Understanding of the mean-field theory of spin glasses

Václav Janiš

FZÚ AV ČR, v. v. i.

June 10, 2008

Collaborators: Antonín Klíč, Matouš Ringel, Lenka Zdeborová

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Outline

Paragon mean-field spin-glass model

- Ising Hamiltonian (classical spins) $S_I = \pm 1$

$$H[J,S] = \sum_{i<j} J_{ij} S_i S_j + h \sum_i S_i$$

- Long-range random spin couplings J_{ij} Gaussian random variables

$$N \left\langle J_{ij} \right\rangle_{av} = \sum_{j=1}^{N} J_{ij} = 0, \quad N \left\langle J_{ij}^2 \right\rangle_{av} = \sum_{j=1}^{N} J_{ij}^2 = J^2$$

- Free energy (self-averaging) – summation over lattice sites
  $\Leftrightarrow$ averaging over spin couplings (ergodic theorem)

$$F = -\frac{1}{\beta} \lim_{N\to\infty} \ln \mathrm{Tr}_S \left[\exp\left\{-\beta H[J,S]\right\}\right] = -\frac{1}{\beta} \lim_{N\to\infty} \left\langle \ln \mathrm{Tr}_S \left[\exp\left\{-\beta H[J,S]\right\}\right] \right\rangle_{av}$$

Averaging the logarithm (quenched disorder) – complicated

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Averaging the logarithm (quenched disorder) – complicated

Replica symmetry breaking

Averaging the partition function (annealed disorder) – straightforward

Replica trick

$$\beta F_{av} = - \lim_{n \to 0} \left(Z^n - 1 \right) / n$$

$$Z^n = \int D[J]\mu[J] \prod_{a=1}^{n} \prod_{i=1}^{N} d[S_i^a]\rho[S_i^a] \exp\left\{ -\beta \sum_{a=1}^{n} H[J, S^a] \right\}$$

Averaging over J_{ij} – coupling of spin replicas

Replica symmetric ansatz: $Q_{a\beta} = \langle S_i^a S_i^\beta \rangle = q$ for $a \neq \beta$

results in the SK solution (*inconsistent*)

Parisi RSB scheme – ansatz for a replica symmetry breaking

- $Q_{a\beta}$ has a hiearachical structure
- Analytic continuation $n \to 0$ ($n < 1$)

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Replica trick – analytic continuation

- Only specific matrices $n \times n$ allow for analytic continuation to real n
- The most general case – hierarchical orthogonal embeddings ($K = 2$)

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\begin{pmatrix}
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Ultrametric structure — only bloc matrices of identical elements
— larger blocks multiples of smaller blocks
— hierarchy of embeddings around diagonal

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Parisi RSB scheme solution – implicit representation

Infinite-many hierarchical levels K & Continuous limit
(sums in the limit $n \to 0$ go over to integrals (continuous functions))

$$\Delta q_l = q_{l+1} - q_l \xrightarrow[K\to\infty]{} dq, \quad \Delta m_l = m_{l-1} - m_l \xrightarrow[K\to\infty]{} dm$$

$$f[q] = -\frac{\beta}{4}\left[1 - 2q(1) + \int_0^1 dm q(m)^2\right] - \frac{1}{\beta}\int_{-\infty}^{\infty}\frac{d\eta}{\sqrt{2\pi}} e^{\eta^2/2} f\left(0, h + \sqrt{q(0)}\eta\right)$$

$$\frac{\partial f}{\partial m} = -\frac{1}{2}\frac{dq}{dm}\left[\frac{\partial^2 f}{\partial h^2} + m\left(\frac{\partial f}{\partial h}\right)^2\right], \quad f(1,h) = \ln\cosh(\beta h)$$

$$f_T = \max_{q(x)} f[q]$$

Unclear aspects of the Parisi construction

Order parameters

- What is the meaning of the order-parameter function $q(m)$?
- Where do the order parameters come from?
- Are thermal or random fluctuations responsible for RSB?

Parisi's solution

- What is the phase space on which we have to maximize $f[q]$?
- How does the stationarity equation for $q(m)$ look like?
- Is the Parisi continuous RSB exact?

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Homogeneity of thermodynamic potentials

- Homogeneity in the phase space

$$S(E) = k_B \ln \Gamma(E) = k_B \frac{1}{\nu} \ln \Gamma(E)^{\nu} = k_B \frac{1}{\nu} \ln \Gamma(\nu E)$$

$$F(T) = -k_B T \frac{1}{\nu} \left\langle \ln \left[\mathrm{Tr} e^{-\beta H} \right]^{\nu} \right\rangle_{av}$$

- Homogeneity of thermodynamic potentials (Euler)

$$a F(T, V, N, \ldots, X_i, \ldots) = F(T, aV, aN, \ldots, aX_i, \ldots)$$

Density of the free energy $f = F/N$
– function of only densities of extensive variables X_i/N

The existence and uniqueness of the thermodynamic limit $N \to \infty$ are guaranteed

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Real replicas – stability w.r.t. phase-space scalings

Real replicas – means to probe thermodynamic homogeneity

Replicated Hamiltonian: $[H]_\nu = \sum_{a=1}^{\nu} H^a = \sum_{a=1}^{\nu} \sum_{<ij>} J_{ij} S_i^a S_j^a$

Coupling between different replicas: $\Delta H(\mu) = \frac{1}{2} \sum_{a \neq b} \sum_i \mu^{ab} S_i^a S_i^b$

Averaged replicated free energy with coupled replicas

$$F_\nu(\mu) = -k_B T \frac{1}{\nu} \left\langle \ln \mathrm{Tr} \exp \left\{ -\beta \sum_a^{\nu} H^a - \beta \Delta H(\mu) \right\} \right\rangle_{av}$$

Eventually – analytic continuation of the replicated free energy to real ν

Stability w.r.t. phase space scaling: $\lim_{\mu \to 0} \frac{dF_\nu(\mu)}{d\nu} = 0$

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TAP free energy

Free energy for one (typical) configuration of spin couplings

$$F_{TAP} = \sum_i \left\{ m_i \eta_i^0 - \frac{1}{\beta} \ln 2 \cosh[\beta(h + \eta_i^0)] \right\} - \frac{1}{2} \sum_{ij} \left[J_{ij} m_i m_j + \frac{1}{2} \beta J_{ij}^2 (1 - m_i^2)(1 - m_j^2) \right]$$

Stationarity equations

$$m_i = \tanh[\beta(h + \eta_i^0)] \,, \qquad \eta_i^0 = \sum_j J_{ij} m_j - m_i \sum_j \beta J_{ij}^2 (1 - m_j^2)$$

Standard averaging over disorder – *Gaussian randomness*

$$\langle \eta_i \eta_j \rangle_{av} = \beta^2 J^2 \delta_{ij} \langle m_i^2 \rangle_{av}$$

results in the SK solution (unstable)

Properties of TAP theory

- Multiple solutions (unstable, metastable, nonexistent)

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Properties of TAP theory

- *Multiple solutions* (unstable, metastable, nonexistent)
- Thermodynamically inhomogeneous
- Lack of convergence in the thermodynamic limit (equlibrium state not uniquely defined)
- Specific rules for averaging over random spin couplings (accounting for many TAP solutions via the replica trick)

TAP with ν real replicas

Replicated free energy for one configuration of J_{ij} (linear reponse to $\mu \longrightarrow 0$) :

$$F_\nu = \frac{1}{\nu}\sum_{a=1}^{\nu}\left\{\sum_i m_i^a\left[\eta_i^a + \beta J^2 \sum_{b=1}^{a-1}\chi^{ab} m_i^b\right] + \frac{\beta J^2 N}{2}\sum_{b=1}^{a-1}(\chi^{ab})^2 - \frac{1}{2}\sum_{i,j} J_{ij} m_i^a m_j^a - \frac{1}{4}\sum_{i,j}\beta J_{ij}^2\left[1-(m_i^a)^2\right]\left[1-(m_j^a)^2\right]\right\}$$
$$- \frac{1}{\beta}\sum_i \ln \mathrm{Tr}\exp\left\{\frac{1}{2}(\beta J)^2\sum_{a\neq b}^{\nu}\chi^{ab} S_i^a S_i^b + \beta\sum_{a=1}^{\nu}(h+\eta_i^a)\, S_i^a\right\}$$

New averaged order parameters: $\chi^{ab} = N^{-1}\sum_i\left[\langle S_i^a S_i^b\rangle - m_i^a m_i^b\right]$

Gaussian fluctuating fileds $\eta_i^a = \sum_j J_{ij} m_j^a - \sum_{b=1}^{\nu} m_i^b \sum_j \beta J_{ij}^2 \chi_{jj}^{ab}$

covariance $\langle \eta_i^a \eta_j^b\rangle_{av} = \delta_{i,j}\sum_l J_{il}^2 m_l^a m_l^b = \delta_{i,j} J^2 q^{ab}$

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Replicated TAP – equivalence of replicas I

Equivalence of spin replicas – hierarcical ordering of classes of TAP solutions

$$m_i^a \equiv \langle S_i^a \rangle_T = m_i \ ,$$
$$\chi^{ab} = \chi^{ba} \ ,$$
$$\{\chi^{a1}, \ldots, \chi^{a\nu}\} = \{\chi^{b1}, \ldots, \chi^{b\nu}\}$$

Example: One RSB TAP free energy – two hierarchies of TAP solutions (*exchanging energy*)

$$F_1(\chi,\nu) = -\frac{1}{4}\sum_{i,j}\beta J_{ij}^2(1-m_i^2)(1-m_j^2) - \frac{1}{2}\sum_{i,j} J_{ij} m_i m_j + \frac{\beta J^2 N}{4}\chi[(\nu-1)\chi+2]$$
$$+\sum_i m_i\left[\eta_i + \frac{1}{2}\beta J^2(\nu-1)\chi m_i\right] - \frac{1}{\beta\nu}\sum_i \ln\int \mathcal{D}\lambda_i\left[2\cosh[\beta(h+\lambda_i J\sqrt{\chi}+\eta_i)]\right]^\nu$$

Replicated TAP – equivalence of replicas II

Local magnetization

$$m_i = \left\langle \rho^{(\nu)}(h + \eta_i; \lambda, \chi) \tanh[\beta(h + \eta_i + \lambda J\sqrt{\chi})] \right\rangle_\lambda \equiv \langle \rho_i^\nu t_i \rangle_\lambda$$

where

$$\rho_i^\nu \equiv \rho^{(\nu)}(h + \eta_i; \lambda, \chi) = \frac{\cosh^\nu[\beta(h + \eta_i + \lambda J\sqrt{\chi})]}{\left\langle \cosh^\nu[\beta(h + \eta_i + \lambda J\sqrt{\chi})] \right\rangle_\lambda}$$

$\mathcal{D}\lambda_l \equiv d\lambda_l \; e^{-\lambda_l^2/2}/\sqrt{2\pi}$, $t \equiv \tanh\left[\beta\left(h + \eta\sqrt{q} + \sum_{l=1}^K \lambda_l \sqrt{\Delta\chi_l}\right)\right]$ with $\langle X(\lambda_l) \rangle_{\lambda_l} \equiv \int_{-\infty}^{\infty} \mathcal{D}\lambda_l \; X(\lambda_l)$

Gaussian fluctuating field – Legendre conjugate variable to m_i

$$\eta_i = \sum_j J_{ij} m_j - m_i \left[\beta J^2(\nu - 1)\chi + \sum_j \beta J_{ij}^2 (1 - m_j^2)\right]$$

Replicated TAP – equivalence of replicas III

Local susceptibility

$$\chi = \frac{1}{N}\sum_i \left[\langle \rho_i^\nu t_i^2\rangle_\lambda - \langle \rho_i^\nu t_i\rangle_\lambda^2\right]$$

RSB parameter – Legnedre conjugate to χ

$$\frac{\beta^2 J^2}{4}\chi(2q+\chi)\nu$$
$$= \frac{1}{N}\sum_i \left[\langle \ln\cosh[\beta(h+\eta_i+\lambda J\sqrt{\chi})]\rangle_\lambda - \ln\langle \cosh^\nu[\beta(h+\eta_i+\lambda J\sqrt{\chi})]\rangle_\lambda^{1/\nu}\right]$$

Stability of TAP with 2 hierarchies

TAP stability – convergence criterion: $1 \geq \dfrac{\beta^2 J^2}{N} \sum_i (1 - m_i^2)^2$

Stability criteria – when RSB parameters relevant?

$$1 \geq \frac{\beta^2 J^2}{N} \sum_i \left\langle \rho_i^\nu (1 - t_i^2)^2 \right\rangle_\lambda$$

$$1 \geq \frac{\beta^2 J^2}{N} \sum_i \left[1 - (1-\nu) \left\langle \rho_i^\nu t_i^2 \right\rangle_\lambda - \nu \left\langle \rho_i^\nu t_i \right\rangle_\lambda^2 \right]^2$$

Overlap susceptibility: $\chi \propto \beta^2 J^2 \left\langle (1 - m_i^2)^2 \right\rangle_{av} - 1 > 0$

Replication parameter (at AT instability line): $\nu_0 = \dfrac{2 \left\langle m_i^2 (1 - m_i^2)^2 \right\rangle_{av}}{\left\langle (1 - m_i^2)^3 \right\rangle_{av}}$

Standard averaging of TAP	$\Rightarrow$	SK (RS) solution
Standard averaging of 1RSB-TAP	$\Rightarrow$	1RSB

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Standard averaging of 1RSB-TAP $\Rightarrow$ 1RSB

Hierarchical TAP theory

TAP with K hierarchies of real replicas

$$F_K(m_i,\eta_i;\chi_1,\nu_1,\ldots,\chi_K,\nu_K) = -\frac{1}{4}\sum_{i,j}\beta J_{ij}^2(1-m_i^2)(1-m_j^2) - \frac{1}{2}\sum_{i,j} J_{ij} m_i m_j$$
$$+\sum_i m_i\left[\eta_i + \frac{1}{2}\beta J^2 m_i \sum_{l=1}^{K}(\nu_l-\nu_{l-1})\chi_l\right] + \frac{\beta J^2 N}{4}\sum_{l=1}^{K}(\nu_l-\nu_{l-1})\chi_l^2 + \frac{\beta J^2 N}{2}\chi_1$$
$$-\frac{1}{\beta\nu_K}\sum_i \ln\left[\int_{-\infty}^{\infty}\mathcal{D}\lambda_K\left\{\ldots\int_{-\infty}^{\infty}\mathcal{D}\lambda_1\left\{Z_0\right\}^{\nu_1}\ldots\right\}^{\nu_K/\nu_{K-1}}\right]$$

Initial local partition sum

$$Z_0 = 2\cosh\left[\beta\left(h+\eta_i+\sum_{l=1}^{K}\lambda_l\sqrt{\chi_l-\chi_{l+1}}\right)\right]$$

Averaging of hierarchical TAP – discrete RSB scheme

Averaged TAP free energy density with K hierarchies of replicas

$$f_K(q;\Delta\chi_1,\ldots,\Delta\chi_K,\nu_1,\ldots,\nu_K) = -\frac{\beta}{4}\left(1-q-\sum_{l=1}^{K}\Delta\chi_l\right)^2 - \frac{1}{\beta}\ln 2 + \frac{\beta}{4}\sum_{l=1}^{K}\nu_l\Delta\chi_l\left[2\left(q+\sum_{i=l}^{K}\Delta\chi_i\right)-\Delta\chi_l\right] - \frac{1}{\beta}\int_{-\infty}^{\infty}\mathcal{D}\eta\ \ln\ Z_K$$

$\Delta\chi_l = \chi_l - \chi_{l+1} \geq \Delta\chi_{l+1} \geq 0,\quad \nu_l$ – arbitrary positive

Hierarchical local partition sums $Z_l = \left[\int_{-\infty}^{\infty}\mathcal{D}\lambda_l\ Z_{l-1}^{\nu_l}\right]^{1/\nu_l}$

Initial condition $Z_0 = \cosh\left[\beta\left(h+\eta\sqrt{q}+\sum_{l=1}^{K}\lambda_l\sqrt{\Delta\chi_l}\right)\right]$

Properties of the hierarchical solution

Degeneracy in the hierarchical free energy ($\chi_{K+1} = 0, \nu_0 = 1$)

$$f_K(\Delta\chi_{K-1} = 0) = f_{K-1}, \quad f_K(\chi_K = 0) = f_{K-1}$$
$$f_K(\nu_K = \nu_{K-1}) = f_{K-1}, \quad f_K(\nu_K = 0) = f_{K-1},$$
$$\frac{\partial}{\partial \nu_K} f_K(\nu_K = \nu_{K-1}) \leq 0, \quad \frac{\partial}{\partial \nu_K} f_K(\nu_K = 0) \geq 0$$

- *Local thermodynamic homogeneity*: $\dfrac{\partial f_K}{\partial \nu_l} = 0$
- Global thermodynamic homogeneity: $\chi_K = 0$
- $\nu_l > 1$ – free energy minimized
- $\nu_l < 1$ – free energy maximized

Only if $1 > \nu_1 > \ldots \nu_K \geq 0$ then $\chi_l > \chi_{l+1} \geq 0$

Hierarchical scheme converges

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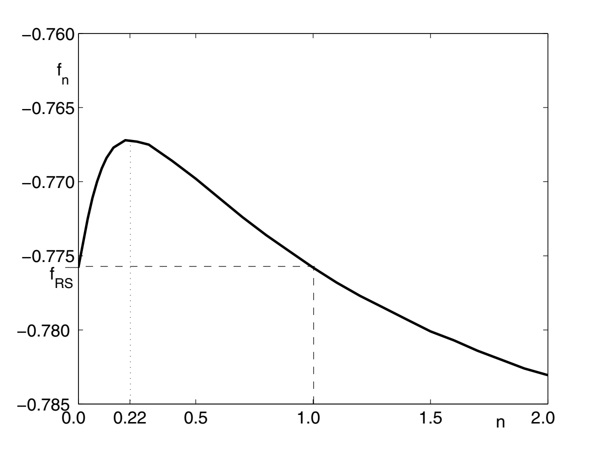
1RSB thermodynamic inhomogeneity



Stability conditions

- Nonnegativity of eigenvalues of the nonlocal susceptibility – resolvent

$$G(z) = \frac{1}{N}\mathrm{Tr}\left[z\widehat{1} - \widehat{\chi}^{-1}\right]^{-1}$$

- Relation to the homogeneous and spin-glass susceptibilities (zero magnetic field)

$$\chi = \frac{1}{N}\sum_i \chi_{ii} = -G(0)\,, \qquad \chi_{SG} = \frac{1}{N}\sum_{ij}\chi_{ij}^2 = -\left.\frac{dG(z)}{dz}\right|_{z=0} \geq 0$$

- $K+1$ stability conditions for a solution with K hierarchies

$$\Lambda_l = \beta^2\left\langle\left\langle\left\langle 1 - t^2 + \sum_{i=1}^{l}\nu_i\left(\langle t\rangle_{i-1}^2 - \langle t\rangle_i^2\right)\right\rangle_l^2\right\rangle_K\right\rangle_\eta \geq 0$$

$l = 0, 1, \ldots, K$, $\langle t\rangle_l(\eta, \lambda_K, \ldots, \lambda_{l+1}) \equiv \langle\rho_l \ldots \langle\rho_1 t\rangle_{\lambda_1}\ldots\rangle_{\lambda_l}$

One-step RSB

- Free energy ($J^2 = 1$)

$$f(q;\chi,\nu) = -\frac{\beta}{4}(1-q)^2 + \frac{\beta}{4}(\nu-1)\chi(2q+\chi) + \frac{\beta}{2}\chi - \frac{1}{\beta\nu}\int_{-\infty}^{\infty}\mathcal{D}\eta \ln \int_{-\infty}^{\infty}\mathcal{D}\lambda \left\{2\cosh\left[\beta\left(h+\eta\sqrt{q}+\lambda\sqrt{\chi}\right)\right]\right\}^{\nu}$$

- Stationarity equations

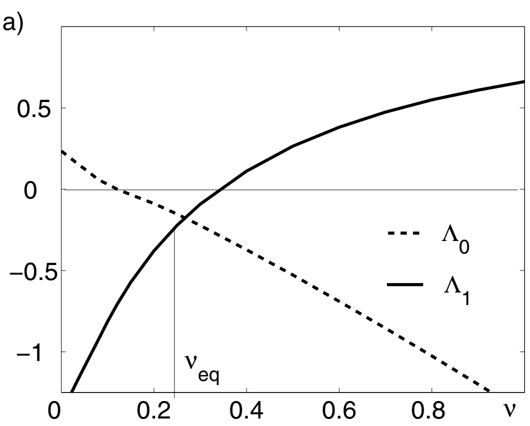
$$\begin{aligned} q &= \left\langle \langle t\rangle_\lambda^2 \right\rangle_\eta \\ q_{EA} = q+\chi &= \left\langle \langle t^2\rangle_\lambda \right\rangle_\eta \\ \beta^2\chi(2q+\chi)\nu &= \Big[\langle \ln\cosh[\beta(h+\eta\sqrt{q}+\lambda\sqrt{\chi})]\rangle_\lambda \\ &\quad - \ln\left\langle \cosh^\nu[\beta(h+\eta\sqrt{\eta}+\lambda\sqrt{\chi})]\right\rangle_\lambda^{1/\nu}\Big] \end{aligned}$$

Stability of 1RSB

Stability conditions

$$\Lambda_1 = 1 - \beta^2 \langle \langle (1 - t^2)^2 \rangle_\lambda \rangle_\eta$$
$$\Lambda_0 = 1 - \beta^2 \langle \langle \left(1 - (1-\nu) t^2 - \nu \langle t \rangle_\lambda^2 \right) \rangle_\lambda^2 \rangle_\eta$$

Stability of 1 RSB



Continuous limit of the discrete RSB scheme

- Discrete RSB solution unstable for any finite K, hence $K \longrightarrow \infty$
- Continuous ansatz (homogeneous distribution) $\Delta\chi_l = \chi_1/K$ (checked explicitly near the AT instability line)
- Continuous index variable $x = \lim_{K\to\infty}(K-l)/K$ ($x^P = \lim_{K\to\infty} l/K$)
- Gaussian integrals only (linear approximation) $g_l \equiv \ln \mathcal{Z}_l$

$$g_l = \ln \left\langle \mathcal{Z}_{l-1}^{\nu_l} \right\rangle_{\lambda_l}^{1/\nu_l} = g_{l-1} + \frac{\Delta\chi_l}{2}\left(g''_{l-1} + \nu_l g'^2_{l-1}\right) + O(\Delta\chi_l^2)$$

$$g'_l \equiv \frac{\partial g_l}{\partial h}$$

Parisi's differential equation (opposite overall sign)

$$\frac{\partial g(x,h)}{\partial x} = \frac{\dot\chi(x)}{2}\left[\frac{\partial^2 g(x,h)}{\partial h^2} + m(x)\left(\frac{\partial g(x,h)}{\partial h}\right)^2\right]$$

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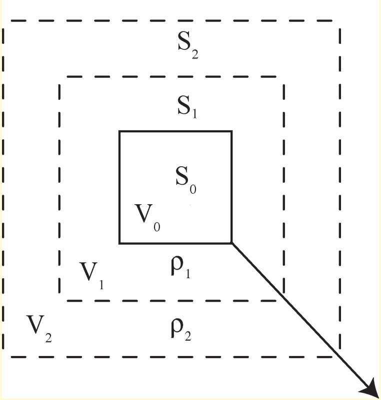
Physical interpretation of the RSB order parameters



- $\nu_l N$ active spins in the volume $\nu_l V$ affected by the replicated spins from the next hierarchy

$$\frac{N}{V} \ln Z_{l-1}(\beta, \overline{h}_l) \longrightarrow \frac{N}{\nu_l V} \ln \int \mathcal{D}\lambda_l \, Z^{\nu_l}_{l-1}\left(\beta, \overline{h}_l + \lambda_l \sqrt{\Delta\chi_l}\right)$$

- λ_l – effective magnetic fields, $\Delta\chi_l$ – interaction strength
- Effective weight of replicated spins in thermal averaging

$$\rho_l = \frac{Z^{\nu_l}_{l-1}}{\left\langle Z^{\nu_l}_{l-1} \right\rangle_{\lambda_l}}$$

Integral representation of the Parisi free energy

- *Continuous limit* – independently of the stability of the discrete scheme

$$f(q,X;m(\lambda)) = -\frac{\beta}{4}(1-q-X)^2 - \frac{1}{\beta}\ln 2 + \frac{\beta X}{2}\int_0^1 d\lambda\, m(\lambda)\left[q + X(1-\lambda)\right] - \frac{1}{\beta}\left\langle g(1, h+\eta\sqrt{q})\right\rangle_\eta$$

- Integral representation of the intracting part

$$g(1,h) = \mathbb{E}_0(X,h;1,0)\circ g_0(h) \equiv \mathbb{T}_\lambda \exp\left\{\frac{X}{2}\int_0^1 d\lambda\left[\partial^2_{\bar h} + m(\lambda)g'(\lambda;h+\bar h)\partial_{\bar h}\right]\right\} g_0(h+\bar h)\Big|_{\bar h=0}$$

- $\mathbb{T}$-product from quantum many-body PT

$$\mathbb{T}_\lambda \exp\left\{\int_0^1 d\lambda \widehat{O}(\lambda)\right\} \equiv 1 + \sum_{n=1}^{\infty}\int_0^1 d\lambda_1 \int_0^{\lambda_1}\ldots\int_0^{\lambda_{n-1}} d\lambda_n \widehat{O}(\lambda_1)\ldots\widehat{O}(\lambda_n)$$

Stationarity equations

- Number order parameters ($h_\eta = h + \eta\sqrt{q}$)

$$q = \frac{1}{\beta^2}\left\langle g'(1,h_\eta)^2\right\rangle_\eta \}$$

$$X = \frac{1}{\beta^2}\left[\left\langle \mathbb{E}(X,h_\eta;1,0)\circ g_0'(h_\eta)^2\right\rangle_\eta - \left\langle g'(1,h_\eta)^2\right\rangle_\eta\right]$$

- Functional order parameter

$$\lambda = \frac{1}{\beta^2 X}\left[\left\langle \mathbb{E}(X,h_\eta;1,0)\circ g_0'(h_\eta)^2\right\rangle_\eta - \left\langle \mathbb{E}(X,h_\eta;1,\lambda)\circ g'(\lambda,h_\eta)^2\right\rangle_\eta\right]$$

- Integral representation for the derivative

$$g'(\nu,h) = \mathbb{T}_\lambda \exp\left\{X\int_0^\nu d\lambda\left[\frac{1}{2}\partial_{\bar{h}}^2 + m(\lambda)g'(\lambda;h+\bar{h})\partial_{\bar{h}}\right]\right\} g_0'(h+\bar{h})\Big|_{\bar{h}=0}$$

$$\equiv \mathbb{E}(X,h;\nu,0)\circ g_0(h)$$

Stability conditions

- Stability conditions from the continuous limit of the discrete scheme

$$1 \geq \frac{1}{\beta^2} \left\langle \mathbb{E}(X, h_\eta; 1, \lambda) \circ g''(\lambda, h_\eta)^2 \right\rangle_\eta , \quad \forall \lambda \in [0, 1]$$

$$g''(\nu, h) = \mathbb{T}_\lambda \exp\left\{ X \int_0^\nu d\lambda \left[\frac{1}{2}\partial_{\bar{h}}^2 + m(\lambda)\partial_{\bar{h}}\, g'(\lambda; h + \bar{h}) \right] \right\} g_0''(h + \bar{h}) \Big|_{\bar{h}=0}$$

- Derivative of the equation for the functional order parameter

$$1 = \frac{d}{d\lambda} \mathbb{E}(X, h; 1, \lambda) \circ g'(\lambda, h)^2 = -X\mathbb{E}(X, h; 1, \lambda) \circ g''(\lambda, h)^2 \,.$$

- Consequence – *marginal stability in the whole SG phase*

$$\beta^2 = \left\langle \mathbb{E}(X, h_\eta; 1, \lambda) \circ g''(\lambda, h_\eta)^2 \right\rangle_\eta$$

Conclusions I

Mean-field theory of spin glasses

- Free energy self-averaging
- Typical distribution of spin couplings – multitude of solutions (thermodynamic inhomogeneity incurred)
- Generations of real replicas – successive embeddings of spin replicas (TAP solution may interchange energy to reach homogeneity)
- Homogenepous order parameters even without averaging over randomness (regulate interaction between replica generations)
- Standard averaging – within linear response and with FDT

Nonmeasurable order parameters – needed to describe measurable quantities

Conclusions II

Discrete RSB scheme

- Hirarchical structure – number of hierarchies K determined from stability
- Stable or marginally stable solution
- Probably too many order parameters: q, $\{\nu_l, \Delta\chi_l\}_{l=1}^{K}$
- Physical interpretation of the order parameters

Continuous RSB scheme

- Continuous limit of the discrete scheme
- Exists independently of the stability of the discrete scheme
- Only marginally stable solution
- Minimal set of order parameters: q, $q_{EA} = q + X$, $m(\lambda)$, $\lambda \in [0, 1]$

How does the continuous solution look like when 1RSB becomes stable (Potts glass)?