla_j \chi_{jj}\right] - \frac{\langle (1 - m_k^2(1 - 3m_k^2)\rangle_{av}}{\langle m_k^2(1 - m_k^2)\rangle_{av} \langle (1 - m_k^2)^2\rangle_{av}} m_i \chi_{ii} m_j \chi_{jj}$$

Mean-field nonolocal susceptibility III

Decoupled sum with nonlocal TAP susceptibility

$$\frac{1}{N}\sum_{ij}\chi_{ij}m_i(1-m_i^2)m_j = \frac{\beta^2 J^2}{2}\left\langle(1-m_k^2)^2\right\rangle_{av}\left\langle(1-m_k^2)(1-3m_k^2)\right\rangle_{av}$$

Asymptotic solution for the global parameters I

- Asymptotic equation for the overlap susceptibility

$$\beta^2 J^2 \left\langle (1 - m_i^2)^2 \right\rangle_{av} - 1$$
$$= \beta^4 J^4 \chi \Big\{ \left\langle (1 - m_i^2) \left[2 - \nu - 2(5 - 3\nu) m_i^2 + (4 - \nu) m_i^4 \right] \right\rangle_{av}$$
$$+ 8\beta^2 J^2 (1 - \nu) \left\langle m_i^2 (1 - m_i^2) \right\rangle_{av} \left\langle m_i^2 (1 - m_i^2)^2 \right\rangle_{av} \Big\} \quad (4)$$

- Asymptotic equation for the geometric (multilplicity) factor

$$\beta^2 J^2 \left\langle (1 - m_i^2)^2 \right\rangle_{av} - 1$$
$$= \frac{2}{3} \beta^4 J^4 \chi \Big\{ \left\langle (1 - m_i^2) \left[3 - 2\nu - 2(7 - 5\nu) m_i^2 + (5 - 2\nu) m_i^4 \right] \right\rangle_{av}$$
$$+ 12\beta^2 J^2 (1 - \nu) \left\langle m_i^2 (1 - m_i^2) \right\rangle_{av} \left\langle m_i^2 (1 - m_i^2)^2 \right\rangle_{av} \Big\} \quad (5)$$

Asymptotic solution for the global parameters II

- Leading χ asymptotics below and ν_0 along the AT line

$$\nu_0 = \frac{2\langle m_i^2(1-m_i^2)^2\rangle_{av}}{\langle(1-m_i^2)^3\rangle_{av}}$$

- Physics does not depend on ν, if $\chi = 0$
- Solution for ν_0 only for small magnetic field: $\langle m_i^2\rangle_{av} = \langle\tanh^2[\beta(h+\eta_i^0)]\rangle_{av} < 1$

Crossover in magnetic field

Crossover in the asymptotic solution

- Asymptotic solution for χ physical only if positive and $\nu \leq 1$
- Positive solution only for small magnetic fields up to a critical value $\nu_c < 1$ from

$$0 = \left\langle (1 - m_i^2) \left[2 - \nu_c - 2(5 - 3\nu_c) m_i^2 + (4 - \nu_c) m_i^4 \right] \right\rangle_{av} + 8\beta^2 J^2 (1 - \nu_c) \left\langle m_i^2 (1 - m_i^2) \right\rangle_{av} \left\langle m_i^2 (1 - m_i^2)^2 \right\rangle_{av}$$

- Critical magnetic field h_c when ν_0 used for ν_c
- For higher magnetic fields – asymptotic expansion to higher powers of χ

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Summary

Answers to the addressed questions

1. *How RSB from TAP?* – real spin replicas for different TAP solutions, hierarchical (thermodynamically homogeneous) extension of TAP theory
2. *Is TAP exact?* – only in the paramagnetic phase, low-temperature solution unstable (thermodynamically inhomogeneous) $\Rightarrow$ hierarchical TAP
3. *Does TAP produce stable equilibrium?* – standard TAP NO – hierarchical TAP YES
4. *Thermodynamikc limit & self-averaging?* - - only for hierarchical TAP, different TAP states dynamically interact via χ – simply connected phase space with a unique equilibrium state
5. *Origin of RSB order parameters?* – thermal fluctuations responsible for RSB order parameters, randomness harmless

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TAP vs. hierarchical TAP

TAP subet of hierarchical TAP

TAP

- *PM* – single solution
- *SG* – multiple solutions
  - exponentially many independent quasi-equilibrium states
  - a single weighted (composite) equilibrium state
  - locally stable states degenerate with unstable states
  - direct averaging over randomess impossible (incorrect)
  - exclusion of unstable states only by solving numerically inhomogeneous TAP equatons

Hierarchical TAP

- *PM* – single solution (TAP)
- *SG* – single solution
  - TAP states dynamically interact – melt into one
  - single equilibrium state characterized by homogeneous RSB parameters
  - degeneracy of TAP states lifted – unstable states removed
  - self-averaging free energy
  - no need to solve inhomogeneous hierarchical TAP equations numerically

Stability parameters

To the main text