Critical imbeddings with multivariate rearrangements

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Abstract

Our concern in this paper lies with imbeddings of general spaces of Besov and Lizorkin-Triebel type with dominating mixed derivatives in the first critical case. We employ multivariate exponential Orlicz and Lorentz-Orlicz spaces in the role of targets. We study basic properties of the target spaces, in particular, we compare them with usual exponential spaces, showing that in this case the multivariate clones are in fact better adapted to the character of smoothness of the imbedded spaces. Then we prove sharp limiting imbedding theorems and establish estimates for the multivariate growth envelope functions.

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1   Introduction

Various aspects of critical imbeddings have been intensively studied in numerous papers in last years, inspired with the celebrated imbedding theorem [Tru] and its later generalizations.

Here we shall be concerned with the limiting imbeddings of the spaces $S_{pq}^r B(\mathbb{R}^N \times \mathbb{R}^N)$ and $S_{pq}^r F(\mathbb{R}^N \times \mathbb{R}^N)$ in the so called first critical case, that is, when $rp = N$, where $2N$ is the dimension of the underlying Euclidean space. We shall prove the refinement corresponding to the Brézis-Wainger

theorem [BW]. There is a rich literature especially in last years, devoted to limiting imbeddings and we shall make no attempt here to make an account of either the history or the recent research in this area. We refer to [Tri2] for a detailed survey with further references.

In [KS1] and [KS2] we considered critical imbeddings of spaces with dominating mixed derivatives and various clones of special reduced imbeddings. The current paper can be considered as a free continuation to [KS1]; in [KS2] we studied imbeddings of these spaces into spaces of almost Lipschitz continuous functions (that is, in the second critical case, $rp = N + p$). Note that some of these results have been quite recently generalized to the case of vector-valued function spaces without the *UMD* property(see e.g. [KS3]).

The target spaces for the critical imbeddings in the first critical case are the exponential Orlicz spaces or their Orlicz-Lorentz refinements. In both cases the targets are characterized with help of non-increasing rearrangements. If the imbedded spaces have more complicated structure as in the case of dominating mixed derivatives, then there is a natural question whether the target spaces can be improved or refined considering other spaces in the role of targets.

Recall that for a measurable $f : \mathbb{R}^M \to \mathbb{C}$ the *non-increasing rearrangement of f* is the function

$$f^*(t) = \inf\{\lambda > 0 : \mu_f(\lambda) \leq t\},$$

where

$$\mu_f(\lambda) = |\{x \in \mathbb{R}^M : |f(x)| > \lambda\}|.$$

One of the tools used recently for refinements of subcritical Sobolev imbeddings are multivariate rearrangements (see [K]), which turn out to be appropriate for our goal as well, and we shall prove the corresponding imbedding theorem in the first critical case (Theorem 3.1).

The concept of the growth envelope function and of the growth envelope, introduced by Triebel and Haroske (see [H], [Tri2]) show that if we use the usual rearrangements, then the use of the exponential Orlicz and exponential Lorentz-Orlicz spaces in the role of the target is quite natural: the "optimal functions" permit the growth just fitting the biggest possible growth of these standard rearrangements. A closer look at the critical imbeddings in [KS1] and considerations concerning the sharpness of the imbeddings there reveal in turn that one can expect better imbeddings when passing from usual rearrangements to the multivariate ones because they are better adopted to measure the unboundedness of functions with dominating mixed smoothness properties.

We shall generally adhere to common notations, more specific notions will be defined in the following. The constant c in various estimates is

a "generic" constant: its values might change from line to line but it is independent of particular choice of functions appearing in the formulas.

2 Rearrangements, extrapolation, spaces

2.1 Rearrangements

The multivariate rearrangements were studied in several papers; even more general questions of Banach spaces with mixed norms were considered. For the Lebesgue and Lorentz spaces in this spirit we refer to Blozinski [B], Kolyada [K] and Yatsenko [Ya].

Let us fix notions and notations. Let $f : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{C}$ be measurable. Put

$$
\begin{aligned}
(R_1 f)(s, y) &= [f(., y)]^*(s), & s > 0, \ , y \in \mathbb{R}^N, \\
(R_2 f)(x, t) &= [f(x, .)]^*(t), & x \in \mathbb{R}^N, \ t > 0,
\end{aligned}
\tag{2.1}
$$

and define the *multivariate non-increasing rearrangement of f* by

$$
(R_{12}) f(s, t) = (R_2 \circ R_1 f)(s, t) = [(R_1 f)(s, .)]^*(t), \qquad s, t > 0. \tag{2.2}
$$

Sometimes it is more convenient to work with the averages of the above operators. Let

$$
\begin{aligned}
(A_1 f)(s, y) &= \frac{1}{s} \int_0^s (R_1 f)(u, y) \, du, & s > 0, \ y \in \mathbb{R}^N, \\
(A_2 f)(x, t) &= \frac{1}{t} \int_0^t (R_2 f)(x, v) \, dv, & x \in \mathbb{R}^N, \ t > 0,
\end{aligned}
$$

and define the *multivariate average of f* by

$$
\begin{aligned}
(A_{12} f)(s, t) &= (A_2 \circ A_1 f)(s, t) \\
&= \frac{1}{t} \int_0^t (R_2(A_1 f))(s, v) \, dv \\
&= \frac{1}{t} \int_0^t R_2 \left(\frac{1}{s} \int_0^s (R_1 f)(u, .) \, du \right)(v) \, dv \\
&= \frac{1}{st} \int_0^t R_2 \left(\int_0^s (R_1 f)(u, .) \, du \right)(v) \, dv, \qquad s, t > 0.
\end{aligned}
\tag{2.3}
$$

One can consider the multivariate rearrangement R_{21} and the multivariate average A_{21} defined in an obvious way.

In accordance with (2.1) we should write extra parentheses around R_2 and its inner function on the third and fourth line of the last display; since

no misunderstanding can occur we shall often use such a little abuse of the notations to simplify the formulae.

On the other hand one can consider the operator

$$f^{**}(s,t) = \frac{1}{st} \int_0^t \int_0^s (R_{12}f)(\sigma,\tau) \, d\sigma d\tau$$

and one can ask about the relation between these various expressions. Plainly

$$(R_{12}f)(s,t) \leq (A_{12}f)(s,t) \leq f^{**}(s,t). \qquad (2.4)$$

By virtue of the Fubini theorem and properties of the non-increasing rearrangements the function $R_{12}f$ is equimeasurable with $R_1 f$ and the latter is in turn equimeasurable with f. Hence

$$\int_0^\infty \int_0^\infty (R_{12})f(s,t) \, dsdt = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f(x,y) \, dxdy$$
$$= \int_0^\infty R_1 \left(\int_0^\infty (R_2 f)(x,.) \, dt \right)(s) \, ds. \qquad (2.5)$$

Analogous equality hold for R_{21}.

The superposition of the rearrangements in (2.2) and of the averages in (2.3) is worth of a comment: It should be pointed out that the functions $R_{12}f$ and $A_{12}f$ are non-increasing with respect to both variables. This means the following: If for a fixed x we rearrange a function $f = f(x,y)$ with respect to the variable y, then the result is the non-increasing function $t \mapsto (R_2 f)(x,t)$, $t > 0$. Now, if we rearrange $(R_2 f)(x,t)$ as a function of the variable x for every fixed t, it can generally happen that the resulting function will be rearranged in a different way in the direction of the axis t, nevertheless, the monotonicity in t will be preserved! This remarkable property of multivariate rearrangements can be seen and understood well for instance considering a rearrangement of a matrix, where the application of R_{12} consists in subsequent non-increasing rearrangements of the elements in columns and rows. Hence

$$R_2(R_{12}f)(s,t) = R_1(R_{12}f)(s,t) = (R_{12}f)(s,t), \qquad s,t,>0. \qquad (2.6)$$

Further, one can easily check that

$$(R_{12}(f+g))(s_1+s_2, t_1+t_2) \leq (R_{12}f)(s_1,t_1) + (R_{12}g)(s_2,t_2).$$

The operator A_{12} is subadditive whereas R_{12} and f^{**} are not.

The symbol $L^{p,q}$ ($1 \leq p,q \leq \infty$) will denote the usual *Lorentz spaces* equipped with the (quasi)norm

$$\|f|L^{p,q}\| = \left(\frac{q}{p} \int_0^\infty \left[t^{1/p} f^*(t) \right] \frac{dt}{t} \right)^{1/q}$$

(with usual modifications if $p = \infty$ or $q = \infty$).

The *multivariate Lorentz spaces* can be introduced as follows: Consider for simplicity the product $\mathbb{R}^{2N} = \mathbb{R}^N \times \mathbb{R}^N$. Then for $1 < p, q < \infty$ we define the (quasi)norm in $L_R^{p,q} = L_R^{p,q}(\mathbb{R}^N \times \mathbb{R}^N)$ by

$$\|f|L_R^{p,q}\| = \left(\frac{q}{p} \int_0^\infty \frac{q}{p} \int_0^\infty \left[(st)^{1/p} (R_{12}f)(s,t) \right]^q \frac{ds}{s} \frac{dt}{t} \right)^{1/q}. \qquad (2.7)$$

Similarly one can use R_{21}. Relations between $L_R^{p,q}$ and the standard Lorentz spaces $L^{p,q}$ were studied in [Ya] where it was shown, for instance, that

$$L_R^{p,q} \hookrightarrow L^{p,q} \quad \text{if } q < p. \qquad (2.8)$$

The domain $\mathbb{R}^N \times \mathbb{R}^N$ will be considered to simplify the notations and the exposition. We shall not pursue the technicalities connected with the more general domain $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ and the corresponding spaces with mixed norms.

Let us recall the sharp form of Calderón's lemma (see [SW, Chapter 5]), stating that if $p \geq q$, then $\|f|L_p\| \leq \|f|L^{p,q}\|$, i.e. the norm of the corresponding imbedding does not exceed one. This combined with (2.8) and with the equimeasurability properties (2.5) immediately yields

$$\|f|L_p(\mathbb{R}^{2N})\| \leq c\|f|L_R^{p,q}(\mathbb{R}^N \times \mathbb{R}^N)\| \qquad \text{if } q < p,$$

where c is the norm of the imbedding in (2.8). Applying the sharp version of Calderón's lemma in a more careful and elaborate way the last estimate can be improved: There holds

$$\|f|L_p(\mathbb{R}^{2N})\| \leq \|f|L_R^{p,q}(\mathbb{R}^N \times \mathbb{R}^N)\| \qquad \text{if } q < p, \qquad (2.9)$$

that is, the norm of the imbedding $L_R^{p,q}(\mathbb{R}^N \times \mathbb{R}^N) \hookrightarrow L_p(\mathbb{R}^{2N})$ does not exceed one. Indeed, let $f = f(x,y)$, $x, y \in \mathbb{R}^N$, and $q < p$. Then

$$\|f|L_p(\mathbb{R}^N \times \mathbb{R}^N)\| \leq \left\| \left(\frac{q}{p} \int_0^\infty \left[s^{1/p} R_1 f(s,y) \right]^q \frac{ds}{s} \right)^{1/q} \Big| L_p(\mathbb{R}^N) \right\|$$
$$\text{(by Calderón's lemma)}$$
$$= \left\| \frac{q}{p} \int_0^\infty \left[s^{1/p} R_1 f(s,y) \right]^q \frac{ds}{s} \Big| L_{p/q}(\mathbb{R}^N) \right\|^{1/q}$$
$$\leq \left(\frac{q}{p} \int_0^\infty \left(\int_{\mathbb{R}^N} \left[s^{1/p} R_1 f(s,y) \right]^p dy \right)^{q/p} \frac{ds}{s} \right)^{1/q}$$
$$\text{(by generalized Minkowki's inequality)}$$
$$= \left(\frac{q}{p} \int_0^\infty \|s^{1/p} R_1 f(s,\cdot)|L_p(\mathbb{R}^N)\|^q \frac{ds}{s} \right)^{1/q}$$

$$\leq \left(\frac{q}{p} \int_0^\infty \left(\frac{q}{p} \int_0^\infty \left[s^{1/p} R_{12} f(s,t) t^{1/p} \right]^q \frac{dt}{t} \right) \frac{ds}{s} \right)^{1/q}$$

(by Calderón's lemma)

$$= \| f | L_R^{p,q}(\mathbb{R}^N \times \mathbb{R}^N) \|$$

and (2.9) is proved.

Since we shall work with atomic decompositions it will be useful to introduce further shorthands. Let $b : \mathbb{Z}^N \times \mathbb{Z}^N \to \mathbb{C}$ and denote the elements of b by $b(m,n)$. Put

$$(R_1 b)(\alpha, n) = [b(.,n)]^*(\alpha), \qquad \alpha \in \mathbb{N}, \ n \in \mathbb{Z}^N,$$

$$(R_2 b)(m, \beta) = [b(m,.)]^*(\beta), \qquad m \in \mathbb{Z}^N, \ \beta \in \mathbb{N},$$

$$(R_{12} b)(\alpha, \beta) = (R_2 \circ R_1 b)(\alpha, \beta) = [(R_1 b)(\alpha, .)]^*(\beta), \quad \alpha, \beta \in \mathbb{N},$$

$$(A_1 b)(\mu, n) = \frac{1}{\mu} \sum_{\alpha=1}^{\mu} (R_1 b)(\alpha, n), \qquad \mu \in \mathbb{N}, \ n \in \mathbb{Z}^N,$$

$$(A_2 b)(m, \nu) = \frac{1}{\nu} \sum_{\beta=1}^{\nu} (R_2 b)(m, \beta), \qquad m \in \mathbb{Z}^N, \ \nu \in \mathbb{N},$$

and

$$b^{**}(\mu, \nu) = \frac{1}{\mu\nu} \sum_{\alpha=1}^{\mu} \sum_{\beta=1}^{\nu} (R_{12} b)(\alpha, \beta), \qquad \mu \in \mathbb{N}, \ \nu \in \mathbb{N}.$$

Next we collect a couple of inequalities, which will be useful in the sequel. The inequalities in the following Lemma are standard and can be found e.g. in the classical monograph [HLP].

Lemma 2.1 (Hardy's discrete inequality). *Let $1 < q \leq \infty$ and let (b_κ) or $(b_{\kappa,\lambda})$, $\kappa, \lambda \in \mathbb{N}$, be sequences of non-negative real numbers. Then*

$$\sum_{k=1}^{\infty} \left(\frac{1}{k} \sum_{\kappa=1}^{k} b_\kappa \right)^q \leq c \sum_{\kappa=1}^{\infty} b_\kappa^q$$

and

$$\sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} \left(\frac{1}{k\ell} \sum_{\kappa=1}^{k} \sum_{\lambda=1}^{\ell} b_{\kappa,\lambda} \right)^q \leq c \sum_{\kappa=1}^{\infty} \sum_{\lambda=1}^{\infty} b_{\kappa,\lambda}^q$$

(with appropriate modification for $q = \infty$).

A consequence of the previous Lemma is the following maximal inequality, which holds for any sequence $b = (b^{m,n})$, $m, n \in \mathbb{Z}^N$,

$$\sum_{\mu=1}^{\infty} \sum_{\nu=1}^{\infty} [b^{**}(\mu, \nu)]^q \leq c \sum_{\alpha=1}^{\infty} \sum_{\beta=1}^{\infty} (R_{12} b)(\alpha, \beta)^q = c \sum_{m \in \mathbb{Z}^N} \sum_{n \in \mathbb{Z}^N} |b^{(m,n)}|^q.$$

2.2   Extrapolation

In the following we shall need various refinements of Orlicz spaces. For the purpose of a unified terminology and notations we give a short survey and several definitions. To avoid unnecessary technicalities (arising from considering spaces on sets with infinite measure) we tacitly shall assume that the spaces in the sequel are defined on some open set Ω in $\mathbb{R}^N$, $|\Omega| = 1$. We shall suppose that the reader is familiar with *Orlicz spaces* L^Φ, where Φ is a Young function. Important in applications are the *Zygmund spaces*, $L^p(\log L)^a$, i.e. the Orlicz spaces with the generating Young function $\Phi(t) = t^p \log(1 + t)^a$, $1 \le p < \infty$, $a \in \mathbb{R}$. Note that the Orlicz spaces are rearrangement invariant and that $f \in L^p(\log L)^a$ is equivalent with $\int_0^1 [f^*(t)]^p (\log e/t)^a \, dt < \infty$ (see e.g. [BR] or [BS]). Hence if $p = \infty$, it is well justified to consider $L^\infty(\log L)^a$ $(a < 0)$ as the space with the finite quasinorm $\sup_{0<t\le1} f^*(t)(\log e/t)^{1/a}$. This also naturally agrees with the extrapolation properties (2.11) and (2.12) below so that in this case we get the exponential Orlicz space $L^{\exp t^{-a}}$.

In [BR] there were studied the so called *Lorentz-Zygmund spaces* $L^{p,q;a}$, $1 \le p, q \le \infty$, $a \in \mathbb{R}$, which—in this notations—include the Zygmund spaces $L^p(\log L)^a$ $(1 \le p < \infty)$, the exponential Orlicz spaces $L^{\exp t^a}$, $a > 0$ (which are nothing but the Zygmund spaces $L^\infty(\log L)^{-1/a}$) and also, for $a = 0$, the Lorentz spaces $L^{p,q}$. The general formula for the (quasi)norm in the space $L^{p,q;a}$, $1 \le p \le \infty$, $1 \le q \le \infty$ reads

$$\|f|L^{p,q;a}\| = \left(\frac{q}{p} \int_0^1 \left[t^{1/p} f^*(t)(\log e/t)^a \right]^q \, dt/t \right)^{1/q}$$

with appropriate modifications for $p = \infty$ or $q = \infty$: If $1 \le p < \infty$ and $q = \infty$, then we get the weak spaces with the quasinorm $\sup_{0<t\le1} t^{1/p} f^*(t)(\log e/t)^a$. If $p = \infty$, $1 \le q < \infty$, and $a \ne 0$, then we have the space $L^{\infty,q;-a}$ with the norm

$$\left(\int_0^1 \left[f^*(t) \, (\log e/t)^a \right]^q \, dt/t \right)^{1/q}. \tag{2.10}$$

The case $a = -1$ will be of particular importance for us; this is the case of the exponential Orlicz-Lorentz space, sometimes called the BW_q-*space* because of its prominent role in critical imbeddings—we refer to the celebrated paper [BW]. See also comments later in this section.

To have a more complete picture we would like to recall several facts about extrapolation of Lebesgue and Lorentz spaces. It is a "folklore fact" that the space $L^{\exp t^q}$, i.e. the Orlicz space with the generating function $t \mapsto \exp t^q - 1$ can be characterized as the space of all functions f which

satisfy

$$\sup_{k \in \mathbb{N}} \frac{\|f|L_k\|}{k^{1/q}} < \infty \tag{2.11}$$

or

$$\sup_{0 < t < 1} \frac{f^*(t)}{(\log e/t)^{1/q}} < \infty \tag{2.12}$$

or

$$\sup_{0 < t < 1} \frac{\|f|L^{k,\infty}\|}{k^{1/q}} < \infty$$

(further refinements in terms of Lorentz spaces $L^{k,r}$ are possible—see e.g. [EK]). Let us point out that all the three formulas above give equivalent (quasi)norms.

On the other hand, Lorentz type clones of Orlicz spaces, $L^{\Phi,\Psi}$ in self-explaining notations (Ψ and Φ are Young functions), have been studied in a number of papers, see e.g. [M-S]. If $\Phi(t) = \exp t^{q'} - 1$, $\Psi(t) = t^q$, $q' = q/(q-1)$ with $1 < q < \infty$ (see [EK] for an analysis of this case), then $L^{\exp t^{q'}-1, t^q}$ (or just $L^{\exp t^{q'}, t^q}$ for brevity) can be shown to coincide with the *Lorentz-Zygmund spaces* $L^{\infty,q,-1}$ with the quasinorm

$$\left(\int_0^1 \left[\frac{f^*(t)}{\log e/t} \right]^q \frac{dt}{t} \right)^{1/q}. \tag{2.13}$$

Let us point out that these spaces are nothing but BW_q of [BR] mentioned earlier.

Discretizing the integral on the left (e.g. using dyadic intervals) leads to various extrapolation characterizations of these spaces; see [CUK]. Note in passing that the space $L^{\exp t^{q'}, t^q}$, $1 < q < \infty$, can be characterized by the condition

$$\left(\sum_{k=1}^{\infty} \left[\frac{\|f|L_k\|}{k} \right]^q \right)^{1/q} < \infty \tag{2.14}$$

and that another equivalent (quasi)norm in $L^{\exp t^{q'}, t^q}$ can be taken as

$$\left(\sum_{k=1}^{\infty} \left[\frac{\|f|L^{k,r}\|}{k} \right]^q \right)^{1/q}$$

with any $1 < r < \infty$. We refer to [EK] and [CUK]) for a systematic treatment of these and "localized" blowup conditions for Lebesgue and Lorentz norms.

In the current paper we deal with a multivariate variant of (2.10) and (2.13). Quasinorms of this type will be used to handle the critical imbeddings—see e.g. (3.1) in Theorem 3.1. Although our main objective in this paper is an improvement of critical imbeddings and we shall not

systematically study the corresponding multivariate spaces themselves, we shall present and prove various relevant comparisons with the spaces based on usual rearrangements (this section and later in Section 4 and 5).

Let us also observe that the condition (3.1) expresses an imbedding between the appropriate spaces provided all functions live in a bounded extension domain; generally it should be understood as a measurement of the large values of $|f|$ in terms of integrability properties of its rearrangement near the origin.

Note also that Theorem 3.1 can be viewed as a certain multivariate Lorentz refinement of the critical imbeddings in [KS1], in the same spirit as the Sobolev space $W^{k,p}$ in the sublimiting case is imbedded into the Lorentz space $L^{p^*,p}$, where $p^* = Np/(N - kp)$ is the Sobolev exponent, that is, into a "better" space than the standard target L_{p^*}.

To be more specific, we shall consider the *multivariate Lorentz-Zygmund spaces* $L_R^{\infty\,q}(\log L)^{-a}(\log L)^{-b} = L_R^{\infty\,q}(\log L)^{-a}(\log L)^{-b}(\Omega_1 \times \Omega_2)$, where $\Omega_1 \times \Omega_2 \subset \mathbb{R}^N \times \mathbb{R}^N$, $|\Omega_1| = |\Omega_2| = 1$, consisting of functions f with the finite quasinorm

$$\|f|L^{\infty\,q}(\log L)^{-a}(\log L)^{-b}\|$$
$$= \left(\int_0^1 \int_0^1 \left[\frac{(R_{12}f)(s,t)}{(\log e/s)^a\,(\log e/t)^b}\right]^q \frac{ds}{s}\frac{dt}{t}\right)^{1/q} \quad (2.15)$$

in the case adopted to our splitting $\mathbb{R}^{2N} = \mathbb{R}^N \times \mathbb{R}^N$. Let us observe that in the same spirit as in [M-S] and [EK] one can introduce and consider more general *multivariate Lorentz-Orlicz spaces* $L_R^{(\Phi_1,\Psi_1),(\Phi_2,\Psi_2)}$, whose special case $L_R^{(\exp t^{q'},t^q),(\exp t^{q'},t^q)}$ would correspond to (2.15) with $a = b = 1$ (cf. the comments before (2.13)).

If $q = \infty$, then we have the space $L_R^{\infty,\infty}(\log L)^{-a}(\log L)^{-b}$ for $a, b \neq 0$, with the quasinorm

$$\sup_{0 < s,t \leq 1} \frac{(R_{12}f)(s,t)}{(\log e/s)^{1/a}(\log e/t)^{1/b}}.$$

Proposition 2.2. *Let* $1 < q < \infty$, $a, b > 0$, $\Omega_1, \Omega_2 \subset \mathbb{R}^N$, $|\Omega_1| = |\Omega_2| = 1$, *and let* f *be a measurable function in* $\Omega_1 \times \Omega_2$. *Then*

$$\sum_{k=1}^{\infty} \left[\frac{\|f|L_k(\Omega_1 \times \Omega_2)\|}{k^{a+b}}\right]^q \leq c \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \left[\frac{(R_{12}f)(2^{-i},2^{-j})}{i^a j^b}\right]^q$$
$$\leq c \int_0^1 \int_0^1 \left[\frac{(R_{12}f)(s,t)}{(\log e/s)^a\,(\log e/t)^b}\right]^q \frac{ds}{s}\frac{dt}{t}. \tag{2.16}$$

Proof. Let $k \geq q$. Then $L^{k,q} \hookrightarrow L_k$ with the norm of the imbedding not exceeding one (see [SW, Chapter V, Theorem 3.11]).

It follows from (2.9) and (2.7) that

$$\|f|L_k(\Omega_1 \times \Omega_2)\|^q \leq \frac{q^2}{k^2} \int_0^1 \int_0^1 \left[(st)^{1/k}(R_{12}f)(s,t)\right]^q \frac{ds}{s}\frac{dt}{t}$$

$$\leq \frac{cq^2}{k^2} \sum_{i=0}^\infty \sum_{j=0}^\infty \int_{2^{-j-1}}^{2^{-j}} \int_{2^{-i-1}}^{2^{-i}} \left[2^{-i/k}2^{-j/k}(R_{12}f)(2^{-i-1},2^{-j-1})\right]^q \frac{ds}{s}\frac{dt}{t}$$

$$\leq \frac{cq^2}{k^2} \sum_{i=0}^\infty \sum_{j=0}^\infty 2^{-qi/k}2^{-qj/k} \left[(R_{12}f)(2^{-i-1},2^{-j-1})\right]^q.$$

Hence

$$\sum_{k=[q]+1}^\infty \left[\frac{\|f|L_k(\Omega_1 \times \Omega_2)\|}{k^{a+b}}\right]^q$$

$$\leq cq^2 \sum_{i=0}^\infty \sum_{j=0}^\infty \left[(R_{12}f)(2^{-i-1},2^{-j-1})\right]^q \sum_{k=[q]+1}^\infty \left[k^{-aq-1}2^{-qi/k}\right]\left[k^{-bq-1}2^{-qj/k}\right].$$

Denote the last sum by $S(a,b,q)$; there holds

$$S(a,b,q) \leq \left(\sum_{k=[q]+1}^\infty k^{-aq-1}2^{-qi/k}\right)\left(\sum_{k=[q]+1}^\infty k^{-bq-1}2^{-qj/k}\right).$$

Replacing the first sum on the right hand side by an integral and changing variables, we get for $i > 0$,

$$\sum_{k=1}^\infty k^{-bq-1}2^{-qi/k} \sim \int_1^\infty \xi^{-aq} \exp\left(-\frac{qi}{\xi}\log 2\right)\frac{d\xi}{\xi}$$

$$\sim \int_0^{qi\log 2} \xi^{aq-1}\left(\frac{1}{qi\log 2}\right)^{aq} e^{-\xi}\,d\xi$$

$$= \left(\frac{1}{qi\log 2}\right)^{aq}\Gamma(aq)$$

$$\leq ci^{-aq}.$$

An analogous estimate holds true for the second sum. Hence we get, for $i,j > 0$,

$$S(a,b,q) \leq ci^{-aq}j^{-bq},$$

which proves the first inequality in (2.16).

To prove the right wing inequality in (2.16) consider

$$2^{-k-1} \leq s \leq 2^{-k} \quad \text{and} \quad 2^{-\ell-1} \leq t \leq 2^{-\ell}.$$

By virtue of the monotonicity properties of $R_{1,2}f$ and of log we have

$$\int_{2^{-k-1}}^{2^{-k}} \int_{2^{-\ell-1}}^{2^{-\ell}} \left[\frac{(R_{12}f)(s,t)}{(\log e/s)^a (\log e/t)^b} \right]^q \frac{ds}{s} \frac{dt}{t} \geq c \left[\frac{(R_{12}f)(2^{-k}, 2^{-\ell})}{k^a \ell^b} \right]^q$$

and the assertion follows by summing over k and ℓ. $\qquad\square$

Corollary 2.3. *Let the assumptions on Ω_j, $j = 1, 2$, and f from the previous Proposition be satisfied. Then*

$$\int_0^1 \left[\frac{f^*(t)}{(\log e/t)^2} \right]^r \frac{dt}{t} \leq c \int_0^1 \int_0^1 \left[\frac{(R_{12}f)(s,t)}{(\log e/s)\,(\log e/t)} \right]^r \frac{ds}{s} \frac{dt}{t} \qquad (2.17)$$

for any $1 < r < \infty$.

Proof. Discretizing the integral on the left hand side of (2.17) we can write

$$\int_0^1 \left[\frac{f^*(t)}{(\log e/t)^2} \right]^r \frac{dt}{t} \leq c \sum_{k=1}^{\infty} \frac{f^*(2^{-k})^r}{k^{2r}}$$

$$\leq c \sum_{k=1}^{\infty} \frac{1}{k^{2r}} \left(\int_{2^{-k+1}}^{2^{-k}} f^*(t)^k \, dt \right)^{r/k} \qquad (2.18)$$

$$\leq c \sum_{k=1}^{\infty} \left[\frac{\|f|L_k\|}{k^2} \right]^r .$$

Now (2.17) follows on invoking Proposition 2.2 with $a = b = 1$. $\qquad\square$

Remark 2.4. The left hand side of (2.18) equals to $\|f|L^{\infty,r;-2}\|^r$; see (2.10). The above proof shows that $\|f|L^{\infty,r;-2}\|$ can be estimated from above by a series similar to (2.14) (with k in the denominator replaced by k^2). One can prove that the converse estimate holds, too, hence there is in fact an equivalence. An analogous equivalence can be proved for all the spaces $\|f|L^{\infty,r;-a}\|$ ($1 < r < \infty$, $a > 0$). We omit the details and refer to [CUK] for the idea of the proof.

Proposition 2.5. *Let the assumptions on Ω_j, $j = 1, 2$, a, b, and f from the previous Proposition be satisfied. Then*

$$\sup_k \frac{\|f|L_k(\Omega_1 \times \Omega_2)\|}{k^{a+b}} \leq c \sup_{0<s<1} \sup_{0<t<1} \frac{(R_{12}f)(s,t)}{(\log e/s)^a (\log e/t)^b}. \qquad (2.19)$$

Proof. Let the right hand side of (2.19) be finite, otherwise there is nothing to prove. Then,

$$(R_{12}f)(s,t) \leq \gamma \left(\log \frac{e}{s} \right)^a \left(\log \frac{e}{t} \right)^b .$$

for some constant $\gamma > 0$. After integration (because f is equimeasurable with $R_{12}f$),

$$\|f|L_k(\Omega_1 \times \Omega_2)\| = \left(\int_0^1 \int_0^1 [(R_{12}f)(s,t)]^k \, dsdt \right)^{1/k}$$

$$\leq \gamma \left(\int_0^1 \left[\log \frac{e}{s} \right]^{ak} ds \right)^{1/k} \left(\int_0^1 \left[\log \frac{e}{t} \right]^{bk} dt \right)^{1/k}.$$

The integrals with the logarithmic function can be estimated as follows. We have, after changes of variables,

$$\int_0^1 \left[\log \frac{e}{s} \right]^{ak} ds = e \int_e^\infty (\log x)^{ak} \frac{dx}{x^2}$$

$$= e \int_1^\infty y^{ak} e^{-y} \, dy \leq e\Gamma(ak+1)$$

$$= eak\Gamma(ak),$$

where Γ denotes the Gamma function. Hence

$$\left(\int_0^1 \left[\log \frac{e}{s} \right]^{ak} ds \right)^{1/k} \leq (eak\Gamma(ak))^{1/k} \leq c(\Gamma(ak))^{1/k}.$$

By Stirling's formula,

$$\Gamma(ak)^{1/k} \leq \left[(ak)^{ak-1/2} e^{-ak} \right]^{1/k} \leq ck^{a-1/(2k)}$$

$$\leq ck^a.$$

This gives

$$\left(\int_0^1 \left[\log \frac{e}{s} \right]^{ak} ds \right)^{1/k} \leq ck^a$$

so that

$$\|f|L_k(\Omega_1 \times \Omega_2)\| \leq c\gamma k^{a+b}.$$

$\square$

Next we compare the usual exponential spaces with their multivariate "diagonal case".

Corollary 2.6. *Under the assumptions of Proposition 2.2,*

$$\sup_{0<t\leq 1} \frac{f^*(t)}{(\log e/t)^{2a}} \leq c \sup_{0<t\leq 1} \sup_{0<s\leq 1} \frac{(R_{12}f)(s,t)}{(\log e/s)^a (\log e/t)^a}. \qquad (2.20)$$

Proof. It is enough to use the fact that the left hand side of (2.20) is equivalent to the left hand side of (2.19) when $a = b$ (see (2.11) and (2.12)); note for completeness that the left hand side of (2.20) is equivalent to $\|f|L^{\exp t^{1/(2a)}}\|$. $\qquad\square$

Remark 2.7. It is possible to give an alternative proof of (2.20) without invoking the equivalence of various extrapolation characterizations of the exponential Orlicz spaces in question. Indeed, let

$$\varphi(s,t) = (\log e/s)(\log e/t), \qquad s,t > 0,$$

and assume that the right hand side of (2.20) is finite. Then

$$(R_{12}f)(s,t) \leq \gamma \left[\varphi(s,t)\right]^a$$

with some $0 < \gamma < \infty$. However, it is not difficult to show that

$$\varphi^*(\tau) \sim (\log e/\tau)^2.$$

Since f and $R_{12}f$ are equimeasurable it follows that

$$f^*(\tau) = (R_{12}f)^*(\tau) \leq [\varphi^*(\tau)]^a \leq c_1 (\log e/\tau)^{2a},$$

which proves the Corollary.

Remark 2.8. Denote, for brevity, by $Q_{\infty,a,b}(f)$ the right hand side of Proposition 2.5, that is, $\|f|L_R^{\infty,\infty}(\log L)^{-a}(\log L)^{-b}\|$. Then

$$Q_{\infty,a,b}(f) \sim \| \, \|f|L^{\exp t^{1/b}}(y)\| \, | \, L^{\exp t^{1/a}}(x)\| \, \|,$$

where on the right hand side there is a mixed norm of the function $f = f(x,y)$, first taken with respect to the variable y and then with respect to x (which is indicated here by the occurrence of the variable in the symbol for the space). Because of the equimeasurability one can alternatively write R_2 in the inner norm, the rearrangement R_1 of it, and finally the norm with respect to the variable x. In other words, $Q_{\infty,a,b}(f)$ can be viewed as a certain "mixed iterated quasinorm".

Indeed,

$$\frac{(R_{12}f)(s,t)}{(\log e/s)^a(\log e/t)^b} = \frac{1}{(\log e/s)^a} R_1 \left(\frac{(R_2 f)(x,.)(t)}{\log e/t)^b} \right)(s)$$

$$\leq \frac{1}{(\log e/s)^a} R_1 \left(\sup_{0<t\leq 1} \frac{(R_2 f)(x,.)(t)}{(\log e/t)^b} \right)(s)$$

$$\leq \frac{c}{(\log e/s)^a} R_1 \left(\|f(x,.)|L^{\exp t^{1/b}}\| \right)(s)$$

$$\leq c \sup_{0<s\leq 1} \frac{1}{(\log e/s)^a} \, R_1 \left(\|f(x,.)|L^{\exp t^{1/b}}\| \right)(s)$$

$$\leq c \left\| \, \|f(x,.)|L^{\exp t^{1/b}}\| \, \left| L^{\exp s^{1/a}} \right\| \right. .$$

On the other hand, there exists $c > 0$ such that for every x there exists $t = t_x$ such that

$$\frac{(R_2 f)(x,t_x)}{(\log e/t_x)^b} \geq \frac{c}{2} \|R_2 f(x,.)|L^{\exp t^{1/b}}\|.$$

Hence also

$$\sup_{0<s\leq 1} \frac{1}{(\log e/s)^a} \, R_1 \left(\sup_{0<t\leq 1} \frac{(R_2 f)(x,t)}{(\log e/t)^b} \right)(s)$$

$$\geq \sup_{0<s\leq 1} \frac{1}{(\log e/s)^a} R_1 \left(\frac{c}{2}\|R_2 f(x,.)|L^{\exp t^{1/b}}\| \right)(s)$$

$$\geq \frac{c^2}{4} \, Q_{\infty,a,b}(f).$$

The inequality (2.20) can therefore be understood as a comparison of of the usual Orlicz exponential space with the diagonal case of the mixed norm space with the multivariate rearrangements, the latter space being nothing but a special (diagonal) case of a more general exponential mixed norm space. Hence there is a natural question whether the diagonal case gives the usual norm, i.e. whether there is an equivalence in (2.20). Rather surprisingly this is not true. Let us give a simple example. For sake of clarity let us temporarily use the variable in the symbols for the spaces. Let $N = 1$, $a = 1$, and consider $g = g(y)$, $0 < y \leq 1$, such that $\|g|L_k(y)\| \sim k^2$, $k \in \mathbb{N}$. Then $g \notin L^{\exp t}(y)$; equivalently $\sup_k \|g|L_k(y)\|/k = \infty$. Put

$$f(x,y) = g(y), \qquad 0 < x,y \leq 1.$$

Plainly

$$\|f|L_k(x,y)\| \sim k^2$$

hence the left hand side of (2.20) is finite whereas the right hand side is infinite.

Similar analysis can be carried out for other estimates in this section; as agreed earlier our major in this paper concern will be the critical imbeddings.

In the last two assertions we shall compare the multivariate expressions of the exponential Orlicz and Lorentz-Orlicz type (cf. [Tri2, Section 12]).

Proposition 2.9. *Let* $1 < q < \infty$ *and* $a, b, > 0$. *Then*

$$\sup_{0<s<1} \sup_{0<t<1} \frac{(R_{12}f)(s,t)}{(\log e/s)^a (\log e/t)^b}$$

$$\leq c \left(\int_0^1 \int_0^1 \left[\frac{(R_{12}f)(s,t)}{(\log e/s)^a (\log e/t)^b} \right]^q \frac{ds}{s \log e/s} \frac{dt}{t \log e/t} \right)^{1/q}.$$

Proof. We have

$$\left(\left[\log \frac{e}{s} \right]^{-aq} \right) = -aq \left(\log \frac{e}{s} \right)^{-aq-1} \left(-\frac{e}{s} \right)$$

$$= eaq \frac{1}{(\log e/s)^{aq}} \frac{1}{s \log e/s}.$$

Thus

$$\left(\log \frac{e}{s} \right)^{-aq} \leq eaq \int_0^s \frac{1}{(\log e/u)^{aq}} \frac{du}{u \log e/u}$$

and

$$\left(\log \frac{e}{s} \right)^{-aq} \left(\log \frac{e}{t} \right)^{-bq}$$

$$\leq e^2 abq^2 \int_0^s \int_0^t \frac{1}{(\log e/u)^{aq} (\log e/v)^{bq}} \frac{du}{u \log e/u} \frac{dv}{v \log e/v}.$$

Since $R_{12}f$ is non-increasing with respect to the both variables this implies

$$\frac{(R_{12}f)(s,t)}{(\log e/s)^a (\log e/t)^b}$$

$$\leq c \left(\int_0^s \int_0^t \left[\frac{(R_{12}f)(u,v)}{(\log e/u)^a (\log e/v)^b} \right]^q \frac{du}{u \log e/u} \frac{dv}{v \log(e/v)} \right)^{1/q}$$

$$\leq c \left(\int_0^1 \int_0^1 \left[\frac{(R_{12}f)(u,v)}{(\log e/u)^a (\log e/v)^b} \right]^q \frac{du}{u \log e/u} \frac{dv}{v \log e/v} \right)^{1/q},$$

which proves the claim. □

Corollary 2.10. *Let* $1 < q < \infty$. *Then*

$$\sup_{0<s<1} \sup_{0<t<1} \frac{(R_{12}f)(s,t)}{[(\log e/s)(\log e/t)]^{1/d'}}$$

$$\leq c \left(\int_0^1 \int_0^1 \left[\frac{(R_{12}f)(s,t)}{(\log e/s)(\log e/t)} \right]^d \frac{dt}{s} \frac{dt}{t} \right)^{1/d} \qquad (2.21)$$

$$\leq c \left(\int_0^1 \int_0^1 \left[\frac{(R_{12}f)(s,t)}{(\log e/s)(\log e/t)} \right]^r \frac{dt}{s} \frac{dt}{t} \right)^{1/r}$$

for every $1 < r < d < \infty$, $\frac{1}{d} + \frac{1}{d'} = 1$.

Proof. Let $1 < d < \infty$. Then the first inequality in (2.21) follows immediately from Proposition 2.9 on putting $a = b = 1/d'$.

If $1 < r < d$, then discretizing the middle term in (2.21) we get

$$\left(\int_0^1 \int_0^1 \left[\frac{(R_{12}f)(s,t)}{(\log e/s)(\log e/t)}\right]^d \frac{dt}{s}\frac{dt}{t}\right)^{1/d} \sim \left(\sum_{k=1}^\infty \sum_{\ell=1}^\infty \left[\frac{(R_{12}f)(2^{-k},2^\ell)}{k\ell}\right]^d\right)^{1/d}$$

$$\leq \left(\sum_{k=1}^\infty \sum_{\ell=1}^\infty \left[\frac{(R_{12}f)(2^{-k},2^\ell)}{k\ell}\right]^r\right)^{1/r}$$

$$\leq c\left(\int_0^1 \int_0^1 \left[\frac{(R_{12}f)(s,t)}{(\log e/s)(\log e/t)}\right]^r \frac{dt}{s}\frac{dt}{t}\right)^{1/r}.$$

$\square$

Remark 2.11. Using the Orlicz-Lorentz type notations mentioned earlier, the inequalities in (2.21) express imbeddings

$$L_R^{\exp t^{r'},t^r} \hookrightarrow L_R^{\exp t^{d'},t^d} \hookrightarrow L_R^{\infty,\infty}(\log L)^{-1/d'}(\log L)^{-1/d'},$$

which are nothing but Orlicz type generalizations of monotonicity properties of usual Lorentz spaces with respect to the parameters. Here, one has to replace inequalities for parameters of Lorentz spaces by the usual ordering of Young functions: It is $\Psi_1 \prec \Psi_2$ if there exists $c > 0$ such that $\Psi_1(t) \leq \Psi_2(ct)$ for large values of t. Similar interpretation, giving a more transparent idea of behaviour of the multivariate exponential spaces, can be carried out in the case of the other inequalities above, too.

2.3 Spaces with dominating mixed smoothness

Let us recall definitions of the spaces that we shall work with. Let ψ be a real-valued infinitely differentiable function such that $0 \leq \psi(x) \leq 1$, $\psi(x) = 1$ if $|x| \leq 1$, and $\psi(x) = 0$ if $|x| \geq 2$. Put

$$\varphi_0(x) = \psi(x),$$
$$\varphi_1(x) = \varphi_0(x/2) - \varphi_0(x)$$
$$\varphi_j(x) = \varphi_1(2^{-j+1}x) \qquad j = 2,3,\dots.$$

Plainly

$$\sum_{j=0}^\infty \varphi_j(x) = 1, \qquad x \in \mathbb{R}^N.$$

The system of functions $\{\varphi_j\}$ is the *smooth dyadic decomposition of the unity.*

Now we give the definitions of the spaces with dominating mixed derivatives. We shall restrict ourselves to the special splitting $\mathbb{R}^{2N} = \mathbb{R}^N \times \mathbb{R}^N$.

Let $\{\varphi_j(x)\}_{j=0}^{\infty}$ be a smooth dyadic resolution of unity in $\mathbb{R}^N$. For $f \in \mathcal{S}'(\mathbb{R}^{2N})$ we put

$$f_{j,k}(x,y) := [\mathcal{F}^{-1}\varphi_j(\xi)\varphi_k(\eta)\mathcal{F}f](x,y)$$

$(j,k = 0, 1, \dots)$.

Definition 2.12. (i) Let $0 < p, q \leq \infty$, $r \in \mathbb{R}^1$. Then

$$S_{pq}^r B(\mathbb{R}^N \times \mathbb{R}^N) = \{f \in \mathcal{S}'(\mathbb{R}^{2N}) : \|f|S_{pq}^r B(\mathbb{R}^N \times \mathbb{R}^N)\|$$
$$= \|2^{rj+rk}f_{j,k}(x,y)|L_p|\ell_q\| < \infty\}.$$

(ii) Let additionally $0 < p < \infty$. Then

$$S_{pq}^r F(\mathbb{R}^N \times \mathbb{R}^N) = \{f \in \mathcal{S}'(\mathbb{R}^{2N}) : \|f|S_{pq}^r F(\mathbb{R}^N \times \mathbb{R}^N)\|$$
$$= \|2^{rj+rk}f_{j,k}(x,y)|\ell_q|L_p\| < \infty\}.$$

Of course the mixed norms in the definition of $S_{pq}^r B(\mathbb{R}^N \times \mathbb{R}^N)$ should be understood in the sense

$$\|2^{rj+rk}f_{j,k}(x,y)|L_p(\mathbb{R}^N \times \mathbb{R}^N)|\ell_q(\mathbb{N}_0 \times \mathbb{N}_0)\|,$$

similarly in (ii).

For the definition of the usual Besov and Lizorkin-Triebel spaces $B_{pq}^r(\mathbb{R}^M)$ and $F_{pq}^r(\mathbb{R}^M)$ we refer to [Tri1].

Remark 2.13. The spaces defined above are special cases of those introduced and studied in [ScT, Chapter 2] and one can find further references there. Up to equivalent (quasi)norms these spaces do not depend on a particular choice of the dyadic resolutions of unity. In [ScT] there are given further equivalent characterizations, in particular, those in terms of differences and derivatives.

Remark 2.14. We recall important properties of the *Sobolev spaces with dominating mixed smoothness* (see [ScT, Chapter 2]). Let $1 < p < \infty$. Then

$$S_{p2}^r F(\mathbb{R}^N \times \mathbb{R}^N) = S_p^r H(\mathbb{R}^N \times \mathbb{R}^N) := \{f \in \mathcal{S}'(\mathbb{R}^{2N}) : \|f|S_p^r H(\mathbb{R}^N \times \mathbb{R}^N)\|$$
$$= \|\mathcal{F}^{-1}\left[(1+|\xi|^2)^{r/2}(1+|\eta|^2)^{r/2}\mathcal{F}f\right]|L_p(\mathbb{R}^n \times \mathbb{R}^N)\| < \infty\}.$$
$$(2.22)$$

If additionally $r = 0, 1, 2, \dots$, then $S_p^r H(\mathbb{R}^N \times \mathbb{R}^N) = S_p^r W(\mathbb{R}^{2N})$, where

$$S_p^r W(\mathbb{R}^{2N}) := \{f \in L_p(\mathbb{R}^{2N}) : D^{(\alpha^1, \alpha^2)}f \in L_p, \ |\alpha^j| \leq r \ (j=1,2)\},$$

and where $D^{(\alpha^1, \alpha^2)}$ stands for $D_x^{\alpha^1} D_y^{\alpha^2}$, with equivalence of the corresponding norms. Observe also that

$$\|f|S_p^r W(\mathbb{R}^{2N})\| \sim \|f|L_p(\mathbb{R}^{2N})\| + \sum_{j=1}^{N} \left\| \frac{\partial^r f}{\partial x_j^r} \Big| L_p(\mathbb{R}^{2N}) \right\|$$
$$+ \sum_{j=1}^{N} \left\| \frac{\partial^r f}{\partial y_j^r} \Big| L_p(\mathbb{R}^{2N}) \right\| + \sum_{j,k=1}^{N} \left\| \frac{\partial^{2r} f}{\partial x_j^r \partial y_k^r} \Big| L_p(\mathbb{R}^{2N}) \right\|,$$

(2.23)

which is a very illustrative justification for the name of the spaces: there are no "pure" derivatives of order bigger than r, the mixed derivatives of order $2r$ "dominate".

Note that $S_p^0 H(\mathbb{R}^N \times \mathbb{R}^N) = L_p(\mathbb{R}^{2N})$.

Remark 2.15. The following sharp imbeddings hold (see [ScT] and [ScSi] for details):

$$S_{p_0\, q_0}^{r_0} B(\mathbb{R}^N \times \mathbb{R}^N) \hookrightarrow S_{p\, q}^{r} B(\mathbb{R}^N \times \mathbb{R}^N)$$

if

$$0 < p_0 < p \le \infty, \quad 0 < q_0 \le q \le \infty, \quad r < r_0,$$
$$r_0 - \frac{N}{p_0} = r - \frac{N}{p},$$

and

$$S_{p_0\, q_0}^{r_0} F(\mathbb{R}^N \times \mathbb{R}^N) \hookrightarrow S_{p\, p_0}^{r} B(\mathbb{R}^N \times \mathbb{R}^N)$$

if

$$0 < p_0 < p < \infty, \quad 0 < q_0 \le \infty, \quad r < r_0.$$

3   The main theorem

We shall investigate the imbeddings of the spaces $S_{p\,q}^{N/p} B(\mathbb{R}^N \times \mathbb{R}^N)$ for $0 < p < \infty$, $1 < q \le \infty$, as well as of $S_{p\,q}^{N/p} F(\mathbb{R}^N \times \mathbb{R}^N)$ for $1 < p < \infty$, $0 < q \le \infty$, into multivariate Lorentz-Orlicz spaces. Our approach will be based on atomic characterization (see [Tri2], [H]), which permits a reduction of the integral inequalities to corresponding problems in sequence spaces, which are easier to handle.

Our main result reads as follows:

Theorem 3.1. *Let $0 < p < \infty$ and $1 < q \le \infty$. Then*

$$\int_0^1 \int_0^1 \left[\frac{(A_{12}f)(s,t)}{(\log e/s)\,(\log e/t)} \right]^q \frac{ds}{s} \frac{dt}{t} \le c\|f|S_{p\,q}^{N/p} B(\mathbb{R}^N \times \mathbb{R}^N)\|^q \qquad (3.1)$$

and

$$\sup_{0<s\leq 1}\sup_{0<t\leq 1}\frac{(A_{12}f)(s,t)}{[(\log e/s)(\log e/t)]^{1/q'}}\leq c\|f|S_{pq}^{N/p}B(\mathbb{R}^N\times\mathbb{R}^N)\| \qquad (3.2)$$

holds for all $f\in S_{pq}^{N/p}B(\mathbb{R}^N\times\mathbb{R}^N)$ *(with an appropriate modification if* $q=\infty$*). Analogous inequalities holds for* $A_{21}f$*.*

Corollary 3.2. *Let* $1<p<\infty$ *and* $0<q\leq\infty$*. Then*

$$\int_0^1\int_0^1\left(\frac{(A_{12}f)(s,t)}{(\log e/s)(\log e/t)}\right)^p\frac{ds}{s}\frac{dt}{t}\leq\|f|S_{pq}^{N/p}F(\mathbb{R}^N\times\mathbb{R}^N)\|^p. \qquad (3.3)$$

Proof. Using the imbeddings in Remark 2.15 we get

$$S_{pq}^{N/p}F(\mathbb{R}^N\times\mathbb{R}^N)\hookrightarrow S_{up}^{N/u}B(\mathbb{R}^N\times\mathbb{R}^N)$$

if $1<p<u<\infty$. Now (3.3) is a consequence of (3.1). □

Remark 3.3. Taking into account the equivalences (2.22) and (2.23), Corollary 3.2 applies to the spaces $S_p^{N/p}H(\mathbb{R}^N\times\mathbb{R}^N)$ and $S_p^{N/p}W(\mathbb{R}^{2N})$, too.

A natural question arises about the case $0<q\leq 1$ in Theorem 3.1 and about $0<p\leq 1$ in Corollary 3.2. Invoking imbedding theorems in [ScT] and the sharp imbeddings from Remark 2.15 one can infer that in both cases functions in these spaces are uniformly continuous and bounded in $\mathbb{R}^{2N}$.

Remark 3.4. The Sobolev imbeddings in the subcritical case have been recently refined in [K] with use of multivariate rearrangements. For simplicity, let us restrict ourselves to the case $N=2$ and assume that $r=1$ and $1<p<2$. Then

$$W_p^1(\mathbb{R}^2)=H_p^1(\mathbb{R}^2)\hookrightarrow L_R^{p^*,p}(\mathbb{R}^2),$$

where $H_p^1(\mathbb{R}^2)$ is the space of Bessel potentials and

$$p^*=\frac{2p}{2-p}$$

(see (2.7) for the definition of the target space).

An example of critical imbeddings for the "standard" Besov spaces in $\mathbb{R}^{2N}$ and of $S_{pq}^{N/p}B(\mathbb{R}^N\times\mathbb{R}^N)$, throwing some more light on the structure of the spaces with dominating mixed derivatives, can be found after the proof of the main theorem.

Note for completeness that the multivariate rearrangements have been recently also used to strengthen subcritical imbeddings of anisotropic Lipschitz spaces in [P].

Proof of Theorem 3.1. Step 1. Since $A_{12}f$ is non-increasing with respect to s and t we can discretize the integral on the left hand side of (3.1). For any $D \geq 2^N$ we obtain

$$\int_0^1 \int_0^1 \left[\frac{(A_{12}f)(s,t)}{(\log e/s)(\log e/t)} \right]^q \frac{ds}{s} \frac{dt}{t}$$

$$\leq \sum_{k=1}^{\infty} \int_{2^{-(k+1)N}D}^{2^{-kN}D} \sum_{\ell=k_0}^{\infty} \int_{2^{-(\ell+1)N}D}^{2^{-\ell N}D} \left[\frac{(A_{12}f)(s,t)}{(\log e/s)(\log e/t)} \right]^q \frac{ds}{s} \frac{dt}{t} \quad (3.4)$$

$$\leq c \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} \left[\frac{(A_{12}f)(2^{-kN}D, 2^{-\ell N}D)}{k\ell} \right]^q.$$

Note that the constant c may depend on the choice of D.

Step 2. By virtue of the first imbedding in Remark 2.15 we can suppose that $p > 1$. Recall (see [R] and [V]) that $f \in S_{pq}^{N/p}B(\mathbb{R}^N \times \mathbb{R}^N)$ if and only if there is a decomposition

$$f = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^N} \sum_{n \in \mathbb{Z}^N} b_{i,j}^{m,n} a_{i,j}^{m,n}(x,y), \quad (3.5)$$

where $a_{i,j}^{m,n}(x,y)$ are atoms, that is,

$a_{i,j}^{m,n} \in C_0^{\infty}(\mathbb{R}^N \times \mathbb{R}^N)$,

supp $a_{i,j}^{m,n} \subset dQ_i(m) \times dQ_j(n)$ with some $d > 1$,

where $Q_i(m) = \{y: 2^{-i}m_r \leq y_r < 2^{-i}(m_r+1), r = 1, \dots, N\}$,

$|D^{(\alpha,\beta)} a_{i,j}^{m,n}(x,y)| \leq 2^{i|\alpha|+j|\beta|}$,

and

$$\|b\|_{p,q}^q := \sum_{(i,j) \in \mathbb{N}_0 \times \mathbb{N}_0} \left(\sum_{(m,n) \in \mathbb{Z}^N \times \mathbb{Z}^N} |b_{i,j}^{m,n}|^p \right)^{q/p} < \infty. \quad (3.6)$$

Moreover,

$$\inf_b \|b\|_{p,q} \sim \|f|S_{pq}^{N/p}B(\mathbb{R}^N \times \mathbb{R}^N)\|.$$

Step 3. Let f be decomposed as in (3.5). We wish to estimate the right hand side of (3.1) by (3.6) multiplied by a constant independent of the particular choice of the coefficients $b_{i,j}^{m,n}$ and of the atoms $a_{i,j}^{m,n}(x,y)$. Put

$$f_{i,j}(x,y) = \sum_{m \in \mathbb{Z}^N} \sum_{n \in \mathbb{Z}^N} b_{i,j}^{m,n} a_{i,j}^{m,n}(x,y), \qquad (i,j) \in \mathbb{N}_0 \times \mathbb{N}_0. \quad (3.7)$$

By (3.5), (3.7), and by the subadditivity of A_{12} we have

$$\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}\left[\frac{(A_{12}f)(2^{-kN}D, 2^{-\ell N}D)}{k\ell}\right]^q$$
$$\leq \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}\left[\frac{1}{k\ell}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(A_{12}f_{i,j})(2^{-kN}D, 2^{-\ell N}D)\right]^q. \tag{3.8}$$

Denote by $RH(f)$ the expression on the right hand side of (3.8). We estimate $RH(f)$ by four terms as follows:

$$RH(f) \leq \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}\left[\frac{1}{k\ell}\sum_{i=0}^{k}\sum_{j=0}^{l}(A_{12}f_{i,j})(2^{-kN}D, 2^{-\ell N}D)\right]^q$$
$$+ \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}\left[\frac{1}{k\ell}\sum_{i=k+1}^{\infty}\sum_{j=\ell+1}^{\infty}(A_{12}f_{i,j})(2^{-kN}D, 2^{-\ell N}D)\right]^q$$
$$+ \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}\left[\frac{1}{k\ell}\sum_{i=0}^{k}\sum_{j=\ell+1}^{\infty}(A_{12}f_{i,j})(2^{-kN}D, 2^{-\ell N}D)\right]^q \tag{3.9}$$
$$+ \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}\left[\frac{1}{k\ell}\sum_{i=k+1}^{\infty}\sum_{j=0}^{\ell}(A_{12}f_{i,j})(2^{-kN}D, 2^{-\ell N}D)\right]^q$$
$$= \sum\nolimits_{1} + \sum\nolimits_{2} + \sum\nolimits_{3} + \sum\nolimits_{4}.$$

Step 4. We claim that there exist constants $c > 0$ and $D \geq 2^N$ such that

$$(A_{12}f_{i,j})(s,t) \leq \frac{c}{\mu\nu}\sum_{\alpha=1}^{\mu}\sum_{\beta=1}^{\nu}(R_{12}b_{i,j})(\alpha,\beta) \tag{3.10}$$
$$= b_{i,j}^{**}(\mu,\nu)$$

provided

$$(s,t) \in (2^{-iN}D(\mu-1), 2^{-iN}D\mu] \times (2^{-jN}D(\nu-1), 2^{-iN}D\nu], \ \mu,\nu \in \mathbb{N}. \tag{3.11}$$

To show this recall that

$$f_{i,j}(x,y) = \sum_{m\in\mathbb{Z}^N}\sum_{n\in\mathbb{Z}^N} b_{i,j}^{m,n} a_{i,j}^{m,n}(x,y)$$

and since the atoms are bounded by 1 from above we have

$$|f_{i,j}(x,y)| \leq \sum_{\text{fin}}\sum_{m\in\mathbb{Z}^N}\sum_{n\in\mathbb{Z}^N} |b_{i,j}^{m,n}|\chi_{i,j}^{m+\tilde{m},n+\tilde{n}}(x,y),$$

where $\chi_{i,j}^{m+\widetilde{m},n+\widetilde{n}}$ is the characteristic function of the cube $Q_{i,j}(m+\widetilde{m},n+\widetilde{n}) = Q_i(m+\widetilde{m}) \times Q_j(n+\widetilde{n})$ and by virtue of the properties of atoms (compact supports) the first sum is taken over a finite index set, whose cardinality is independent of i and j. Hence there exists a constant $c_1 \in (0, 1/2]$ such that

$$(R_{12}f_{i,j})(s,t) \leq \sum_{\mathrm{fin}} R_{12}\bigg(\sum_{m\in\mathbb{Z}^N} \sum_{n\in\mathbb{Z}^N} |b_{i,j}^{m,n}| \chi_{i,j}^{m+\widetilde{m},n+\widetilde{n}} \bigg)(c_1 s, c_1 t)$$

and this is

$$\leq c R_{12}\bigg(\sum_{m\in\mathbb{Z}^N} \sum_{n\in\mathbb{Z}^N} |b_{i,j}^{m,n}| \chi_{i,j}^{m,n} \bigg)(c_1 s, c_1 t)$$

$$= c \sum_{\mu=1}^{\infty} \sum_{\nu=1}^{\infty} (R_{12}b_{i,j})(\mu,\nu)(\chi_i^{\mu})(s)(\chi_j^{\nu})(t),$$

where χ_i^{μ} stands for the (one-dimensional) characteristic function of the interval $[2^{-iN}c_1^{-N}(\mu-1), 2^{-iN}c_1^{-N}\mu)$. Therefore if (3.11) holds with $D = c_1^{-N} \geq 2^N$, we have, invoking (2.4), that

$$(A_{12}f_{i,j})(s,t) \leq \frac{1}{st} \int_0^t \int_0^s (R_{12}f_{i,j})(u,v)\,du\,dv$$

$$\leq c 2^{iN+jN} \frac{1}{\mu\nu} \sum_{\alpha=1}^{\mu} \sum_{\beta=1}^{\nu} (R_{12}b_{i,j})(\alpha,\beta) \int_0^t \chi_i^{\alpha}(u)\,du \int_0^s \chi_j^{\beta}(v)\,dv.$$

Since two last integrals are $\leq 2^{-iN}D$ and $\leq 2^{-jN}D$, resp., this is

$$\leq \frac{c}{\mu\nu} \sum_{\alpha=1}^{\mu} \sum_{\beta=1}^{\nu} (R_{12}b_{i,j})(\alpha,\beta).$$

Summarizing the above estimates we have (3.10) for all (s,t) satisfying (3.11).

Step 5. We estimate $\sum_1$. Since $i \leq k$ and $j \leq \ell$ we have

$$(2^{-kN}D, 2^{-\ell N}D) \in (0, 2^{-iN}D] \times (0, 2^{-jN}D].$$

The estimate (3.10) with $\mu = \nu = 1$ yields

$$(A_{12}f_{i,j})(2^{-kN}D, 2^{-\ell N}D) \leq c(R_{12}b_{i,j})(1,1).$$

Hence by the maximal inequality from Lemma 2.1 we obtain, for any $1 < q \leq \infty$ and $1 < p < \infty$,

$$\sum_1 \leq c \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} \bigg[\frac{1}{k\ell} \sum_{i=0}^{k} \sum_{j=0}^{\ell} (R_{12}b_{i,j})(1,1) \bigg]^q$$

$$\leq c \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} [(R_{12}b_{i,j})(1,1)]^q$$

$$\leq c \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left[\max_{m,n} |b_{i,j}^{m,n}| \right]^q$$

$$\leq c \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left(\sum_{m \in \mathbb{Z}^N} \sum_{n \in \mathbb{Z}^N} |b_{i,j}^{m,n}|^p \right)^{q/p}$$

$$= c\|b\|_{p,q}^q.$$

Step 6. We shall estimate the term $\sum_2$. Fix $i \in \{k+1, \dots\}$ and $j \in \{\ell+1, \dots\}$. It is

$$2^{-kN}D = 2^{-iN}D2^{(i-k)N},$$
$$2^{-\ell N}D = 2^{-jN}D2^{(j-\ell)N}.$$

The estimate (3.10) with $\mu = 2^{(i-k)N}$ and $\nu = 2^{(j-\ell)N}$ implies

$$(A_{12}f_{i,j})(2^{-kN}D, 2^{-\ell N}D)$$
$$\leq c \frac{1}{2^{(i-k)N}} \frac{1}{2^{(j-\ell)N}} \sum_{\alpha=1}^{2^{(i-k)N}} \sum_{\beta=1}^{2^{(j-\ell)N}} (R_{12}b_{i,j})(\alpha, \beta). \qquad (3.12)$$

We have

$$\left[b_{i,j}^{**}(2^{\kappa N}, 2^{\lambda N}) \right]^p \leq \left[b_{i,j}^{**}(\rho, \sigma) \right]^p \qquad (3.13)$$

for $\kappa, \lambda = 1, 2, \dots$, and

$$2^{(\kappa-1)N} < \rho \leq 2^{\kappa N},$$
$$2^{(\lambda-1)N} < \sigma \leq 2^{\lambda N}.$$

Summing up (3.13) over such ρ and σ we get (the number of the terms on the left is $2^{(\kappa-1)N+(\lambda-1)N}(2^N-1)^2$)

$$2^{(\kappa-1)N+(\lambda-1)N} \left[b_{i,j}^{**}(2^{\kappa N}, 2^{\lambda N}) \right]^p \leq \sum_{\rho=2^{(\kappa-1)N}+1}^{2^{\kappa N}} \sum_{\sigma=2^{(\lambda-1)N}+1}^{2^{\lambda N}} \left[b_{i,j}^{**}(\rho, \sigma) \right]^p.$$

It follows that

$$\left[b_{i,j}^{**}(2^{\kappa N}, 2^{\lambda N}) \right]^p \leq c2^{-(\kappa+\lambda)N} \sum_{\rho=1}^{\infty} \sum_{\sigma=1}^{\infty} \left[b_{i,j}^{**}(\rho, \sigma) \right]^p$$

After applying the maximal inequality for sequences this is

$$
\begin{aligned}
&\leq c2^{-(\kappa+\lambda)N}\sum_{\alpha=1}^{\infty}\sum_{\beta=1}^{\infty}[(R_{12})(\alpha,\beta)]^{p}\\
&= c2^{-(\kappa+\lambda)N}\sum_{m\in\mathbb{Z}^{N}}\sum_{n\in\mathbb{Z}^{N}}|b_{i,j}^{m,n}|^{p} \qquad\qquad (3.14)\\
&= c2^{-(\kappa+\lambda)N}\||b_{i,j}|\ell_{p}\|^{p}.
\end{aligned}
$$

Putting $\kappa = i-k$ and $\lambda = j-\ell$ we get from (3.14) and (3.12) that

$$
(A_{12}f_{i,j})(2^{-kN}D, 2^{-\ell N}D) \leq c2^{-(i-k)N/p-(j-\ell)N/p}\||b_{i,j}|\ell_{p}\|.
$$

This implies

$$
\sum\nolimits_{2} \leq c\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}\left[\frac{1}{k^{q}\ell^{q}}\sum_{i=k+1}^{\infty}\sum_{j=\ell+1}^{\infty}2^{-(i-k)N/p-(j-\ell)N/p}\||b_{i,j}|\ell_{p}\|\right]^{q}.
$$

By Hölder's inequality it follows for $q > 1$ that

$$
\sum\nolimits_{2} \leq c\|b\|_{p,q}^{q}.
$$

Step 7. We estimate the term $\sum_{3}$. Let $i = 0, 1, \ldots, k$ and $j = \ell+1, \ell+2, \ldots$ be fixed. Then $2^{-kN}D \in (0, 2^{-iN}D]$ and $2^{-\ell N}D = 2^{-jN}D2^{(j-\ell)N}$. The estimate (3.10) from Step 4 with $\mu = 1$ and $\nu = 2^{(j-\ell)N}$ yields

$$
(A_{12}f_{i,j})(2^{-kN}D, 2^{-\ell N}D) \leq \frac{c}{2^{(j-\ell)N}}\sum_{\beta=1}^{2^{(j-\ell)N}}(R_{12}b_{i,j})(1,\beta). \qquad (3.15)
$$

Arguing as before in Step 6 we get for $p > 1$ and $\lambda = 1, 2\ldots$,

$$
\begin{aligned}
\left[\frac{1}{2^{\lambda N}}\sum_{\beta=1}^{2^{\lambda N}}(R_{12}b_{i,j})(1,\beta)\right]^{p} &= \left[(A_{2}\circ R_{1}b_{i,j})(1, 2^{\lambda N})\right]^{p}\\
&\leq c2^{-\kappa N}\sum_{\sigma=1}^{\infty}[(A_{2}\circ R_{1}b_{i,j})(1,\sigma)]^{p}
\end{aligned}
$$

and by the maximal inequality this is

$$
\begin{aligned}
&\leq c2^{-\lambda N}\sum_{\beta=1}^{\infty}[(R_{2}\circ R_{1}b_{i,j})(1,\beta)]^{p}\\
&\leq c2^{-\lambda N}\sum_{n\in\mathbb{Z}^{N}}|(R_{1}b_{i,j})(1,n)|^{p}.
\end{aligned}
$$

Invoking (3.15) we obtain

$$(A_{12}f_{i,j})(2^{-kN}D, 2^{-\ell N}D) \le c2^{-(j-\ell)N/p}\bigg(\sum_{m\in\mathbb{Z}^N} |(R_1 b_{i,j})(1,n)|^p\bigg)^{1/p}$$

$$\le c2^{-(j-\ell)N/p}\bigg(\sum_{m\in\mathbb{Z}^N}\sum_{n\in\mathbb{Z}^N} |b_{i,j}^{m,n}|^p\bigg)^{1/p} \qquad (3.16)$$

$$= c2^{-(j-\ell)N/p}\|b_{i,j}|\ell_p\|.$$

Inserting (3.15) and (3.16) into $\sum_3$ we get

$$\sum_3 \le c\sum_{k=1}^\infty\sum_{\ell=1}^\infty\bigg(\frac{1}{k\ell}\sum_{i=0}^k\sum_{j=\ell+1}^\infty 2^{-(j-\ell)N/p}\|b_{i,j}|\ell_p\|\bigg)^q.$$

Denoting $c_{i,\ell} = \sum_{j=\ell+1}^\infty 2^{-(j-\ell)N/p}\|b_{i,j}|\ell_p\|$ this becomes

$$= c\sum_{\ell=1}^\infty\frac{1}{\ell^q}\sum_{k=1}^\infty\bigg(\frac{1}{k}\sum_{i=0}^k c_{i,\ell}\bigg)^q$$

and by the maximal inequality this is

$$\le c\sum_{\ell=1}^\infty\frac{1}{\ell^q}\sum_{i=0}^\infty c_{i,\ell}^q$$

$$= c\sum_{i=0}^\infty\sum_{\ell=1}^\infty\frac{1}{\ell^q}\bigg(\sum_{j=\ell+1}^\infty 2^{-(j-\ell)N/p}\|b_{i,j}|\ell_p\|\bigg)^q.$$

By Hölder's inequality the last line above is

$$\le c\sum_{i=0}^\infty\sum_{\ell=1}^\infty\frac{1}{\ell^q}\bigg(\sum_{j=\ell+1}^\infty 2^{-(j-\ell)Nq'/p}\bigg)^{q/q'}\sum_{j=0}^\infty\|b_{i,j}|\ell_p\|^q$$

$$\le c\bigg(\sum_{\ell=1}^\infty\frac{1}{\ell^q}\bigg)\sum_{i=0}^\infty\sum_{j=0}^\infty\|b_{i,j}|\ell_p\|^q$$

$$= c\|b\|_{p,q}^q.$$

Step 8. It remains to establish an estimate for $\sum_4$. Let $i = k+1,\dots$ and $j = 0,\dots,\ell$ be fixed. Then $2^{-kN}D = 2^{-iN}D2^{(i-k)N}$ and $2^{-\ell N}D \in (0, 2^{-jN}D]$. Employing (3.10) with $\mu = 2^{(i-k)N}$ and $\nu = 1$ and arguing as in Step 6, we can write

$$(A_{12}f_{i,j})(2^{-kN}D, 2^{-\ell N}D)$$

$$\leq \frac{c}{2^{(i-k)N}} \sum_{\alpha=1}^{2^{(i-k)N}} (R_{12}b_{i,j})(\alpha, 1)$$

$$= c(b_{i,j})^{**}(2^{(i-k)N}, 1)$$

$$\leq c2^{-(i-k)N/p} \left(\sum_{\rho=1}^{\infty} [(b_{i,j})^{**}(\rho, 1)]^p \right)^{1/p},$$

which is, after application of the maximal inequality,

$$\leq c2^{-(i-k)N/p} \left(\sum_{\alpha=1}^{\infty} \sum_{\beta=1}^{\infty} |(R_{12}b_{i,j})(\alpha, \beta)|^p \right)^{1/p} \tag{3.17}$$

$$= c2^{-(i-k)N/p} \|b_{i,j}|\ell_p\|.$$

Inserting (3.17) into $\sum_4$ gives

$$\sum_4 \leq c \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \left(\frac{1}{k\ell} \sum_{i=k+1}^{\infty} \sum_{j=0}^{\ell} 2^{-(i-k)N/p} \|b_{i,j}|\ell_p\| \right)^q$$

$$= c \sum_{k=0}^{\infty} \frac{1}{k^q} \sum_{\ell=0}^{\infty} \left[\frac{1}{\ell} \sum_{j=0}^{\ell} \left(\sum_{i=k+1}^{\infty} 2^{-(i-k)N/p} \|b_{i,j}|\ell_p\| \right) \right]^q.$$

Denote

$$d_{k,j} = \sum_{i=k+1}^{\infty} 2^{-(i-k)N/p} \|b_{i,j}|\ell_p\|.$$

Then by Hölder's inequality,

$$d_{k,j} \leq c \left(\sum_{i=0}^{\infty} \|b_{i,j}|\ell_p\|^q \right)^{1/q} =: e_j$$

independently of k. Applying once more the maximal inequality we arrive at

$$\sum_4 \leq \sum_{\ell=0}^{\infty} \left(\frac{1}{\ell} \sum_{j=0}^{\ell} e_j \right)^q \leq c \sum_{j=0}^{\infty} e_j^q$$

$$= c \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \|b_{i,j}|\ell_p\|^q$$

$$= c\|b\|_{p,q}^q.$$

Finally, the estimate (3.2) follows directly from (3.1) and Corollary 2.10. The theorem is proved. $\square$

4 Sharpness and envelopes

In this section we discuss the sharpness of estimates (3.1) and (3.3). At first we observe that

$$\|f_1 \otimes f_2|S^r_{pq}B(\mathbb{R}^N \times \mathbb{R}^N)\| = \|f_1|B^r_{pq}(\mathbb{R}^N)\| \, \|f_2|B^r_{pq}(\mathbb{R}^N)\| \qquad (4.1)$$

and

$$\|g_1 \otimes g_2|S^r_{pq}F(\mathbb{R}^N \times \mathbb{R}^N)\| = \|g_1|F^r_{pq}(\mathbb{R}^N)\| \, \|g_2|F^r_{pq}(\mathbb{R}^N)\| \qquad (4.2)$$

for all $f_1, f_2 \in B^r_{pq}(\mathbb{R}^N)$ and $g_1, g_2 \in F^r_{pq}(\mathbb{R}^N)$, resp. Note in passing that $((R_{12})(f \otimes g))(s,t) = f^*(s)g^*(t)$ for measurable $f = f(x)$ and $g = g(y)$, $x, y \in \mathbb{R}^N$.

Theorem 4.1. (1) *Let $0 < p < \infty$ and $1 < q \leq \infty$. Then*

$$\sup\{(R_{12}f)(s,t) : \|f|S^{N/p}_{pq}B(\mathbb{R}^N \times \mathbb{R}^N)\| \leq 1\}$$
$$\sim (\log e/s)^{1/q'} (\log e/t)^{1/q'} \qquad (4.3)$$

for all s, t, $0 < s \leq 1$, $0 < t \leq 1$.
(2) *Let $1 < p < \infty$ and $0 < q \leq \infty$. Then*

$$\sup\{(R_{12}f)(s,t) : \|f|S^{N/p}_{pq}F(\mathbb{R}^N \times \mathbb{R}^N)\| \leq 1\}$$
$$\sim (\log e/s)^{1/q'} (\log e/t)^{1/q'} \qquad (4.4)$$

for all s, t, $0 < s \leq 1$, $0 < t \leq 1$.

Proof. One direction follows from (3.2) in the case (1) and from (3.2) combined with Corollary 3.2 in the case (2).

The converse claim follows using (4.1), (4.2), and tensor products of extremal functions in $B^{N/p}_{pq}(\mathbb{R}^N)$, which were introduced in [ET], [Tri2], and [H]. We do not give details here since the proof follows an analogous assertion in [KS3]. $\qquad \square$

Theorem 4.2. (1) *Let $0 < p < \infty$ and $1 < q \leq \infty$. If there exists $c > 0$ such that*

$$\int_0^1 \int_0^1 \left[\frac{(R_{12}f)(s,t)}{(\log e/s)^{1/q'} (\log e/t)^{1/q'}}\right]^u \frac{ds}{s \log e/s} \frac{dt}{t \log e/t}$$
$$\leq c\|f|S^{N/p}_{pq}B(\mathbb{R}^N \times \mathbb{R}^N)\|^u$$

for all $f \in S^{N/p}_{pq}B(\mathbb{R}^N \times \mathbb{R}^N)$, then $q \leq u$.
(2) *Let $1 < p < \infty$ and $0 < q \leq \infty$. If there exists $c > 0$ such that*

$$\int_0^1 \int_0^1 \left[\frac{(R_{12}f)(s,t)}{(\log e/s)^{1/p'} (\log e/t)^{1/p'}}\right]^u \frac{ds}{s \log e/s} \frac{dt}{t \log e/t}$$
$$\leq c\|f|S^{N/p}_{pq}F(\mathbb{R}^N \times \mathbb{R}^N)\|^u$$

for all $f \in S^{N/p}_{pq}F(\mathbb{R}^N \times \mathbb{R}^N)$, then $p \leq u$.

The *proof* is again an appropriate modification of the arguments from Section 5 of [KS3]. One uses tensor products of extremal functions, (4.1) and (4.2).

Remark 4.3. The left hand side of (4.3) and (4.4) can be considered as a *local multivariate growth envelope function* in the spirit of [Tri2] or [H]. In this sense Theorem 4.1 and Theorem 4.2 establish the *local multivariate growth envelopes* for the spaces $S_{pq}^{N/p}B(\mathbb{R}^N \times \mathbb{R}^N)$ and $S_{pq}^{N/p}F(\mathbb{R}^N \times \mathbb{R}^N)$, respectively.

5 The standard case revisited, comparisons

We shall discuss the question what can be said about the behaviour of the normal rearrangement $f^*(t)$. Moreover, it is of interest to compare our results with those for isotropic spaces of the same "order of smoothness", close to the spaces with dominating mixed smoothness. While it is clear that for a non-negative integer r the space $W_p^{2r}(\mathbb{R}^{2N})$ is imbedded into $S_p^r W(\mathbb{R}^{2N})$ the analogous claim for the Besov spaces is not immediate. Nevertheless, such an imbedding is true and we state both imbeddings as the following

Proposition 5.1. (1) *If* $1 < p < \infty$ *and* $r \in \mathbb{N}_0$, *then*

$$W_p^{2r}(\mathbb{R}^{2N}) \hookrightarrow S_p^r W(\mathbb{R}^{2N}).$$

(2) *If* $1 \le p \le \infty$, $0 < q \le \infty$, *and* $0 < r < \infty$, *then*

$$B_{p,q}^{2r}(\mathbb{R}^{2N}) \hookrightarrow S_{pq}^r B(\mathbb{R}^N \times \mathbb{R}^N).$$

Proof. As observed above the statement in (1) is trivial. As to (2) we have—somewhat surprisingly—to dig much deeper. Let $(\varphi_j(\xi))$ be a smooth dyadic resolution of unity in $\mathbb{R}^N$ and put

$$\psi_\ell(\xi,\eta) = \sum_{\max(j,k)=\ell} \varphi_j(\xi)\varphi_k(\eta), \qquad \ell = 1, 2, \ldots,$$

$$\psi_0(\xi,\eta) = \varphi_0(\xi)\varphi_0(\eta).$$

Then

$$\sum_{\ell=0}^{\infty} \psi_\ell \equiv 1$$

and (ψ_ℓ) is a dyadic resolution of unity in $\mathbb{R}^{2N}$, subordinated to "dyadic corridors", and we have

$$\|f|B_{p,q}^{2r}(\mathbb{R}^{2N})\| \sim \left(\sum_{\ell=0}^{\infty} 2^{2rq}\|\mathcal{F}^{-1}\psi_\ell \mathcal{F}f|L_p(\mathbb{R}^{2N})\|^q \right)^{1/q}.$$

Put

$$\Psi^0 = \psi_0 + \psi_1$$
$$\Psi^\ell = \psi_{\ell-1} + \psi_\ell + \psi_{\ell+1}, \qquad \ell = 1, 2, \dots .$$

If $\max(j,k) = \ell$, then

$$\varphi_j(\xi)\varphi_k(\eta) = \varphi_j(\xi)\varphi_k(\eta)\Psi^\ell(\xi,\eta)$$

since $\Psi^\ell \equiv 1$ on $\operatorname{supp} \varphi_j \otimes \varphi_k$. Moreover, if $\max(j,k) = \ell$, then by Young's inequality,

$$\|\mathcal{F}^{-1}[\varphi_j \otimes \varphi_k \mathcal{F}f]|L_p(\mathbb{R}^{2N})\| = \|\mathcal{F}^{-1}[\varphi_j \otimes \varphi_k \mathcal{F}(\mathcal{F}^{-1}\Psi^\ell \mathcal{F}f)]|L_p(\mathbb{R}^{2N})\|$$
$$\leq c\|\mathcal{F}^{-1}(\varphi_j \otimes \varphi_k)|L_1(\mathbb{R}^{2N})\| \, \|\mathcal{F}^{-1}(\Psi^\ell \mathcal{F}f)|L_p(\mathbb{R}^{2N})\|$$
$$\leq c\|\mathcal{F}^{-1}(\Psi^\ell \mathcal{F}f)|L_p(\mathbb{R}^{2N})\|,$$

where the constant c is independent of j and k. Hence

$$\|f|S_{pq}^r B(\mathbb{R}^N \times \mathbb{R}^N)\|^q \leq \sum_{\ell=0}^\infty \sum_{\max(j,k)=\ell} 2^{(j+k)rq} \|\mathcal{F}^{-1}\varphi_j \otimes \varphi_k \mathcal{F}f|L_p(\mathbb{R}^{2N})\|^q$$

$$\leq c \sum_{\ell=0}^\infty \|\mathcal{F}^{-1}\Psi^\ell \mathcal{F}f|L_p(\mathbb{R}^{2N})\|^q \sum_{\max(j,k)=\ell} 2^{(j+k)rq}$$

$$\leq c \sum_{\ell=0}^\infty \|\mathcal{F}^{-1}\Psi^\ell \mathcal{F}f|L_p(\mathbb{R}^{2N})\|^q \left(\sum_{j=0}^\ell 2^{(\ell+j)rq} + \sum_{k=0}^{\ell-1} 2^{(\ell+k)rq} \right)$$

$$\leq c \sum_{\ell=0}^\infty \|\mathcal{F}^{-1}\Psi^\ell \mathcal{F}f|L_p(\mathbb{R}^{2N})\|^q \left(2^{2\ell rq+1} \sum_{j=0}^\ell 2^{(j-\ell)rq} \right)$$

and the last sum on the right hand side is $\leq c$ iff $r > 0$ and it is $\leq c\ell$ for $r = 0$. We can conclude that

$$\|f|S_{pq}^r B(\mathbb{R}^N \times \mathbb{R}^N)\|^q \leq c \sum_{\ell=0}^\infty 2^{2r\ell q} \|\mathcal{F}^{-1}\Psi^\ell \mathcal{F}f|L_p(\mathbb{R}^{2N})\|^q$$

$$\leq c \sum_{\ell=0}^\infty 2^{2r\ell q} \|\mathcal{F}^{-1}\psi_\ell \mathcal{F}f|L_p(\mathbb{R}^{2N})\|^q.$$

$\square$

Remark 5.2. If $r = 0$, then

$$\|f|S_{pq}^0 B(\mathbb{R}^N \times \mathbb{R}^N)\| \leq c\left(\sum_{\ell=0}^\infty \left[\|(\ell+1)^{1/q}\|\mathcal{F}^{-1}\psi_\ell \mathcal{F}f|L_p(\mathbb{R}^{2N})\| \right]^q \right)^{1/q}$$

$$= c\|f|B_{p,q}^{0,1/q}(\mathbb{R}^{2N})\|;$$

there is a space with generalized smoothness on the right hand side. For properties of these spaces we refer to [L].

In the subcritical case the following imbeddings hold:

$$W_p^{2r}(\mathbb{R}^{2N}) \hookrightarrow S_p^r W(\mathbb{R}^N \times \mathbb{R}^N) \hookrightarrow L_{p^*}(\mathbb{R}^N \times \mathbb{R}^N)$$

and

$$B_{pq}^{2r}(\mathbb{R}^{2N}) \hookrightarrow S_{pq}^r B(\mathbb{R}^N \times \mathbb{R}^N) \hookrightarrow L_{p^*}(\mathbb{R}^N \times \mathbb{R}^N),$$

where $1 < p < \infty$, $0 < q \le \infty$, and

$$r - \frac{N}{p} = -\frac{N}{p^*} \quad \left(\text{that is,} \quad p^* = \frac{Np}{N - rp} \right).$$

None of these imbeddings can be improved with respect to p^*. Now we consider the critical case $r = N/p$. Then all spaces under consideration contain unbounded functions and the question about a measurement of their unboundedness makes sense. This turns out to be different in these cases. It is known from the theory of the growth envelopes (see [H], [Tri2]) that for small ε,

$$\sup\{f^*(t) : \|f|W_p^{2N/p}(\mathbb{R}^{2N})\| \le 1\} \sim |\log t|^{1/p'}, \qquad 0 < t < \varepsilon, \qquad (5.1)$$

and

$$\int_0^\varepsilon \left[\frac{f^*(t)}{|\log t|} \right]^p \frac{dt}{t} \le c\|f|W_p^{2N/p}(\mathbb{R}^{2N})\|^p. \qquad (5.2)$$

The exponent p on the left hand side is sharp. Moreover, if $0 < p < \infty$ and $1 < q \le \infty$, then

$$\sup\{f^*(t) : \|f|B_{p,q}^{2N/p}(\mathbb{R}^{2N})\| \le 1\} \sim |\log t|^{1/q'} \qquad (5.3)$$

and

$$\int_0^\varepsilon \left[\frac{f^*(t)}{|\log t|} \right]^q \frac{dt}{t} \le c\|f|B_{p,q}^{2N/p}(\mathbb{R}^{2N})\|^q, \qquad (5.4)$$

where again the exponent q on the left hand side is sharp.

Theorem 5.3. (1) *Let* $1 < p < \infty$ *and let* $f \in S_p^{N/p} W(\mathbb{R}^{2N})$ *be supported in* $\Omega_1 \times \Omega_2 \subset \mathbb{R}^N \times \mathbb{R}^N$, $|\Omega_1| = |\Omega_2| = 1$. *Then*

$$\sup\{f^*(t) : \|f|S_p^{N/p} W(\mathbb{R}^{2N})\| \le 1\} \sim \left(\log \frac{e}{t} \right)^{2/p'} \qquad (5.5)$$

for all t, $0 < t \le 1$, *and*

$$\int_0^1 \left[\frac{f^*(t)}{(\log e/t)^2} \right]^p \frac{dt}{t} \le c\|f|S_p^{N/p} W(\mathbb{R}^{2N})\|^p. \qquad (5.6)$$

Moreover, $S_p^{N/p}W$ in (5.5) and (5.6) can be replaced by $S_{pq}^{N/p}F$ with an arbitrary q, $0 < q \leq \infty$.

(2) Let $1 \leq p < \infty$, $1 < q < \infty$, and let $f \in S_{pq}^{N/p}B(\mathbb{R}^N \times \mathbb{R}^N)$ be supported in $\Omega_1 \times \Omega_2 \subset \mathbb{R}^N \times \mathbb{R}^N$, $|\Omega_1| = |\Omega_2| = 1$. Then

$$\sup\{f^*(t) : \|f|S_{pq}^{N/p}B(\mathbb{R}^N \times \mathbb{R}^N)\| \leq 1\} \sim \left(\log \frac{e}{t}\right)^{2/q'} \qquad (5.7)$$

for all t, $0 < t \leq 1$, and

$$\int_0^1 \left[\frac{f^*(t)}{(\log e/t)^2}\right]^q \frac{dt}{t} \leq c\|f|S_{pq}^{N/p}B(\mathbb{R}^N \times \mathbb{R}^N)\|^q. \qquad (5.8)$$

Proof. To prove the equivalence in (5.5) and (5.7) we use again the arguments from [KS3, Theorem 5.4] (extremal functions), now in combination with the arguments in Remark 2.7. This gives the estimates of the envelope function from below. The remaining part of (1) follows from Corollary 3.2, Corollary 2.10, Corollary 2.6, and Corollary 2.3. As to (2) the remaining estimate is a consequence of (3.2), Corollary 2.6, and Corollary 2.3. $\qquad \square$

Remark 5.4. We point out the interesting appearance of different exponents: $2/p'$ and $2/q'$ in (5.5)-(5.8), and $1/p'$ and $2/q'$ in (5.1)-(5.4).

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