Introduction to Exactly Solvable 1D Quantum Lattice Models of Solid-State Physics

Ladislav Šamaj

Institute of Physics, Slovak Academy of Sciences, Dúbravská cesta 9, 84511 Bratislava Slovakia; E-mail: Ladislav.Samaj@savba.sk

Content

1. Heisenberg model

1.1 Definitions and notations

We introduce the following notations for special 2×2 matrices: the unity matrix

$$1 \equiv s^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \tag{1.1}$$

the Pauli matrices $\{s^\alpha\}$, $\alpha = x(1), y(2), z(3)$

$$s^x \equiv s^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad s^y \equiv s^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad s^z \equiv s^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.2}$$

The Pauli matrices fulfill the product relations

$$(s^\alpha)^2 = 1, \qquad s^\alpha \cdot s^\beta = \mathrm{i}\epsilon_{\alpha\beta\gamma} s^\gamma \qquad \text{for } \alpha \neq \beta, \tag{1.3}$$

where $\epsilon_{\alpha\beta\gamma}$ is the antisymmetric tensor ($\epsilon_{123} = 1, \epsilon_{213} = -1$, etc.). Consequently,

$$[s^\alpha, s^\beta] = 2\mathrm{i}\epsilon_{\alpha\beta\gamma} s^\gamma, \qquad \{s^\alpha, s^\beta\} = 2\delta_{\alpha\beta}, \tag{1.4}$$

where $[A, B] = A \cdot B - B \cdot A$ denotes the commutation operator and $\{A, B\} = A \cdot B + B \cdot A$ denotes the anticommutation operator.

One usually works in the basis formed by the eigenvectors of s^z:

$$s^z e^+ = e^+, \qquad e^+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \qquad s^z e^- = -e^-, \qquad e^- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{1.5}$$

It is useful to introduce the matrices

$$s^+ = \frac{1}{2}(s^x + \mathrm{i}s^y) \equiv \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad s^- = \frac{1}{2}(s^x - \mathrm{i}s^y) \equiv \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{1.6}$$

which act on the vectors $e^\pm$ as follows

$$s^+ e^+ = s^- e^- = 0; \qquad s^+ e^- = e^+, \quad s^- e^+ = e^-. \tag{1.7}$$

The tensor product of two matrices A and B is defined as

$$(\mathbf{A} \otimes \mathbf{B})^{\sigma_1\sigma_2}_{\sigma'_1\sigma'_2} = A^{\sigma_1}_{\sigma'_1} B^{\sigma_2}_{\sigma'_2} \equiv A_{\sigma_1\sigma'_1} B_{\sigma_2\sigma'_2} = \begin{pmatrix} A_{11}\mathbf{B} & \dots & A_{1n}\mathbf{B} \\ \vdots & & \vdots \\ A_{m1}\mathbf{B} & \dots & A_{mn}\mathbf{B} \end{pmatrix}. \tag{1.8}$$

Let us consider a chain of N lattice sites $n = 1, 2, \dots, N$. We introduce the lattice Pauli operators (matrices) of dimension $2^N \times 2^N$ on this chain as follows

$$\mathbf{s}^\alpha_n = \underbrace{1 \otimes \dots \otimes 1}_{n-1} \otimes \underbrace{s^\alpha}_{n} \otimes 1 \otimes \dots \otimes \underbrace{1}_{N}. \tag{1.9}$$

Explicitly,

$$(\mathbf{s}^\alpha_n)^{\sigma_1 \dots \sigma_N}_{\sigma'_1 \dots \sigma'_N} = (s^\alpha)_{\sigma_n\sigma'_n} \delta_{\sigma_1\sigma'_1} \dots \delta_{\sigma_{n-1}\sigma'_{n-1}} \delta_{\sigma_{n+1}\sigma'_{n+1}} \dots \delta_{\sigma_N\sigma'_N}. \tag{1.10}$$

There exists an important general rule concerning the ordinary product of two $2^N \times 2^N$ matrices, each of which is given by a direct product of N 2×2 matrices: one has to perform the matrix products of the 2×2 matrices at the corresponding sites $n = 1, 2, \ldots, N$ and the resulting $2^N \times 2^N$ matrix is simply their tensor product. As an example, we write down the multiplication of two Pauli operators $\mathbf{s}_n^\alpha$ and $\mathbf{s}_{n'}^{\alpha'}$ for two different sites $n \neq n'$:

$$\mathbf{s}_n^\alpha \mathbf{s}_{n'}^{\alpha'} = \underbrace{1}_{1} \otimes \ldots \otimes 1 \otimes \underbrace{s^\alpha}_{n} \otimes 1 \otimes \ldots \otimes \underbrace{s^{\alpha'}}_{n'} \otimes 1 \otimes \ldots \otimes \underbrace{1}_{N}. \tag{1.11}$$

It stands to reason that

$$[\mathbf{s}_n^\alpha, \mathbf{s}_{n'}^{\alpha'}] = 0 \qquad \text{for } n \neq n'. \tag{1.12}$$

The corresponding Hilbert space is $V_1 \otimes V_2 \otimes \ldots \otimes V_N$, where V_n is isomorphic to C^2. We shall choose as the basis for each V_n 2×1 orthonormal vectors e^+ (spin up) and e^- (spin down) defined in Eq. (1.5). The vector basis for Pauli operators is generated by all possible 2^N tensor products of these 2×1 vectors on the chain. Let us adopt the convention according to which a basis vector $|n_1, n_2, \ldots, n_M\rangle$ corresponds to the tensor product of M e^- vectors put on the ordered lattice sites

$$n_1 < n_2 < \ldots < n_M \tag{1.13}$$

and $(N - M)$ e^+ vectors put on all remaining sites:

$$|n_1, n_2, \ldots, n_M\rangle = \underbrace{e^+}_{1} \otimes e^+ \otimes \ldots e^+ \otimes \underbrace{e^-}_{n_1} \otimes e^+ \otimes \ldots e^+ \otimes \underbrace{e^-}_{n_2} \otimes \ldots \otimes \underbrace{e^+}_{N}. \tag{1.14}$$

Equivalently,

$$|n_1, n_2, \ldots, n_M\rangle = \mathbf{s}_{n_1}^- \mathbf{s}_{n_2}^- \ldots \mathbf{s}_{n_M}^- |0\rangle, \tag{1.15}$$

where $|0\rangle$ is the tensor product of N e^+ vectors. Since the total number of these orthogonal vectors is

$$\binom{N}{0} + \binom{N}{1} + \ldots + \binom{N}{N} = 2^N,$$

they form a complete basis of the 2^N-dimensional Hilbert space.

1.2 Hamiltonian and its symmetries

For quantum mechanical reasons (exchange force), the coupling between two magnetic dipoles may cause nearest-neighbours to have lowest energy when they are parallel or antiparallel. This phenomenon is described by the Heisenberg model with nearest-neighbour interactions of quantum Pauli spin-1/2 operators defined on a lattice. In the case of the one-dimensional (1D) periodic chain of $n = 1, 2, \ldots N$ sites, the most general form of the Heisenberg Hamiltonian reads

$$H = -\frac{1}{2} \sum_{n=1}^{N} \left(J_x \mathbf{s}_n^x \mathbf{s}_{n+1}^x + J_y \mathbf{s}_n^y \mathbf{s}_{n+1}^y + J_z \mathbf{s}_n^z \mathbf{s}_{n+1}^z \right), \tag{1.16}$$

where J_x, J_y, J_z are coupling constants and the chain periodicity is ensured by setting $\mathbf{s}_{N+1}^\alpha = \mathbf{s}_1^\alpha$.

There are three possibilities for the coupling constants:

- $J_x = J_y = J_z = J$: the isotropic XXX Heisenberg model, solved by Bethe [1] (1931).
- $(J_x = J_y = J) \neq J_z$: the XXZ Heisenberg model, solved by C. N. Yang & C. P. Yang [2,3] (1966).
- $(J_x \neq J_y) \neq J_z$: the XYZ Heisenberg model, solved by Baxter [4] (1972).

We shall concentrate on the XXZ Heisenberg model defined by the Hamiltonian

$$H(J, J_z) = -\frac{1}{2}\sum_{n=1}^{N}\left[J\left(\mathbf{s}_n^x\mathbf{s}_{n+1}^x + \mathbf{s}_n^y\mathbf{s}_{n+1}^y\right) + J_z\mathbf{s}_n^z\mathbf{s}_{n+1}^z\right]. \tag{1.17}$$

For a bipartite chain with $N =$ even number, the set of lattice sites can be divided into two subsets A and B. Then, due to the relations for the Pauli matrices $s^z s^x s^z = -s^x$ and $s^z s^y s^z = -s^y$, the unitary transformation with $U = \prod_{n\in A}\mathbf{s}_n^z$ leaves H unchanged, except for the replacement $J \to -J$:

$$UH(J, J_z)U^\dagger = H(-J, J_z). \tag{1.18}$$

Thus, without any loss of generality we can take $J = 1$ and consider the Hamiltonian

$$H(\Delta) \equiv \frac{1}{J}H(J, J_z) = -\frac{1}{2}\sum_{n=1}^{N}\left(\mathbf{s}_n^x\mathbf{s}_{n+1}^x + \mathbf{s}_n^y\mathbf{s}_{n+1}^y + \Delta\mathbf{s}_n^z\mathbf{s}_{n+1}^z\right), \tag{1.19}$$

where the only parameter $\Delta = J_z/J$ can take an arbitrary real value. The case $\Delta = 1$ ($\Delta = -1$) corresponds to the ferromagnetic (antiferromagnetic) isotropic chain. All other values of Δ correspond to the anisotropic XXZ Heisenberg chain.

There exists another useful symmetry of the XXZ model for $N =$ even number. Introducing the unitary operator $V = \exp(\mathrm{i}\pi\sum_{n=1}^{N} n\mathbf{s}_n^z)$, the unitary transformation of H reads:

$$VH(\Delta)V^\dagger = -H(-\Delta). \tag{1.20}$$

In this way, the energy spectra of the Hamiltonians $H(\Delta)$ and $H(-\Delta)$ are related by the reflection around $E = 0$. It is therefore sufficient to study the energy spectra of the Hamiltonians say with $\Delta \geq 0$.

Finally, using the identity

$$\mathbf{s}_n^x\mathbf{s}_{n+1}^x + \mathbf{s}_n^y\mathbf{s}_{n+1}^y = 2\left(\mathbf{s}_n^+\mathbf{s}_{n+1}^- + \mathbf{s}_n^-\mathbf{s}_{n+1}^+\right) \tag{1.21}$$

it can be readily shown that the Hamiltonian (1.11) commutes with the total spin along the anisotropy axis,

$$[H(\Delta), \sum_{n=0}^{N}\mathbf{s}_n^z] = 0. \tag{1.22}$$

This is no longer true for the XYZ Heisenberg chain.

1.3 Schrödinger equation

Since the Hamiltonian commutes with the total spin along the anisotropy axis, see relation (1.22), one can look for the solution of the Schrödinger equation

$$H(\Delta)\psi_M = E\psi_M \tag{1.23}$$

as a superposition of all vectors in the Hilbert space with the fixed number M of down spins

$$\psi_M = \sum_{\{n\}} a(n_1, n_2, \ldots, n_M)|n_1, n_2, \ldots, n_M\rangle, \tag{1.24}$$

where the summation goes over all possible sets of M ordered sites (1.13).

In order to proceed, we have to know how the Hamiltonian acts on a given Hilbert vector $|n_1, n_2, \ldots, n_M\rangle$. Using (1.21), we can rewrite the Hamiltonian (1.19) as follows

$$H(\Delta) = -N\frac{\Delta}{2} + \sum_{n=1}^{N} H_{n,n+1}, \qquad -H_{n,n+1} = \mathbf{s}_n^+\mathbf{s}_{n+1}^- + \mathbf{s}_n^-\mathbf{s}_{n+1}^+ + \frac{1}{2}\Delta\left(\mathbf{s}_n^z\mathbf{s}_{n+1}^z - 1\right). \tag{1.25}$$

The component $H_{n,n+1}$ acts as the unity operator on each site, except for the couple of nearest neighbours $n, n+1$. Since it holds

$$\left(\mathbf{s}_n^+\mathbf{s}_{n+1}^- + \mathbf{s}_n^-\mathbf{s}_{n+1}^+\right)\begin{pmatrix} e_n^\pm \otimes e_{n+1}^\pm \\ e_n^\pm \otimes e_{n+1}^\mp \end{pmatrix} = \begin{pmatrix} 0 \\ e_n^\mp \otimes e_{n+1}^\pm \end{pmatrix} \tag{1.26}$$

and

$$\mathbf{s}_n^z\mathbf{s}_{n+1}^z\begin{pmatrix} e_n^\pm \otimes e_{n+1}^\pm \\ e_n^\pm \otimes e_{n+1}^\mp \end{pmatrix} = \begin{pmatrix} e_n^\pm \otimes e_{n+1}^\pm \\ -e_n^\pm \otimes e_{n+1}^\mp \end{pmatrix}, \tag{1.27}$$

we have

$$H_{n,n+1}\begin{pmatrix} e_n^\pm \otimes e_{n+1}^\pm \\ e_n^\pm \otimes e_{n+1}^\mp \end{pmatrix} = \Delta\begin{pmatrix} 0 \\ e_n^\pm \otimes e_{n+1}^\mp \end{pmatrix} - \begin{pmatrix} 0 \\ e_n^\mp \otimes e_{n+1}^\pm \end{pmatrix}. \tag{1.28}$$

With regard to the representation (1.25), it holds

$$\left[H(\Delta) + N\frac{\Delta}{2}\right]|n_1, n_2, \ldots, n_M\rangle = N_a\Delta|n_1, n_2, \ldots, n_M\rangle - \sum_{\{n'\}}|n_1', n_2', \ldots, n_M'\rangle. \tag{1.29}$$

Here, the configuration of sites $\{n'\}$ is obtained from $\{n\}$ by the interchange of just one nearest-neighbour pair of antiparallel spins

$$n_1' = n_1, \quad n_2' = n_2, \ldots, \quad n_\alpha' = n_\alpha \pm 1, \ldots, \quad n_M' = n_M, \tag{1.30}$$

under the condition that the site configuration $\{n'\}$ will be the allowed one fulfilling the ordering condition (1.13), and

$$N_a = \sum_{\{n'\}} 1 \tag{1.31}$$

is the number of the nearest-neighbour antiparallel spins in the site configuration $\{n\}$.

Finally, the condition that ψ_M (1.24) is the eigenfunction (more precisely, eigenvector) of the Hamiltonian $H(\Delta)$, see the Schrödinger Eq. (1.23), can be expressed as

$$\bar{E}a\{n\} = \sum_{\{n'\}} \left(\Delta\, a\{n\} - a\{n'\}\right), \tag{1.32}$$

where $\bar{E} = E + N\Delta/2$ is the shifted energy. In order to apply the interchange $\{n\} \to \{n'\}$ (1.30) in the whole interval of site coordinates n_α, we have to extend this interval beyond N by identifying sites n with $N + n$. The periodic boundary conditions for the a-amplitudes are written in the form

$$a(n_1, n_2, \ldots, n_M) = a(n_2, n_3, \ldots, n_M, n_1 + N) \tag{1.33}$$

respecting the prescribed ordering of sites (1.13). The next task is to solve the set of equations (1.32), complemented by the periodic boundary conditions (1.33), for each sector of the Hilbert space with $M = 0, 1, \ldots, N$ down spins.

1.4 Coordinate Bethe ansatz

1.4.1 M=0,1

The case $M = 0$ is trivial. The vector $|0\rangle$ with all sites in the spin-up state e^+ is the eigenvector of the Hamiltonian $H(\Delta)$ with the energy $E = -N\Delta/2$.

In the sector with one spin down $M = 1$, Eq. (1.32) reads

$$\bar{E}a(n) = 2\Delta a(n) - a(n-1) - a(n+1). \tag{1.34}$$

The solution of this equation is the plane wave

$$a(n) = A\exp(\mathrm{i}kn). \tag{1.35}$$

The wave number k is quantized according to the periodicity condition (1.33), $a(n) = a(n + N)$, as follows

$$\exp(\mathrm{i}kN) = 1 \Longleftrightarrow Nk = 2\pi I, \qquad I = 0, 1, \ldots, N-1. \tag{1.36}$$

The interval of I-values can be shifted by an arbitrary integer. The energy E is obtained by substituting the solution (1.35) for $a(n)$ into (1.34), with the result

$$E = -N\frac{\Delta}{2} + 2\left(\Delta - \cos k\right). \tag{1.37}$$

1.4.2 M=2

For $M = 2$ one has to distinguish between two cases: sites n_1 and n_2 are either nearest neighbours or they are not.

Let us start with the case when they are not nearest neighbours, i.e. $n_2 \neq n_1 + 1$. The equation (1.32) then takes the form

$$\begin{aligned}\bar{E}a(n_1, n_2) = {} & 4\Delta a(n_1, n_2) - a(n_1 - 1, n_2) - a(n_1 + 1, n_2) \\ & - a(n_1, n_2 - 1) - a(n_1, n_2 + 1).\end{aligned} \tag{1.38}$$

The general solution of this equation reads

$$a(n_1, n_2) = A_{12}\mathrm{e}^{\mathrm{i}(k_1 n_1 + k_2 n_2)} + A_{21}\mathrm{e}^{\mathrm{i}(k_2 n_1 + k_1 n_2)}, \tag{1.39}$$

$$E = -N\frac{\Delta}{2} + 2\left(\Delta - \cos k_1\right) + 2\left(\Delta - \cos k_2\right). \tag{1.40}$$

The coefficients A_{12} and A_{21} are as-yet free.

When the sites n_1 and n_2 are nearest neighbours, i.e. $n_2 = n_1 + 1$, Eq. (1.32) takes the form

$$\bar{E}a(n, n+1) = 2\Delta a(n, n+1) - a(n-1, n+1) - a(n, n+2). \tag{1.41}$$

We shall look for the solution of this equation in the same form (1.39) as in the previous case, where the coefficients A_{12} and A_{21} will be constrained by a condition. We can do that due the fact that Eq. (1.41) can be represented as the previous one (1.38) provided that there is a constraint on $a(n_1, n_2)$. For this purpose, we extend the definition of $a(n_1, n_2)$ to identical sites $n_1 = n_2$ and put formally $n_1 = n, n_2 = n + 1$ in (1.38):

$$\begin{aligned}\bar{E}a(n, n+1) = {} & 4\Delta a(n, n+1) - a(n-1, n+1) - a(n+1, n+1) \\ & -a(n, n) - a(n, n+2).\end{aligned} \tag{1.42}$$

The two equations (1.41) and (1.42) are equivalent if it holds

$$a(n+1, n+1) - 2\Delta a(n, n+1) + a(n, n) = 0. \tag{1.43}$$

The insertion of the solution (1.39) into this equation leads to the following relation between the A-coefficients:

$$\frac{A_{12}}{A_{21}} = -\frac{\mathrm{e}^{\mathrm{i}(k_1+k_2)} - 2\Delta\mathrm{e}^{\mathrm{i}k_1} + 1}{\mathrm{e}^{\mathrm{i}(k_1+k_2)} - 2\Delta\mathrm{e}^{\mathrm{i}k_2} + 1} = \exp\left(\mathrm{i}\theta_{12}\right). \tag{1.44}$$

The phase factor is readily shown to be given by

$$\mathrm{ctg}\left(\frac{\theta_{12}}{2}\right) = \Delta\frac{\mathrm{ctg}(k_1/2) - \mathrm{ctg}(k_2/2)}{(\Delta+1) + (\Delta-1)\mathrm{ctg}(k_1/2)\mathrm{ctg}(k_2/2)}. \tag{1.45}$$

It is antisymmetric with respect to the exchange of indices,

$$\theta_{12} = -\theta_{21}. \tag{1.46}$$

Setting the common prefactor to unity, the coefficients are expressible simply as

$$A_{12} = \exp\left(\frac{\mathrm{i}}{2}\theta_{12}\right), \qquad A_{21} = \exp\left(\frac{\mathrm{i}}{2}\theta_{21}\right). \tag{1.47}$$

We note that the wave numbers must be unequal, $k_1 \neq k_2$. In the opposite case $k_1 = k_2 = k$, it follows from Eq. (1.44) that $A_{12} = -A_{21}$ and consequently the amplitude $a(n_1, n_2) = (A_{12} + A_{21})\exp[\mathrm{i}k(n_1 + n_2)]$ vanishes.

The wave numbers k_1 and k_2 are quantized according to the periodic boundary condition (1.33), $a(n_1, n_2) = a(n_2, n_1 + N)$, as follows

$$A_{12} = A_{21}\mathrm{e}^{\mathrm{i}k_1 N}, \qquad A_{21} = A_{12}\mathrm{e}^{\mathrm{i}k_2 N}. \tag{1.48}$$

With regard to (1.44), these conditions can be rewritten in a more convenient form

$$Nk_1 = 2\pi I_1 + \theta_{12}, \qquad Nk_2 = 2\pi I_2 + \theta_{21}, \tag{1.49}$$

where I_1 and I_2 are sequences of N consecutive integers such that $k_1 \neq k_2$.

1.4.3 M=3

The treatment of the sector $M = 3$ allows to perform a generalization of the formalism to arbitrary M. It is necessary to consider all possibilities of nearest-neighbour positions for three sites, ordered as $n_1 < n_2 < n_3$:

(a) $n_2 \neq n_1 + 1, \quad n_3 \neq n_2 + 1,$
(b) $n_2 = n_1 + 1, \quad n_3 \neq n_2 + 1,$
(c) $n_2 \neq n_1 + 1, \quad n_3 = n_2 + 1,$
(d) $n_2 = n_1 + 1, \quad n_3 = n_2 + 1.$

In the case (a) with no nearest neighbours, the Schrödinger Eq. (1.32) takes the form

$$\begin{aligned}\bar{E}a(n_1,n_2,n_3) = {} & 6\Delta a(n_1,n_2,n_3) - a(n_1-1,n_2,n_3) - a(n_1+1,n_2,n_3) \\ & -a(n_1,n_2-1,n_3) - a(n_1,n_2+1,n_3) \\ & -a(n_1,n_2,n_3-1) - a(n_1,n_2,n_3+1). \end{aligned} \tag{1.50}$$

Its solution is represented as a superposition of plane waves

$$\begin{aligned}a(n_1,n_2,n_3) = {} & A_{123}\mathrm{e}^{\mathrm{i}(k_1n_1+k_2n_2+k_3n_3)} + A_{132}\mathrm{e}^{\mathrm{i}(k_1n_1+k_3n_2+k_2n_3)} \\ & +A_{213}\mathrm{e}^{\mathrm{i}(k_2n_1+k_1n_2+k_3n_3)} + A_{231}\mathrm{e}^{\mathrm{i}(k_2n_1+k_3n_2+k_1n_3)} \\ & +A_{312}\mathrm{e}^{\mathrm{i}(k_3n_1+k_1n_2+k_2n_3)} + A_{321}\mathrm{e}^{\mathrm{i}(k_3n_1+k_2n_2+k_1n_3)}. \end{aligned} \tag{1.51}$$

The corresponding energy is given by

$$E = -N\frac{\Delta}{2} + 2\left(\Delta - \cos k_1\right) + 2\left(\Delta - \cos k_2\right) + 2\left(\Delta - \cos k_3\right). \tag{1.52}$$

In the presence of nearest-neighbour sites, we shall use the same trick as in the $M = 2$ sector. The case (b) implies

$$\frac{A_{123}}{A_{213}} = \mathrm{e}^{\mathrm{i}\theta_{12}}, \quad \frac{A_{132}}{A_{312}} = \mathrm{e}^{\mathrm{i}\theta_{13}}, \quad \frac{A_{231}}{A_{321}} = \mathrm{e}^{\mathrm{i}\theta_{23}}, \tag{1.53}$$

where $\theta_{\alpha\beta}$ with $\alpha, \beta = 1, 2, 3$ is the obvious generalization of the function given by Eqs. (1.44) and (1.45). The case (c) implies

$$\frac{A_{123}}{A_{132}} = \mathrm{e}^{\mathrm{i}\theta_{23}}, \quad \frac{A_{213}}{A_{231}} = \mathrm{e}^{\mathrm{i}\theta_{13}}, \quad \frac{A_{312}}{A_{321}} = \mathrm{e}^{\mathrm{i}\theta_{12}}. \tag{1.54}$$

The consideration of the case (d) does not imply any new relations among the A-coefficients. Using the symmetry relation (1.46) for the phases $\theta_{\alpha\beta}$, the solution of 6 homogeneous relations in Eqs. (1.53) and (1.54) for 6 coefficients can be written in the form

$$\begin{aligned} A_{123} &= \exp\left[\frac{\mathrm{i}}{2}\left(\theta_{12}+\theta_{13}+\theta_{23}\right)\right], \\ A_{213} &= \exp\left[\frac{\mathrm{i}}{2}\left(\theta_{21}+\theta_{23}+\theta_{13}\right)\right], \\ A_{321} &= \exp\left[\frac{\mathrm{i}}{2}\left(\theta_{32}+\theta_{31}+\theta_{21}\right)\right], \end{aligned} \tag{1.55}$$

etc., with the obvious formal structure.

The periodic boundary condition (1.33), $a(n_1, n_2, n_3) = a(n_2, n_3, n_1 + N)$, implies

$$A_{123} = A_{231}\mathrm{e}^{\mathrm{i}k_1 N}, \quad A_{213} = A_{132}\mathrm{e}^{\mathrm{i}k_2 N}, \quad A_{312} = A_{123}\mathrm{e}^{\mathrm{i}k_3 N}. \tag{1.56}$$

The wave numbers k_1, k_2 and k_3, necessarily distinct, are thus quantized as follows

$$\begin{aligned} Nk_1 &= 2\pi I_1 + \theta_{12} + \theta_{13}, \\ Nk_2 &= 2\pi I_2 + \theta_{21} + \theta_{23}, \\ Nk_3 &= 2\pi I_3 + \theta_{31} + \theta_{32}, \end{aligned} \tag{1.57}$$

where each of I_1, I_2 and I_3 is a sequence of N consecutive integers.

1.4.4 Arbitrary M

We have seen that the solution in the sector $M = 3$ was constructed explicitly using the information gain from the $M = 2$ sector, namely the phase function $\theta_{\alpha\beta}$ defined in (1.44) or (1.45). This property is maintained also for higher sectors $M = 4, 5, \ldots, N$.

Let us introduce the symmetric group S_M of all $M!$ permutations of M number $(1, 2, \ldots, M)$. We shall denote an element of the group S_M by P, $P\alpha$ with $\alpha = 1, 2, \ldots, M$ will denote the number at the αth position in P. Like for instance, for $M = 3$ and the permutation $P = (3, 1, 2)$ one has $P1 = 3$, $P2 = 1$ and $P3 = 2$.

The Bethe ansatz has the form

$$a(n_1, n_2, \ldots, n_M) = \sum_{P \in S_M} A_P \exp\left(\mathrm{i}\sum_{\alpha=1}^{M} k_{P\alpha} n_\alpha\right). \tag{1.58}$$

This solution certainly fulfills the Schrödinger equation when there are no nearest-neighbour sites, i.e. $n_{\alpha+1} \neq n_\alpha + 1$ for each $\alpha = 1, 2, \ldots, M$. The corresponding energy is given by

$$E = -N\frac{\Delta}{2} + \sum_{\alpha=1}^{M} 2(\Delta - \cos k_\alpha). \tag{1.59}$$

When two sites are the nearest neighbours, say $n_{\alpha+1} = n_\alpha + 1$ with a given α, the counterpart of the "consistency" equation (1.43) is

$$a(\ldots, n_\alpha + 1, n_\alpha + 1, \ldots) - 2\Delta a(\ldots, n_\alpha, n_\alpha + 1, \ldots) + a(\ldots, n_\alpha, n_\alpha, \ldots) = 0. \tag{1.60}$$

Inserting the Bethe ansatz (1.58) to this consistency equation leads to

$$\begin{aligned} \sum_{P \in S_M} A_P \Big[\mathrm{e}^{\mathrm{i}(k_{P\alpha} + k_{P(\alpha+1)})} - 2\Delta \mathrm{e}^{\mathrm{i}k_{P(\alpha+1)}} + 1\Big] \\ \times \mathrm{e}^{\mathrm{i}k_{P1}n_1 + \cdots + \mathrm{i}(k_{P\alpha} + k_{P(\alpha+1)})n_\alpha + \cdots \mathrm{i}k_{PM}n_M} = 0. \end{aligned} \tag{1.61}$$

In the summation, each permutation P is coupled with the permutation $P(\alpha, \alpha+1)$ which is generated from P by the transposition of the nearest-neighbours $P\alpha$ and $P(\alpha+1)$, i.e. if $P = (P1, \ldots, P\alpha, P(\alpha+1), \ldots, PM)$ then $P(\alpha, \alpha+1) = (P1, \ldots, P(\alpha+1), P\alpha, \ldots, PM)$.

Since the corresponding A-coefficients are multiplied by the same $\{n\}$-dependent exponential in (1.61), it holds

$$A_P \left[\mathrm{e}^{\mathrm{i}(k_{P\alpha}+k_{P(\alpha+1)})} - 2\Delta \mathrm{e}^{\mathrm{i}k_{P(\alpha+1)}} + 1 \right] + A_{P(\alpha,\alpha+1)} \left[\mathrm{e}^{\mathrm{i}(k_{P\alpha}+k_{P(\alpha+1)})} - 2\Delta \mathrm{e}^{\mathrm{i}k_{P\alpha}} + 1 \right] = 0. \tag{1.62}$$

With respect to the definition (1.44) of θ phase, one thus has

$$A_{P(\alpha,\alpha+1)} = A_P \exp\left(-\mathrm{i}\theta_{P\alpha,P(\alpha+1)}\right). \tag{1.63}$$

This implies

$$A_P = \exp\left(\frac{\mathrm{i}}{2} \sum_{\substack{\alpha,\beta=1 \\ (\alpha<\beta)}}^{M} \theta_{P\alpha,P\beta} \right). \tag{1.64}$$

The periodic boundary condition (1.33) is equivalent to the conditions

$$A_P = A_{PC} \mathrm{e}^{\mathrm{i}k_{P1}N} \quad \text{for arbitrary } P, \tag{1.65}$$

where PC is the cyclic transposition of P, i.e. when $P = (P1, P2, \ldots, PM)$ then $PC = (P2, \ldots, PM, P1)$. With respect to the result (1.64) for the A-amplitudes, one has

$$\mathrm{e}^{\mathrm{i}k_{P1}N} = \frac{A_P}{A_{PC}} = \exp\left(\mathrm{i} \sum_{\alpha=2}^{M} \theta_{P1,P\alpha} \right) \quad \text{for arbitrary } P. \tag{1.66}$$

We conclude that the wave numbers k_1, k_2, …, k_M are quantized according to the set of M Bethe equations

$$Nk_\alpha = 2\pi I_\alpha + \sum_{\substack{\beta=1 \\ (\beta\neq\alpha)}}^{M} \theta_{\alpha\beta}, \qquad \alpha = 1, 2, \ldots, M. \tag{1.67}$$

Each I_α belongs to a sequence of N consecutive integers such that arbitrary two wave numbers $k_\alpha \neq k_\beta$.

Especially simple is the case $\Delta = 0$. In that case, the phase factor (1.44) is given by $\exp(\mathrm{i}\theta_{\alpha\beta}) = -1$. The boundary conditions (1.66) then implies

$$\mathrm{e}^{\mathrm{i}k_\alpha N} = (-1)^{M-1}, \qquad E = -2\sum_{\alpha=1}^{M} \cos k_\alpha. \tag{1.68}$$

1.5 Orbach parametrization

The θ-function, given by Eq. (1.45), is a complicated function of wave k-numbers. For reasons which will be clearer later, it is useful to parametrize the k-numbers by "rapidities" λ, $k = k(\lambda)$, in such a way that the θ-function will depend only on the difference of the corresponding rapidities: $\theta_{\alpha\beta} = \theta(\lambda_\alpha - \lambda_\beta)$. The form of the parametrization depends on the value of the anisotropy parameter Δ: $\Delta > 1$, $\Delta = 1$, $-1 < \Delta < 1$, $\Delta = -1$, $\Delta < -1$. For simplicity, in what follows we shall restrict ourselves to the ferromagnetic ($\Delta = 1$) and antiferromagnetic ($\Delta = -1$) isotropic chains.

1.5.1 $\Delta = 1$

For $\Delta = 1$, the relation (1.45) takes the form

$$\mathrm{ctg}\left(\frac{\theta_{\alpha\beta}}{2}\right) = \frac{1}{2}\left[\mathrm{ctg}\left(\frac{k_\alpha}{2}\right) - \mathrm{ctg}\left(\frac{k_\beta}{2}\right)\right]. \tag{1.69}$$

The needed parametrization is then

$$\lambda_\alpha = \frac{1}{2}\mathrm{ctg}\left(\frac{k_\alpha}{2}\right), \qquad \mathrm{ctg}\left(\frac{\theta_{\alpha\beta}}{2}\right) = \lambda_\alpha - \lambda_\beta. \tag{1.70}$$

The inverse relations read

$$k_\alpha = \frac{1}{\mathrm{i}}\ln\left(\frac{2\lambda_\alpha + \mathrm{i}}{2\lambda_\alpha - \mathrm{i}}\right), \qquad \theta_{\alpha\beta} = \frac{1}{\mathrm{i}}\ln\left(\frac{\lambda_\alpha - \lambda_\beta + \mathrm{i}}{\lambda_\alpha - \lambda_\beta - \mathrm{i}}\right). \tag{1.71}$$

In general, the quantities k_α, and consequently the rapidities λ_α, can be complex numbers. For real values, they are localized in intervals

$$0 < k_\alpha < 2\pi, \qquad -\infty < \lambda_\alpha < \infty. \tag{1.72}$$

Inserting relations (1.71) into the Bethe equations (1.66) results in

$$\left(\frac{\lambda_\alpha + \mathrm{i}/2}{\lambda_\alpha - \mathrm{i}/2}\right)^N = \prod_{\substack{\beta=1 \\ (\beta\neq\alpha)}}^{M} \frac{\lambda_\alpha - \lambda_\beta + \mathrm{i}}{\lambda_\alpha - \lambda_\beta - \mathrm{i}}, \qquad \alpha = 1, 2, \ldots, M. \tag{1.73}$$

The energy (1.59) is expressible in terms of rapidities as follows

$$E = -\frac{N}{2} + \sum_{\alpha=1}^{M} \frac{1}{\lambda_\alpha^2 + 1/4}. \tag{1.74}$$

1.5.2 $\Delta = -1$

For $\Delta = -1$, the relation (1.45) takes the form

$$\mathrm{ctg}\left(\frac{\theta_{\alpha\beta}}{2}\right) = \frac{1}{2}\left[\mathrm{tg}\left(\frac{k_\beta}{2}\right) - \mathrm{tg}\left(\frac{k_\alpha}{2}\right)\right]. \tag{1.75}$$

The needed parametrization is then

$$\lambda_\alpha = \frac{1}{2}\mathrm{tg}\left(\frac{k_\alpha}{2}\right), \qquad \mathrm{ctg}\left(\frac{\theta_{\alpha\beta}}{2}\right) = \lambda_\beta - \lambda_\alpha. \tag{1.76}$$

The inverse relations read

$$k_\alpha = \frac{1}{\mathrm{i}}\ln\left(\frac{\mathrm{i} - 2\lambda_\alpha}{\mathrm{i} + 2\lambda_\alpha}\right), \qquad \theta_{\alpha\beta} = \frac{1}{\mathrm{i}}\ln\left(\frac{\lambda_\beta - \lambda_\alpha + \mathrm{i}}{\lambda_\beta - \lambda_\alpha - \mathrm{i}}\right). \tag{1.77}$$

If k_α and λ_α take real values, they are localized in intervals

$$-\pi < k_\alpha < \pi, \qquad -\infty < \lambda_\alpha < \infty. \tag{1.78}$$

Bethe equations (1.66) now take the form

$$\left(\frac{\mathrm{i}/2-\lambda_\alpha}{\mathrm{i}/2+\lambda_\alpha}\right)^N = \prod_{\substack{\beta=1\\(\beta\neq\alpha)}}^{M} \frac{\lambda_\alpha-\lambda_\beta-\mathrm{i}}{\lambda_\alpha-\lambda_\beta+\mathrm{i}}, \qquad \alpha=1,2,\ldots,M. \tag{1.79}$$

The energy (1.59) is expressible in terms of rapidities as follows

$$E = \frac{N}{2} - \sum_{\alpha=1}^{M} \frac{1}{\lambda_\alpha^2+1/4}. \tag{1.80}$$

When the number of lattice sites N = even number, Eqs. (1.73) and (1.79) coincide while the respective energies (1.74) and (1.80) differ only by the sign, which is in agreement with the equivalence of the spectra of conjugate Hamiltonians H_Δ and $-H_{-\Delta}$.

1.6 Ground state and its energy

When $\Delta \geq 1$, the ground state of the Heisenberg chain is evidently ferromagnetic, i.e. $M = 0$, with the energy

$$E_0 = -N\frac{\Delta}{2}, \qquad \Delta \geq 1. \tag{1.81}$$

In the region of $\Delta < 1$, the "global" ground state has the total S^z-spin equal to zero, i.e. belong to the sector with $M = N/2$ (N even). To analyze the Bethe equations (1.67), one has to specify the values of integers I_α which correspond to the ground state. It has been shown in Refs. [2, 3] that in the given M-sector the ground-state integers $\{I_\alpha\}_{\alpha=1}^{M}$, are distinct and symmetrically distributed around 0 with the unity step, i.e.

$$-I_{\max}, -I_{\max}+1, \ldots, I_{\max}-1, I_{\max}.$$

Since $2I_{\max}+1 = M$, one has

$$I_1, I_2, \ldots, I_M = -\frac{M-1}{2}, -\frac{M-1}{2}+1, \ldots, \frac{M-1}{2}. \tag{1.82}$$

This choice of I-values was first observed for the $\Delta = 0$ case (1.68): here, $Nk_\alpha = 2\pi I_\alpha$ with $\{I_\alpha\}$ given by (1.82) indeed minimizes the energy and simultaneously ensures that $k_\alpha \neq k_\beta$.

The Bethe equations (1.67) with the I-values (1.82) in the zero-spin sector $M = N/2$

$$k_\alpha = 2\pi\frac{I_\alpha}{N} - \frac{1}{N}\sum_{\substack{\beta=1\\(\beta\neq\alpha)}}^{N/2} \theta_{\alpha\beta}, \qquad \alpha=1,2,\ldots,N/2. \tag{1.83}$$

lead, in the continuum $N \to \infty$ limit, to *real* wave numbers $\{k_\alpha\}$. Setting $I_\alpha/N = x$ ($-1/4 < x < 1/4$), one gets from (1.83)

$$k[\lambda(x)] = 2\pi x + \int_{-1/4}^{1/4} \mathrm{d}y\ \theta[\lambda(x)-\lambda(y)]. \tag{1.84}$$

Here, the wave numbers k and the θ-function are parametrized by the continuous rapidities $\lambda(x)$. Let us introduce the density of rapidities around the given value by the relation $\mathrm{d}x = \rho(\lambda)\mathrm{d}\lambda$. Differentiation of Eq. (1.84) with respect to λ then leads to

$$\frac{\mathrm{d}k(\lambda)}{\mathrm{d}\lambda} = 2\pi\rho(\lambda) - \int \mathrm{d}\lambda'\ \rho(\lambda')\frac{\partial}{\partial\lambda}\theta(\lambda-\lambda'), \tag{1.85}$$

where the range of integration depends on Δ.

For the antiferromagnetic isotropic case $\Delta = -1$, the Orbach parametrization (1.76), or equivalently (1.77), gives

$$\frac{\mathrm{d}k(\lambda)}{\mathrm{d}\lambda} = \frac{4}{1+4\lambda^2}, \qquad \frac{\partial}{\partial\lambda}\theta(\lambda-\lambda') = \frac{2}{1+(\lambda-\lambda')^2}. \tag{1.86}$$

The rapidity in (1.78) takes real values on the whole axis. Eq. (1.85) is then written as

$$\frac{2}{1+4\lambda^2} = \pi\rho(\lambda) + \int_{-\infty}^{\infty} \mathrm{d}\lambda'\ \frac{\rho(\lambda')}{1+(\lambda-\lambda')^2}. \tag{1.87}$$

Applying in this equation the Fourier transform

$$\rho(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\xi\ \mathrm{e}^{-\mathrm{i}\lambda\xi}\hat{\rho}(\xi), \tag{1.88}$$

and subsequently using the integral formula

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\lambda \frac{\mathrm{e}^{-\mathrm{i}\lambda\xi}}{1+\lambda^2} = \frac{1}{2}\mathrm{e}^{-|\xi|} \tag{1.89}$$

(the residuum theorem), one ends up with

$$\hat{\rho}(\xi) = \frac{1}{2\cosh(\xi/2)}, \qquad \rho(\lambda) = \frac{1}{2\cosh(\pi\lambda)}. \tag{1.90}$$

The ground-state energy (1.80) then reads

$$E_0 = \frac{N}{2} - N\int_{-\infty}^{\infty} \mathrm{d}\lambda\ \rho(\lambda)\frac{1}{\lambda^2+1/4} = \frac{N}{2} - N2\ln 2. \tag{1.91}$$

1.7 Excited states

The general analysis of Bethe equations is complicated for finite N, but simplifies substantially in the limit $N \to \infty$.

Let us consider for simplicity the isotropic $\Delta = 1$ ferromagnetic chain. Its Orbach parametrization is presented in Section 1.5.1, Eqs. (1.69) - (1.74). As was mentioned, the ground state corresponds to all spins up ($M = 0$) with the energy $E_0 = -N/2$.

For the first excited states in the sector $M = 1$, the equations for λ and k read

$$\left(\frac{\lambda+\mathrm{i}/2}{\lambda-\mathrm{i}/2}\right)^N = 1, \qquad \mathrm{e}^{\mathrm{i}kN} = 1. \tag{1.92}$$

The corresponding energy is $E = E_0 + 2(1 - \cos k)$. In the limit $N \to \infty$, the wave numbers k cover continuously the whole interval $\langle 0, 2\pi)$ and the rapidities cover the real axis $-\infty < \lambda < \infty$. The excitations of this type are called magnons. The energy of a magnon with the wave number k is $\epsilon_k = 2(1 - \cos k)$.

In the sector with two spin downs ($M = 2$), Bethe Eqs. (1.73) read

$$\left(\frac{\lambda_1 + \mathrm{i}/2}{\lambda_1 - \mathrm{i}/2}\right)^N = \frac{\lambda_1 - \lambda_2 + \mathrm{i}}{\lambda_1 - \lambda_2 - \mathrm{i}}, \qquad \left(\frac{\lambda_2 + \mathrm{i}/2}{\lambda_2 - \mathrm{i}/2}\right)^N = \frac{\lambda_2 - \lambda_1 + \mathrm{i}}{\lambda_2 - \lambda_1 - \mathrm{i}}. \tag{1.93}$$

Let us first study *real* solutions of these equations and denote $(\lambda_1 - \lambda_2 + \mathrm{i})/(\lambda_1 - \lambda_2 - \mathrm{i}) = \exp(\mathrm{i}\varphi)$, $\varphi \in R$. Then,

$$\mathrm{e}^{\mathrm{i}k_1 N} = \mathrm{e}^{\mathrm{i}\varphi}, \qquad \mathrm{e}^{\mathrm{i}k_2 N} = \mathrm{e}^{-\mathrm{i}\varphi}. \tag{1.94}$$

In the limit $N \to \infty$, k_1 and k_2 once again cover continuously the interval $\langle 0, 2\pi)$ and one has the state of two independent magnons with the energy $E = E_0 + \epsilon_{k_1} + \epsilon_{k_2}$.

However, the system of two equations in (1.93) exhibits also complex solutions

$$\lambda_1 = x_1 + \mathrm{i}y_1, \qquad \lambda_2 = x_2 + \mathrm{i}y_2. \tag{1.95}$$

Comparing the modulus of lhs and rhs of the first equation in (1.93), one gets

$$\left[\frac{x_1^2 + (y_1 - 1/2)^2}{x_1^2 + (y_1 + 1/2)^2}\right]^N = \frac{(x_1 - x_2)^2 + (y_1 - y_2 - 1)^2}{(x_1 - x_2)^2 + (y_1 - y_2 + 1)^2}. \tag{1.96}$$

Let us assume that $y_1 > 0$. As $N \to \infty$, the lhs of (1.96) goes exponentially to 0. The rhs thus implies

$$x_1 = x_2, \qquad y_1 - y_2 = 1. \tag{1.97}$$

The same findings are obtained from the second equation in (1.93). Multiplying the two equations in (1.93) results in the relation

$$\left[\frac{x_1 + \mathrm{i}(y_1 + 1/2)}{x_1 + \mathrm{i}(y_1 - 3/2)}\right]^N = 1, \tag{1.98}$$

from which $y_1 = 1/2$ and $-\infty < x_1 < \infty$. Finally,

$$\lambda_1 = x + \mathrm{i}/2, \qquad \lambda_2 = x - \mathrm{i}/2. \tag{1.99}$$

This is the bound state of two magnons. Its energy $\epsilon_{k_1,k_2} = 1 - \cos(k_1 + k_2)$ is always lower than the sum of energies for two independent magnons $\epsilon_{k_1} + \epsilon_{k_2}$.

For an arbitrary value of Δ and in the limit $N \to \infty$, the states of the Heisenberg chain exhibit in the space of rapidities a string structure. Namely, the states are grouped into complex strings

$$\lambda_\mu = x + \mathrm{i}\mu, \qquad \mu = -m, -m+1, \ldots, m-1, m. \tag{1.100}$$

Since $2m + 1$ is the number of states and as such it has to be a positive integer, one has $m = 0, 1/2, 1, 3/2$, etc. The statistics of complex strings enables one to derive the thermodynamics (the free energy, etc.) of the Heisenberg chain at arbitrary temperature T, see monographs [M2, M3].

2. Quantum inverse scattering method

2.1 Definition of the S-matrix

The Heisenberg XXZ chain is the simplest system for which the exact solution is obtained by using the coordinate Bethe ansatz, see the representation (1.58) for the wave-function amplitudes. In this case, one has M "particles", namely down-spins in the sea of up-spins, which have no internal degrees of freedom. The particles are specified only by their positions on the chain and these positions are ordered, $n_1 < n_2 < \ldots < n_M$. The scattering of particles is defined by the relation between amplitude $A_{21} = S_{12}A_{12}$ where S_{12} is a scalar.

There exist physical systems of particles possessing internal degrees of freedom $\sigma = 1, \ldots, l$, sometimes called "colours". Like for example, for an electron σ denotes one of the two spin states $\{\uparrow, \downarrow\}$. Let us consider N particles on a chain of sites $x = 1, 2, \ldots, L$. The particles are characterized by the couples of data $(\sigma_1, x_1), (\sigma_2, x_2), \ldots, (\sigma_N, x_N)$. Now, mutual positions among arbitrarily coloured particles are relevant. For a given configuration of particles we define the permutation $Q = (Q1, Q2, \ldots, QN) \in S_N$ as follows

$$X_Q = \{x_{Q1} \leq x_{Q2} \leq \ldots \leq x_{QN}\}. \tag{2.1}$$

For example, if three particles are ordered as follows $x_3 < x_1 < x_2$ one has $Q = (3, 1, 2)$. The permutation Q carries the information about the positional ordering of the particles (from the left to the right) on the chain: $(\sigma_{Q1}, x_{Q1}), (\sigma_{Q2}, x_{Q2}), \ldots, (\sigma_{QN}, x_{QN})$.

In section 3, we shall solve the 1D Hubbard model of N interacting electrons with the spin $\sigma \in \{\uparrow, \downarrow\}$. The solution for the wave function will be found in the form of the generalized Bethe ansatz:

$$\psi_{\sigma_1\sigma_2\ldots\sigma_N}(x_1, x_2, \ldots, x_N) = \sum_{P\in S_N} \mathrm{sign}(PQ) A(Q|P) \exp\left(\mathrm{i}\sum_{\alpha=1}^{N} k_{P\alpha} x_{Q\alpha}\right), \tag{2.2}$$

where

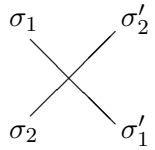
$$A(Q|P) \equiv A_{\sigma_{Q1}\sigma_{Q2}\ldots\sigma_{QN}}(k_{P1}, k_{P2}, \ldots, k_{PN}) \tag{2.3}$$

denotes the amplitude and $\mathrm{sign}(PQ)$, which is equivalent to $(-1)^{\eta_P+\eta_Q}$ with η_P (η_Q) being the parity of the permutation P (Q), ensures the antisymmetry of the wave function ψ with respect to the interchange of any pair of particles. As before, the summation over all permutations P "distributes" the given wave numbers $(k_1, k_2, \ldots, k_N)$ among all particles.

The A-amplitudes are related by the scattering of particles. Namely, for $N = 2$ one has

$$A_{\sigma_j\sigma_i}(k_v, k_u) = \sum_{\sigma_i'\sigma_j'} S^{\sigma_i\sigma_i'}_{\sigma_j\sigma_j'}(k_u, k_v) A_{\sigma_i'\sigma_j'}(k_u, k_v), \tag{2.4}$$

where $(i, j), (u, v) \in \{(1, 2); (2, 1)\}$ and S denotes the two-particle scattering matrix of dimension l^2. It is seen that the scattering is elastic, i.e. not only the total momentum but also both individual momenta are conserved. Note however that in the scattering process particles can change their σ-colours. The two-particle S-matrix is usually represented graphically as follows

$S^{\sigma_1\sigma_1'}_{\sigma_2\sigma_2'}(k_1, k_2) =$ 

. (2.5)

For an arbitrary particle number N, the two-particle S-matrix relates the A-amplitudes $A(Q|P)$ and $A(\tilde{Q}|\tilde{P})$ which differ from one another by the transposition of a pair of nearest-neighbour particles, i.e. $\tilde{Q} = Q(\alpha, \alpha+1)$ and $\tilde{P} = P(\alpha, \alpha+1)$ [see the definition after Eq. (1.61)]. Schematically, one can write

$$A_{\ldots\sigma_j\sigma_i\ldots}(\ldots k_v, k_u \ldots) = \sum_{\sigma_i'\sigma_j'} S^{\sigma_i\sigma_i'}_{\sigma_j\sigma_j'}(k_u, k_v) A_{\ldots\sigma_i'\sigma_j'\ldots}(\ldots k_u, k_v \ldots). \tag{2.6}$$

Applying successively this nearest-neighbour transposition rule, one can convert an arbitrary amplitude $A(Q|P)$ to the one with $Q = I$, where $I = (1, 2, \ldots, N)$ is the identity permutation corresponding to the coordinate sector

$$X_I = \{x_1 \leq x_2 \leq \ldots \leq x_N\}. \tag{2.7}$$

In this way, the scattering of N particles factorizes into a product of two-particle scatterings which is the fundamental property of integrable systems.

2.2 Yang-Baxter equation

The S-matrices of integrable systems are not arbitrary. There exists a general relation among the elements of the S-matrix which can be deduced from the scattering of three particles.

Let us study the scattering process of three particles which starts from the initial state $x_3 \leq x_2 \leq x_1$, corresponding to $Q = (3, 2, 1)$, and ends in the final state $x_1 \leq x_2 \leq x_3$, corresponding to $Q = I \equiv (1, 2, 3)$. There are two possible realizations of this three-particle scattering in terms of the two-particle scatterings:

$$(a) \qquad (3,2,1) \to (3,1,2) \to (1,3,2) \to (1,2,3);$$

$$(b) \qquad (3,2,1) \to (2,3,1) \to (2,1,3) \to (1,2,3).$$

Using the prescription (2.6), the (a) sequence of two-particle scatterings is expressible as

$$\begin{aligned} A_{\sigma_3\sigma_2\sigma_1}(k_3, k_2, k_1) = & \sum_{\sigma_1'\sigma_2'} S^{\sigma_1\sigma_1'}_{\sigma_2\sigma_2'}(k_1, k_2) \underbrace{A_{\sigma_3\sigma_1'\sigma_2'}(k_3, k_1, k_2)}\\ & \sum_{\sigma_3'\sigma_1''} S^{\sigma_1'\sigma_1''}_{\sigma_3\sigma_3'}(k_1, k_3) \underbrace{A_{\sigma_1''\sigma_3'\sigma_2'}(k_1, k_3, k_2)}\\ & \sum_{\sigma_2''\sigma_3''} S^{\sigma_2'\sigma_2''}_{\sigma_3'\sigma_3''}(k_2, k_3) A_{\sigma_1''\sigma_2''\sigma_3''}(k_1, k_2, k_3), \end{aligned} \tag{2.8}$$

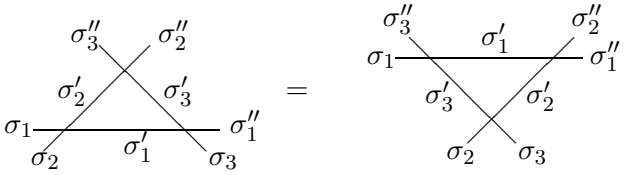
while the (b) sequence corresponds to

$$\begin{aligned} A_{\sigma_3\sigma_2\sigma_1}(k_3, k_2, k_1) = & \sum_{\sigma_2'\sigma_3'} S^{\sigma_2\sigma_2'}_{\sigma_3\sigma_3'}(k_2, k_3) \underbrace{A_{\sigma_2'\sigma_3'\sigma_1}(k_2, k_3, k_1)}\\ & \sum_{\sigma_1'\sigma_3''} S^{\sigma_1\sigma_1'}_{\sigma_3'\sigma_3''}(k_1, k_3) \underbrace{A_{\sigma_2'\sigma_1'\sigma_3''}(k_2, k_1, k_3)}\\ & \sum_{\sigma_1''\sigma_2''} S^{\sigma_1'\sigma_1''}_{\sigma_2'\sigma_2''}(k_1, k_2) A_{\sigma_1''\sigma_2''\sigma_3''}(k_1, k_2, k_3). \end{aligned} \tag{2.9}$$

The final result must be the same for both sequences of two-particle scatterings. Using the Orbach parametrization of wave numbers in terms of rapidities (spectral parameters) $\{k_\alpha = k_\alpha(\lambda_\alpha); S(k_\alpha, k_\beta) = S(\lambda_\alpha - \lambda_\beta)\}$, the equivalence of the processes (2.8) and (2.9) implies the following constraint for the elements of the S-matrix:

$$\sum_{\sigma_1'\sigma_2'\sigma_3'} S^{\sigma_1\sigma_1'}_{\sigma_2\sigma_2'}(\lambda-\mu) S^{\sigma_1'\sigma_1''}_{\sigma_3\sigma_3'}(\lambda) S^{\sigma_2'\sigma_2''}_{\sigma_3'\sigma_3''}(\mu) = \sum_{\sigma_1'\sigma_2'\sigma_3'} S^{\sigma_2\sigma_2'}_{\sigma_3\sigma_3'}(\mu) S^{\sigma_1\sigma_1'}_{\sigma_3'\sigma_3''}(\lambda) S^{\sigma_1'\sigma_1''}_{\sigma_2'\sigma_2''}(\lambda-\mu). \qquad (2.10)$$

This Yang-Baxter relation is of the form $S_{12}(\lambda-\mu)S_{13}(\lambda)S_{23}(\mu) = S_{23}(\mu)S_{13}(\lambda)S_{12}(\lambda-\mu)$ where the Yang-Baxter matrices S_{12}, S_{13} and S_{23} act on the spaces $V_1 \otimes V_2$, $V_1 \otimes V_3$ and $V_2 \otimes V_3$, respectively. It can be represented graphically as follows



. (2.11)

This graphical representation of the Yang-Baxter relation is applicable to the mathematical theory of knots [5]. The solutions of the Yang-Baxter equation are related to quantum groups and Lie algebra (Dynkin diagrams).

The Yang-Baxter equation (2.10) ensures the equivalence of all possible multi-particle scattering processes, independently of the order in which the two-particle scatterings are performed.

2.3 Transfer and monodromy matrices

Let us now forget about the origin of the "small" scattering S-matrix of dimension l^2 and use it as the building element of "large" matrices formulated on the chain of N sites $n = 1, 2, \ldots, N$. In this part, we introduce a hierarchy of large matrices and derive for them the analogies of the Yang-Baxter Eq. (2.10). The method, known as the algebraic Bethe ansatz or the quantum inverse scattering method (QISM), was worked out by L. D. Faddeev and his coworkers [6, 7].

- The *transfer matrix* T is defined as follows

$$\mathrm{T}(\lambda)^{\sigma_1\ldots\sigma_N}_{\sigma_1'\ldots\sigma_N'} = \sum_{\gamma_1,\ldots,\gamma_N} S^{\gamma_1\gamma_2}_{\sigma_1\sigma_1'}(\lambda) S^{\gamma_2\gamma_3}_{\sigma_2\sigma_2'}(\lambda) \cdots S^{\gamma_N\gamma_1}_{\sigma_N\sigma_N'}(\lambda). \qquad (2.12)$$

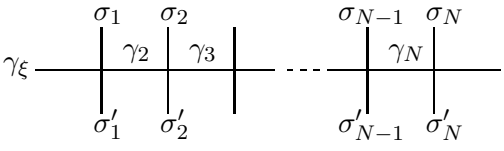
Since each of the indices $\{\sigma_1, \ldots, \sigma_N\}$ or $\{\sigma_1', \ldots, \sigma_N'\}$ can take l^N values, the dimension of T is l^N.

- The *monodromy matrix* $\mathcal{T}_\xi$ of dimension l^{N+1} is defined by

$$\mathcal{T}_\xi(\lambda)^{\gamma_\xi\sigma_1\ldots\sigma_N}_{\gamma_\xi'\sigma_1'\ldots\sigma_N'} = \sum_{\gamma_2,\ldots,\gamma_N} S^{\gamma_\xi\gamma_2}_{\sigma_1\sigma_1'}(\lambda) S^{\gamma_2\gamma_3}_{\sigma_2\sigma_2'}(\lambda) \cdots S^{\gamma_N\gamma_\xi'}_{\sigma_N\sigma_N'}(\lambda), \qquad (2.13)$$

where ξ is an auxiliary site with state indices $(\gamma_\xi, \gamma_\xi')$. The monodromy matrix can be represented graphically as follows

$\mathcal{T}_\xi(\lambda)^{\gamma_\xi\sigma_1\ldots\sigma_N}_{\gamma_\xi'\sigma_1'\ldots\sigma_N'} =$



. (2.14)

The transfer matrix (2.12) is obtained from this representation by connecting the free ends, i.e. setting $\gamma_\xi = \gamma'_\xi \equiv \gamma_1$ and summing over γ_1, creating in this way the circle. Algebraically, one has

$$\mathrm{T}(\lambda) = \mathrm{Tr}_\xi \ \mathcal{T}_\xi(\lambda), \tag{2.15}$$

where $\mathrm{Tr}_\xi \cdots \equiv \sum_{\gamma_\xi,\gamma'_\xi} \delta_{\gamma_\xi\gamma'_\xi} \cdots$ is the trace in the auxiliary ξ-space.

• It is useful to introduce also the $\mathcal{L}_{\xi n}$-matrices ($n = 1, 2, \ldots, N$) of dimension l^{N+1}:

$$\mathcal{L}_{\xi n}(\lambda)^{\gamma_\xi \sigma_1 \ldots \sigma_N}_{\gamma'_\xi \sigma'_1 \ldots \sigma'_N} = S^{\gamma_\xi \gamma'_\xi}_{\sigma_n \sigma'_n}(\lambda) \delta_{\sigma_1 \sigma'_1} \ldots \delta_{\sigma_{n-1} \sigma'_{n-1}} \delta_{\sigma_{n+1} \sigma'_{n+1}} \ldots \delta_{\sigma_N \sigma'_N}. \tag{2.16}$$

It can be shown by the explicit evaluation of matrix products that the monodromy matrix is expressible as the following product of $\mathcal{L}_{\xi n}$-matrices

$$\mathcal{T}_\xi(\lambda) = \mathcal{L}_{\xi 1}(\lambda) \mathcal{L}_{\xi 2}(\lambda) \cdots \mathcal{L}_{\xi N}(\lambda). \tag{2.17}$$

• Besides the auxiliary site ξ is useful to introduce another auxiliary site η and define in the (ξ, η) space the $\mathcal{L}_{\xi\eta}$-matrix of dimension l^2:

$$\mathcal{L}_{\xi\eta}(\lambda)^{\gamma_\xi \gamma_\eta}_{\gamma'_\xi \gamma'_\eta} \equiv S^{\gamma_\xi \gamma'_\xi}_{\gamma_\eta \gamma'_\eta}(\lambda). \tag{2.18}$$

The Yang-Baxter Eq. (2.10) for the S-matrix can be transcribed in terms of $\mathcal{L}$-matrices as follows

$$\mathcal{L}_{\xi\eta}(\lambda - \mu) \mathcal{L}_{\xi n}(\lambda) \mathcal{L}_{\eta n}(\mu) = \mathcal{L}_{\eta n}(\mu) \mathcal{L}_{\xi n}(\lambda) \mathcal{L}_{\xi\eta}(\lambda - \mu). \tag{2.19}$$

Here, the products of $\mathcal{L}$ matrices are performed in the space of N ordinary chain sites $1, 2, \ldots, N$ and two auxiliary ξ, η sites; for example, the matrix element on the lhs has to be understood in the following way:

$$\begin{aligned}\{\mathcal{L}_{\xi\eta}(\lambda - \mu) \mathcal{L}_{\xi n}(\lambda) \mathcal{L}_{\eta n}(\mu)\}^{\gamma_\xi \gamma_\eta \{\sigma_1 \ldots \sigma_N\}}_{\gamma'_\xi \gamma'_\eta \{\sigma'_1 \ldots \sigma'_N\}} = \sum_{\gamma''_\xi, \gamma''_\eta} \sum_{\{\sigma''_1, \ldots, \sigma''_N\}} & \mathcal{L}_{\xi\eta}(\lambda - \mu)^{\gamma_\xi \gamma_\eta}_{\gamma''_\xi \gamma''_\eta} \\ & \times \mathcal{L}_{\xi n}(\lambda)^{\gamma''_\xi \{\sigma_1 \ldots \sigma_N\}}_{\gamma'_\xi \{\sigma''_1 \ldots \sigma''_N\}} \mathcal{L}_{\eta n}(\mu)^{\gamma''_\eta \{\sigma''_1 \ldots \sigma''_N\}}_{\gamma'_\eta \{\sigma'_1 \ldots \sigma'_N\}}.\end{aligned} \tag{2.20}$$

One can obtain from Eq. (2.19) an analogous relation for the monodromy matrix:

$$\mathcal{L}_{\xi\eta}(\lambda - \mu) \mathcal{T}_\xi(\lambda) \mathcal{T}_\eta(\mu) = \mathcal{T}_\eta(\mu) \mathcal{T}_\xi(\lambda) \mathcal{L}_{\xi\eta}(\lambda - \mu). \tag{2.21}$$

To prove this relation, we take advantage of the fact that the matrices $\mathcal{L}_{\xi n}$ and $\mathcal{L}_{\eta m}$ commute for $n \neq m$ and write down

$$\mathcal{T}_\xi(\lambda) \mathcal{T}_\eta(\mu) = \mathcal{L}_{\xi 1}(\lambda) \mathcal{L}_{\eta 1}(\mu) \cdots \mathcal{L}_{\xi N}(\lambda) \mathcal{L}_{\eta N}(\mu). \tag{2.22}$$

Multiplying this relation from the left by $\mathcal{L}_{\xi\eta}(\lambda - \mu)$ and then commuting successively $\mathcal{L}_{\xi\eta}$ by using Eq. (2.19) results in the relation (2.21).

Equations (2.19) and (2.21) can be rewritten in more convenient forms by applying from the left the permutation operator of indices ξ and η, $\Pi_{\xi\eta}$, with the elements $(\Pi_{\xi\eta})^{\gamma_\xi\gamma_\eta}_{\gamma'_\xi\gamma'_\eta} = \delta_{\gamma_\xi\gamma'_\eta}\delta_{\gamma'_\xi\gamma_\eta}$. Introducing the R matrix of dimension l^2

$$R(\lambda) = \Pi_{\xi\eta}\mathcal{L}_{\xi\eta}(\lambda), \tag{2.23}$$

Eq. (2.19) transforms to

$$R(\lambda-\mu)\left[\mathcal{L}_n(\lambda)\otimes\mathcal{L}_n(\mu)\right] = \left[\mathcal{L}_n(\mu)\otimes\mathcal{L}_n(\lambda)\right]R(\lambda-\mu) \tag{2.24}$$

and Eq. (2.21) transforms to

$$R(\lambda-\mu)\left[\mathcal{T}(\lambda)\otimes\mathcal{T}(\mu)\right] = \left[\mathcal{T}(\mu)\otimes\mathcal{T}(\lambda)\right]R(\lambda-\mu). \tag{2.25}$$

Here, the tensor products are considered in the (ξ,η) space; for example, the lhs of Eq. (2.24) has to be understood in the following way

$$\sum_{\gamma''_\xi,\gamma''_\eta} R(\lambda-\mu)^{\gamma_\xi\gamma_\eta}_{\gamma''_\xi\gamma''_\eta}\mathcal{L}_n(\lambda)^{\gamma''_\xi}_{\gamma'_\xi}\mathcal{L}_n(\mu)^{\gamma''_\eta}_{\gamma'_\eta}. \tag{2.26}$$

Finally, on the base of Eq. (2.25) we shall prove the commutation property of transfer matrices $\{\mathrm{T}(\lambda)\}$. Multiplying both sides of Eq. (2.25) from the right by the inverse matrix $R^{-1}(\lambda-\mu)$ results in

$$R(\lambda-\mu)\left[\mathcal{T}(\lambda)\otimes\mathcal{T}(\mu)\right]R^{-1}(\lambda-\mu) = \mathcal{T}(\mu)\otimes\mathcal{T}(\lambda). \tag{2.27}$$

Let us trace both sides of this equation in the auxiliary ξ and η spaces. The lhs then reads

$$\sum_{\gamma_\xi,\gamma_\eta}\sum_{\substack{\alpha,\alpha'\\ \beta,\beta'}} R(\lambda-\mu)^{\gamma_\xi\gamma_\eta}_{\alpha\beta}\mathcal{T}(\lambda)^{\alpha}_{\alpha'}\mathcal{T}(\mu)^{\beta}_{\beta'}R^{-1}(\lambda-\mu)^{\alpha'\beta'}_{\gamma_\xi\gamma_\eta} = \sum_{\substack{\alpha,\alpha'\\ \beta,\beta'}}\mathcal{T}(\lambda)^{\alpha}_{\alpha'}\mathcal{T}(\mu)^{\beta}_{\beta'}\delta_{\alpha\alpha'}\delta_{\beta\beta'} = \mathrm{T}(\lambda)\mathrm{T}(\mu), \tag{2.28}$$

while the rhs is expressible as

$$\sum_{\gamma_\xi,\gamma_\eta}\{\mathcal{T}(\mu)\otimes\mathcal{T}(\lambda)\}^{\gamma_\xi\gamma_\eta}_{\gamma_\xi\gamma_\eta} = \sum_{\gamma_\xi,\gamma_\eta}\mathcal{T}(\mu)^{\gamma_\xi}_{\gamma_\xi}\mathcal{T}(\lambda)^{\gamma_\eta}_{\gamma_\eta} = \mathrm{T}(\mu)\mathrm{T}(\lambda). \tag{2.29}$$

We conclude that

$$[\mathrm{T}(\lambda),\mathrm{T}(\mu)] = 0 \qquad \text{for arbitrary } \lambda \text{ and } \mu. \tag{2.30}$$

The existence of the infinite family of commuting transfer matrices is of primary importance: *the eigenvectors of transfer matrices* $\mathrm{T}(\lambda)$ *are common and do not depend on the spectral parameter* λ.

2.4 S-matrix of the Heisenberg model

The S-matrix associated with the XXX Heisenberg chain reads

$$S^{\sigma_1\sigma'_1}_{\sigma_2\sigma'_2}(\lambda) = \sum_{j=0}^{3} w_j(\lambda)s^{j}_{\sigma_1\sigma'_1}s^{j}_{\sigma_2\sigma'_2}, \qquad w_j(\lambda) = w(\lambda) \text{ for } j=1,2,3, \tag{2.31}$$

where s^0 is the 2×2 unity matrix (1.1) and $\{s^\alpha\}$ with $\alpha=1,2,3$ are the Pauli matrices (1.2). The S-matrix (2.31) has dimension 4 and is of the form

$$S(\lambda)=\begin{pmatrix} a(\lambda) & 0 & 0 & 0\\ 0 & c(\lambda) & b(\lambda) & 0\\ 0 & b(\lambda) & c(\lambda) & 0\\ 0 & 0 & 0 & a(\lambda)\end{pmatrix},\quad \text{where}\quad \begin{array}{l} a(\lambda)=w_0(\lambda)+w(\lambda)\\ b(\lambda)=2w(\lambda)\\ c(\lambda)=w_0(\lambda)-w(\lambda)\end{array}. \tag{2.32}$$

The permutation operator Π and the R-matrix (2.23) are expressible as follows

$$\Pi=\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix},\qquad R(\lambda)=\Pi S(\lambda)=\begin{pmatrix} a(\lambda) & 0 & 0 & 0\\ 0 & b(\lambda) & c(\lambda) & 0\\ 0 & c(\lambda) & b(\lambda) & 0\\ 0 & 0 & 0 & a(\lambda)\end{pmatrix}. \tag{2.33}$$

The $\mathcal{L}_{\xi n}$-matrix, defined by (2.16), reads

$$\mathcal{L}_{\xi n}(\lambda)^{\gamma_\xi\sigma_1\ldots\sigma_N}_{\gamma'_\xi\sigma'_1\ldots\sigma'_N}=\sum_{j=0}^{3}w_j(\lambda)s^j_{\gamma_\xi\gamma'_\xi}(\mathbf{s}^j_n)^{\sigma_1\ldots\sigma_N}_{\sigma'_1\ldots\sigma'_N}=\sum_{j=0}^{3}w_j(\lambda)s^j\otimes\mathbf{s}^j_n. \tag{2.34}$$

In the space of the auxiliary ξ space, the $\mathcal{L}_{\xi n}$-matrix takes the form

$$\mathcal{L}_n(\lambda)=\begin{pmatrix} w_0(\lambda)\mathbf{s}^0_n+w(\lambda)\mathbf{s}^z_n & 2w(\lambda)\mathbf{s}^-_n\\ 2w(\lambda)\mathbf{s}^+_n & w_0(\lambda)\mathbf{s}^0_n-w(\lambda)\mathbf{s}^z_n\end{pmatrix}\equiv\begin{pmatrix}\hat\alpha_n(\lambda)&\hat\beta_n(\lambda)\\ \hat\gamma_n(\lambda)&\hat\delta_n(\lambda)\end{pmatrix}. \tag{2.35}$$

Having the explicit forms of the R-matrix (2.33) and the $\mathcal{L}_{\xi n}$-matrix (2.35), one can look for the solution of the Yang-Baxter equation (2.24) in the auxiliary (ξ,η) space. For example, for the $(1,2)$ matrix element one gets the constraint

$$\begin{aligned} a(\lambda-\mu)\left[w_0(\lambda)\mathbf{s}^0_n+w(\lambda)\mathbf{s}^z_n\right]2w(\mu)\mathbf{s}^-_n \;=\; &\left[w_0(\mu)\mathbf{s}^0_n+w(\mu)\mathbf{s}^z_n\right]2w(\lambda)\mathbf{s}^-_n b(\lambda-\mu)\\ &+2w(\mu)\mathbf{s}^-_n\left[w_0(\mu)\mathbf{s}^0_n+w(\mu)\mathbf{s}^z_n\right]c(\lambda-\mu). \end{aligned} \tag{2.36}$$

Substituting a,b,c from (2.32) and using the relations $s^zs^-=-s^-$, $s^-s^z=s^-$, one gets

$$w_0(\lambda)w(\mu)w(\lambda-\mu)-w(\lambda)w_0(\mu)w(\lambda-\mu)+w(\lambda)w(\mu)w(\lambda-\mu)-w(\lambda)w(\mu)w_0(\lambda-\mu)=0. \tag{2.37}$$

It is interesting that the same equation is obtained for all nontrivial matrix elements in the Yang-Baxter equation (2.24). Eq. (2.37) is therefore the only constraint. Its division by $w(\lambda)w(\mu)w(\lambda-\mu)$ results in

$$\frac{c(\lambda)}{b(\lambda)}=\frac{c(\mu)}{b(\mu)}+\frac{c(\lambda-\mu)}{b(\lambda-\mu)}. \tag{2.38}$$

The simplest solution of this equation is $c(\lambda)/b(\lambda)=\alpha+\beta\lambda$. Choosing $c(\lambda)/b(\lambda)=\lambda/\mathrm{i}$ one ends up with

$$a(\lambda):c(\lambda):b(\lambda)=(\lambda+\mathrm{i}):\lambda:\mathrm{i}. \tag{2.39}$$

It is still necessary to establish the relationship between the Hamiltonian of the XXX Heisenberg chain and the transfer matrix built from the S-matrix defined by relations

(2.32) and (2.39). When $\lambda = 0$, the S-matrix is proportional to the permutation matrix $S(\lambda = 0) = \mathrm{i}\Pi$. Thus,

$$\mathrm{T}(\lambda = 0)^{\sigma_1 \ldots \sigma_N}_{\sigma'_1 \ldots \sigma'_N} = \mathrm{i}^N \delta_{\sigma_1 \sigma'_2} \delta_{\sigma_2 \sigma'_3} \cdots \delta_{\sigma_{N-1} \sigma'_N} \delta_{\sigma_N \sigma'_1} \tag{2.40}$$

and

$$\mathrm{T}^{-1}(\lambda = 0)^{\sigma_1 \ldots \sigma_N}_{\sigma'_1 \ldots \sigma'_N} = \mathrm{i}^{-N} \delta_{\sigma_1 \sigma'_N} \delta_{\sigma_2 \sigma'_1} \cdots \delta_{\sigma_{N-1} \sigma'_{N-2}} \delta_{\sigma_N \sigma'_{N-1}}. \tag{2.41}$$

For the derivative of the logarithm of the transfer matrix (2.12) with respect to λ,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \ln \mathrm{T}(\lambda) = \mathrm{T}^{-1}(\lambda) \frac{\mathrm{d}}{\mathrm{d}\lambda} \mathrm{T}(\lambda),$$

taken at $\lambda = 0$, we then get

$$\left\{ \frac{\mathrm{d}}{\mathrm{d}\lambda} \ln \mathrm{T}(\lambda) \right\}^{\sigma_1 \ldots \sigma_N}_{\sigma'_1 \ldots \sigma'_N} \Bigg|_{\lambda=0} = \frac{1}{\mathrm{i}} \sum_{n=1}^{N} \delta_{\sigma_1 \sigma'_1} \cdots \delta_{\sigma_{n-1} \sigma'_{n-1}} \frac{\mathrm{d}}{\mathrm{d}\lambda} S^{\sigma_n \sigma'_{n+1}}_{\sigma_{n+1} \sigma'_n}(\lambda) \Big|_{\lambda=0} \delta_{\sigma_{n+2} \sigma'_{n+2}} \cdots \delta_{\sigma_N \sigma'_N}. \tag{2.42}$$

The considered S-matrix, given by Eqs. (2.32) and (2.39), can be reexpressed in an equivalent form

$$S^{\sigma_1 \sigma'_1}_{\sigma_2 \sigma'_2}(\lambda) = \sum_{j=0}^{3} p_j(\lambda) s^j_{\sigma_1 \sigma'_2} s^j_{\sigma_2 \sigma'_1}, \quad \text{where} \;\; p_0 = \frac{\lambda}{2} + \mathrm{i} \;\; \text{and} \;\; p_j = \frac{\lambda}{2} \;\; \text{for} \; j = 1, 2, 3. \tag{2.43}$$

Thus,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} S^{\sigma_n \sigma'_{n+1}}_{\sigma_{n+1} \sigma'_n}(\lambda) \Big|_{\lambda=0} = \frac{1}{2} \delta_{\sigma_n \sigma'_n} \delta_{\sigma_{n+1} \sigma'_{n+1}} + \frac{1}{2} \sum_{j=1}^{3} s^j_{\sigma_n \sigma'_n} s^j_{\sigma_{n+1} \sigma'_{n+1}}. \tag{2.44}$$

Inserting this relation into (2.42) results in

$$H_{\mathrm{XXX}} \equiv -\frac{1}{2} \sum_{n=1}^{N} \left(\mathbf{s}^x_n \mathbf{s}^x_{n+1} + \mathbf{s}^y_n \mathbf{s}^y_{n+1} + \mathbf{s}^z_n \mathbf{s}^z_{n+1} \right) = \frac{N}{2} \mathrm{I}_N - \mathrm{i} \frac{\mathrm{d}}{\mathrm{d}\lambda} \ln \mathrm{T}(\lambda) \Big|_{\lambda=0}, \tag{2.45}$$

where I_N is the unity matrix of dimension 2^N. In this way, the diagonalization of the XXX Heisenberg Hamiltonian is related to the diagonalization of the transfer matrix.

The generalization of the formalism to the XXZ and XYZ Heisenberg chains is straightforward. The general form of the S-matrix is the one presented in Eq. (2.31) with no constraints on w_j parameters, i.e.

$$S^{\sigma_1 \sigma'_1}_{\sigma_2 \sigma'_2}(\lambda) = \begin{pmatrix} a(\lambda) & 0 & 0 & d(\lambda) \\ 0 & c(\lambda) & b(\lambda) & 0 \\ 0 & b(\lambda) & c(\lambda) & 0 \\ d(\lambda) & 0 & 0 & a(\lambda) \end{pmatrix}, \quad \text{where} \quad \begin{array}{l} a(\lambda) = w_0(\lambda) + w_3(\lambda) \\ b(\lambda) = w_1(\lambda) + w_2(\lambda) \\ c(\lambda) = w_0(\lambda) - w_3(\lambda) \\ d(\lambda) = w_1(\lambda) - w_2(\lambda) \end{array}. \tag{2.46}$$

The Yang-Baxter Eq. (2.24) is satisfied when the system of equations

$$\begin{aligned} & w_m(\lambda) w_l(\mu) w_j(\lambda - \mu) - w_l(\lambda) w_m(\mu) w_k(\lambda - \mu) \\ & + w_k(\lambda) w_j(\mu) w_l(\lambda - \mu) - w_j(\lambda) w_k(\mu) w_m(\lambda - \mu) = 0 \end{aligned} \tag{2.47}$$

holds for an arbitrary permutation (j,k,l,m) of $(0,1,2,3)$. For the XXZ model with the anisotropy $\Delta = \cos(2\nu)$, one has $w_1 = w_2$ and the equation (2.47) implies the following parametrization

$$a(\lambda) : c(\lambda) : b(\lambda) : d(\lambda) = \sin(\lambda+\nu) : \sin(\lambda-\nu) : \sin(2\nu) : 0. \tag{2.48}$$

For the XYZ model with $d(\lambda) \neq 0$, the parametrization of a, b, c and d is done in terms of the elliptic sine function.

2.5 Diagonalization of the transfer matrix

Let us restrict ourselves to the case of the XXX Heisenberg chain. We start with the monodromy matrix $\mathcal{T}_\xi$, represented in the auxiliary ξ-space as follows

$$\mathcal{T}(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}, \tag{2.49}$$

where each of the matrices A, B, C and D acts in the 2^N-dimensional space of indices $\alpha_1 \dots \alpha_N$. Considering this representation in the Yang-Baxter equation (2.25), one obtains the following commutation rules:

$$[A(\lambda), A(\mu)] = [B(\lambda), B(\mu)] = [C(\lambda), C(\mu)] = [D(\lambda), D(\mu)] = 0, \tag{2.50}$$

$$A(\lambda)B(\mu) = \frac{a(\mu-\lambda)}{c(\mu-\lambda)} B(\mu)A(\lambda) - \frac{b(\mu-\lambda)}{c(\mu-\lambda)} B(\lambda)A(\mu), \tag{2.51}$$

$$D(\lambda)B(\mu) = \frac{a(\lambda-\mu)}{c(\lambda-\mu)} B(\mu)D(\lambda) - \frac{b(\lambda-\mu)}{c(\lambda-\mu)} B(\lambda)D(\mu). \tag{2.52}$$

Our task is to find the spectrum of the transfer matrix

$$\mathrm{T}(\lambda) = \mathrm{Tr}_\xi \, \mathcal{T}_\xi(\lambda) = A(\lambda) + D(\lambda). \tag{2.53}$$

Let us choose as the "generating" vector the tensor product of e^+ vectors [see Eq. (1.5)] on the chain of N sites

$$\Omega = \underbrace{e^+}_{1} \otimes \dots \otimes \underbrace{e^+}_{n} \otimes \dots \otimes \underbrace{e^+}_{N}. \tag{2.54}$$

It follows from the representation (2.35) of the $\mathcal{L}_{\xi n}$-matrix in the auxiliary ξ-space that the operators $\hat\alpha_n, \hat\beta_n, \hat\gamma_n$ and $\hat\delta_n$ act on the local vector e_n^+ as follows:

$$\begin{aligned} &\hat\alpha_n(\lambda) e_n^+ = a(\lambda) e_n^+, \qquad \hat\beta_n(\lambda) e_n^+ = b(\lambda) e_n^-, \\ &\hat\gamma_n(\lambda) e_n^+ = 0, \qquad\qquad \hat\delta_n(\lambda) e_n^+ = c(\lambda) e_n^+. \end{aligned} \tag{2.55}$$

These relations can be written in a compact form

$$\mathcal{L}_n(\lambda) e_n^+ = \begin{pmatrix} a(\lambda) & [\cdots] \\ 0 & c(\lambda) \end{pmatrix} e_n^+, \tag{2.56}$$

where the symbol $[\cdots]$ denotes an operator matrix element, transforming the vector e_n^+ to e_n^-, whose explicit form will be irrelevant. With regard to the representation (2.17) of the monodromy matrix and the triangle character of the matrix on the rhs of Eq. (2.56), one finds

$$\mathcal{T}(\lambda)\Omega = \begin{pmatrix} a^N(\lambda) & [\cdots] \\ 0 & c^N(\lambda) \end{pmatrix} \Omega. \tag{2.57}$$

Using the representation (2.49), we obtain the action of the elements of the monodromy matrix on the generating vector Ω:

$$A(\lambda)\Omega = a^N(\lambda)\Omega, \quad C(\lambda)\Omega = 0, \quad D(\lambda)\Omega = c^N(\lambda)\Omega. \tag{2.58}$$

The action of the operator $B(\lambda)$ on the vector Ω is unknown, however, the action of this operator is determined uniquely by the commutation relations with $A(\lambda)$ (2.51) and with $D(\lambda$ (2.52).

The eigenvectors of the transfer matrix will be searched in the ansatz form

$$\phi(\lambda_1, \ldots, \lambda_M) = \prod_{\alpha=1}^{M} B(\lambda_\alpha)\Omega, \tag{2.59}$$

where $M = 0, 1, \ldots, N$ and the parameters $(\lambda_1, \lambda_2, \ldots, \lambda_M)$ are as-yet unspecified; note that the eigenvectors are not supposed to depend on the spectral parameter λ which is in agreement with the previous analysis. The eigenfunction equation reads

$$\mathrm{T}(\lambda)\phi(\lambda_1, \ldots, \lambda_M) = [A(\lambda) + D(\lambda)] \prod_{\alpha=1}^{M} B(\lambda_\alpha)\Omega. \tag{2.60}$$

Since the action of the operator B on Ω is not known, it is necessary to commute this operator with the operators A and D, whose actions on Ω are known [see Eq. (2.58)], with the aid of the commutation relations (2.51) and (2.52). Let us consider the expression containing the A operator and start with the first expansion step

$$A(\lambda) \prod_{\alpha=1}^{M} B(\lambda_\alpha)\Omega = \left[\frac{a(\lambda_1 - \lambda)}{c(\lambda_1 - \lambda)} B(\lambda_1)A(\lambda) - \frac{b(\lambda_1 - \lambda)}{c(\lambda_1 - \lambda)} B(\lambda)A(\lambda_1)\right] \prod_{\alpha=2}^{M} B(\lambda_\alpha)\Omega. \tag{2.61}$$

In the next step, the rhs of this equation is expanded as

$$\begin{aligned} \Big[\frac{a(\lambda_1 - \lambda)}{c(\lambda_1 - \lambda)} \frac{a(\lambda_2 - \lambda)}{c(\lambda_2 - \lambda)} B(\lambda_1)B(\lambda_2)A(\lambda) - \frac{a(\lambda_1 - \lambda)}{c(\lambda_1 - \lambda)} \frac{b(\lambda_2 - \lambda)}{c(\lambda_2 - \lambda)} B(\lambda_1)B(\lambda)A(\lambda_2) \\ - \frac{b(\lambda_1 - \lambda)}{c(\lambda_1 - \lambda)} \frac{a(\lambda_2 - \lambda_1)}{c(\lambda_2 - \lambda_1)} B(\lambda)B(\lambda_2)A(\lambda_1) + \frac{b(\lambda_1 - \lambda)}{c(\lambda_1 - \lambda)} \frac{b(\lambda_2 - \lambda_1)}{c(\lambda_2 - \lambda_1)} B(\lambda)B(\lambda_1)A(\lambda_2)\Big] \\ \times \prod_{\alpha=3}^{M} B(\lambda_\alpha)\Omega. \end{aligned} \tag{2.62}$$

Since the operators $\{B(\lambda_\alpha)\}$ commute with each other, the result must be symmetric with respect to the interchange $\lambda_1 \leftrightarrow \lambda_2$. From the point of view of Eq. (2.62), this is equivalent to saying that the following relation

$$\frac{b(\lambda_1 - \lambda)}{c(\lambda_1 - \lambda)} \frac{b(\lambda_2 - \lambda_1)}{c(\lambda_2 - \lambda_1)} - \frac{a(\lambda_1 - \lambda)}{c(\lambda_1 - \lambda)} \frac{b(\lambda_2 - \lambda)}{c(\lambda_2 - \lambda)} = -\frac{b(\lambda_2 - \lambda)}{c(\lambda_2 - \lambda)} \frac{a(\lambda_1 - \lambda_2)}{c(\lambda_1 - \lambda_2)} \tag{2.63}$$

must hold. It is easy to check that within the parametrization (2.39) this equation is indeed fulfilled. Thus, Eq. (2.62) can be reexpressed as follows

$$\Big[\frac{a(\lambda_1-\lambda)}{c(\lambda_1-\lambda)}\frac{a(\lambda_2-\lambda)}{c(\lambda_2-\lambda)}B(\lambda_1)B(\lambda_2)A(\lambda)-\frac{b(\lambda_1-\lambda)}{c(\lambda_1-\lambda)}\frac{a(\lambda_2-\lambda_1)}{c(\lambda_2-\lambda_1)}B(\lambda)B(\lambda_2)A(\lambda_1)$$
$$-\frac{b(\lambda_2-\lambda)}{c(\lambda_2-\lambda)}\frac{a(\lambda_1-\lambda_2)}{c(\lambda_1-\lambda_2)}B(\lambda)B(\lambda_1)A(\lambda_2)\Big]\prod_{\alpha=3}^{M}B(\lambda_\alpha)\Omega. \tag{2.64}$$

Since $c(\lambda)=\lambda$, there must hold $\lambda_1\neq\lambda_2$. Proceeding in this way, also with the expression in (2.60) containing the operator D, one ends up with the result

$$\mathrm{T}(\lambda)\phi(\lambda_1,\ldots,\lambda_M)=\Lambda(\lambda;\lambda_1,\ldots,\lambda_M)\prod_{\alpha=1}^{M}B(\lambda_\alpha)\Omega$$
$$+\sum_{\alpha=1}^{M}\Lambda_\alpha(\lambda;\lambda_1,\ldots,\lambda_M)B(\lambda)\prod_{\substack{\beta=1\\(\beta\neq\alpha)}}^{M}B(\lambda_\beta)\Omega, \tag{2.65}$$

where

$$\Lambda(\lambda;\lambda_1,\ldots,\lambda_M)=a^N(\lambda)\prod_{\alpha=1}^{M}\frac{a(\lambda_\alpha-\lambda)}{c(\lambda_\alpha-\lambda)}+c^N(\lambda)\prod_{\alpha=1}^{M}\frac{a(\lambda-\lambda_\alpha)}{c(\lambda-\lambda_\alpha)} \tag{2.66}$$

and

$$\Lambda_\alpha(\lambda;\lambda_1,\ldots,\lambda_M)=-\frac{b(\lambda_\alpha-\lambda)}{c(\lambda_\alpha-\lambda)}\left[a^N(\lambda_\alpha)\prod_{\substack{\beta=1\\(\beta\neq\alpha)}}^{M}\frac{a(\lambda_\beta-\lambda_\alpha)}{c(\lambda_\beta-\lambda_\alpha)}-c^N(\lambda_\alpha)\prod_{\substack{\beta=1\\(\beta\neq\alpha)}}^{M}\frac{a(\lambda_\alpha-\lambda_\beta)}{c(\lambda_\alpha-\lambda_\beta)}\right]. \tag{2.67}$$

The condition for $\phi(\lambda_1,\ldots,\lambda_M)$ (2.59) to be an eigenvector of the transfer matrix $T(\lambda)$ is the nullity of all Λ_α (2.67), i.e. the system of equations

$$\left[\frac{a(\lambda_\alpha)}{c(\lambda_\alpha)}\right]^N=\prod_{\substack{\beta=1\\(\beta\neq\alpha)}}^{M}\frac{a(\lambda_\alpha-\lambda_\beta)}{a(\lambda_\beta-\lambda_\alpha)}\frac{c(\lambda_\beta-\lambda_\alpha)}{c(\lambda_\alpha-\lambda_\beta)}\qquad\alpha=1,2,\ldots,M \tag{2.68}$$

which determines the set of distinct parameters $\lambda_1,\lambda_2,\ldots,\lambda_M$. With respect to the parametrization (2.38), this set of equations can be written as follows

$$\left(\frac{\lambda_\alpha+\mathrm{i}}{\lambda_\alpha}\right)^N=\prod_{\substack{\beta=1\\(\beta\neq\alpha)}}^{M}\frac{\lambda_\alpha-\lambda_\beta+\mathrm{i}}{\lambda_\alpha-\lambda_\beta-\mathrm{i}},\qquad\alpha=1,2,\ldots,M. \tag{2.69}$$

The corresponding eigenvalue of $\mathrm{T}(\lambda)$ is Λ given by Eq. (2.66). With respect to the parametrization (2.38), Λ is expressible as

$$\Lambda(\lambda;\lambda_1,\ldots,\lambda_M)=(\lambda+\mathrm{i})^N\prod_{\alpha=1}^{M}\frac{\lambda_\alpha-\lambda+\mathrm{i}}{\lambda_\alpha-\lambda}+\lambda^N\prod_{\alpha=1}^{M}\frac{\lambda-\lambda_\alpha+\mathrm{i}}{\lambda-\lambda_\alpha}. \tag{2.70}$$

Using the relationship (2.45) between the XXX Hamiltonian and the transfer matrix, the Hamiltonian eigenvalue E is given by

$$\begin{aligned} E &= -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\lambda}\ln\Lambda(\lambda;\lambda_1,\ldots,\lambda_M)\Big|_{\lambda=0} + \frac{N}{2} \\ &= -\frac{N}{2} - \mathrm{i}\sum_{\alpha=1}^{M}\left(\frac{1}{\lambda_\alpha} - \frac{1}{\lambda_\alpha+\mathrm{i}}\right). \end{aligned} \tag{2.71}$$

Finally, performing the shift $\lambda_\alpha = \lambda'_\alpha - \mathrm{i}/2$ for all $\alpha = 1, 2, \ldots, M$, one recovers the Bethe equations (1.73) and (1.74) of the XXX Heisenberg chain.

The advantage of the outlined QISM consists in its general applicability to an arbitrary integrable model. At first stage, it is necessary to establish an S-matrix which fulfills the Yang-Baxter equation. Then, the transfer matrix constructed from this S-matrix is shown to be related to a quantum Hamiltonian. Finally, the diagonalization of the transfer matrix proceeds along the above lines. In this way one can solve XYZ Heisenberg model whose Hamiltonian does not commute with the total spin in the z-direction and therefore there is no hope to apply the coordinate Bethe ansatz. This model exhibits the phase transitions and its singular behavior around the critical points is nonuniversal, i.e. the critical indices *depend* on the model's parameters.

The quantity $\mathrm{T}(\lambda)$ with a free parameter λ is in fact the true transfer matrix of some *classical* two-dimensional statistical (usually vertex) models and its diagonalization implies the free energy of these models [M1].

Another application of the inhomogeneous version of the QISM will be documented in the treatment of boundary conditions for the Hubbard and Kondo models.

3. Hubbard model

3.1 Hamiltonian and its symmetries

The 1D Hubbard model was solved by Lieb and Wu [8], for a reminiscence and some new rigorous results see Ref. [9].

We consider the 1D Hubbard model on a periodic chain of $l = 1, 2, \ldots, L$ sites. Its Hamiltonian in the second quantization may be written as

$$H = T \sum_{l=1}^{L} \sum_{\sigma=\uparrow,\downarrow} (c_{l\sigma}^{\dagger} c_{l+1\sigma} + c_{l+1\sigma}^{\dagger} c_{l\sigma}) + U \sum_{l=1}^{L} n_{l\uparrow} n_{l\downarrow}, \tag{3.1}$$

where $c_{l\sigma}^{\dagger}$ and $c_{l\sigma}$ are creation and annihilation operators of electrons (the chain periodicity is ensured by setting $c_{L+1\sigma} = c_{1\sigma}$), $n_{l\sigma} = c_{l\sigma}^{\dagger} c_{l\sigma}$ is the occupation number operator for electron of spin $\sigma \in \{\uparrow, \downarrow\}$ at site l and $U > 0$ is the Coulomb coupling constant.

For a bipartite chain with $L =$ even number, the set of lattice sites can be divided into two subsets, A and B, such that there is no hopping between A sites or B sites. Then, the unitary transformation $U^{\dagger} H U$ with $U = \exp[\mathrm{i}\pi \sum_{l \in A} (n_{l\uparrow} + n_{l\downarrow})]$ leaves H unchanged, except for the replacement $T \to -T$. Without any loss of generality we can take $T = -1$.

The commutation relations

$$[\sum_{l} n_{l\uparrow}, H] = [\sum_{l} n_{l\downarrow}, H] = 0 \tag{3.2}$$

imply that the numbers of down-spin electrons M and up-spin electrons M' are conserved, and therefore they are good quantum numbers (the conservation of the total number of electrons $N = M + M'$ is obvious). Thus, we can label the Fock eigenstates of the Hamiltonian (3.1) by M and M' and write the Schrödinger equation as follows

$$H|M, M'\rangle = E(M, M')|M, M'\rangle. \tag{3.3}$$

There exists a symmetry between particles and holes. Introducing fermion operators

$$d_{l\sigma} = c_{l\sigma}^{\dagger}, \qquad d_{l\sigma}^{\dagger} = c_{l\sigma} \tag{3.4}$$

and using the relation $n_{l\sigma} = 1 - d_{l\sigma}^{\dagger} d_{l\sigma}$, one obtains the identity

$$E(M, M') = -(L - N)U + E(L - M, L - M'). \tag{3.5}$$

Since $N \geq L$ if and only if $(L - M) + (L - M') \leq L$, we can restrict ourselves to the case of at most "half-filled band"

$$N \leq L. \tag{3.6}$$

In addition, due to the spin-up and spin-down symmetry, it is sufficient to consider

$$M \leq M'. \tag{3.7}$$

3.2 Nested Bethe ansatz

The Fock eigenstates of the Hubbard model are expressible as follows

$$|M, M'\rangle = \sum_{\{\sigma_j\}} \sum_{\{x_k\}} \psi_{\sigma_1 \dots \sigma_N}(x_1, \dots, x_N)\, c^\dagger_{x_1\sigma_1} \dots c^\dagger_{x_N\sigma_N}|0\rangle, \tag{3.8}$$

where $\sum_{\{\sigma_j\}}$ denotes summation over all $N!/(M!M'!)$ possible spin configurations and $|0\rangle$ denotes the particle vacuum. Due to the anticommutation relations between the Fermion operators, the amplitudes ψ are totally antisymmetric under simultaneous exchange of spin and space variables:

$$\psi_{\sigma_{Q1} \dots \sigma_{QN}}(x_{Q1}, \dots, x_{QN}) = \mathrm{sign}(Q)\, \psi_{\sigma_1 \dots \sigma_N}(x_1, \dots, x_N), \tag{3.9}$$

where $Q = (Q1, Q2, \dots, QN)$ is a permutation of the labels $\{1, 2, \dots, N\}$, i.e. an element of the symmetric group S_N. The antisymmetry property (3.9) implies that the summation over spin configurations in (3.8) is redundant and one has

$$|M, M'\rangle = \frac{N!}{M!M'!} \sum_{\{x_k\}} \psi_{\sigma_1 \dots \sigma_N}(x_1, \dots, x_N)\, c^\dagger_{x_1\sigma_1} \dots c^\dagger_{x_N\sigma_N}|0\rangle, \tag{3.10}$$

where $(\sigma_1, \dots, \sigma_N)$ is an arbitrary configuration of M electrons with spin down and M' electrons with spin up. Inserting (3.10) into the eigenvalue equation, one gets the "first quantized" version of the Schrödinger equation for the wave function ψ:

$$\begin{aligned} -\sum_{j=1}^{N} \sum_{\epsilon=\pm1} \psi_{\sigma_1 \dots \sigma_N}(x_1, \dots, x_j + \epsilon, \dots, x_N) + U \sum_{j<k} \delta(x_j, x_k)\, \psi_{\sigma_1 \dots \sigma_N}(x_1, \dots, x_N) \\ = E \psi_{\sigma_1 \dots \sigma_N}(x_1, \dots, x_N). \end{aligned} \tag{3.11}$$

Here, $\delta(x, x')$ denotes the Kronecker delta.

3.2.1 Two electrons

For the case of $N = 2$ electrons, the Schrödinger equation (3.11) reads

$$\begin{aligned} -\psi_{\sigma_1\sigma_2}(x_1 - 1, x_2) - \psi_{\sigma_1\sigma_2}(x_1 + 1, x_2) - \psi_{\sigma_1\sigma_2}(x_1, x_2 - 1) - \psi_{\sigma_1\sigma_2}(x_1, x_2 + 1) \\ + U\delta(x_1, x_2)\, \psi_{\sigma_1\sigma_2}(x_1, x_2) = E \psi_{\sigma_1\sigma_2}(x_1, x_2). \end{aligned} \tag{3.12}$$

Let $Q = (Q1, Q2) \in S_2$ be a permutation of the labels of particle coordinates which defines the sector X_Q of mutual particle positions as follows

$$X_Q = \{x_{Q1} \le x_{Q2}\}. \tag{3.13}$$

In particular, $Q = (1, 2)$ for $x_1 \le x_2$ and $Q = (2, 1)$ for $x_2 \le x_1$.

When $x_1 < x_2$ or $x_1 > x_2$, (3.12) reduces to the Schrödinger equation for free electrons on the chain and its solution is a superposition of plane waves. The "nested" Bethe ansatz form for the wave function reads

$$\psi_{\sigma_1\sigma_2}(x_1, x_2) = \sum_{P \in S_2} \mathrm{sign}(PQ)\, A_{\sigma_{Q1}\sigma_{Q2}}(k_{P1}, k_{P2}) \exp\left(\mathrm{i} \sum_{\alpha=1}^{2} k_{P\alpha} x_{Q\alpha} \right), \tag{3.14}$$

where k_1 and k_2 are the momenta of the two electrons. The prefactor sign(PQ) ensures the antisymmetry property of the wave function with respect to the interchange of particles:

$$\psi_{\sigma_1\sigma_2}(x_1,x_2) = -\psi_{\bar{\sigma}_1\bar{\sigma}_2}(\bar{x}_1,\bar{x}_2)\,, \tag{3.15}$$

where $\bar{\sigma}_1 = \sigma_2, \bar{x}_1 = x_2$ and $\bar{\sigma}_2 = \sigma_1, \bar{x}_2 = x_1$. To prove this fact, let us consider the case $x_1 < x_2$. In the ansatz (3.14) for $\psi_{\sigma_1\sigma_2}(x_1,x_2)$ we then have $Q = (1,2)$ and so

$$\psi_{\sigma_1\sigma_2}(x_1,x_2) = A_{\sigma_1\sigma_2}(k_1,k_2)\mathrm{e}^{\mathrm{i}(k_1x_1+k_2x_2)} - A_{\sigma_1\sigma_2}(k_2,k_1)\mathrm{e}^{\mathrm{i}(k_2x_1+k_1x_2)}\,. \tag{3.16}$$

On the other hand, for $\psi_{\bar{\sigma}_1\bar{\sigma}_2}(\bar{x}_1,\bar{x}_2)$ one has $\bar{x}_1 > \bar{x}_2$, i.e. $Q = (2,1)$, and the Bethe ansatz (3.14) yields

$$\psi_{\bar{\sigma}_1\bar{\sigma}_2}(\bar{x}_1,\bar{x}_2) = -A_{\bar{\sigma}_2\bar{\sigma}_1}(k_1,k_2)\mathrm{e}^{\mathrm{i}(k_1\bar{x}_2+k_2\bar{x}_1)} + A_{\bar{\sigma}_2\bar{\sigma}_1}(k_2,k_1)\mathrm{e}^{\mathrm{i}(k_2\bar{x}_2+k_1\bar{x}_1)}. \tag{3.17}$$

It is seen that Eqs. (3.16) and (3.17) imply the antisymmetry relation (3.15).

Substituting the ansatz (3.14) into Eq. (3.12) with $x_1 \neq x_2$ leads to the energy

$$E = -2(\cos k_1 + \cos k_2). \tag{3.18}$$

When $x_1 = x_2$, the electrons occupy the same site and interact with one another through a scattering process. The Bethe ansatz (3.14) for the wave function requires the scattering to be purely elastic, which means that the momenta k_1 and k_2 of the two electrons are individually conserved (the electrons either keep or exchange their momenta). The scattering process is determined by two conditions. Firstly, we have to "match" the wave function defined in the two sectors $Q = (1,2)$ and $Q = (2,1)$ when $x_1 = x_2 = x$. This yields the condition

$$\begin{aligned}\psi_{\sigma_1\sigma_2}(x,x) &= [A_{\sigma_1\sigma_2}(k_1,k_2) - A_{\sigma_1\sigma_2}(k_2,k_1)]\exp[\mathrm{i}(k_1+k_2)x] \\ &= [A_{\sigma_2\sigma_1}(k_2,k_1) - A_{\sigma_2\sigma_1}(k_1,k_2)]\exp[\mathrm{i}(k_1+k_2)x].\end{aligned} \tag{3.19}$$

Secondly, the Schrödinger equation (3.12) has to be fulfilled for $x_1 = x_2 = x$, which implies

$$\begin{aligned}&-\mathrm{e}^{-\mathrm{i}k_1}A_{\sigma_1\sigma_2}(k_1,k_2) + \mathrm{e}^{-\mathrm{i}k_2}A_{\sigma_1\sigma_2}(k_2,k_1) + \mathrm{e}^{\mathrm{i}k_2}A_{\sigma_2\sigma_1}(k_1,k_2) - \mathrm{e}^{\mathrm{i}k_1}A_{\sigma_2\sigma_1}(k_2,k_1) \\ &-\mathrm{e}^{\mathrm{i}k_2}A_{\sigma_1\sigma_2}(k_1,k_2) + \mathrm{e}^{\mathrm{i}k_1}A_{\sigma_1\sigma_2}(k_2,k_1) + \mathrm{e}^{-\mathrm{i}k_1}A_{\sigma_2\sigma_1}(k_1,k_2) - \mathrm{e}^{-\mathrm{i}k_2}A_{\sigma_2\sigma_1}(k_2,k_1) \\ &+[U + 2(\cos k_1 + \cos k_2)][A_{\sigma_1\sigma_2}(k_1,k_2) - A_{\sigma_1\sigma_2}(k_2,k_1)] = 0.\end{aligned} \tag{3.20}$$

With the aid of Eqs. (3.19) and (3.20) we can express any two of the four amplitudes $A_{\sigma_{Q1}\sigma_{Q2}}(k_{P1},k_{P2})$ in terms of the other two. Simple algebra gives

$$A_{\sigma_2\sigma_1}(k_2,k_1) = \sum_{\sigma_1',\sigma_2'} S^{\sigma_1\sigma_1'}_{\sigma_2\sigma_2'}(k_1,k_2)A_{\sigma_1'\sigma_2'}(k_1,k_2), \tag{3.21}$$

where S is the two-particle scattering matrix with elements

$$S^{\sigma_1\sigma_1'}_{\sigma_2\sigma_2'}(k_1,k_2) = \frac{\sin k_1 - \sin k_2}{\sin k_1 - \sin k_2 + \mathrm{i}U/2} I^{\sigma_1\sigma_1'}_{\sigma_2\sigma_2'} + \frac{\mathrm{i}U/2}{\sin k_1 - \sin k_2 + \mathrm{i}U/2}\Pi^{\sigma_1\sigma_1'}_{\sigma_2\sigma_2'}. \tag{3.22}$$

Here, I is the identity operator with elements $I^{\sigma_1\tau_1}_{\sigma_2\tau_2} = \delta_{\sigma_1\sigma_1'}\delta_{\sigma_2\sigma_2'}$ and Π is the permutation operator $\Pi^{\sigma_1\sigma_1'}_{\sigma_2\sigma_2'} = \delta_{\sigma_1\sigma_2'}\delta_{\sigma_2\sigma_1'}$. The natural parametrization of momenta reads

$$\sin k_n = \lambda_n^0 \qquad n = 1, 2; \tag{3.23}$$

the necessity of introducing the upperscript 0 in λ_n^0 will be clear later. Within this parametrization, the S-matrix (3.22) can be expressed as

$$S_{12}(\lambda = \lambda_1^0 - \lambda_2^0) = \frac{\lambda + \mathrm{i}(U/2)\Pi_{12}}{\lambda + \mathrm{i}(U/2)}. \tag{3.24}$$

This matrix has the form of the S-matrix for the XXX Heisenberg model (2.32), where the matrix elements a, b and c are

$$a(\lambda) = 1, \qquad b(\lambda) = \frac{\mathrm{i}U/2}{\lambda + \mathrm{i}U/2}, \qquad c(\lambda) = \frac{\lambda}{\lambda + \mathrm{i}U/2}. \tag{3.25}$$

Since it holds

$$a(\lambda) : c(\lambda) : b(\lambda) = (\lambda + \mathrm{i}U/2) : \lambda : \mathrm{i}U/2, \tag{3.26}$$

Eq. (2.38), which is a direct consequence of the Yang-Baxter equation, is automatically fulfilled.

We impose periodic boundary conditions on the wave function:

$$\psi_{\sigma_1\sigma_2}(0, x_2) = \psi_{\sigma_1\sigma_2}(L, x_2), \qquad \psi_{\sigma_1\sigma_2}(x_1, 0) = \psi_{\sigma_1\sigma_2}(x_1, L); \tag{3.27}$$

$$\psi_{\sigma_1\sigma_2}(1, x_2) = \psi_{\sigma_1\sigma_2}(L+1, x_2), \qquad \psi_{\sigma_1\sigma_2}(x_1, 1) = \psi_{\sigma_1\sigma_2}(x_1, L+1). \tag{3.28}$$

Inserting the Bethe ansatz (3.14) into these conditions yields

$$\exp(\mathrm{i}k_{P1}L)\, A_{\sigma_{Q2}\sigma_{Q1}}(k_{P2}, k_{P1}) = A_{\sigma_{Q1}\sigma_{Q2}}(k_{P1}, k_{P2}), \tag{3.29}$$

where the permutations $P, Q \in S_2$ are arbitrary. Choosing say $Q = (2, 1)$, one gets explicitly

$$\exp(\mathrm{i}k_{P1}L)\, A_{\sigma_1\sigma_2}(k_{P2}, k_{P1}) = A_{\sigma_2\sigma_1}(k_{P1}, k_{P2}) = \sum_{\sigma_1', \sigma_2'} S^{\sigma_1\sigma_1'}_{\sigma_2\sigma_2'}(k_{P2}, k_{P1}) A_{\sigma_1'\sigma_2'}(k_{P2}, k_{P1}). \tag{3.30}$$

In the sector of both electrons with spin up, it follows from the explicit form of the S-matrix (3.22) that

$$\exp(\mathrm{i}k_{P1}L)\, A_{\uparrow\uparrow}(k_{P2}, k_{P1}) = A_{\uparrow\uparrow}(k_{P1}, k_{P2}) = A_{\uparrow\uparrow}(k_{P2}, k_{P1}). \tag{3.31}$$

The periodic boundary conditions thus take the simple form

$$\exp(\mathrm{i}k_n L) = 1 \qquad n = 1, 2. \tag{3.32}$$

Similarly as in the case of the Heisenberg chain, the wave numbers must be distinct, $k_1 \neq k_2$, in order to prevent the nullity of the wave function. The same result is obtained in the sector of both electrons with spin down.

In the sector of one electron with spin up and the other one with spin down, the diagonalized form of Eq. (3.30) reads

$$\left\{ \mathrm{e}^{\mathrm{i}k_{P1}L} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{\sin k_{P2} - \sin k_{P1} - \mathrm{i}U/2}{\sin k_{P2} - \sin k_{P1} + \mathrm{i}U/2} & 0 \\ 0 & 1 \end{pmatrix} \right\} \begin{pmatrix} A_{\uparrow\downarrow}(k_{P2}, k_{P1}) - A_{\downarrow\uparrow}(k_{P2}, k_{P1}) \\ A_{\uparrow\downarrow}(k_{P2}, k_{P1}) + A_{\downarrow\uparrow}(k_{P2}, k_{P1}) \end{pmatrix} = 0. \tag{3.33}$$

This eigenvalue equation has two solutions. The first quantization condition, corresponding to $A_{\uparrow\downarrow}(k_{P2}, k_{P1}) = -A_{\downarrow\uparrow}(k_{P2}, k_{P1})$, takes the form

$$\exp(\mathrm{i}k_{P1}L) = \frac{\sin k_{P2} - \sin k_{P1} - \mathrm{i}U/2}{\sin k_{P2} - \sin k_{P1} + \mathrm{i}U/2}. \tag{3.34}$$

Introducing the quantity

$$\lambda_1 = \frac{1}{2}\left(\sin k_1 + \sin k_2\right), \tag{3.35}$$

Eq. (3.34) can be reexpressed in a more symmetric way

$$\exp(\mathrm{i}k_n L) = \frac{\lambda_1 - \sin k_n - \mathrm{i}U/4}{\lambda_1 - \sin k_n + \mathrm{i}U/4} \qquad n = 1, 2. \tag{3.36}$$

Since it follows from (3.34) that $\exp(\mathrm{i}k_1 L)\exp(\mathrm{i}k_2 L) = 1$, the parameter λ_1 is determined by the relation

$$\prod_{n=1}^{2} \frac{\lambda_1 - \sin k_n - \mathrm{i}U/4}{\lambda_1 - \sin k_n + \mathrm{i}U/4} = 1. \tag{3.37}$$

The second condition, corresponding to $A_{\uparrow\downarrow}(k_{P2}, k_{P1}) = A_{\downarrow\uparrow}(k_{P2}, k_{P1}) = A_{\uparrow\downarrow}(k_{P1}, k_{P2})$, is equivalent to the previous one (3.32).

3.2.2 N electrons

The generalization of the above results to the case of N electrons is straightforward. The Bethe ansatz for the solution ψ of the Schrödinger equation (3.11) in the sector Q with

$$X_Q = \{x_{Q1} \leq x_{Q2} \leq \ldots \leq x_{QN}\} \tag{3.38}$$

is

$$\begin{aligned} \psi_{\sigma_1\sigma_2\ldots\sigma_N}(x_1, x_2, \ldots, x_N) = & \sum_{P \in S_N} \mathrm{sign}(PQ) A_{\sigma_{Q1}\sigma_{Q2}\ldots\sigma_{QN}}(k_{P1}, k_{P2}, \ldots, k_{PN}) \\ & \times \exp\left(\mathrm{i}\sum_{\alpha=1}^{N} k_{P\alpha} x_{Q\alpha}\right). \end{aligned} \tag{3.39}$$

Substituting this ansatz into (3.11) for the case $x_n \neq x_m$ $(n, m = 1, \ldots, N; n \neq m)$, the energy is obtained as follows

$$E = -2\sum_{n=1}^{N} \cos k_n. \tag{3.40}$$

Note that by construction the Bethe ansatz wave function (3.39) is antisymmetric under simultaneous exchange of spin and space variables. This fact assures the Schrödinger

equation (3.11) to be satisfied when three or more electrons are occupying the same site. The only non-trivial case to consider is the presence of two electrons on the same site. Using the single valuedness of the wave function and solving the matching conditions at the Q-sector boundaries, one gets the nearest-neighbour electron scattering between the amplitudes

$$A_{\ldots\sigma_j\sigma_i\ldots}(\ldots k_v, k_u \ldots) = \sum_{\sigma_i'\sigma_j'} S^{\sigma_i\sigma_i'}_{\sigma_j\sigma_j'}(k_u, k_v) A_{\ldots\sigma_i'\sigma_j'\ldots}(\ldots k_u, k_v \ldots). \tag{3.41}$$

We impose periodic boundary conditions on the wave function:

$$\begin{aligned}\psi_{\sigma_1\ldots\sigma_N}(x_1, \ldots, x_{n-1}, 0, x_{n+1}, \ldots, x_N) &= \psi_{\sigma_1\ldots\sigma_N}(x_1, \ldots, x_{n-1}, L, x_{n+1}, \ldots, x_N) \\ \psi_{\sigma_1\ldots\sigma_N}(x_1, \ldots, x_{n-1}, 1, x_{n+1}, \ldots, x_N) &= \psi_{\sigma_1\ldots\sigma_N}(x_1, \ldots, x_{n-1}, L+1, x_{n+1}, \ldots, x_N),\end{aligned} \tag{3.42}$$

where $n = 1, \ldots, N$. Inserting the Bethe ansatz (3.39) into these conditions yields

$$\exp(\mathrm{i}k_{P1}L)\, A_{\sigma_{Q2}\ldots\sigma_{QN}\sigma_{Q1}}(k_{P2}, \ldots, k_{PN}, k_{P1}) = A_{\sigma_{Q1}\sigma_{Q2}\ldots\sigma_{QN}}(k_{P1}, k_{P2}, \ldots, k_{PN}). \tag{3.43}$$

Let us consider the special case $P = Q = (1, N, N-1, \ldots, 2)$, for which Eq. (3.43) takes the form

$$\exp(\mathrm{i}k_1 L)\, A_{\sigma_N\ldots\sigma_2\sigma_1}(k_N, \ldots, k_2, k_1) = A_{\sigma_1\sigma_N\ldots\sigma_2}(k_1, k_N, \ldots k_2). \tag{3.44}$$

Let us apply the two-particle scattering formula (3.41) to the amplitude on the lhs of this equation in order to "commute" successively k_1 with $k_2, k_3, \ldots, k_N$:

$$\begin{aligned}A_{\sigma_N\ldots\sigma_2\sigma_1}(k_N, \ldots, k_2, k_1) &= \sum_{\sigma_2', \tau_2} S^{\sigma_1\tau_2}_{\sigma_2\sigma_2'}(k_1, k_2)\; A_{\sigma_N\ldots\tau_2\sigma_2'}(k_N, \ldots, k_1, k_2) \\ &= \sum_{\substack{\sigma_2', \ldots, \sigma_N' \\ \tau_2, \ldots, \tau_N}} S^{\sigma_1\tau_2}_{\sigma_2\sigma_2'}(k_1, k_2)\; S^{\tau_2\tau_3}_{\sigma_3\sigma_3'}(k_1, k_3) \cdots \\ &\quad \times S^{\tau_{N-1}\tau_N}_{\sigma_N\sigma_N'}(k_1, k_N)\; A_{\tau_N\sigma_N'\ldots\sigma_2'}(k_1, k_N, \ldots, k_2).\end{aligned} \tag{3.45}$$

Identifying $\tau_N \equiv \sigma_1'$, the combination of Eqs. (3.44) and (3.45) yields

$$\exp(-\mathrm{i}k_1 L)\, A_{\sigma_1\sigma_N\ldots\sigma_2}(k_1, k_N, \ldots, k_2) = \sum_{\sigma_1', \sigma_2', \ldots, \sigma_N'} \mathrm{T}_1{}^{\sigma_1\sigma_N\ldots\sigma_2}_{\sigma_1'\sigma_N'\ldots\sigma_2'} A_{\sigma_1'\sigma_N'\ldots\sigma_2'}(k_1, k_N, \ldots, k_2), \tag{3.46}$$

where

$$\mathrm{T}_1{}^{\sigma_1\sigma_N\ldots\sigma_2}_{\sigma_1'\sigma_N'\ldots\sigma_2'} = \sum_{\tau_2, \ldots, \tau_{N-1}} S^{\sigma_1\tau_2}_{\sigma_2\sigma_2'}(k_1, k_2)\; S^{\tau_2\tau_3}_{\sigma_3\sigma_3'}(k_1, k_2) \cdots S^{\tau_{N-1}\sigma_1'}_{\sigma_N\sigma_N'}(k_1, k_2). \tag{3.47}$$

In this way the task of finding the quantization condition for k_1 is transformed to the eigenvalue problem of the T_1-matrix.

Let us introduce the inhomogeneous transfer matrix

$$\mathrm{T}(\lambda; \lambda_1^0, \ldots, \lambda_N^0)^{\sigma_1\ldots\sigma_N}_{\sigma_1'\ldots\sigma_N'} = \sum_{\gamma_1, \ldots, \gamma_N} S^{\gamma_1\gamma_2}_{\sigma_1\sigma_1'}(\lambda - \lambda_1^0)\; S^{\gamma_2\gamma_3}_{\sigma_2\sigma_2'}(\lambda - \lambda_2^0) \cdots S^{\gamma_N\gamma_1}_{\sigma_N\sigma_N'}(\lambda - \lambda_N^0). \tag{3.48}$$

At the point $\lambda = \lambda_1^0$, using the equality $S_{\sigma_1\sigma_1'}^{\gamma_1\gamma_2}(0) = \delta_{\sigma_1\gamma_2}\delta_{\sigma_1'\gamma_1}$ for our S-matrix (3.24) and identifying $\gamma_n \equiv \tau_{n-1}$ $(n = 1, \ldots, N)$ one finds that

$$\mathrm{T}_1 = \mathrm{T}(\lambda = \lambda_1^0; \lambda_1^0, \ldots, \lambda_N^0). \tag{3.49}$$

Performing the above procedure for $P1 = n$ $(n = 1, \ldots, N)$, one gets the following eigenvalue equation for k_n momentum:

$$\exp(-\mathrm{i}k_n L)\ A = \mathrm{T}_n\ A, \qquad \mathrm{T}_n = \mathrm{T}(\lambda = \lambda_n^0; \lambda_1^0, \ldots, \lambda_N^0). \tag{3.50}$$

3.3 Boundary conditions within the inhomogeneous QISM

The transfer matrix (3.48) is by the structure very similar to the one defined in the QISM by Eq. (2.12). The important difference is that the spectral parameters of the S-matrices in (3.48) are inhomogeneous, dependent on the sites $n = 1, \ldots, N$. As we shall see, this complication does not prevent from the diagonalization of the transfer matrix (3.48) by using an inhomogeneous version of the QISM described in section 2.

We recall that the S-matrices in the definition of the transfer matrix (3.48) are equivalent to the one of the XXX Heisenberg chain (2.32) with the matrix elements a, b and c defined in Eq. (3.23). As before, the transfer matrix (3.48) is the trace of the monodromy matrix $\mathcal{T}$ in the auxiliary ξ-space,

$$\mathrm{T}(\lambda; \lambda_1^0, \ldots, \lambda_N^0) = \mathrm{Tr}_\xi\ \mathcal{T}_\xi(\lambda; \lambda_1^0, \ldots, \lambda_N^0). \tag{3.51}$$

The monodromy matrix is expressible as the product of local $\mathcal{L}_{\xi n}$-matrices

$$\mathcal{T}_\xi(\lambda; \lambda_1^0, \ldots, \lambda_N^0) = \mathcal{L}_{\xi 1}(\lambda - \lambda_1^0)\mathcal{L}_{\xi 2}(\lambda - \lambda_2^0)\cdots\mathcal{L}_{\xi N}(\lambda - \lambda_N^0). \tag{3.52}$$

Yang-Baxter equations, obtained within the homogeneous QISM (section 2), now take the following forms. Considering $\mathcal{L}_{\xi\eta}(\lambda)$ defined by (2.18), the counterpart of the Yang-Baxter Eq. (2.19) reads

$$\mathcal{L}_{\xi\eta}(\lambda - \mu)\mathcal{L}_{\xi n}(\lambda - \lambda_n^0)\mathcal{L}_{\eta n}(\mu - \lambda_n^0) = \mathcal{L}_{\eta n}(\mu - \lambda_n^0)\mathcal{L}_{\xi n}(\lambda - \lambda_n^0)\mathcal{L}_{\xi\eta}(\lambda - \mu). \tag{2.19$'$}$$

Then, the analogy of the Yang-Baxter Eq. (2.21) is

$$\begin{aligned}&\mathcal{L}_{\xi\eta}(\lambda - \mu)\mathcal{T}_\xi(\lambda; \lambda_1^0, \ldots, \lambda_N^0)\mathcal{T}_\eta(\mu; \lambda_1^0, \ldots, \lambda_N^0)\\ &= \mathcal{T}_\eta(\mu; \lambda_1^0, \ldots, \lambda_N^0)\mathcal{T}_\xi(\lambda; \lambda_1^0, \ldots, \lambda_N^0)\mathcal{L}_{\xi\eta}(\lambda - \mu).\end{aligned} \tag{2.21$'$}$$

Introducing $R(\lambda) = \Pi_{\xi\eta}\mathcal{L}_{\xi\eta}$, this equation takes an equivalent form

$$\begin{aligned}&R(\lambda - \mu)\left[\mathcal{T}(\lambda; \lambda_1^0, \ldots, \lambda_N^0) \otimes \mathcal{T}(\mu; \lambda_1^0, \ldots, \lambda_N^0)\right]\\ &= \left[\mathcal{T}(\mu; \lambda_1^0, \ldots, \lambda_N^0) \otimes \mathcal{T}(\lambda; \lambda_1^0, \ldots, \lambda_N^0)\right] R(\lambda - \mu).\end{aligned} \tag{2.25$'$}$$

In analogy with Eq. (2.35), the $\mathcal{L}_{\xi n}$-matrix is expressible in the ξ-space as follows

$$\mathcal{L}_n(\lambda - \lambda_n^0) = \begin{pmatrix} \hat{\alpha}_n(\lambda - \lambda_n^0) & \hat{\beta}_n(\lambda - \lambda_n^0) \\ \hat{\gamma}_n(\lambda - \lambda_n^0) & \hat{\delta}_n(\lambda - \lambda_n^0) \end{pmatrix}. \tag{3.53}$$

The operator elements act on the local vector e_n^+ in the following way:

$$\begin{aligned} &\hat{\alpha}_n(\lambda-\lambda_n^0)e_n^+ = a(\lambda-\lambda_n^0)e_n^+, \quad && \hat{\beta}_n(\lambda-\lambda_n^0)e_n^+ = b(\lambda-\lambda_n^0)e_n^-, \\ &\hat{\gamma}_n(\lambda-\lambda_n^0)e_n^+ = 0, && \hat{\delta}_n(\lambda-\lambda_n^0)e_n^+ = c(\lambda-\lambda_n^0)e_n^+. \end{aligned} \tag{3.54}$$

These relations can be written in a compact form

$$\mathcal{L}_n(\lambda-\lambda_n^0)e_n^+ = \begin{pmatrix} a(\lambda-\lambda_n^0) & [\cdots] \\ 0 & c(\lambda-\lambda_n^0) \end{pmatrix} e_n^+. \tag{3.55}$$

The monodromy matrix is expressed in the auxiliary ξ-space as

$$\mathcal{T}(\lambda;\lambda_1^0,\ldots,\lambda_N^0) = \begin{pmatrix} A(\lambda;\lambda_1^0,\ldots,\lambda_N^0) & B(\lambda;\lambda_1^0,\ldots,\lambda_N^0) \\ C(\lambda;\lambda_1^0,\ldots,\lambda_N^0) & D(\lambda;\lambda_1^0,\ldots,\lambda_N^0) \end{pmatrix}. \tag{3.56}$$

The Yang-Baxter equation (2.25') implies the commutation rules of type (2.50)-(2.52) for the matrices A, B, C and D. Note that the shifts of spectral parameters λ and μ by $\{\lambda_n^0\}$ are "canceled" in parameters $a(\mu-\lambda)$, $b(\mu-\lambda)$ and $c(\mu-\lambda)$, i.e. the inhomogeneity does not enter into the commutation procedure of the matrix B with A or D. When the monodromy matrix acts on the generating vector Ω (the tensor product of e^+ vectors on the chain of N sites), the representation (3.52) and Eq. (3.55) imply that

$$\mathcal{T}(\lambda;\lambda_1^0,\ldots,\lambda_N^0)\Omega = \begin{pmatrix} \prod_{n=1}^N a(\lambda-\lambda_n^0) & [\cdots] \\ 0 & \prod_{n=1}^N c(\lambda-\lambda_n^0) \end{pmatrix} \Omega. \tag{3.57}$$

Comparing Eqs. (3.56) and (3.57) with one another, the action of the elements of the monodromy matrix on the generating vector Ω reads:

$$A\Omega = \prod_{n=1}^N a(\lambda-\lambda_n^0)\Omega, \quad C\Omega = 0, \quad D\Omega = \prod_{n=1}^N c(\lambda-\lambda_n^0)\Omega. \tag{3.58}$$

The eigenvectors of the transfer matrix are searched in the ansatz form

$$\phi(\lambda_1^0,\ldots,\lambda_N^0;\lambda_1,\ldots,\lambda_M) = \prod_{\alpha=1}^M B(\lambda_\alpha)\Omega, \tag{3.59}$$

where $\lambda_1,\ldots,\lambda_M$ are free spectral parameters. Performing the whole commutation procedure between Eqs. (2.61)-(2.64), one ends up with the crucial relation (2.65), where the eigenvalue of the transfer matrix (2.66) is replaced by

$$\Lambda(\lambda;\lambda_1^0,\ldots,\lambda_N^0;\lambda_1,\ldots,\lambda_M) = \prod_{n=1}^N a(\lambda-\lambda_n^0)\prod_{\alpha=1}^M \frac{a(\lambda_\alpha-\lambda)}{c(\lambda_\alpha-\lambda)} + \prod_{n=1}^N c(\lambda-\lambda_n^0)\prod_{\alpha=1}^M \frac{a(\lambda-\lambda_\alpha)}{c(\lambda-\lambda_\alpha)} \tag{3.60}$$

and the Bethe equations (2.68) for the spectral parameters $\lambda_1,\ldots,\lambda_M$ take the form

$$\prod_{n=1}^N \frac{a(\lambda_\alpha-\lambda_n^0)}{c(\lambda_\alpha-\lambda_n^0)} = \prod_{\substack{\beta=1 \\ (\beta\neq\alpha)}}^M \frac{a(\lambda_\alpha-\lambda_\beta)}{a(\lambda_\beta-\lambda_\alpha)}\frac{c(\lambda_\beta-\lambda_\alpha)}{c(\lambda_\alpha-\lambda_\beta)} \qquad \alpha = 1,2,\ldots,M. \tag{3.61}$$

Eq. (3.50) is thus equivalent to the quantization condition

$$\exp(-\mathrm{i}k_n L) = \prod_{\alpha=1}^{M} \frac{\lambda_\alpha - \sin k_n + \mathrm{i}U/2}{\lambda_\alpha - \sin k_n} \qquad n = 1, \ldots, N. \tag{3.62}$$

The Bethe equations (3.61) take the explicit form

$$\prod_{n=1}^{N} \frac{\lambda_\alpha - \sin k_n + \mathrm{i}U/2}{\lambda_\alpha - \sin k_n} = \prod_{\substack{\beta=1 \\ (\beta\neq\alpha)}}^{M} (-1) \frac{\lambda_\alpha - \lambda_\beta + \mathrm{i}U/2}{\lambda_\beta - \lambda_\alpha + \mathrm{i}U/2} \qquad \alpha = 1, \ldots, M. \tag{3.63}$$

Finally, shifting $\lambda_\alpha \to \lambda'_\alpha - \mathrm{i}U/4$ and after some simple manipulations one gets the results in the well-known symmetric form

$$\exp(\mathrm{i}k_n L) = \prod_{\alpha=1}^{M} \frac{\lambda_\alpha - \sin k_n - \mathrm{i}U/4}{\lambda_\alpha - \sin k_n + \mathrm{i}U/4} \qquad n = 1, \ldots, N \tag{3.64}$$

and

$$\prod_{n=1}^{N} \frac{\lambda_\alpha - \sin k_n - \mathrm{i}U/4}{\lambda_\alpha - \sin k_n + \mathrm{i}U/4} = \prod_{\substack{\beta=1 \\ (\beta\neq\alpha)}}^{M} \frac{\lambda_\alpha - \lambda_\beta - \mathrm{i}U/2}{\lambda_\alpha - \lambda_\beta + \mathrm{i}U/2} \qquad \alpha = 1, \ldots, M. \tag{3.65}$$

These general results are in agreement with the respective Eqs. (3.36) and (3.37), obtained for the special case $N = 2$ and $M = 1$.

3.4 Ground state and its energy

Logarithming Eqs. (3.64) and (3.65) one arrives at

$$k_n L = 2\pi I_n + \sum_{\beta=1}^{M} 2 \operatorname{arctg}\left[\frac{4}{U}(\lambda_\beta - \sin k_n)\right], \tag{3.66}$$

$$\sum_{n=1}^{N} 2 \operatorname{arctg}\left[\frac{4}{U}(\lambda_\alpha - \sin k_n)\right] = 2\pi J_\alpha + \sum_{\substack{\beta=1 \\ (\beta\neq\alpha}}^{M} 2 \operatorname{arctg}\left[\frac{2}{U}(\lambda_\alpha - \lambda_\beta)\right], \tag{3.67}$$

where I_n and J_α are integers or half-integers. Here, we have used the formula

$$\frac{x - \mathrm{i}y}{x + \mathrm{i}y} = \mathrm{e}^{\mathrm{i}(\Theta - \pi)} \qquad \Longrightarrow \qquad \Theta = 2 \operatorname{arctg}\left(\frac{x}{y}\right). \tag{3.68}$$

In the thermodynamics limit $L \to \infty$, $N \to \infty$, $M \to \infty$ with the finite particle density N/L and spin M/L, the numbers $\{I_n\}$ and $\{J_\alpha\}$ are consecutive sequences $I_{n+1} = I_n + 1$, $J_{\alpha+1} = J_\alpha + 1$ symmetrically distributed around 0. The corresponding real numbers $\{k_n\}$ and $\{\lambda_\alpha\}$ are situated in the intervals

$$-Q \leq k_n \leq Q, \qquad -B \leq \lambda_\alpha \leq B. \tag{3.69}$$

In the continuum limit $k_n \to k(x)$, $\lambda_\alpha \to \lambda(x)$ and $I_n/N \to x$, the density of k is $\rho(k)$ and the density of λ is $\sigma(\lambda)$. Eq. (3.66) implies

$$k(x) = 2\pi x + \int_{-B}^{B} \mathrm{d}\lambda\ \sigma(\lambda)\ 2\ \mathrm{arctg}\left[\frac{4}{U}(\lambda - \sin k)\right]. \tag{3.70}$$

The differentiation of this equation with respect to k yields

$$2\pi\rho(k) = 1 + \cos k \int_{-B}^{B} \mathrm{d}\lambda\ \sigma(\lambda)\frac{8U}{U^2 + 16(\lambda - \sin k)^2}. \tag{3.71}$$

The second equation is obtained from Eq. (3.67) by differentiating with respect to λ:

$$\int_{-Q}^{Q} \mathrm{d}k\ \rho(k)\frac{8U}{U^2 + 16(\lambda - \sin k)^2} = 2\pi\sigma(\lambda) + \int_{-B}^{B} \mathrm{d}\lambda'\ \sigma(\lambda')\frac{4U}{U^2 + 4(\lambda - \sin k)^2}. \tag{3.72}$$

The parameters Q and B are determined by the conditions

$$\int_{-Q}^{Q} \mathrm{d}k\ \rho(k) = \frac{N}{L}, \qquad \int_{-B}^{B} \mathrm{d}\lambda\sigma(\lambda) = \frac{M}{L}. \tag{3.73}$$

The energy (3.40) is expressible, for real k_n and in the continuum limit, as follows

$$E(M, M') = -2L \int_{-Q}^{Q} \mathrm{d}k\ \rho(k)\cos k. \tag{3.74}$$

The global ground state corresponds to a half-filled band $N = L$ with zero total spin $M = M' = N/2$ (N is even). This corresponds to $B = \infty$ and $Q = \pi$ [9]. With these values of B and Q, Eqs. (3.71) and (3.72) can be solved by using the Fourier method, in particular, the following Fourier transforms:

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\mathrm{d}\lambda}{2\pi} \mathrm{e}^{\mathrm{i}\omega\lambda} \frac{8U}{U^2 + 16\lambda^2} &= \mathrm{e}^{-U|\omega|/4}, \\ \int_{-\infty}^{\infty} \frac{\mathrm{d}\lambda}{2\pi} \mathrm{e}^{\mathrm{i}\omega\lambda} \frac{4U}{U^2 + 4\lambda^2} &= \mathrm{e}^{-U|\omega|/2}. \end{aligned} \tag{3.75}$$

The final results read

$$\sigma(\lambda) = \frac{1}{2\pi} \int_0^{\infty} \mathrm{d}\omega \frac{J_0(\omega)\cos(\omega\lambda)}{\cosh(\omega U/4)}, \tag{3.76}$$

$$\rho(k) = \frac{1}{2\pi} + \frac{\cos k}{\pi} \int_0^{\infty} \mathrm{d}\omega \frac{J_0(\omega)\cos(\omega \sin k)}{1 + \exp(\omega U/2)}, \tag{3.77}$$

$$E_0(N/2, N/2) = -4L \int_0^{\infty} \mathrm{d}\omega \frac{J_0(\omega)J_1(\omega)}{\omega[1 + \exp(\omega U/2)]}, \tag{3.78}$$

where $J_0(\omega)$ and $J_1(\omega)$ are the Bessel functions.

The complete thermodynamics of the 1D Hubbard model is reviewed in Ref. [10].

3.5 Absence of Mott's transition conductor-insulator

When an interaction parameter of electrons is varied, the system of electrons in the ground state can undergo a conducting-insulating Mott transition. In the Hubbard model one expects that the Mott transition occurs at some critical $U_c > 0$. In what follows we show that this is not the case of the 1D Hubbard model in which there exists no Mott transition for all $U > 0$.

In order to determine whether the ground state is conducting or insulating, it is necessary to evaluate the chemical potentials

$$\mu_+ = E_0(M+1, M; U) - E_0(M, M; U), \qquad \mu_- = E_0(M, M; U) - E_0(M-1, M; U). \quad (3.79)$$

Due to the symmetry (3.5), the chemical potentials fulfil the relation

$$\mu_+ + \mu_- = U. \quad (3.80)$$

The system is conducting is $\mu_+ = \mu_-$ and insulating if $\mu_+ > \mu_-$; this is related to the fact that in the insulating phase there are bounded pairs of $\uparrow\downarrow$ electrons while in the conducting phase the interaction is screened. It was shown in ref. [9] that

$$\mu_-(U) = 2 - 4\int_0^\infty \mathrm{d}\omega \frac{J_1(\omega)}{\omega[1+\exp(\omega U/2)]}. \quad (3.81)$$

Consequently, $\mu_+(U) > \mu_-(U)$ and the 1D Hubbard model is insulating for all $U > 0$. There is no conducting-insulating transition in the ground state of the 1D Hubbard model, except at $U = 0$.

4. Kondo effect

4.1 Hamiltonian

The Kondo model describes the interaction of a conduction band with a localized spin impurity. The conduction band is defined by the Hamiltonian

$$H_0 = \sum_{\mathbf{k},\sigma} \epsilon(k) c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma}, \tag{4.1}$$

where $c_{\mathbf{k}\sigma}$ ($c^\dagger_{\mathbf{k}\sigma}$) is the annihilation (creation) operator of an electron with momentum $\mathbf{k}$ and spin component $\sigma \in \{\uparrow, \downarrow\}$. The conduction band is coupled via spin exchange interaction to a spin $\mathbf{s}_0$ localized at $\mathbf{r} = 0$,

$$H_I = J \sum_{\sigma,\sigma'} \Psi^*_\sigma(\mathbf{r}=0) \mathbf{s}_{\sigma\sigma'} \Psi_{\sigma'}(\mathbf{r}=0) \cdot \mathbf{s}_0. \tag{4.2}$$

Here, $\Psi_\sigma(\mathbf{r}=0)$ [$\Psi^*_\sigma(\mathbf{r}=0)$] is the Fourier transform of $c_{\mathbf{k}\sigma}$ ($c^\dagger_{\mathbf{k}\sigma}$). For simplicity, the impurity spin $\mathbf{s}_0$ will be considered to be the spin-1/2, but the exact analysis can be performed for any spin $S = 1/2, 1, 3/2, \ldots$.

The system is rotationally invariant and therefore it is useful to expand the electron operators $c_{\mathbf{k}\sigma}$ and $c^\dagger_{\mathbf{k}\sigma}$ in the basis of the spherical functions:

$$c_{\mathbf{k}\sigma} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_{lm}\left(\frac{\mathbf{k}}{k}\right) c_{klm,\sigma}, \qquad c^\dagger_{\mathbf{k}\sigma} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y^*_{lm}\left(\frac{\mathbf{k}}{k}\right) c^\dagger_{klm,\sigma}. \tag{4.3}$$

One assumes for simplicity that from the angular modes only the s-wave modes with $l = m = 0$ have nonzero coupling to the impurity. The total Hamiltonian $H = H_0 + H_I$ then reads

$$H = \sum_{k,l,m,\sigma} \epsilon(k) c^\dagger_{klm,\sigma} c_{klm,\sigma} + J \sum_{\substack{k,k' \\ \sigma,\sigma'}} c^\dagger_{k00,\sigma} \mathbf{s}_{\sigma\sigma'} c_{k'00,\sigma'} \cdot \mathbf{s}_0. \tag{4.4}$$

Let us restrict ourselves to momenta k close to the Fermi surface, $k = k_F + q$ with $q \ll D$ where D is a cut-off of the order of k_F, and consider the linear dispersion law

$$\epsilon(k) = \epsilon_F + v_F(k - k_F). \tag{4.5}$$

We shift the energy by ϵ_F, set $v_F = 1$ and leave in the free-electron part of H only the relevant electrons with $l = m = 0$. Using the notation $c_{k00,\sigma} \equiv c_{k\sigma}$ and $c^\dagger_{k00,\sigma} \equiv c^\dagger_{k\sigma}$, one thus gets

$$H = \sum_{k,\sigma} k c^\dagger_{k\sigma} c_{k\sigma} + J \sum_{\substack{k,k' \\ \sigma,\sigma'}} c^\dagger_{k\sigma} \mathbf{s}_{\sigma\sigma'} c_{k'\sigma'} \cdot \mathbf{s}_0. \tag{4.6}$$

This Hamiltonian is effectively one-dimensional. In the coordinate representation in terms of the operators $c_\sigma(x) = \int \mathrm{d}k \, \exp(\mathrm{i}kx) c_{k\sigma}$, it takes the form

$$H = \int \mathrm{d}x \, \left[-\mathrm{i} \sum_\sigma c^\dagger_\sigma(x) \frac{\partial}{\partial x} c_\sigma(x) + J\delta(x) \sum_{\sigma,\sigma'} c^\dagger_\sigma(x) \mathbf{s}_{\sigma\sigma'} c_{\sigma'}(x) \cdot \mathbf{s}_0 \right]. \tag{4.7}$$

In the first quantization, the Schrödinger equation for N electrons $(\sigma_1, x_1), \ldots, (\sigma_N, x_N)$ and one impurity $(\sigma_0, x_0 \equiv 0)$ is written as follows

$$\left(-\mathrm{i}\sum_{n=1}^{N}\frac{\partial}{\partial x_n} - E\right)\psi_{\sigma_1\ldots\sigma_N,\sigma_0}(x_1,\ldots,x_N) + J\sum_{n=1}^{N}\delta(x_n) \times \sum_{\sigma_n',\sigma_0'} \mathbf{s}_{\sigma_n\sigma_n'}\cdot\mathbf{s}_{\sigma_0\sigma_0'}\psi_{\sigma_1\ldots\sigma_n'\ldots\sigma_N,\sigma_0'}(x_1,\ldots,x_N) = 0. \tag{4.8}$$

4.2 S-matrices

The Kondo model was solved by Andrei [11] and Wiegmann [12], for reviews see Ref. [13,14].

Let us first study the Schrödinger Eq. (4.8) for a single electron interacting with the impurity:

$$\left(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x} - E\right)\psi_{\sigma,\sigma_0}(x) + J\delta(x)\sum_{\sigma',\sigma_0'}\mathbf{s}_{\sigma\sigma'}\cdot\mathbf{s}_{\sigma_0\sigma_0'}\psi_{\sigma',\sigma_0'}(x) = 0. \tag{4.9}$$

The wave function is searched in the Bethe-ansatz form

$$\psi_{\sigma,\sigma_0}(x) = \exp(\mathrm{i}kx)\left[A_{\sigma\sigma_0}\theta(-x) + A_{\sigma_0\sigma}\theta(x)\right]. \tag{4.10}$$

For $x \neq 0$, this function obviously satisfies Eq. (4.9) and implies the eigenvalue

$$E = k. \tag{4.11}$$

At $x = 0$, the Schrödinger Eq. (4.9) leads to

$$\delta(0)\left\{-\mathrm{i}\left(A_{\sigma_0\sigma} - A_{\sigma\sigma_0}\right) + \frac{1}{2}J\mathbf{s}\cdot\mathbf{s}_0\left(A_{\sigma\sigma_0} + A_{\sigma_0\sigma}\right)\right\} = 0. \tag{4.12}$$

Here, we used the equality $\theta(x) + \theta(-x) = 1$ and the consequent renormalization prescription

$$\theta(x)\delta(x) = \theta(-x)\delta(x) = \frac{1}{2}\delta(x). \tag{4.13}$$

For the scattering S-matrix of the electron with number $n = 1, 2, \ldots, N$, which relates the amplitudes according to the standard prescription

$$A_{\sigma_0\sigma_n} = \sum_{\sigma_n',\sigma_0'} S^{\sigma_n\sigma_n'}_{\sigma_0\sigma_0'} A_{\sigma_n'\sigma_0'}, \tag{4.14}$$

we thus get

$$S_{n0} = \frac{\mathrm{i} + J\mathbf{s}_n\cdot\mathbf{s}_0/2}{\mathrm{i} - J\mathbf{s}_n\cdot\mathbf{s}_0/2} = \frac{\mathrm{i} - (J/2) + J\Pi_{n0}}{\mathrm{i} + (J/2) - J\Pi_{n0}}. \tag{4.15}$$

Here, we took the advantage of the fact that the permutation operators $\Pi_{n0} = (I_{n0} + \mathbf{s}_n \cdot \mathbf{s}_0)/2$. Since $\Pi^2 = I$ the S-matrix (4.15) can be transformed to a more convenient form

$$S_{n0} = \exp(-\mathrm{i}\phi)\frac{I_{n0} - \mathrm{i}c\ \Pi_{n0}}{1 - \mathrm{i}c}, \tag{4.16}$$

where

$$c = \frac{2J}{1 - (3J^2/4)}, \qquad \exp(\mathrm{i}\phi) = \frac{1 - (3J^2/4) + 2\mathrm{i}J}{1 + (3J^2/4) - \mathrm{i}J}. \tag{4.17}$$

As concerns the scattering S-matrix for two electrons n and m away from impurity, since the Hamiltonian does not contain any interaction term one might be tempted to consider $S_{nm} = I_{nm}$. However, this choice would not satisfy the "mixed" Yang-Baxter equation for the scattering of two electrons and the impurity

$$S_{nm}S_{n0}S_{m0} = S_{m0}S_{n0}S_{nm} \tag{4.18}$$

due to the non-commutativity of S_{n0} and S_{m0}. The solution of this problem consists in the fact that for free non-interacting electrons an arbitrary electron-electron S-matrix is allowed. Being inspired by the Hubbard model, we consider

$$S_{nm} = \Pi_{nm}. \tag{4.19}$$

This choice evidently satisfies, besides the standard Yang-Baxter equation for three electrons

$$S_{nm}S_{nk}S_{mk} = S_{mk}S_{nk}S_{nm}, \tag{4.20}$$

also the mixed one (4.18) for two electrons and the impurity.

Let us introduce the "state variables" $\lambda_n^0 = 1$ or 0 depending on whether n refers to an electron ($n = 1, 2, \ldots, N$) or to the impurity ($n = 0$), respectively. Then, the expressions for the S-matrices (4.16) and (4.19) can be unified:

$$S_{nm} = \exp\left[-\mathrm{i}(\lambda_n^0 - \lambda_m^0)\phi\right] \frac{(\lambda_n^0 - \lambda_m^0)I_{nm} - \mathrm{i}c\ \Pi_{nm}}{(\lambda_n^0 - \lambda_m^0) - \mathrm{i}c}. \tag{4.21}$$

The Bethe ansatz for the amplitude of N electrons and the impurity is analogous to that of the Hubbard model, the corresponding energy depends on the momenta of electrons as follows

$$E = \sum_{n=1}^{N} k_n. \tag{4.22}$$

4.3 Boundary conditions within the inhomogeneous QISM

Let us consider N electrons localized on the line of length L. Imposing periodic boundary conditions for the electrons leads to N eigenvalue equations for their momenta analogous to the ones (3.50) for the Hubbard model:

$$\exp(-\mathrm{i}k_n L)A = \mathrm{T}_n A \qquad n = 1, \ldots, N. \tag{4.23}$$

Here, the operator T_n, defined by the previous Eq. (3.47), is now built with the S-matrices of the Kondo model (4.21):

$$\mathrm{T}_n = S_{n,n-1} \ldots S_{n1}S_{n0}S_{nN} \ldots S_{n,n+1}. \tag{4.24}$$

We define the transfer matrix

$$\mathrm{T}(\lambda; \lambda_0^0, \lambda_1^0, \ldots, \lambda_N^0) = \mathrm{Tr}_\xi\ \mathcal{T}_\xi(\lambda; \lambda_0^0, \lambda_1^0, \ldots, \lambda_N^0) \tag{4.25}$$

with the monodromy matrix expressed as the product of local $\mathcal{L}_{\xi n}$-matrices

$$\mathcal{T}_\xi(\lambda;\lambda_0^0,\lambda_1^0,\ldots,\lambda_N^0)=\mathcal{L}_{\xi 0}(\lambda-\lambda_0^0)\mathcal{L}_{\xi 1}(\lambda-\lambda_1^0)\cdots\mathcal{L}_{\xi N}(\lambda-\lambda_N^0). \tag{4.26}$$

The operator of interest T_n is determined simply by

$$\mathrm{T}_n=T(\lambda=\lambda_n^0;\lambda_0^0,\lambda_1^0,\ldots,\lambda_N^0)\qquad n=1,\ldots,N. \tag{4.27}$$

Comparing the S-matrices of the Kondo model (4.21) with those of the Hubbard model (3.24) it is clear that they are equivalent if one identifies $U/2\to -c$, up to a trivial exponential prefactor which appears in each T_n just once. We can therefore adopt the final results (3.64) and (3.65) of the inhomogeneous QISM applied to the Hubbard model, with the identification $\sin k_n\equiv\lambda_n^0$ $(n=1,\ldots,N)$ and with the inclusion of the $n=0$ impurity term and the corresponding λ_0^0. Namely, in the sector with M $(0\le M\le N)$ down-spin electrons, the electron momenta are given by

$$\exp(\mathrm{i}k_nL)=\exp(\mathrm{i}\phi)\prod_{\alpha=1}^{M}\frac{\lambda_\alpha-\lambda_n^0+\mathrm{i}c/2}{\lambda_\alpha-\lambda_n^0-\mathrm{i}c/2}\qquad n=1,\ldots,N \tag{4.28}$$

and the spectral parameters $\lambda_1,\ldots,\lambda_M$ are determined by the Bethe equations

$$\prod_{n=0}^{N}\frac{\lambda_\alpha-\lambda_n^0+\mathrm{i}c/2}{\lambda_\alpha-\lambda_n^0-\mathrm{i}c/2}=\prod_{\substack{\beta=1\\(\beta\neq\alpha)}}^{M}\frac{\lambda_\alpha-\lambda_\beta+\mathrm{i}c}{\lambda_\alpha-\lambda_\beta-\mathrm{i}c}\qquad \alpha=1,\ldots,M. \tag{4.29}$$

After the substitution of the state variables $\lambda_0^0=0$ and $\lambda_n^0=1$ for $n=1,\ldots,N$, Eq. (4.28) takes the form

$$\exp(\mathrm{i}k_nL)=\exp(\mathrm{i}\phi)\prod_{\alpha=1}^{M}\frac{\lambda_\alpha-1+\mathrm{i}c/2}{\lambda_\alpha-1-\mathrm{i}c/2}\qquad n=1,\ldots,N, \tag{4.30}$$

while the formal inversion of Eq. (4.29) results in

$$\begin{aligned}\left(\frac{\lambda_\alpha-1-\mathrm{i}c/2}{\lambda_\alpha-1+\mathrm{i}c/2}\right)^N\left(\frac{\lambda_\alpha-\mathrm{i}c/2}{\lambda_\alpha+\mathrm{i}c/2}\right)&=\prod_{\substack{\beta=1\\(\beta\neq\alpha)}}^{M}\frac{\lambda_\alpha-\lambda_\beta-\mathrm{i}c}{\lambda_\alpha-\lambda_\beta+\mathrm{i}c}\\&=-\prod_{\beta=1}^{M}\frac{\lambda_\alpha-\lambda_\beta-\mathrm{i}c}{\lambda_\alpha-\lambda_\beta+\mathrm{i}c}\qquad \alpha=1,\ldots,M.\end{aligned} \tag{4.31}$$

Note that the equations for the momenta $\{k_n\}$ and for the spin spectral parameters $\{\lambda_\alpha\}$ have decoupled. The ϕ-parameter in Eq. (4.30) can be omitted since it only shifts trivially k_n by the uniform ϕ/L.

4.4 Ground state and its energy

Using a formula similar to the one (3.68), one gets from Eq. (4.30) (taken with $\phi\equiv 0$) that

$$k_n=\frac{2\pi}{L}I_n+\frac{1}{L}\sum_{\beta=1}^{M}\left[\Theta(2\lambda_\beta-2)-\pi\right]. \tag{4.32}$$

Here, I_n is an integer and

$$\Theta(x) = -2 \text{ arctg}\left(\frac{x}{c}\right). \tag{4.33}$$

The expression for the energy (4.22) becomes

$$E = \sum_{n=1}^{N} \frac{2\pi}{L} I_n + n_e \sum_{\beta=1}^{M} \left[\Theta(2\lambda_\beta - 2) - \pi\right], \tag{4.34}$$

where $n_e = N/L$ is the electron density. Taking the logarithm of Eqs. (4.31) one finds that the spectral parameters $\{\lambda_\alpha\}_{\alpha=1}^{M}$ satisfy the following set of coupled equations

$$N\Theta(2\lambda_\alpha - 2) + \Theta(2\lambda_\alpha) = -2\pi J_\alpha + \sum_{\beta=1}^{M} \Theta(\lambda_\alpha - \lambda_\beta), \qquad \alpha = 1, \ldots, M, \tag{4.35}$$

where the numbers $\{I_\alpha\}_{\alpha=1}^{M}$ are distinct integers or half-integers.

Let us now concentrate on the ground state determined by specific choices of quantum numbers $\{I_n\}_{n=1}^{N}$ and $\{J_\alpha\}_{\alpha=1}^{M}$.

- Since the integers $\{I_n\}_{n=1}^{N}$ in the energy (4.34) can take arbitrarily large and negative values, the energy spectrum is unbounded from below. It is therefore needed to introduce a cutoff K, say as follows

$$\left|\frac{2\pi}{L} I_n\right| < K. \tag{4.36}$$

  In the ground state, the quantum numbers $\{I_n\}_{n=1}^{N}$ take their minimum values allowed by the cutoff. Since they have to be distinct and run from $-KL/(2\pi)$ upwards with the unity step, setting $\epsilon_F = 0$ they read

$$I_1, I_2, \ldots, I_N = -N, -N+1, \ldots, 0. \tag{4.37}$$

  Consequently, $K = 2\pi N/L = 2\pi n_e$.

- As concerns the quantum numbers $\{J_\alpha\}_{\alpha=1}^{M}$ in Eq. (4.35), they also have to be distinct and, since $|\Theta(x)| \leq \pi$, they are bounded by the general restrictions

$$J^-(N,M) = -\frac{1}{2}(N - M - 1) \leq J_\alpha \leq \frac{1}{2}(N - M - 1) = J^+(N,M). \tag{4.38}$$

  The state with the lowest energy is a spin singlet, $M = N/2$, induced by consecutive numbers $J_{\alpha+1} = J_\alpha + 1$ which fill the interval

$$J^- \leq J_\alpha \leq J^+, \qquad J^\pm = \pm\frac{1}{2}\left(\frac{N}{2} - 1\right). \tag{4.39}$$

In the thermodynamic limit $N, L \to \infty$ with $n_e = N/L$ held fixed, we define the density $\sigma(\lambda)$ of the solutions $\{\lambda_\alpha\}$ of Eq. (4.35). All λ-solutions are real in the ground state and thus from Eq. (4.34) the ground-state energy E_0 becomes

$$E_0 = \sum_{n=1}^{N} \frac{2\pi}{L} I_n + n_e \int_{-\infty}^{\infty} \mathrm{d}\lambda\ \sigma(\lambda) \left[\Theta(2\lambda - 2) - \pi\right]. \tag{4.40}$$

The relation for $\sigma(\lambda)$ is obtained from Eq. (4.35) by the standard procedure, with the final result

$$\sigma(\lambda) = \frac{2c}{\pi}\left[\frac{N}{c^2+4(\lambda-1)^2} + \frac{1}{c^2+4\lambda^2}\right] - \int_{-\infty}^{\infty} \frac{\mathrm{d}\lambda'}{\pi} \frac{c}{c^2+(\lambda-\lambda')^2}\sigma(\lambda'). \tag{4.41}$$

The solution of this equation is obtained with the aid of Fourier transforms:

$$\sigma(\lambda) = \frac{1}{2c}\left\{\frac{N}{\cosh[\pi(\lambda-1)/c]} + \frac{1}{\cosh(\pi\lambda/c)}\right\}. \tag{4.42}$$

Finally, inserting $\{I_n\}_{n=1}^N$ from (4.37) and $\sigma(\lambda)$ from (4.42) into the relation for the energy (4.40), one arrives at

$$E_0 = -\frac{\pi}{2L}N^2 - \mathrm{i}n_e \ln\left[\frac{\Gamma(1+\mathrm{i}c)\Gamma(\frac{1}{2}-\mathrm{i}c)}{\Gamma(1-\mathrm{i}c)\Gamma(\frac{1}{2}+\mathrm{i}c)}\right]. \tag{4.43}$$

The complete thermodynamics of the Kondo model is reviewed in Refs. [13, 14].

All studied models possess short-ranged particle interactions, on the same site or point (Hubbard, Kondo) or the nearest-neighbour one (Heisenberg). The Bethe ansatz for systems with long-ranged interactions among particles is reviewed in the monograph [M4].

Integrable models in the (1+1)-dimensional Field Theory (e.g. sine-Gordon) are solved by using the Thermodynamic Bethe ansatz.

5. References

Monographs

[M1] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Academic Press, London, 1982).

[M2] M. Gaudin, *La Fonction d'Onde de Bethe* (Masson, Paris, 1983).

[M3] Yu. A. Izjumov and Yu. N. Skryabin, *Statistical mechanics of magnetically ordered systems* (Consultants Bureau, New York, 1988); translated from Russian (Nauka, 1987).

[M4] B. Sutherland, *Beautiful Models: 70 Years of Exactly Solved Quantum Many-Body Problems* (World Scientific, Singapore, 2004).

Articles

Heisenberg model

[1] H. A. Bethe, *Z. Phys.* 71 (1931) 205.

[2] C. N. Yang and C. P. Yang, *Phys. Rev.* 150 (1966) 321, 327.

[3] C. N. Yang and C. P. Yang, *Phys. Rev.* 151 (1966) 258.

[4] R. J. Baxter, *Ann. Phys.* 70 (1972) 323.

QISM

[5] F. Y. Wu, *Rev. Mod. Phys.* 64 (1992) 1099.

[6] L. A. Takhtajan and L. D. Faddeev, *Uspekhi Mat. Nauk* 34 (1979) 13.

[7] E. K. Skljanin, L. A. Takhtajan and L. D. Faddeev, *Theor. Math. Phys.* 40 (1980) 688.

Hubbard model

[8] E. H. Lieb and F. Y. Wu, *Phys. Rev. Lett.* 20 (1968) 1445; Erratum: *Phys. Rev. Lett.* 21 (1968) 192.

[9] E. H. Lieb and F. Y. Wu, *Physica A* 321 (2003) 1.

[10] T. Deguchi et al., cond-mat/9904398.

Kondo effect

[11] N. Andrei, *Phys. Rev. Lett.* 45 (1980) 379.

[12] P. B. Wiegmann, *J. Phys. C* 14 (1981) 1463.

[13] N. Andrei, K. Furuya, and J. H. Lowenstein, *Rev. Mod. Phys.* 55 (1983) 331.

[14] N. Andrei, cond-mat/9408101.