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CONTINUOUS DEPENDENCE OF SOLUTIONS OF GENERALIZED LINEAR DIFFERENTIAL EQUATIONS ON A PARAMETER *

ZDENĚK HALAS† AND MILAN TVRDÝ‡

Dedicated to the memory of Mikhail Drakhlin

Abstract. This contribution deals with systems of linear generalized linear differential equations of the form

$$x(t) = \widetilde{x} + \int_a^t \mathrm{d}[A(s)]\, x(s) + g(t) - g(a), \quad t \in [a,b],$$

where $-\infty < a < b < \infty$, A is an $n \times n$-complex matrix valued function, g is an n-complex vector valued function, A and g have bounded variation on $[a,b]$. The integrals are understood in the Kurzweil-Stieltjes sense.

Our aim is to present some new results on continuous dependence of solutions to linear generalized differential equations on parameters and initial data. In particular, we generalize in several aspects the known result by Ashordia. Our main goal consists in a more general notion of a solution to the given system. In particular, neither g nor x need not be of bounded variation on $[a,b]$ and, in general, they can be regulated functions.

Key Words. Generalized linear differential equation, continuous dependence on a parameter, Perron-Stieltjes integral, Kurzweil-Stieltjes integral.

AMS(MOS) subject classification. 34A37, 45A05, 34A30.

1. Introduction.

Starting with Kurzweil [10], generalized differential equations have been extensively studied by many authors, like e.g. Hildebrandt [7], Schwabik, Tvrdý and Vejvoda [16]–[18], [20]–[22], Hönig [8],

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† Department of Mathematics, Faculty of Education, Palacký University, CZ 771 40 OLOMOUC, Žižkovo náměstí 5, Czech Republic (e-mail: zdenekhalas@seznam.cz)

‡ Mathematical Institute, Academy of Sciences of the Czech Republic, 115 67 PRAHA 1, Žitná 25, Czech Republic (e-mail: tvrdy@math.cas.cz)

Ashordia [2], [3]. In particular, see the monographs [18], [16], [21] and [8] and the references therein. Moreover, during several recent decades, the interest in their special cases like equations with impulses or discrete systems increased considerably, cf. e.g. monographs [12], [24], [4], [15] or [1].

The importance of generalized linear differential equations with regulated solutions consists in the fact that they enable us to treat in a unified way both the continuous and discrete systems and, in addition, also systems with fast oscillating data.

In the paper we keep the following notation:

As usual, $\mathbb{N}$ is the set of natural numbers and $\mathbb{C}$ stands for the set of complex numbers. $\mathbb{C}^{m\times n}$ is the space of complex matrices of the type $m\times n$, $\mathbb{C}^n = \mathbb{C}^{n\times 1}$ and $\mathbb{C}^1 = \mathbb{C}$. For a matrix

$$A = (a_{i,j})_{\substack{i=1,2,\dots,m\\ j=1,2,\dots,n}} \in \mathbb{C}^{m\times n},$$

its norm $|A|$ is defined by

$$|A| = \max_{j=1,2,\dots,n} \sum_{i=1}^{m} |a_{i,j}|.$$

In particular, we have $|x| = \sum_{i=1}^{n} |x_i|$ for $x \in \mathbb{C}^n$. The symbols I and 0 stand respectively for the identity and the zero matrix of the proper type. For an $n\times n$-matrix A, $\det[A]$ denotes its determinant.

If $-\infty < a < b < \infty$, then $[a,b]$ and (a,b) denote the corresponding closed and open intervals, respectively. Furthermore, $[a,b)$ and $(a,b]$ are the corresponding half-open intervals. When the intervals $[a,a)$ and $(b,b]$ occur, they are understood to be empty.

For an arbitrary function $F\colon [a,b] \to \mathbb{C}^{m\times n}$ we set

$$\|F\|_\infty = \sup\{|F(t)|\colon t \in [a,b]\}.$$

The set $D = \{t_0, t_1, \dots, t_p\} \subset [a,b]$, $p \in \mathbb{N}$, is called a division of the interval $[a,b]$ if $a = t_0 < t_1 < \dots < t_p = b$. If, for each $t \in [a,b)$ and $s \in (a,b]$, the function $F : [a,b] \to \mathbb{C}^{m\times n}$ possesses in $\mathbb{C}^{m\times n}$ limits

$$F(t+) := \lim_{\tau\to t+} F(\tau), \quad F(s-) := \lim_{\tau\to s-} F(\tau),$$

we say that the function F is regulated on the interval $[a,b]$. The set of all $k\times n$-matrix valued functions regulated on the interval $[a,b]$ is denoted by $G^{m\times n}[a,b]$. Furthermore, we denote

$$\Delta^+F(t) = F(t+) - F(t) \quad \text{for } t \in [a,b), \quad \Delta^+F(b) = 0$$

and

$$\Delta^- F(s) \;=\; F(s) - F(s-) \quad \text{for} \;\; s \in (a,b], \quad \Delta^- F(a) = 0.$$

It is known that, for each $F \in G^{m\times n}[a,b]$, the set of all points of its discontinuity on the interval $[a,b]$ is at most countable. Moreover, for each $\varepsilon > 0$ there are at most finitely many points $t \in [a,b)$ such that $|\Delta^+ F(t)| \ge \varepsilon$ and at most finitely many points $s \in [a,b]$ such that $|\Delta^- F(s)| \ge \varepsilon$. Clearly, each function regulated on $[a,b]$ is bounded on $[a,b]$, i.e. $\|F\|_\infty < \infty$ for $F \in G^{m\times n}[a,b]$.

For a function $F\colon [a,b] \to \mathbb{C}^{m\times n}$ we denote by $\operatorname{var}_a^b F$ its variation over $[a,b]$. We say that F has a bounded variation on $[a,b]$ if $\operatorname{var}_a^b F < \infty$. The set of $k \times n$-complex matrix valued functions of bounded variation on $[a,b]$ is denoted by $BV^{m\times n}[a,b]$. By $AC^{m\times n}[a,b]$ we denote the set of functions $F\colon [a,b] \to \mathbb{C}^{m\times n}$ such that each component f_{ij}, $i = 1,\dots,k$, $j = 1,\dots,n$, of F is absolutely continuous on the interval $[a,b]$. Similarly, $C^{m\times n}[a,b]$ stands for the set of functions $F : [a,b] \to \mathbb{C}^{m\times n}$ that are continuous on $[a,b]$. Analogously to the spaces of functions of bounded variation, $AC^n[a,b] = AC^{n\times 1}[a,b]$, $G^n[a,b] = G^{n\times 1}[a,b]$ and $C^n[a,b] = C^{n\times 1}[a,b]$. Obviously,

$$AC^{m\times n}[a,b] \subset BV^{m\times n}[a,b] \subset G^{m\times n}[a,b]$$

and $C^{m\times n}[a,b] \subset G^{m\times n}[a,b]$. Finally, a function $f : [a,b] \to \mathbb{C}$ is called a *finite step function on* $[a,b]$ if there is a division $\{\alpha_0, \alpha_1, \dots, \alpha_p\}$ of $[a,b]$ such that f is constant on every interval (α_{j-1}, α_j), $j = 1,\dots,p$. The set of all finite step functions on $[a,b]$ is denoted by $S[a,b]$, $S^{m\times n}[a,b]$ is the set of all $m \times n$-matrix valued functions whose arguments are finite step functions and $S^{n\times 1}[a,b] = S^n[a,b]$. It is known that the set $S^{m\times n}[a,b]$ is dense in $G^{m\times n}[a,b]$ with respect to the supremum norm, i.e.

$$\text{(1.1)} \qquad \begin{cases} \text{for each} \;\; \varepsilon > 0 \;\; \text{and each} \;\; F \in G^{m\times n}[a,b] \\ \text{there is an} \;\; \widetilde{F} \in S^{m\times n}[a,b] \;\; \text{such that} \;\; \|F - \widetilde{F}\|_\infty < \varepsilon. \end{cases}$$

2. Kurzweil-Stieltjes integral. The integrals which occur in this paper are the Perron-Stieltjes ones. For the original definition, see A.J. Ward [23] or S. Saks [14]. We use the equivalent summation definition due to J. Kurzweil [10] (cf. also e.g. [11] or [18]). We call this integral the Kurzweil-Stieltjes integral, in short the KS-integral. For the reader's convenience, let us recall the definition of the KS-integral.

Let $-\infty < a < b < \infty$. For a division D of the integral $[a,b]$, $D = \{\alpha_0, \alpha_1, \dots, \alpha_p\}$, and $\xi = (\xi_1, \xi_2, \dots, \xi_p) \in [a,b]^p$, the couple $P = (D, \xi)$ is called a partition of $[a,b]$ if

$$\alpha_{j-1} \le \xi_j \le \alpha_j \quad \text{for} \;\; j = 1,2,\dots,p.$$

The set of all partitions of the interval $[a,b]$ is denoted by $\mathcal{P}[a,b]$.

An arbitrary positive valued function $\delta\colon [a,b]\to(0,\infty)$ is called a gauge on $[a,b]$. Given a gauge δ on $[a,b]$, the partition

$$P=(D,\xi)=\big(\{\alpha_0,\alpha_1,\dots,\alpha_p\},\,(\xi_1,\xi_2,\dots,\xi_p)\big)\in\mathcal{P}[a,b]$$

is said to be δ-fine, if

$$[\alpha_{j-1},\alpha_j]\subset(\xi_j-\delta(\xi_j),\xi_j+\delta(\xi_j))\quad\text{for}\quad j=1,2,\dots,p.$$

For functions $f,\ g\colon [a,b]\to\mathbb{C}$ and a partition P of $[a,b]$,

$$P=(\{\alpha_0,\alpha_1,\dots,\alpha_p\},\,(\xi_1,\xi_2,\dots,\xi_p))\,,$$

we define

$$\Sigma(f\,\Delta g;P)=\sum_{i=1}^{p}f(\xi_i)\,[g(\alpha_i)-g(\alpha_{i-1})].$$

We say that $I\in\mathbb{C}$ is the KS-integral of f with respect to g from a to b if

$$\begin{cases}\text{for each }\ \varepsilon>0\ \text{ there is a gauge }\ \delta\ \text{ on }\ [a,b]\ \text{ such that}\\ |I-\Sigma(f\,\Delta g;P)|<\varepsilon\quad\text{for all }\ \delta\text{-fine }\ P\in\mathcal{P}[a,b].\end{cases}$$

In such a case we write

$$I=\int_a^b f\ \mathrm{d}[g]\quad\text{or}\quad I=\int_a^b f(t)\ \mathrm{d}[g(t)].$$

It is well-known that the KS-integral $\int_a^b f\ \mathrm{d}[g]$ exists provided $f\in G[a,b]$ and $g\in BV[a,b]$. Taking into account [19, Theorem 2.8] or [21, Theorem 2.3.8], we can formulate the following fundamental assertion.

THEOREM 2.1. *If $f,g\in G[a,b]$ and at least one of the functions f,g has a bounded variation on $[a,b]$, then the integral $\int_a^b f\ \mathrm{d}[g]$ exists. Furthermore,*

$$(2.1)\qquad\begin{cases}\left|\displaystyle\int_a^b f\ \mathrm{d}[g]\right|\le 2\,\big(|f(a)|\,+\,\mathrm{var}_a^b f\big)\ \|g\|_\infty\\ \qquad\qquad\qquad\text{if }\ f\in BV[a,b]\ \text{and } g\in G[a,b],\end{cases}$$

and

$$(2.2)\qquad\begin{cases}\left|\displaystyle\int_a^b f\ \mathrm{d}[g]\right|\le\|f\|_\infty\ \mathrm{var}_a^b\,g\\ \qquad\qquad\qquad\text{if }\ f\in G[a,b]\ \text{and } g\in BV[a,b].\end{cases}$$

Furthermore, if $f \in BV[a,b]$ and $g, g_k \in G[a,b]$ for $k \in \mathbb{N}$, then

$$\lim_{k\to\infty} \|g_k - g\|_\infty = 0 \quad \Rightarrow \quad \lim_{k\to\infty} \left\| \int_a^t f \, \mathrm{d}[g_k - g] \right\|_\infty = 0.$$

Further basic properties of the KS-integral with respect to scalar regulated functions were described in [19] (see also [21]).

Given an $m \times \ell$-matrix valued function F and an $\ell \times n$-matrix valued function G defined on $[a,b]$ and such that all the integrals

$$\int_a^b f_{i,k}(t) \, \mathrm{d}[g_{k,j}(t)] \quad (i = 1,2,\dots,m\,;\ k = 1,2,\dots,\ell;\ j = 1,2,\dots,n)$$

exist, the symbol

$$\int_a^b F(t) \, \mathrm{d}[G(t)] \quad \text{(or more simply} \quad \int_a^b F \, \mathrm{d}[G])$$

stands for the $m \times n$-matrix with the entries

$$\sum_{k=1}^{\ell} \int_a^b f_{i,k} \, \mathrm{d}[g_{k,j}], \quad i = 1,2,\dots,m\,,\ j = 1,2,\dots,n.$$

The extension of the results obtained in [19] or [21] for scalar real valued functions to complex vector valued or matrix valued functions is obvious and hence for the basic facts concerning integrals with respect to regulated functions we will refer to the corresponding assertions from [19] or [21].

The next natural assertion will be very useful for our purposes and, in our opinion, it is not available in literature.

LEMMA 2.2. *Let x, $x_k \in G^n[a,b]$, A, $A_k \in BV^{n\times n}[a,b]$ for $k \in \mathbb{N}$. Furthermore, let*

(2.3) $$\lim_{k\to\infty} \|x_k - x\|_\infty = 0,$$

(2.4) $$\alpha^* := \sup\{\operatorname{var}_a^b A_k \colon\ k \in \mathbb{N}\} < \infty$$

and

(2.5) $$\lim_{k\to\infty} \|A_k - A\|_\infty = 0.$$

Then

$$\lim_{k\to\infty} \left\| \int_a^t \mathrm{d}[A_k]\, x_k - \int_a^t \mathrm{d}[A]\, x \right\|_\infty = 0.$$

Proof. Let $\varepsilon > 0$ be given. By (1.1) and (2.3), we can find $u \in S^n[a,b]$ and $k_0 \in \mathbb{N}$ such that

$$\|x-u\|_\infty < \varepsilon, \quad \|x_k - u\|_\infty < \varepsilon \quad \text{and} \quad \|A_k - A\|_\infty < \varepsilon \quad \text{for} \;\; k \geq k_0.$$

Furthermore, since $\operatorname{var}_a^b u < \infty$, using (2.1) we can see that for $t \in [a,b]$ and $k \geq k_0$ the relations

$$\begin{aligned}
&\left|\int_a^t \mathrm{d}[A_k]\, x_k - \int_a^t \mathrm{d}[A]\, x\right| \\
&\quad = \left|\int_a^t \mathrm{d}[A_k]\,(x_k-u) + \int_a^t \mathrm{d}[A_k - A]\, u + \int_a^t \mathrm{d}[A]\,(u-x)\right| \\
&\quad \leq \alpha^* \varepsilon + 2\,\left(\operatorname{var}_a^b u\right)\,\varepsilon + \alpha^*\varepsilon = 2\,\left(\alpha^* + \operatorname{var}_a^b u\right)\,\varepsilon
\end{aligned}$$

hold, wherefrom our assertion immediately follows. ∎

3. Generalized linear differential equations. Let $A \in BV^{n\times n}[a,b]$, $g \in G^n[a,b]$ and $\widetilde{x} \in \mathbb{C}^n$. Consider an integral equation

$$x(t) = \widetilde{x} + \int_a^t \mathrm{d}[A(s)]\, x(s) + g(t) - g(a). \tag{3.1}$$

We say that a function $x\colon\ [a,b] \to \mathbb{C}^n$ is *a solution* of (3.1) on the interval $[a,b]$ if the integral

$$\int_a^b \mathrm{d}[A(s)]\, x(s)$$

has sense and equality (3.1) is satisfied for all $t \in [a,b]$. Equation (3.1) is usually called a *generalized linear differential equation*. Such equations with solutions having values in the space $\mathbb{R}^n$ of real n-vectors have been thoroughly investigated e.g. in the monographs [16] or [18]. In this section we will describe the basics needed later. Special attention is paid to the features whose extension to the complex case seems not to be so straightforward.

For our purposes the following property is crucial:

$$\det\left[I - \Delta^- A(t)\right] \neq 0 \quad \text{for each} \;\; t \in [a,b]. \tag{3.2}$$

(Recall that we put $\Delta^- A(a) = 0$.) Its importance is well illustrated by the next assertion which is a fundamental existence result for equation (3.1). Its proof follows immediately from [20, Proposition 2.5].

THEOREM 3.1. *Let $A \in BV^{n\times n}[a,b]$ satisfy* (3.2)*. Then, for each $\widetilde{x} \in \mathbb{C}^n$ and each $g \in G^n[a,b]$, equation* (3.1) *has a unique solution x on $[a,b]$ and $x \in G^n[a,b]$. Moreover, $x - g \in BV^n[a,b]$.*

Furthermore, analogously to [18, Theorem III.1.7] where $g \in BV^n[a,b]$, we have:

THEOREM 3.2. *Let* $A \in BV^{n\times n}[a,b]$ *satisfy* (3.2). *Then the relations*

$$c_A := \sup\{|[I - \Delta^- A(t)]^{-1}|\colon\ t \in [a,b]\} < \infty \tag{3.3}$$

and

$$|x(t)| \le c_A\,(|\widetilde{x}| + 2\,\|g\|_\infty)\,\exp(c_A\,\mathrm{var}_a^t A)\ \textit{for}\ t \in [a,b] \tag{3.4}$$

hold for each $\widetilde{x} \in \mathbb{C}^n$, $g \in G^n[a,b]$ *and each solution* x *of* (3.1) *on* $[a,b]$.

Proof. First, notice that for $t \in [a,b]$ such that $|\Delta^- A(t)| < \frac{1}{2}$ we have

$$\big|[I - \Delta^- A(t)]^{-1}\big| = \left|\sum_{k=1}^{\infty}(\Delta^- A(t))^k\right| \le \sum_{k=1}^{\infty}|\Delta^- A(t)|^k = \frac{1}{1 - |\Delta^- A(t)|} < 2.$$

Therefore, (3.3) follows easily from the fact that the set

$$\{t \in [a,b]\colon\ |\Delta^- A(t)| \ge \frac{1}{2}\}$$

has at most finitely many elements.

Now, let x be a solution of (3.1). Put $B(a) = A(a)$ and $B(t) = A(t-)$ for $t \in (a,b]$. Then, as in the proof of [18, Theorem III.1.7], we can deduce that $A - B \in BV^{n\times n}[a,b]$,

$$A(t) - B(t) = \Delta^- A(t) \text{ and } \int_a^t \mathrm{d}[A - B]\,x = \Delta^- A(t) \quad \text{for } \ t \in [a,b].$$

Consequently,

$$x(t) \;=\; [I - \Delta^- A(t)]^{-1}\left(\widetilde{x} + g(t) - g(a) + \int_a^t \mathrm{d}[B]\,x\right)$$

and

$$|x(t)| \;\le\; K_1 + K_2\int_a^t \mathrm{d}[h]\,|x| \quad \text{for } \ t \in [a,b],$$

where

$$K_1 = c_A\,(|\widetilde{x}| + 2\,\|g\|_\infty)\,, \quad K_2 = c_A \quad \text{and} \quad h(t) = \mathrm{var}_a^t B \quad \text{for } \ t \in [a,b].$$

The function h is nondecreasing and, since B is left-continuous on $(a,b]$, h is also left-continuous on $(a,b]$. Therefore we can use the generalized Gronwall inequality (see e.g. [18, Lemma I.4.30] or [16, Corollary 1.43]) to get the estimate (3.4).

Corollary 3.3. *Let $A \in BV^{n\times n}[a,b]$ satisfy* (3.2)*. Then for each $\widetilde{x} \in \mathbb{C}^n$, $g \in G^n[a,b]$ and each solution x of* (3.1) *on $[a,b]$, the estimate*

$$\operatorname{var}_a^b(x-g) \le c_A\,(\operatorname{var}_a^b A)\,(|\widetilde{x}| + 2\,\|g\|_\infty)\,\exp(c_A\,\operatorname{var}_a^b A)$$

is true, where c_A is defined by (3.3).

Proof. By (3.4), we have

$$\|x\|_\infty \le c_A\,(|\widetilde{x}| + 2\,\|g\|_\infty)\,\exp(c_A\,\operatorname{var}_a^b A).$$

Therefore

$$\begin{aligned}\operatorname{var}_a^b(x-g) &\le \big(\operatorname{var}_a^b A\big)\,\|x\|_\infty \\ &\le c_A\,\big(\operatorname{var}_a^b A\big)\,(|\widetilde{x}| + 2\,\|g\|_\infty)\,\exp(c_A\,\operatorname{var}_a^b A).\end{aligned}$$ ∎

Lemma 3.4. *Let $A \in BV^{n\times n}[a,b]$ satisfy* (3.2) *and let c_A be defined by* (3.3)*. Then*

$$c_A = \Big(\inf\Big\{\big|\big[I - \Delta^- A(t)\big]\;x\big|:\ t \in [a,b],\ x \in \mathbb{C}^n,\ |x| = 1\Big\}\Big)^{-1}. \tag{3.5}$$

Proof. We have

$$\begin{aligned} c_A &= \sup\big\{|[I-\Delta^-A(t)]^{-1}|:\ t\in[a,b]\big\} \\ &= \sup\left\{\frac{|[I-\Delta^-A(t)]^{-1}||[I-\Delta^-A(t)]\,x|}{|[I-\Delta^-A(t)]\,x|} : t\in[a,b],\ x\in\mathbb{C}^n,\ |x|=1\right\} \\ &\ge \sup\left\{\frac{|x|}{|[I-\Delta^-A(t)]\,x|}:\ t\in[a,b],\ x\in\mathbb{C}^n,\ |x|=1\right\} \\ &= \sup\left\{\frac{1}{|[I-\Delta^-A(t)]\,x|}:\ t\in[a,b],\ x\in\mathbb{C}^n,\ |x|=1\right\} \\ &= \Big(\inf\big\{|[I-\Delta^-A(t)]\,x|:\ t\in[a,b],\ x\in\mathbb{C}^n,\ |x|=1\big\}\Big)^{-1}. \end{aligned}$$

Thus, it remains to prove that the inequality

$$c_A \le \Big(\inf\Big\{\big|\big[I - \Delta^- A(t)\big]\;x\big|:\ t \in [a,b],\ x \in \mathbb{C}^n,\ |x| = 1\Big\}\Big)^{-1} \tag{3.6}$$

is true, as well. To this aim, first let us notice that for each $t \in [a,b]$ there is a $z \in \mathbb{C}^n$ such that $|z| = 1$ and

$$\big|[I - \Delta^- A(t)]^{-1}\big| = \big|[I - \Delta^- A(t)]^{-1}\,z\big|. \tag{3.7}$$

Indeed, let $t \in [a,b]$ and let $B = [I - \Delta^- A(t)]^{-1}$. Let $i_0 \in \{1,2,\dots,n\}$ be such that $|B| = \sum_{j=1}^n |b_{i_0,j}|$ and let $z \in \mathbb{C}^n$ be such that $z_i = \operatorname{sgn}(b_{i_0,2})$ for $i = 1,2,\dots,n$. Then $|z| = 1$. Furthermore,

$$\begin{aligned} |B\,z| &= \max_{i=1,2,\dots,n} \sum_{j=1}^n |b_{i,j}\, z_j| = \max_{i=1,2,\dots,n} \sum_{j=1}^n |b_{i,j}\, \operatorname{sgn}(b_{i_0,j})| \\ &\le \max_{i=1,2,\dots,n} \sum_{j=1}^n |b_{i,j}| = |B|. \end{aligned}$$

On the other hand, we have

$$|B| = \sum_{j=1}^n |b_{i_0,j}| = \left| \sum_{j=1}^n \operatorname{sgn}(b_{i_0,j})\, b_{i_0,j} \right| \le |B\,z|.$$

Therefore, we can conclude that (3.7) is true.

Now, due to (3.2), there is $w \in \mathbb{C}^n$ such that $z = [I - \Delta^- A(t)]\,w$. Inserting this instead of z into (3.7) and having in mind that $|z| = 1$, we get

$$\begin{aligned} \left| \big[I - \Delta^- A(t)\big]^{-1} \right| &= \frac{|[I - \Delta^- A(t)]^{-1}\,[I - \Delta^- A(t)]\,w|}{|[I - \Delta^- A(t)]\,w|} \\ &= \frac{|w|}{|[I - \Delta^- A(t)]\,w|} = \frac{1}{\left|[I - \Delta^- A(t)]\,\left(\frac{w}{|w|}\right)\right|} \\ &\le \sup\left\{ \frac{1}{|[I - \Delta^- A(t)]\,x|} :\, x \in \mathbb{C}^n,\ |x| = 1 \right\}. \end{aligned}$$

It follows that

$$\begin{aligned} c_A &\le \sup\left\{ \frac{1}{|[I - \Delta^- A(t)]\,x|} :\, t \in [a,b],\ x \in \mathbb{C}^n,\ |x| = 1 \right\} \\ &= \big(\inf\{|[I - \Delta^- A(t)]\,x| :\, t \in [a,b],\ x \in \mathbb{C}^n,\ |x| = 1\big)^{-1}, \end{aligned}$$

i.e. (3.6) is true. This completes the proof. □

4. Continuous dependence of a solution on a parameter. The main result of this paper is provided by Theorem 4.1. It concerns continuous dependence of solutions of generalized linear differential equations on a parameter and generalizes the result due to M. Ashordia [2, Theorem 1]. Unlike [2] and [3], we do not utilize the variation-of-constants formula and therefore we need not assume that, in addition to (3.2), also the condition

$$\det[I + \Delta^+ A(t)] \ne 0 \quad \text{for all} \quad t \in [a,b]$$

is satisfied. Furthermore, both the nonhomogeneous part of the equation and the solutions may be regulated functions (not necessarily of bounded variation).

THEOREM 4.1. *Let* A, $A_k \in BV^{n\times n}[a,b]$, g, $g_k \in G^n[a,b]$, $\widetilde{x}$, $\widetilde{x}_k \in \mathbb{C}^n$ *for* $k \in \mathbb{N}$. *Assume* (2.4), (2.5), (3.2),

$$\lim_{k\to\infty} \|g_k - g\|_\infty = 0 \tag{4.1}$$

and

$$\lim_{k\to\infty} \widetilde{x}_k = \widetilde{x}. \tag{4.2}$$

Then equation (3.1) *has a unique solution* x *on* $[a,b]$. *Furthermore, for each* $k \in \mathbb{N}$ *sufficiently large there exists a unique solution* x_k *on* $[a,b]$ *to the equation*

$$x(t) = \widetilde{x}_k + \int_a^t \mathrm{d}[A_k(s)]\, x(s) + g_k(t) - g_k(a) \tag{4.3}$$

and

$$\lim \|x_k - x\|_\infty = 0. \tag{4.4}$$

Proof. *Step* 1. As in the first part of the proof of [2, Theorem 1], we can show that there is a $k_1 \in \mathbb{N}$ such that

$$\det[I - \Delta^- A_k(t)] \neq 0 \quad \text{on} \quad [a,b]$$

holds for all $k \geq k_1$. In particular, (4.3) has a unique solution x_k for $k \geq k_1$.

Step 2. For $k \geq k_1$, put

$$c_{A_k} := \sup\{\big|[I - \Delta^- A_k(t)]^{-1}\big| \colon t \in (a,b]\}.$$

Then, by Lemma 3.4, we have

$$\begin{aligned} (c_{A_k})^{-1} &= \inf\left\{\big|\big[I - \Delta^- A_k(t)\big]\, x\big| \colon t \in [a,b],\, x \in \mathbb{C}^n,\, |x| = 1\right\} \\ &\geq \inf\left\{\big|\big[I - \Delta^- A(t)\big]\, x\big| \colon t \in [a,b],\, x \in \mathbb{C}^n,\, |x| = 1\right\} \\ &\quad - \sup\left\{\big|\big[\Delta^-(A_k(t) - A(t))\big]\, x\big| \colon t \in [a,b],\, x \in \mathbb{C}^n,\, |x| = 1\right\}. \end{aligned}$$

Since, due to assumption (2.5),

$$\lim_{k\to\infty} \|\Delta^- A_k - \Delta^- A\|_\infty = 0,$$

we conclude that there is a $k_0 \geq k_1$ such that

$$(c_{A_k})^{-1} \geq (c_A)^{-1} - (2\,c_A)^{-1} = (2\,c_A)^{-1} \quad \text{for} \quad k \geq k_0.$$

To summarize:

$$c_{A_k} \leq 2\,c_A < \infty \quad \text{for} \quad k \geq k_0. \tag{4.5}$$

Step 3. Set $w_k = (x_k - g_k) - (x - g)$. Then,

$$w_k(t) = \widetilde{w}_k + \int_a^t \mathrm{d}[A_k]\,w_k + h_k(t) - h_k(a) \quad \text{for} \quad k \geq k_0 \text{ and } t \in [a,b],$$

where $\widetilde{w}_k = (\widetilde{x}_k - g_k(a)) - (\widetilde{x} - g(a))$ and

$$h_k(t) = \int_a^t \mathrm{d}[A_k - A]\,(x-g) + \left(\int_a^t \mathrm{d}[A_k]\,g_k - \int_a^t \mathrm{d}[A]\,g\right).$$

By (4.1) and (4.2) we can see that

$$\lim_{k\to\infty} \widetilde{w}_k = 0. \tag{4.6}$$

Furthermore, since $x - g \in BV^n[a,b]$ and $\lim_{k\to\infty} \|A_k - A\|_\infty = 0$, by Theorem 2.1 we have

$$\lim_{k\to\infty} \left\| \int_a^t \mathrm{d}[A_k - A]\,(x-g) \right\|_\infty = 0$$

and, by Lemma 2.2,

$$\lim_{k\to\infty} \int_a^t \mathrm{d}[A_k]\,g_k = \int_a^t \mathrm{d}[A]\,g.$$

To summarize,

$$\lim_{k\to\infty} \|h_k\|_\infty = 0. \tag{4.7}$$

On the other hand, applying Theorem 3.2 and taking into account the relation (4.5), we get

$$\|w_k\|_\infty \leq 2\,c_A\,(|\widetilde{w}_k| + 2\,\|h_k\|_\infty)\,\exp(2\,c_A\,\alpha^*) \text{ for } t \in [a,b] \text{ and } k \geq k_0,$$

wherefrom, by virtue of (4.6) and (4.7), the relation

$$\lim_{k\to\infty} \|w_k\|_\infty = 0$$

follows. Finally, having in mind assumptions (4.1) and (4.2), we conclude that the relation

$$\lim \|x_k - x\|_\infty = 0$$

is true, as well. This completes the proof.

Now, we want to present an example of application of Theorem 4.1. To this aim let us introduce some additional notation.

NOTATION 4.2. For a $k \in \mathbb{N}$ denote by D_k a division of $[a,b]$ given by

$$\text{(4.8)} \qquad \begin{cases} D_k = \{\alpha_0^k, \alpha_1^k, \dots, \alpha_m^k\}, & \text{where } \; m = 2^k \text{ and} \\ \alpha_i^k = a + \dfrac{i\,(b-a)}{2^k} & \text{for } \; i = 0, 1, \dots, m. \end{cases}$$

For given $A \in BV^{n \times n}[a,b]$, $g \in G^n[a,b]$ and $k \in \mathbb{N}$, we define

$$\text{(4.9)} \qquad A_k(t) = \begin{cases} A(t) & \text{if } t \in D_k, \\ A(\alpha_{i-1}^k) + \dfrac{A(\alpha_i^k) - A(\alpha_{i-1}^k)}{\alpha_i^k - \alpha_{i-1}^k}\,(t - \alpha_{i-1}^k) & \text{if } t \in (\alpha_{i-1}^k, \alpha_i^k) \end{cases}$$

and

$$\text{(4.10)} \qquad g_k(t) = \begin{cases} g(t) & \text{if } \; t \in D_k, \\ g(\alpha_{i-1}^k) + \dfrac{g(\alpha_i^k) - g(\alpha_{i-1}^k)}{\alpha_i^k - \alpha_{i-1}^k}\,(t - \alpha_{i-1}^k) & \text{if } \; t \in (\alpha_{i-1}^k, \alpha_i^k). \end{cases}$$

Then, obviously, $\{A_k\} \subset AC^{n \times n}[a,b]$ and $\{g_k\} \subset AC^n[a,b]$. Moreover, we have

LEMMA 4.3. *Let the sequences* $\{D_k\}$ *and* $\{A_k\}$ *be defined by* (4.8) *and* (4.9), *respectively. Then*

$$\operatorname{var}_a^b A_k \le \operatorname{var}_a^b A \quad \textit{for all} \;\; k \in \mathbb{N}.$$

Proof. Since

$$\operatorname{var}_{\alpha_{\ell-1}^k}^{\alpha_\ell^k} A_k = \big|A(\alpha_\ell^k) - A(\alpha_{\ell-1}^k)\big| \le \operatorname{var}_{\alpha_{\ell-1}^k}^{\alpha_\ell^k} A$$

for each $k \in \mathbb{N}$ and $\ell = 1, 2, \dots, 2^k$, we have

$$\operatorname{var}_a^b A_k = \sum_{\ell=1}^{2^k} \operatorname{var}_{\alpha_{\ell-1}^k}^{\alpha_\ell^k} A_k \le \sum_{\ell=1}^{2^k} \operatorname{var}_{\alpha_{\ell-1}^k}^{\alpha_\ell^k} A = \operatorname{var}_a^b A. \qquad \square$$

Equations (4.3) with A_k and g_k given by (4.9) and (4.10) are just initial value problems for linear ordinary differential systems

$$\text{(4.11)} \qquad x' = A_k'(t)\,x + g_k'(t), \quad x(a) = \widetilde{x}_k.$$

In this view, the next theorem says that the solutions of (3.1) can be uniformly approximated by solutions of linear ordinary differential equations provided the functions A and g are continuous.

THEOREM 4.4. *Assume that* $A \in BV^{n\times n}[a,b] \cap C^{n\times n}[a,b]$ *and* $g \in C^n[a,b]$. *Let* $\widetilde{x}$ *and* $\widetilde{x}_k \in \mathbb{C}^n$, $k \in \mathbb{N}$, *be such that* (4.2) *holds. Furthermore, let the sequence* $\{D_k\}$ *of divisions of the interval* $[a,b]$ *be given by* (4.8) *and let sequences* $\{A_k\} \subset AC^{n\times n}[a,b]$, $\{g_k\} \subset AC^n[a,b]$ *be defined by* (4.9) *and* (4.10), *respectively.*

Then equation (3.1) *has a unique solution* x *on* $[a,b]$. *Furthermore, for each* $k \in \mathbb{N}$, *equation* (4.3) *has a solution* x_k *on* $[a,b]$ *and* (4.4) *holds.*

Proof. Step 1. Since A is uniformly continuous on $[a,b]$, we have:

$$\left\{\begin{array}{l} \text{for each } \ \varepsilon > 0 \text{ there is a } \delta > 0 \text{ such that} \\ |A(t) - A(s)| < \frac{\varepsilon}{2} \\ \text{holds for all } \ t,s \in [a,b] \ \text{ such that } \ |t-s| < \delta\,. \end{array}\right. \tag{4.12}$$

Let $\frac{1}{2^{k_0}} < \delta$ and let t be an arbitrary point of $[a,b]$. Furthermore, let

$$\alpha_{\ell-1}, \alpha_\ell \in P_{k_0} = \{\alpha_0, \alpha_1, \dots, \alpha_{p_{k_0}}\} \text{ and } t \in [\alpha_{\ell-1}, \alpha_\ell].$$

Then

$$|\alpha_\ell - \alpha_{\ell-1}| = \frac{1}{2^{k_0}} < \delta$$

and, according to (4.8), (4.9) and (4.12), we get for $k \geq k_0$

$$\begin{aligned} |A_k(t) - A(t)| &= |A_k(t) - A_k(\alpha_{\ell-1}) + A(\alpha_{\ell-1}) - A(t)| \\ &\leq \left| A(\alpha_{\ell-1}) + [A(\alpha_\ell) - A(\alpha_{\ell-1})] \left[\frac{t - \alpha_{\ell-1}}{\alpha_\ell - \alpha_{\ell-1}}\right] - A(\alpha_{\ell-1}) \right| + \frac{\varepsilon}{2} \\ &= |A(\alpha_\ell) - A(\alpha_{\ell-1})| \left[\frac{t - \alpha_{\ell-1}}{\alpha_\ell - \alpha_{\ell-1}}\right] + \frac{\varepsilon}{2} \leq \varepsilon. \end{aligned}$$

As k_0 was chosen independently of t, we can conclude that (2.5) is true.

Step 2. Analogously we can show that (4.1) holds for $\{g_k\}$ and g.

Step 3. By Lemma 4.3, relation (2.4) holds. Moreover, A and A_k, $k \in \mathbb{N}$, are continuous. Therefore, by Theorem 3.1, equations (3.1) and (4.3) have unique solutions for each $k \in \mathbb{N}$ and we can complete the proof using Theorem 4.1. ∎

REMARK 4.5. Piecewise linear approximations (4.9) and (4.10) of A, g can be constructed also in the general case when A, g need not be continuous. Some results in this direction for homogeneous equations were obtained by M. Pelant in his unpublished thesis [13]. If A and/or g are not continuous on $[a, b]$ the situation is rather more complicated and, in general, the solutions of (4.3) do not converge to the solution of (3.1), but to solutions of a somewhat modified equation. Nevertheless, using matrix logarithms, one can always define a sequence (4.3) of approximating initial value problems for ODE's whose solutions tend to the solution of (3.1). Of course, even if A and g are real valued, such approximating piecewise linear functions will be, in general, complex valued. We intend to expose this topic in more details later.

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