The Phase Transitions in the Random Coloring Problem

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Outline

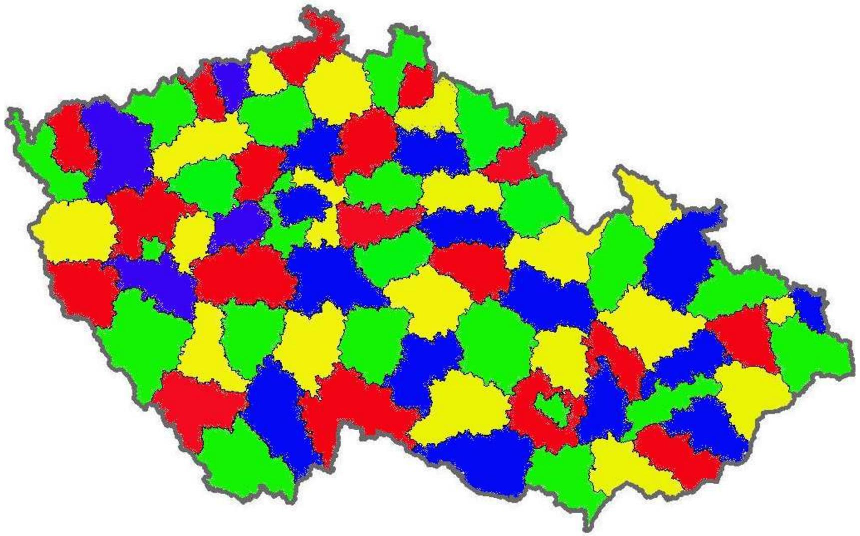
Part I (1-11): Introduction, Model definition,
Motivation from different perspectives;

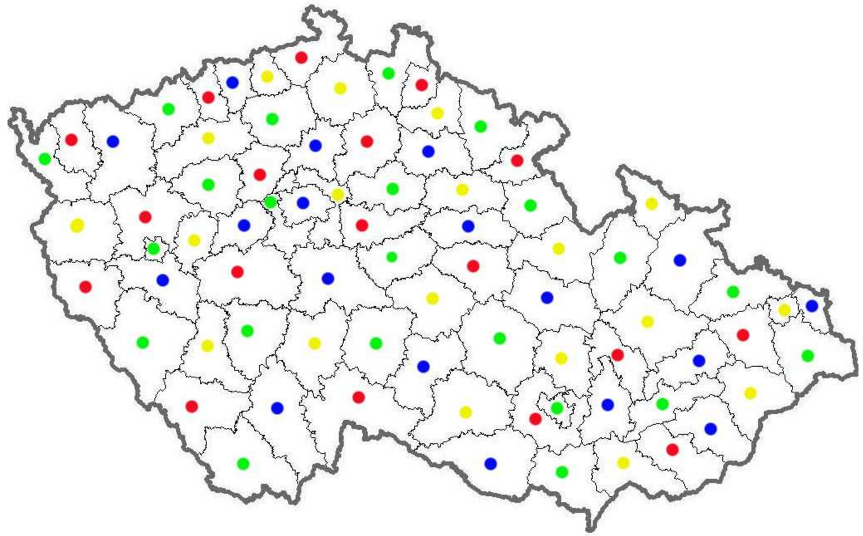
Part II (12-20): Cavity Method: basic ideas (RS and 1RSB);

Part III (21-25): Zoology of the phase transitions in random coloring;

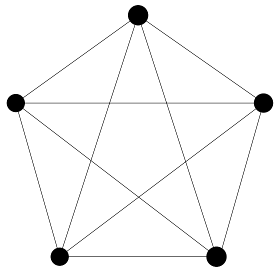
Part IV (26-27): What did we learned, what can we do more?

Definition of graph coloring

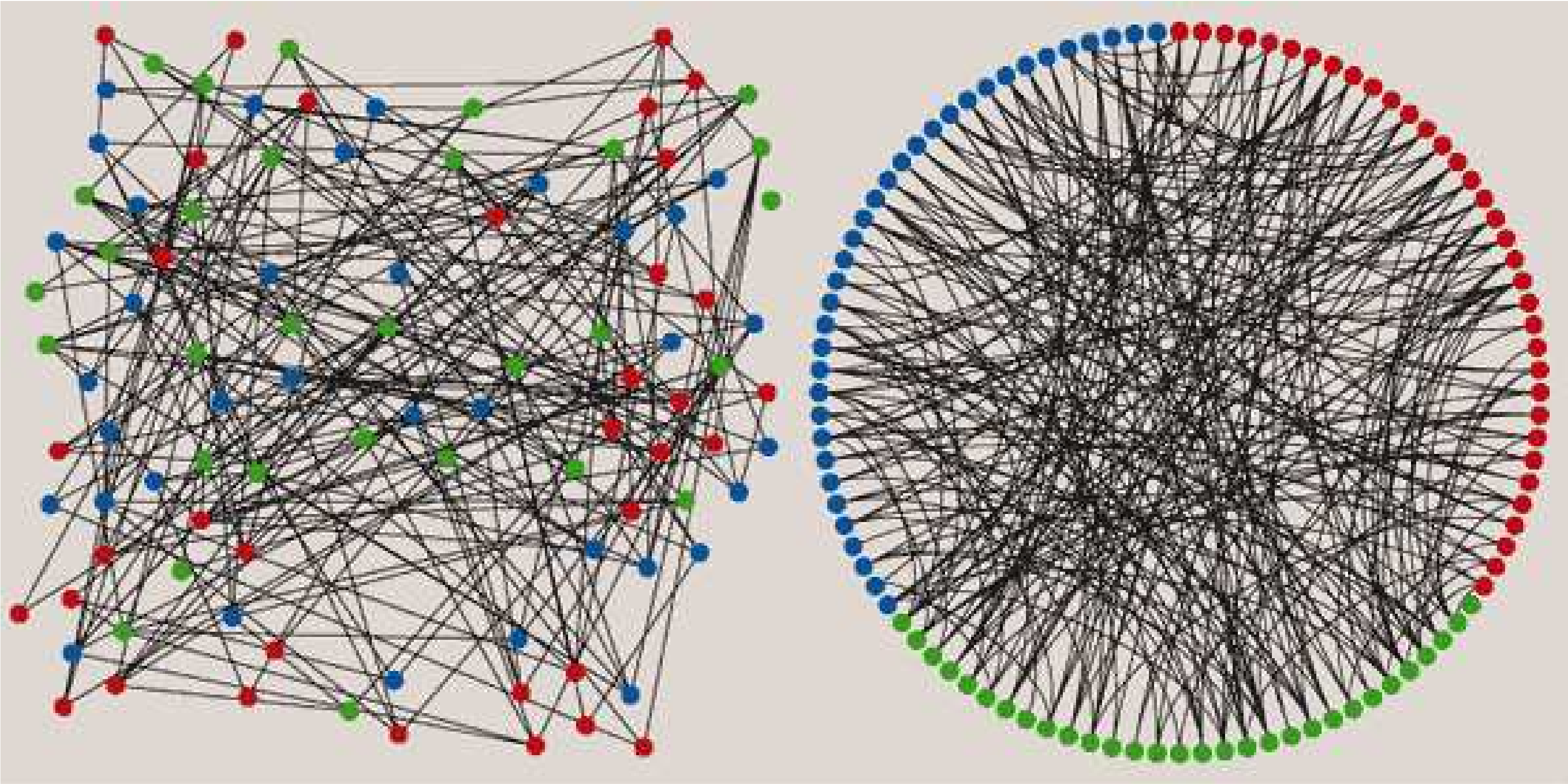




- State: each node has a color
- Rule (energy cost): neighbors have different colors



Less trivial example



How difficult is to color a graph?

Planar graphs:

- 10 minutes by hand for the CZ regions map ...
- Proof of 4-colorability: Appel-Haken (1976) ... probably never entirely checked.
- New proof: Robertson, Sanders, Seymour, Thomas (1994), N^2 algorithm follows.
- Checking 3-colorability for planar graphs is NP-complete, Dailey (1980)

General graphs: given a graph $G(V, E)$, $|V| = N$, and number of colors q

- Is it possible to color the graph? NP-complete

What does it mean NP-complete?

- NP problem: If you give me a solution, I can check it in polynomial time (polynomial in size of the graph)

- P problem: I can find solution in polynomial time for every instance of the problem (for every possible graph)

- NP-complete problem (Cook 1971): If this problem would have a polynomial time solution, all the NP problems do!

The "million" problem: P=NP?
Is there a polynomial algorithm for any of the NP-complete problems?

TOP 3: K-satisfiability, coloring, traveling salesman

Worst versus average

Erdös-Rényi random graphs $G(N,p)$: p probability that two vertexes are connected. Average degree $\alpha = p(N-1)$.

What is the relevant (nontrivial) value of α?

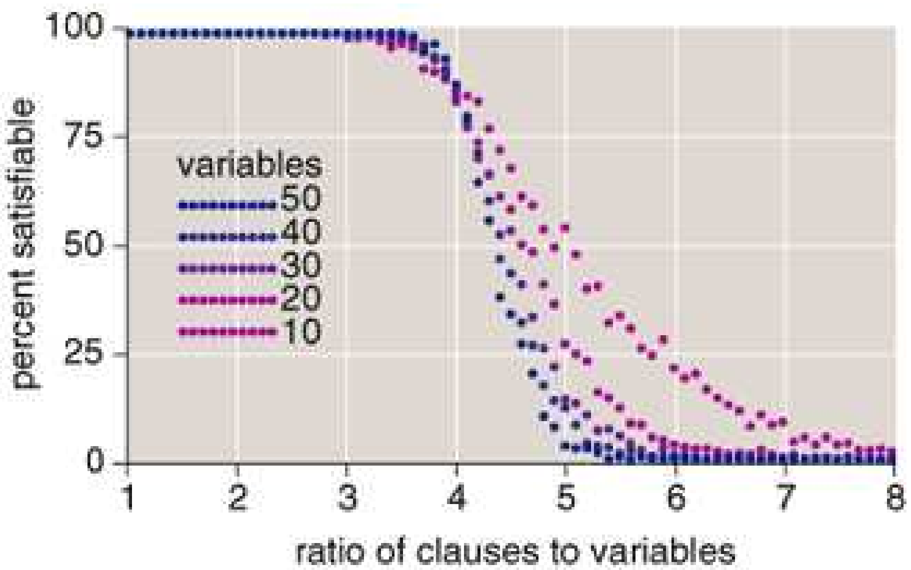
First moment argument: $\langle \mathcal{N} \rangle \geq \text{Prob}(\mathcal{N} > 0)$.

$$\langle \mathcal{N} \rangle = q^N \left(1 - \frac{1}{q}\right)^{pN(N-1)/2} = \exp\left[N\left(\log q + \frac{p(N-1)}{2}\log\frac{q-1}{q}\right)\right]$$

The limit of large graphs $N \to \infty$; interesting region

$$1 < \alpha < \frac{2\log q}{\log q - \log(q-1)} =_{q=3} 5.42.$$

The COL/UNCOL transition



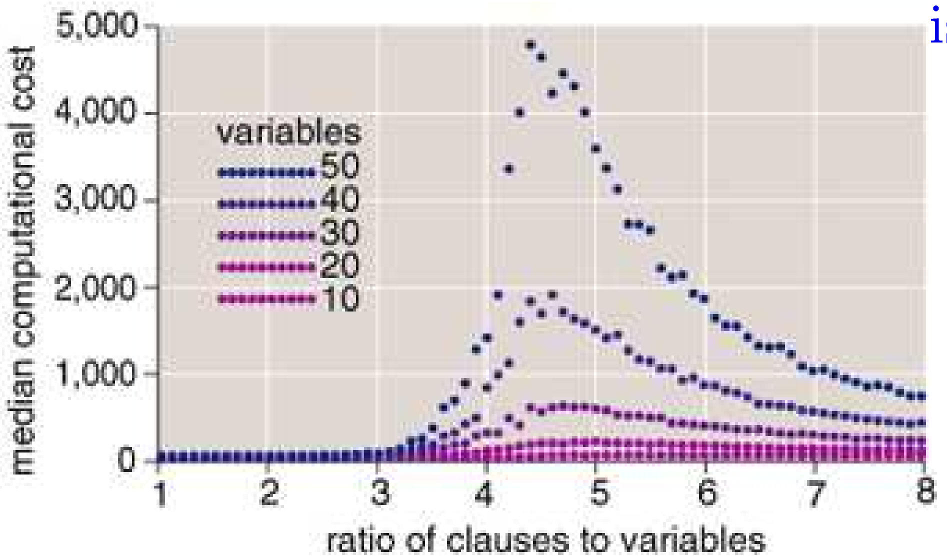
In $N \to \infty$ the transition is sharp. Discontinuous even in entropy.

Idea of phase transitions in purely mathematical problems - back to 1961, Erdös-Rényi – giant component (percolation) in random graphs.

Where the Really Hard Problems are?

Cheeseman, Kanefsky, Taylor (1991); Computational cost of the Davis-Putnam branch and bound algorithm

We want to understand independently of any algorithm!



To know where the hard problems are is useful

- to find them (to test algorithms)
- to avoid them in the real world applications
- to design new algorithms (survey propagation)

Properties of random graphs

- Erdös-Rényi random graph ensemble: each edge present with probability $p = \alpha/N$. For $N \to \infty$, α fixed, the degree distribution is Poissonian $p_k = e^{-\alpha} \frac{\alpha^k}{k!}$
- Regular random graphs: fixed degree r. Special simplification of the cavity equations.
- Both: loops length is or order $\log N$ - locally tree-like structure!

Statistical physics formulation

Hamiltonian (energy function) of antiferromagnetic Potts model

$$\mathcal{H} = \sum_{(i,j)\in E} \delta(s_i, s_j)$$

Graph: quenched disorder.

Average free energy

$$\langle \log Z \rangle = -\beta F(\beta) = -\beta E + S(E)$$

We want to compute average (over graphs) ground state energy

$$E_{\text{gs}} = \lim_{\beta\to\infty} \frac{\partial(\beta F)}{\partial\beta}$$

If E_{gs} also average ground state entropy

$$S_{\text{gs}} = -\lim_{\beta\to\infty} (\beta F)$$

Bethe approximation

- Approximation for lattice models, equivalent to mean field theory.
- Exact for models on random graphs, at least in presence of one or only few phases (e.g. ferromagnet)

Cavity method

Formulation of the Bethe approximation, which is generalizable to glassy systems (many phases, pure states).

Developed by Mézard, Parisi (1999).

Cavity method on trees

Define $\psi_{s_i}^{i\to j}$ as a probability that node i takes color s_i when edge (constraint) (ij) is erased from the graph. Write recursive equations for these probabilities

$$
\begin{aligned}
\psi_{s_i}^{i\to j} &= \frac{1}{Z^{i\to j}} \prod_{k_i n V(i)-j} \sum_{s_k} e^{-\beta\delta_{s_i s_j}} \psi_{s_k}^{k\to i} \\
&= \frac{1}{Z^{i\to j}} \prod_{k_i n V(i)-j} [1-(1-e^{-\beta})\psi_{s_i}^{k\to i}]
\end{aligned}
$$

Cavity free energy on trees

After addition of spin i and all the edges (ik) the free energy is changed by $\Delta F^{i \to j}$, which is given by the normalization

$$Z^{i \to j} = e^{-\beta \Delta F^{i \to j}}$$

In analogy the total free energy is

$$F(\beta) = \sum_i \Delta F^i - \sum_{(ij)} \Delta F^{ij};$$

where ΔF^i is a free energy shift after addition of node i and all the edges around, ΔF^{ij} is shift after addition of edge (ij).

From trees to sparse random graphs

The above equations are also correct on graphs with loops if the loops are long enough so that the clustering property holds (spins k are independent)

$$\psi^{k_1,k_2\to i}_{s_{k_1},s_{k_2}} - \psi^{k_1\to i}_{s_{k_1}}\psi^{k_2\to i}_{s_{k_2}} \to 0$$

Does it hold?

- Math: proof for matching, coloring for $\alpha < q$, SAT for small α etc.
- Physics: local self-consistency (stability) check, computation of the spin glass susceptibility

verage over graphs

Final order parameter is distribution $\mathcal{P}(\psi_{s_i}^{i\to j})$ of $\psi_{s_i}^{i\to j}$ over the graph, that is self-averaging (i.e. large graph is like average over graphs).

The self-consistent equation for $\mathcal{P}(\psi_{s_i}^{i\to j})$ have to be solved numerically in general (population dynamics).

Simplifications for coloring:

- Color symmetry not broken: $\mathcal{P}(\psi_{s_i}^{i\to j})$ symmetric under color permutation.

- Factorization for regular graphs: $\psi_{s_i}^{i\to j}$ the same for every edge, locally every edge have the same neighborhood.

1RSB: General idea

What if the clustering property does not hold?

- Simple case (ferromagnet): the pure phase decompose into few of them (magnetization positive, negative), within those the clustering property holds again!
- Less simple case (1RSB glass): the pure phase decompose into (exponentially) many, within those the clustering property holds again!

Is that correct?

- Math: No proof yet, the standard techniques for thermodynamical limit difficult, since with addition of one spin the system changes a lot. Less standard techniques are not far from success (Montanari, Semerjian, reconstruction on trees).
- Physics: local self-consistency (stability) check, computation of the spin glass susceptibility within states and in between states.

1RSB: What do we compute

Complexity function $\Sigma(F)$ is entropy of states of internal free energy F.
For computational reasons define "replicated" free energy as Legendre transform of the complexity

$$-\beta m \Phi(m,\beta) = -\beta m F(\beta) + \Sigma(F)$$

What is m?

- Legendre parameter, the same as temperature or chemical potential
- The Parisi replica symmetry breaking parameter
- Number of real replicas

What is the value of m?

- To minimize the total free energy of the system $F + \Sigma(F)$ and keep the complexity $\Sigma(F)$ positive $\Rightarrow m = 1$ or maximize the "replicated" free energy Φ.

1RSB: Cavity equations

Order parameter on a single graph is survey (distribution) $P(\psi_{s_i}^{i\to j})$ of probabilities $\psi_{s_i}^{i\to j}$ for every edge (ij). Self-consistent equation

$$P(\psi_{s_i}^{i\to j}) = \frac{1}{Z^{i\to j}} \prod_{k\in V(i)-j} \int d\, P(\psi_{s_i}^{k\to i}) \delta(\psi_{s_i}^{i\to j} - \mathcal{F}(\{\psi_{s_i}^{k\to i}\})) e^{-\beta m \Delta F^{i\to j}}$$

Average over graphs: Distribution of distributions

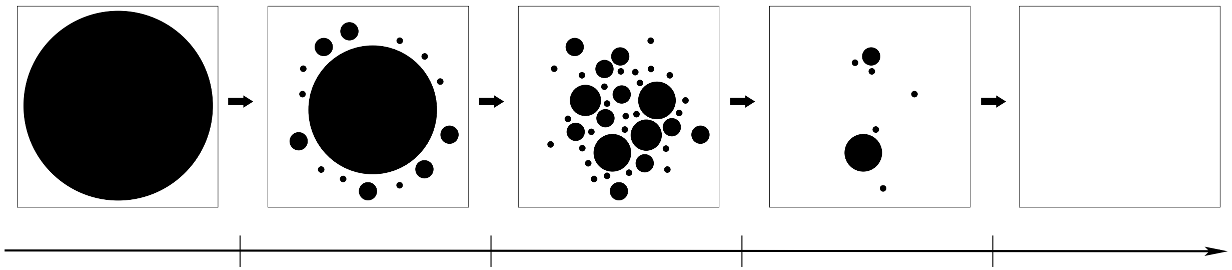
Computational simplifications

- Zero temperature, only energetic terms - integer fields! (Mulet, Pagnani, Weigt, Zecchina, 2002)
- At $m = 1$, analogy with reconstruction on trees.
- Regular graph - factorized case

Hurraaaaayyyyy

End of the Technical Part

Results for coloring



1) Only one cluster, replica symmetry correct

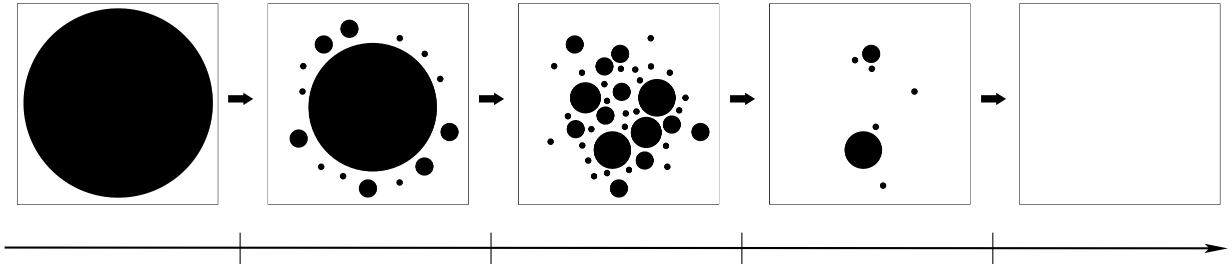
2) Few entropically unimportant clusters appear, replica symmetry still correct

3) The large cluster truly splits into exponentially many small ones; $m = 1$, complexity $\Sigma(m = 1) > 0$, RS free energy still exact, dynamically glassy phase

4) The entropy condensed in a few clusters, $m^* < 1$, complexity $\Sigma(m = m^*) = 0$, the true free energy larger than the RS one

5) No solutions anymore

Algorithmic implications



1)+2) Monte Carlo like (simulated annealing, random walker search) algorithms work

3) Monte Carlo like algorithms fails, Belief Propagation, the RS update of probabilities $\psi_{s_i}^{i \to j}$ works!

4) BP fails, Survey Propagation (Mézard, Zecchina, 2002), 1RSB update, works!

5) No solutions anymore, different strategies for proving nonexistence of solution

Few numbers and large q expansion

Regular graphs

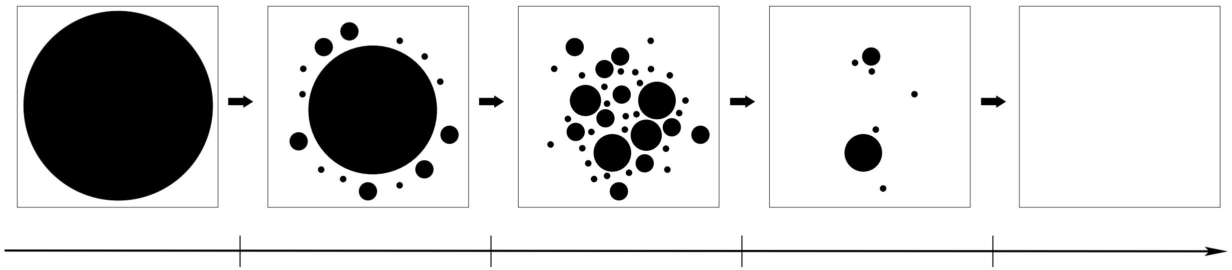
	clust.	cond.	COL
$q=4$	9	10	10
$q=5$	14	14	15
$q=6$	18	19	20

Erdös-Rényi graphs

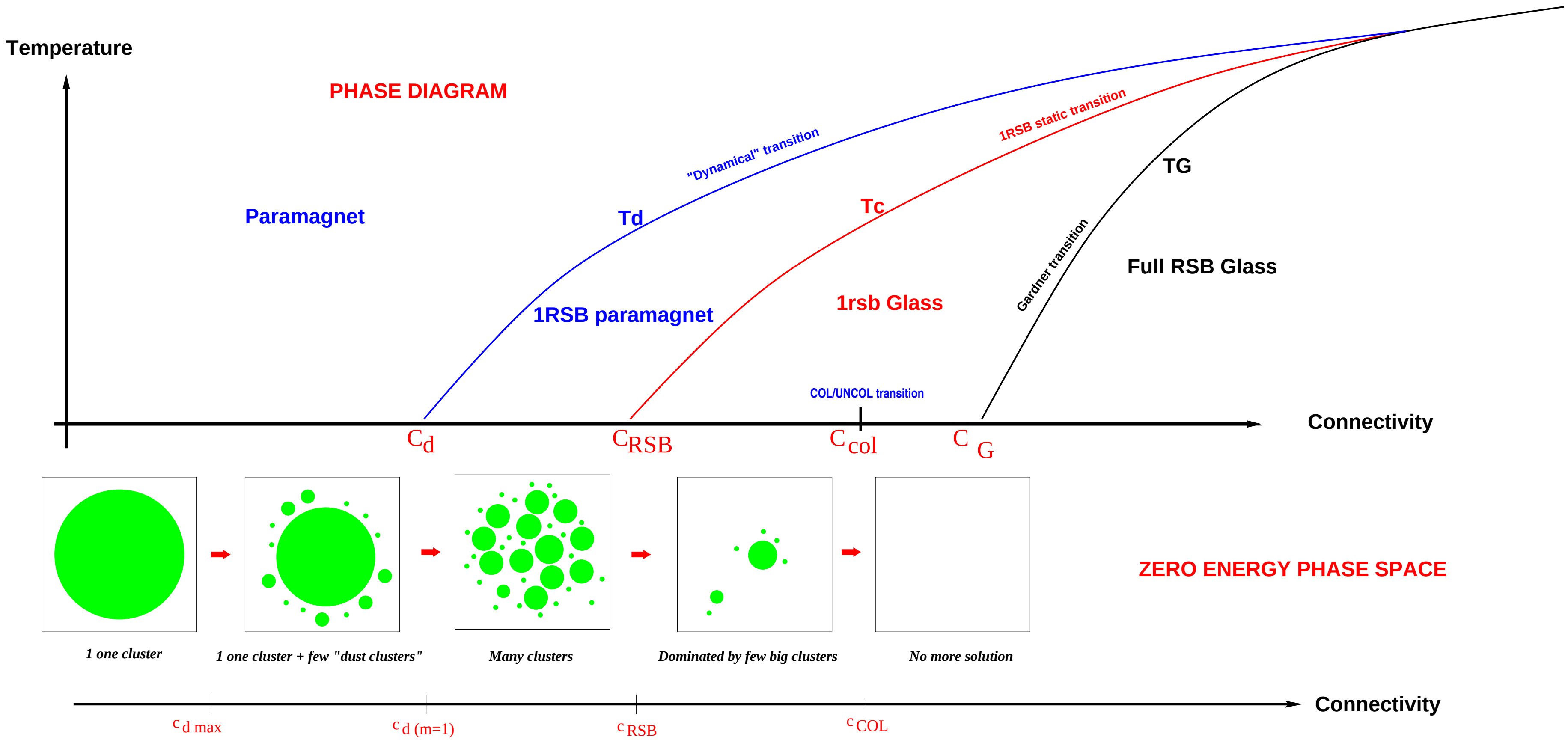
	clust.	cond.	COL
$q=4$	8.36	8.47	8.90
$q=5$	12.84	13.22	13.69

Leading terms in $q \to \infty$

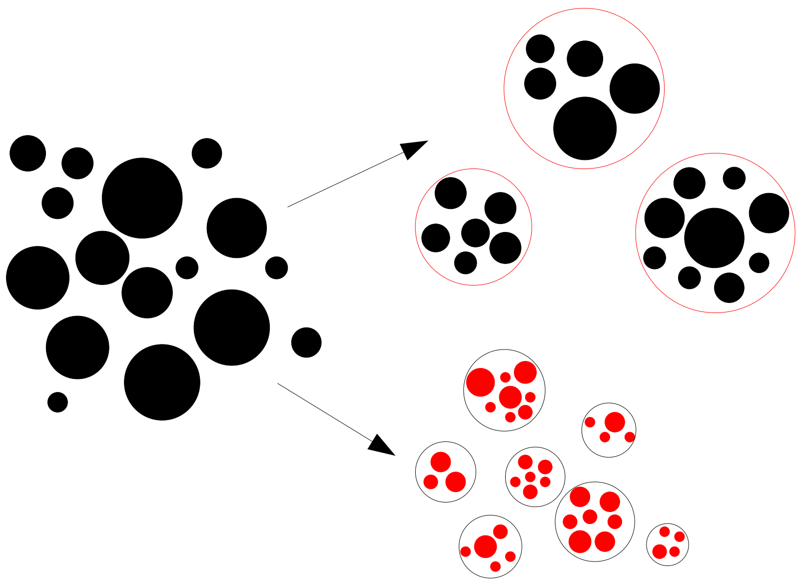
first cl.	clustering	condensation	COL/UNCOL
q	$q \log q + q \log \log q$	$2q \log q - \log q - 2 \log 2$	$2q \log q - \log q - 1$



Results for coloring



Stability towards 2RSB (FRSB)



- Except the case of 3-coloring, the thermodynamically dominant clusters in the COL phase are stable (also at finite temperature)

- Intrinsically simpler than e.g. Sherrington-Kirkpatrick model, where FRSB holds, yet wide variety of unexpected transitions (given above, Back-bone like structures ...)

Things we do not know yet

- Graphs with short loops!
- The region in 3-coloring which is not 1RSB stable
- The dynamics of decimation of the BP or SP
- More efficient solution of the non-simplified functional equations
- Clarify few things about the back-bone (hard fields), whitening procedure

Conclusions

- In coloring (K-SAT etc.) variety of structural phase transitions
- Cavity method describes transitions exactly on random graphs, independently on any algorithm!
- Direct implication for design of efficient algorithms!
- The path towards a rigorous proof is quite advanced.

Reference

Krzakala, Montanari, Ricci-Tersenghi, Semerjian, Zdeborova: *Gibbs States and the Set of Solutions of Random Constraint Satisfaction Problems* : to appear this week on cond-mat, submitted to PNAS.