A generalized anti-maximum principle for the periodic one dimensional p-Laplacian with sign changing potential

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Abstract. It is known that the antimaximum principle holds for the quasilinear periodic problem

$$(|u'|^{p-2}u')' + \mu(t)\,(|u|^{p-2}u) = h(t), \quad u(0) = u(T),\ u'(0) = u'(T),$$

if $\mu \geq 0$ in $[0,T]$ and

$$0 < \|\mu\|_\infty \leq (\pi_p/T)^p\,, \quad \text{where } \ \pi_p = 2\,(p-1)^{1/p}\int_0^1 (1-s^p)^{-1/p}\,\mathrm{d}s,$$

or

$$p = 2 \quad \text{and} \quad 0 < \|\mu\|_\alpha \leq \inf\left\{\frac{\|u'\|_2^2}{\|u\|_\alpha^2} : u \in W_0^{1,2}[0,T]\setminus\{0\}\right\} \text{ for some } \ \alpha,\ 1 \leq \alpha \leq \infty.$$

In this paper we give sharp conditions on the L_α-norm of the potential $\mu(t)$ in order to ensure the validity of the antimaximum principle even in case that $\mu(t)$ can change its sign in $[0,T]$.

Keywords. Antimaximum principle, periodic problem, Dirichlet problem, $p-$Laplacian, singular problem.

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1. Introduction

It is well-known that the second order periodic boundary value problem

$$u'' + \mu\, u = h(t), \quad u(a) = u(b), \quad u'(a) = u'(b), \tag{1.1}$$

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where $-\infty < a < b < \infty$ and $\mu \in \mathbb{R}$, satisfies a *maximum principle* (that is, $h \geq 0$ implies $u \leq 0$) for all $\mu < 0$ and an *anti-maximum principle* (that is, $h \geq 0$ implies $u \geq 0$) for all $0 < \mu \leq (\frac{\pi}{b-a})^2$. Recall that $(\frac{\pi}{b-a})^2$ is the first eigenvalue of the corresponding Dirichlet problem (see [6, Theorem 3.1]) and it is an optimal upper bound for μ in order to get the anti-maximum principle for the periodic problem (see [3, Lemma 2.5]). We notice that an interesting abstract version of the previous fact has been proved by Campos, Mawhin & Ortega [8], for an operator of the form $L+\mu I$, where L is a linear closed Fredholm operator of index zero and I is the identity operator.

The introduction of a nonnegative but non constant potential $\mu \in L_\alpha[a,b]$, where $1 \leq \alpha \leq \infty$, in equation (1.1) makes the problem more difficult to deal with. Recently, Torres & Zhang [19] presented sharp conditions on $\|\mu\|_\alpha$ ensuring the validity of the anti-maximum principle for the problem

$$u'' + \mu(t)\, u = h(t), \quad u(0) = u(2\,\pi), \quad u'(0) = u'(2\,\pi).$$

In particular, in the case $\alpha = \infty$ they recover the classical criterium

$$0 < \|\mu\|_\infty \leq \tfrac{1}{4}. \tag{1.2}$$

On the other hand, Cabada, Lomtatidze & Tvrdý [7, Theorem 3.2], dealing with quasilinear operators, have shown that an antimaximum principle holds for

$$(\phi_p(u'))' + \mu(t)\, \phi_p(u) = h(t), \quad u(0) = u(T), \quad u'(0) = u'(T), \tag{1.3}$$

with

$$0 < \|\mu\|_\infty \leq \left(\frac{\pi_p}{T}\right)^p, \tag{1.4}$$

with π_p defined by

$$\pi_p = \frac{2\,(p-1)^{1/p}}{p}\, B\left(\frac{1}{p}, \frac{1}{p^*}\right). \tag{1.5}$$

Here, as usual, $\phi_p(y) = |y|^{p-2}\, y$ for $y \in \mathbb{R}$ stands for the p-Laplacian with $1 < p < \infty$, $p^* = \frac{p}{p-1}$ and B is the Euler beta function. It is easy to see that the L_∞–estimate (1.4) coincides with (1.2) in the particular case of $p = 2$.

The aim of this paper is to fill the gap between the cases $p = 2$, $1 \leq \alpha \leq \infty$, studied in [19] for $\mu \geq 0$ on $[0,T]$ and in [4] for μ changing sign, and $1 < p < \infty$, $\alpha = \infty$, studied in [7] for $\mu \geq 0$ on $[0,T]$. We will provide sharp L_α-estimates on the potential $\mu(t)$ in order to ensure the validity of the anti-maximum principle for problem (1.3) even in case $\mu(t)$ changes sign in J. Our result extends [19, Corollary 2.5], [4, Theorem 3.2] for arbitrary $1 < p < \infty$ and [7, Theorem 3.2] for any $1 \leq \alpha \leq \infty$, including in this case potentials with zero mean value and solving also the open problem (iii) posed by the authors at the end of [7].

This paper is organized as follows: in section 2 we introduce some preliminary results needed in section 3 to prove our main result. In section 4 we provide some applications to singular differential equations and in section 5 we include some remarks and ideas for further research. Finally, section 6 contains an appendix with the proof of some technical results used in the paper.

2. Preliminaries

Throughout the paper, for $1 \le \alpha \le \infty$ and a bounded interval $[a,b] \subset \mathbb{R}$, we denote by $L_\alpha[a,b]$ the usual Lebesgue space with the corresponding norm $\|\cdot\|_\alpha$ and by α^* the conjugate of α ($\alpha^* = \frac{\alpha}{\alpha-1}$ if $\alpha > 1$, $\alpha^* = \infty$ if $\alpha = 1$ and $\alpha^* = 1$ if $\alpha = \infty$). For $x \in L_1[a,b]$ we denote its mean value by $\bar{x}$, i.e.

$$\bar{x} = \frac{1}{b-a} \int_a^b x(s)\,\mathrm{d}s.$$

Furthermore, for $x \in L_1[a,b]$, we write $x \succ 0$ if $x \ge 0$ a.e. on $[a,b]$ and $\bar{x} > 0$. If $x \in L_1[a,b]$, we denote

$$x_* = \inf_{t\in[a,b]} \text{ess } x(t) \quad \text{and} \quad x^* = \sup_{t\in[a,b]} \text{ess } x(t).$$

As usual, for an arbitrary subinterval I of $\mathbb{R}$ we denote by $C(I)$ the set of functions $x\colon I \to \mathbb{R}$ which are continuous on I. For a bounded interval $J \subset \mathbb{R}$, $C^1(J)$ stands for the set of functions $x \in C(J)$ with the first derivative continuous on J. Further, $AC(J)$ is the set of functions absolutely continuous on J and $AC_{loc}(J)$ is the set of functions absolutely continuous on each compact interval $K \subset J$. For $x \in L_\alpha(J)$, $1 \le \alpha \le \infty$, we put

$$\|x\|_{\alpha,J} = \begin{cases} \left(\displaystyle\int_J |x(t)|^\alpha \,\mathrm{d}t \right)^{1/\alpha} & \text{if } 1 \le \alpha < \infty, \\ \sup\limits_{t\in J} \text{ess } |x(t)| & \text{if } \alpha = \infty. \end{cases}$$

If $1 \le \alpha \le \infty$, then $W^{1,\alpha}(J)$ denotes the set of functions $u \in AC(J)$ such that $u' \in L_\alpha(J)$ and

$$W_0^{1,\alpha} = \{u \in W^{1,\alpha}(J) :\ u = 0 \text{ on } \partial J\}.$$

Finally, if I, $J \subset \mathbb{R}$ are subintervals of $\mathbb{R}$, J bounded, then $Car(J \times I)$ stands for the set of functions satisfying the Carathéodory conditions on $J \times I$, i.e. the set of functions $f : J \times I \to \mathbb{R}$ having the following properties: (i) for each $x \in I$ the function $f(\cdot, x)$ is measurable on J; (ii) for almost every $t \in J$ the function $f(t,\cdot)$ is continuous on I; (iii) for each compact set $K \subset I$, the function $m_{\mathrm{K}}(t) = \sup_{x\in K} |f(t,x)|$ is

Lebesgue integrable on J. Solutions of differential equations are in this paper understood in the Carathéodory sense. In particular, a function $u : J \to \mathbb{R}$ is a *solution* on the interval J to the equation

$$(\phi_p(u'))' = f(t,u), \tag{2.1}$$

if $u \in C^1(J)$, $\phi_p \circ u' \in AC(J)$, $u(t) \in I$ for all $t \in J$ and

$$(\phi_p(u'(t)))' = f(t,u(t)) \quad \text{for a.e.} \quad t \in J.$$

We will consider various boundary value problems consisting of a differential equation of the form (2.1) and of some additional boundary conditions, like the periodic or the Dirichlet conditions. Their solution are, as usual, solutions of the differential equation fulfilling the corresponding boundary condition.

Dirichlet eigenvalues. It is known (see [23]) that the eigenvalue problem

$$(\phi_p(u'))' + (\lambda + \mu(t))\,\phi_p(u) = 0, \quad u = 0 \quad \text{in} \quad \partial J. \tag{2.2}$$

on a bounded interval $J \subset \mathbb{R}$ has a sequence of simple eigenvalues

$$-\infty < \lambda_1^D(\mu,J) < \lambda_2^D(\mu,J) < \cdots < \lambda_n^D(\mu,J) < \cdots .$$

In particular, when the potential μ is constant $(\mu(t) \equiv \mu)$, the eigenvalues are given explicitly by

$$\lambda_n^D(\mu,J) = \left(\frac{n\,\pi_p}{|J|}\right)^p - \mu, \quad n \in \mathbb{N},$$

where $|J|$ denotes the length of the interval J and π_p is defined in (1.5). Using the relationship between the Euler's beta and gamma functions, and, in particular, the relations

$$B(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)} \quad \text{and} \quad \Gamma(x)\,\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}$$

valid for $x \in (0,1)$, we can see that the formula

$$\pi_p := \frac{2\pi(p-1)^{1/p}}{p\,\sin(\pi/p)} \tag{2.3}$$

is true, as well.

For $1 \le \beta \le \infty$, $1 < p < \infty$ and an arbitrary closed bounded subinterval J of $\mathbb{R}$, we denote by $K(\beta,p,J)$ the best Sobolev constant in the inequality

$$C\,\|u\|_{\beta,J}^p \le \|u'\|_{p,J}^p \quad \text{for all } u \in W_0^{1,p}(J),$$

that is

$$K(\beta,p,J)=\inf\left\{\frac{\|u'\|_{p,J}^p}{\|u\|_{\beta,J}^p}\,:\,u\in W_0^{1,p}(J)\setminus\{0\}\right\}.$$

Put

$$\varkappa(\beta,p)=\left(\frac{2\left(1+\frac{\beta}{p^*}\right)^{1/\beta}B(\frac{1}{\beta},\frac{1}{p^*})}{\beta\left(1+\frac{p^*}{\beta}\right)^{1/p}}\right)^p\quad\text{for}\quad\beta\in[1,\infty)\quad\text{and}\quad p\in(1,\infty).$$

It is known, cf. Talenti [17, p. 357], Zhang [22, Theorem 4.1] or Drábek & Manásevich [10, Theorem 5.1], that

$$K(\beta,p,J)=\frac{\varkappa(\beta,p)}{|J|^{p-1+p/\beta}}\quad\text{for}\quad 1<\beta<\infty, \tag{2.4}$$

Let us notice that this result can be derived also from [2, Theorem 2], which seems to be the oldest reference to the relation (2.4). Furthermore, one can show, cf. Lemma 6.1 in Appendix, that the relations

$$K(1,p,J)=\frac{\varkappa(1,p)}{|J|^{2p-1}}\quad\text{and}\quad K(\infty,p,J)=\frac{2^p}{|J|^{p-1}} \tag{2.5}$$

are true, as well. Finally, notice that

$$\begin{aligned}\varkappa(\beta,p)&=\left(\frac{2}{\beta}\right)^p\left(\frac{p^*+\beta}{p^*}\right)^{p/\beta}\left(\frac{\beta}{p^*+\beta}\right)B^p(1/\beta,1/p^*)\\&=\left(\frac{2}{\beta}\right)^p\left(\frac{p^*+\beta}{p^*}\right)^{p/\beta}\left(\frac{\beta}{p^*+\beta}\right)\left(\frac{\Gamma(1/\beta)\,\Gamma(1/p^*)}{\Gamma(1/\beta+1/p^*)}\right)^p\end{aligned}$$

holds for $\beta\in[1,\infty)$ and $p\in(1,\infty)$.

In [24, Theorem 2.4] the following lower bound for the first Dirichlet eigenvalue $\lambda_1^D(\mu,J)$ is established in terms of the L_α–norm of the potential $\mu(t)$ and of the corresponding best Sobolev constants.

2.1. Theorem. *Let J be a bounded interval in $\mathbb{R}$. Furthermore, assume that $\mu\in L_\alpha(J)$ for some $1\le\alpha\le\infty$ and $\|\mu_+\|_{\alpha,J}\le K(p\,\alpha^*,p,J)$. Then*

$$\lambda_1^D(\mu,J)\ge\left(\frac{\pi_p}{|J|}\right)^p\left(1-\frac{\|\mu_+\|_{\alpha,J}}{K(p\,\alpha^*,p,J)}\right)\ge 0.$$

2.2. Remark. One can check that

$$K(p\alpha^*, p, J) = \begin{cases} \dfrac{2^p}{|J|^{p-1}} & \text{if } \alpha = 1, \\[2ex] \dfrac{\varkappa(p\,\alpha^*, p)}{|J|^{p-1/\alpha}} & \text{if } 1 < \alpha < \infty, \\[2ex] \left(\dfrac{\pi_p}{|J|}\right)^p & \text{if } \alpha = \infty, \end{cases} \tag{2.6}$$

where

$$\varkappa(p\,\alpha^*, p) = \left(\frac{2}{p}\right)^p \left(\frac{\alpha-1}{\alpha}\right)^p \frac{\alpha(p-1)}{(\alpha-1)^{1-1/\alpha}} \left(\frac{1}{p\,\alpha-1}\right)^{1/\alpha} B^p\left(\frac{\alpha-1}{p\alpha}, \frac{p-1}{p}\right) \text{ for } 1 < \alpha < \infty.$$

Furthermore, for each $p \in (1, \infty)$, the function $\alpha \to \varkappa(p\,\alpha^*, p)$ is increasing on $(1, \infty)$ and $\lim_{\alpha\to\infty} \varkappa(p\,\alpha^*, p) = \pi_p^p$.

Lower and upper functions. Let $f \in Car(J \times \mathbb{R})$. Then the function $\sigma \in C(J)$ is a *lower function* for problem

$$(\phi_p(u'))' = f(t, u), \quad u = 0 \text{ on } \partial J, \tag{2.7}$$

if $\sigma \le 0$ on ∂J and, for each $t_0 \in \mathrm{int}(J)$, either $\sigma'(t_0-) < \sigma'(t_0+)$, or there exists an open interval $J_0 \subset J$ such that $t_0 \in J_0$, $\sigma \in C^1(J_0)$, $\phi_p \circ \sigma' \in AC(J_0)$ and

$$(\phi_p(\sigma'(t)))' \ge f(t, \sigma(t)) \quad \text{for a.e. } t \in J_0.$$

When all the above inequalities are reversed we call σ an *upper function* for problem (2.7).

Arguing as in the proof of [9, Theorem 5], one can prove the following result, which asserts the solvability of (2.7) in the presence of a pair of well ordered lower and upper solutions.

2.3. Theorem. *Assume the existence of σ_1 and σ_2 a lower and an upper function respectively of problem (2.7) such that $\sigma_1 \le \sigma_2$ on J. Then problem (2.7) has a solution u such that $\sigma_1 \le u \le \sigma_2$ on J.*

Analogously to the Dirichlet problem, we define the lower and upper functions for the periodic problem

$$(\phi_p(u'))' = f(t, u), \quad u(0) = u(T), \quad u'(0) = u'(T). \tag{2.8}$$

We say that the function $\sigma \in C(J)$ is a *lower function* for the periodic boundary value problem (2.8) if $u(0) = u(T)$, $u'(0) \ge u'(T)$ and for each $t_0 \in \mathrm{int}(J)$, either

$\sigma'(t_0-) < \sigma'(t_0+)$, or there exists and open interval $J_0 \subset J$ such that $t_0 \in J_0$, $\sigma \in C^1(J_0)$, $\phi_p \circ \sigma' \in AC(J_0)$ and

$$(\phi_p(\sigma'(t)))' \geq f(t, \sigma(t)) \quad \text{for a.e. } t \in J_0.$$

When all the above inequalities are reversed we call σ an *upper function* for problem (2.8).

3. A generalized anti-maximum principle

The following proposition provides sufficient conditions for a Dirichlet problem to be *non-resonant* (that is, the unique solution of the homogeneous Dirichlet problem is the trivial one).

3.1. Proposition. *Let* $J = [a, b]$, $-\infty < a < b < \infty$ *and* $1 \leq \alpha \leq \infty$. *Furthermore, assume that* $\mu \in L_\alpha[a, b]$ *and*

$$\|\mu_+\|_{\alpha,J} \leq K(p\,\alpha^*, p, J).$$

Then, the Dirichlet problem

$$(\phi_p(u'))' + \mu(t)\,\phi_p(u) = 0 \quad a.e. \ on \ [t_1, t_2], \quad u(t_1) = u(t_2) = 0, \tag{3.1}$$

with $a \leq t_1 < t_2 \leq b$ *has only the trivial solution whenever* $t_2 - t_1 < b - a$.

Proof. Denote $\widetilde{J} = [t_1, t_2]$ and let $\widetilde{\mu}$ stand for the restriction of μ to the interval $\widetilde{J}$. Since $0 < t_2 - t_1 < b - a$ we have

$$\|\widetilde{\mu}_+\|_{\alpha,\widetilde{J}} \leq \|\mu_+\|_{\alpha,J} \leq \frac{\varkappa(p\,\alpha^*, p)}{(b-a)^{p-1/\alpha}} < \frac{\varkappa(p\,\alpha^*, p)}{(t_2-t_1)^{p-1/\alpha}} = K(p\,\alpha^*, p, \widetilde{J})$$

and Theorem 2.1 implies that $\lambda_1^D(\widetilde{\mu}, \widetilde{J}) > 0$ which means that (3.1) is nonresonant. □

3.2. Remark. Notice that the conclusion of Proposition 3.1 is true also if

$$\|\mu_+\|_\alpha < K(p\,\alpha^*, p, J) \quad \textit{and} \quad t_2 - t_1 = b - a.$$

3.3. Definition. Let $0 < T < \infty$. We say that problem (1.3) or

$$(\phi_p(u'))' + \mu(t)\,\phi_p(u) = h(t), \quad u(0) = u(T), \quad u'(0) \geq u'(T), \tag{3.2}$$

fulfils an *antimaximum principle* if, for each $h \in L_1[0, T]$ such that $h \geq 0$ on $[0, T]$, any solution of (1.3) or (3.2) is nonnegative on $[0, T]$, respectively.

Moreover, we say that problems (1.3) or (3.2) fulfil a *strong antimaximum principle* if they fulfil the anti-maximum principle and, in addition, $h \succ 0$ implies that each solution u of problem (1.3) or (3.2) is positive on $[0, T]$, respectively.

3.4. Remark. In other words, problem (3.2) fulfils the antimaximum principle if any lower function of the periodic problem for the quasilinear equation

$$(\phi_p(u'))' + \mu(t)\,\phi_p(u) = 0$$

is nonnegative.

In the next theorem we present our main result. Notice that, unlike [7, 19], the potential $\mu(t)$ need not be nonnegative for a.e. $t \in J$. Instead we suppose only $\bar{\mu} \geq 0$ and $\mu \not\equiv 0$ on J. In addition, we extend also [4, Theorem 3.2] where only the linear case (i.e. $p = 2$) with potential having positive mean value ($\bar{\mu} > 0$) was considered.

3.5. Theorem. *Let* $J = [0,T]$, $0 < T < \infty$, $\mu \in L_\alpha(J)$ *for some* α, $1 \leq \alpha \leq \infty$, $\bar{\mu} \geq 0$ *and*

$$0 < \|\mu_+\|_{\alpha,J} \leq K(p\,\alpha^*, p, J). \tag{3.3}$$

Then problem (3.2) *fulfils the strong anti-maximum principle.*

Proof. Let $h \in L_1(J)$, $h \geq 0$ a.e. on J and let u be an arbitrary solution to (3.2). In particular, $u \in C^1(J)$, $\phi_p \circ u' \in AC(J)$ and

$$(\phi_p(u'(t)))' + \mu(t)\,\phi_p(u(t)) = h(t) \text{ for a.e. } t \in J, \tag{3.4}$$

$$u(0) = u(T), \quad u'(0) \geq u'(T). \tag{3.5}$$

Claim 1. *u does not change its sign on J.*

Suppose, on the contrary, that

$$\left(\min_{t\in J} u(t)\right)\left(\max_{t\in J} u(t)\right) < 0.$$

Let us extend μ, h and u to functions T–periodic on $\mathbb{R}$. Then there are a, b, t_1, $t_2 \in \mathbb{R}$ such that $a < b$, $b - a = T$, $a \leq t_1 < t_2 \leq b$, $t_2 - t_1 < b - a$ and

$$u(t_1) = u(t_2) = 0, \quad u > 0 \text{ on } (t_1, t_2).$$

Denote $\widetilde{J} = [a, b]$ and notice that, due to the periodicity of μ, we have

$$\|\mu_+\|_{\alpha,\widetilde{J}} = \|\mu_+\|_{\alpha,J} \quad \text{and} \quad K(p\,\alpha^*, p, \widetilde{J}) = K(p\,\alpha^*, p, J). \tag{3.6}$$

In general, u need not belong to $C^1[t_1, t_2]$. However, in any case $\sigma_1 := u$ is a lower function for the Dirichlet problem (3.1) and it is positive on (t_1, t_2).

Further, consider the initial value problem

$$(\phi_p(v'))' + \mu(t)\,\phi_p(v) = 0, \quad v(t_1) = 0, \quad v'(t_1) = 1. \tag{3.7}$$

By Lemma 6.2 in Appendix this problem has a solution v defined on the whole $\mathbb{R}$. Moreover, due to (3.3) and (3.6), Proposition 3.1 implies that $v > 0$ on $(t_1, t_2]$.

Define $\sigma_2 := c\,v$, with $c > 0$ so large that $\sigma_2 \ge \sigma_1$ on $[t_1, t_2]$. Since $\sigma_2(t_2) > 0$, σ_2 is an upper function for (3.1). Hence, by Theorem 2.3, problem (3.1) possesses a nontrivial solution. This contradicts Proposition 3.1 and this completes the proof of the claim.

<u>Claim 2</u>. *If $u \le 0$ on J, then $u \equiv 0$ on J.*

First, assume that $u < 0$ on J. Then, dividing the equation (3.4) by $\phi_p(u(t))$ and integrating over J we arrive (after an integration by parts) to the following equality

$$\text{(3.8)}\qquad \begin{cases} (p-1)\displaystyle\int_0^T \left|\frac{u'(t)}{u(t)}\right|^p \mathrm{d}t + \int_0^T \mu(t)\,\mathrm{d}t \\[2ex] \qquad = \displaystyle\int_0^T \frac{h(t)}{\phi_p(u(t))}\,\mathrm{d}t + \left(\frac{\phi_p(u'(0))}{\phi_p(u(0))} - \frac{\phi_p(u'(T))}{\phi_p(u(T))}\right). \end{cases}$$

By our assumptions, both terms on the right-hand side of (3.8) are nonpositive. In particular, we attain a contradiction if $\bar{\mu} > 0$.

If $\bar{\mu} = 0$, then, having in mind that the right-hand side of (3.8) is certainly non-positive on J, we deduce that (3.8) can be true only if $u(t) \equiv u(0) < 0$ and $h = 0$ a.e. on J. On the other hand, in this situation, (3.4) reduces to

$$\mu(t)\,\phi_p(u(0)) = 0 \quad \text{for a.e.} \quad t \in J.$$

By (3.3) and (3.6), μ must be nonzero on a subset of J of a positive measure. This implies $u(0) = 0$, a contradiction.

Now, suppose that $u \le 0$ on J and there is a $t_0 \in J$ such that $u(t_0) = 0$. It is easy to see that if $t_0 \in (0, T)$ then $u'(t_0) = 0$ must be true. Furthermore, if $t_0 = 0$, i.e., in view of the boundary conditions (3.5), $u(0) = u(T) = 0$. This implies that the relations $u'(0) \le 0 \le u'(T)$ has to be satisfied, wherefrom, in view of the boundary conditions (3.5), we get easily that the equalities $u'(0) = 0 = u'(T)$ hold.

Thus, integrating equation (3.4) twice over the interval $[t_0, T]$, we derive the equality

$$u(t) = \int_{t_0}^t \phi_p^{-1}\left(-\int_{t_0}^s \mu(\tau)\,\phi_p(u(\tau))\,\mathrm{d}\tau + \int_{t_0}^s h(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s \quad \text{for} \ \ t \in [t_0, T].$$

Therefore,

$$\begin{aligned} |u(t)| = -u(t) &= \int_{t_0}^t \phi_p^{-1}\left(\int_{t_0}^s \mu(\tau)\,\phi_p(u(\tau))\,\mathrm{d}\tau - \int_{t_0}^s h(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s \\ &\le \int_{t_0}^t \phi_p^{-1}\left(\int_{t_0}^t |\mu(\tau)|\,\phi_p(|u(\tau)|)\,\mathrm{d}\tau\right)\mathrm{d}s \le T\,\phi_p^{-1}\left(\int_{t_0}^t |\mu(\tau)|\,\phi_p(|u(\tau)|)\,\mathrm{d}\tau\right) \end{aligned}$$

and

$$\phi_p(|u(t)|) \le \phi_p(T) \int_{t_0}^{t} |\mu(\tau)|\, \phi_p(|u(\tau)|)\, \mathrm{d}\tau \quad \text{hold for } \ t \in [t_0, T].$$

Hence, making use of Gronwall's lemma, we deduce that

$$|u(t)| = 0 \quad \text{for } \ t \in [t_0, T].$$

In particular, $u(0) = u(T) = 0$, and, repeating the above argument on $[0, t_0]$, we finally conclude that $u \equiv 0$ on J.

Claim 3. Problem (3.2) fulfils the anti-maximum principle.

Indeed, if $h \ge 0$ a.e. on J and u is a solution of (3.2), then, by Claims 1 and 2, $u \ge 0$ on J.

Now we are going to prove that the anti-maximum principle is actually strong.

Claim 4. Let $h \succ 0$ and let u be a solution of (3.2). Then $u > 0$ on J.

By Claim 3 we know that $u \ge 0$ on J and, since $h \succ 0$, u can not vanish on the whole interval J. Suppose that there is $t_0 \in J$ such that $u(t_0) = 0$. As in the proof of Claim 1, let us extend μ, h and u to functions T–periodic on $\mathbb{R}$. Let a, b, t_1, $t_2 \in \mathbb{R}$ be such that $a < b$, $b - a = T$, $a \le t_1 < t_2 \le b$, $t_2 - t_1 < b - a$ and

$$u(t_1) = u(t_2) = 0, \quad u > 0 \text{ on } (t_1, t_2).$$

Denote again $\widetilde{J} = [a, b]$.

If $0 < t_2 - t_1 < T$, then the same argument as that used in the proof of Claim 1 leads us to a contradiction. Therefore, it is $t_1 = a$, $t_2 = b$ and, in particular, $t_2 - t_1 = T$. Let us recall that the periodic extension of u need not belong to $C^1[a, b]$, as it is possible that $k\,T \in (a, b)$ for some $k \in \mathbb{N} \cup \{0\}$. If this is the case, then denoting by c such a point, we obtain $u'(c-) \le u'(c+)$ and u verifies the equality

$$(\phi_p(u'(t)))' + \mu(t)\, \phi_p(u(t)) = h(t)$$

for a.e. $t \in [a, c]$ as well as for a.e. $t \in [c, b]$. Multiplying this equation by u, integrating it over $[a, c]$ and $[c, b]$ and adding both results, we obtain

$$\begin{aligned}
&\int_a^b h(s)\, u(s)\, \mathrm{d}s \\
&= -\int_a^b \phi_p(u'(s))\, u'(s)\, \mathrm{d}s + \int_a^b \mu(s)\, \phi_p(u(s))\, u(s)\, \mathrm{d}s + \big(\phi_p(u'(c^-)) - \phi_p(u'(c^+))\big)\, u(c) \\
&\le -\int_a^b \phi_p(u'(s))\, u'(s)\, \mathrm{d}s + \int_a^b \mu(s)\, \phi_p(u(s))\, u(s)\, \mathrm{d}s.
\end{aligned}$$

Furthermore, having in mind the definition of $K(\beta, p, J)$, and applying the Hölder's inequality, we get

$$0 \le \int_a^b h(s)\, u(s)\, \mathrm{d}s \le -\|u'\|^p_{p,\widetilde{J}} + \|\mu_+\|_{\alpha,\widetilde{J}} \|u\|^p_{\alpha^* p,\widetilde{J}}$$

$$\le \Big(\|\mu_+\|_{\alpha,\widetilde{J}} - K(p\,\alpha^*, p, \widetilde{J})\Big)\, \|u\|^p_{\alpha^*\,p,\widetilde{J}} = (\|\mu_+\|_{\alpha,J} - K(p\,\alpha^*, p, J))\, \|u\|^p_{\alpha^*\,p,J} \le 0.$$

Since $u > 0$ on (a, b), this is possible only if $h \equiv 0$ on $[a, b]$, i.e. $h \equiv 0$ on J, which contradicts our assumption $h \succ 0$. □

Notice that, arguing as in the Claim 4 of the previous proof, we can derive also the following result.

3.6. Corollary. *Under the conditions of Theorem 3.5, $u > 0$ on $[0, T]$ holds for each solution u on $[0, T]$ of the equation*

$$(\phi_p(u'))' + \mu(t)\, \phi_p(u) = 0$$

fulfilling the boundary conditions $u(0) = u(T)$, $u'(0) > u'(T)$.

3.7. Example. Let us consider the sign-changing potential $\mu(t) = a\,(1 + b\,\cos(t))$, where $a > 0$ and $b \in \mathbb{R}$, and the problem

$$(\phi_p(x'))' + a\,(1 + b\,\cos(t))\, \phi_p(x) = h(t), \quad x(0) = x(2\pi), \quad x'(0) \ge x'(2\pi). \tag{3.9}$$

For $p = 2$, $b = 1$ (i.e. potential μ is nonnegative) and periodic boundary conditions (i.e. $x'(0) = x'(2\pi)$), it is known that (3.9) satisfies the strong anti-maximum principle when $0 < a < 0.16448$, which is a key ingredient in [18, Corollary 4.8] to ensure the solvability of the Brillouin electron beam-focusing equation

$$x'' + a\,(1 + \cos(t))\, x = \frac{1}{x^\lambda}, \quad x(0) = x(2\pi), \quad x'(0) = x'(2\pi).$$

(For more information about this problem, see e.g. [1] or [21, Example 4.4].)

Our Theorem 3.5 ensures that problem (3.9) satisfies the strong anti-maximum principle provided that $a > 0$ and

$$0 < a\,\|(1 + b\,\cos(t))_+\|_{\alpha,[0,2\pi]} \le K(p\,\alpha^*, p, [0, 2\pi]) \quad \text{for some} \quad \alpha \in [1, \infty]. \tag{3.10}$$

So, for fixed $p > 1$ and $b \in \mathbb{R}$ condition (3.10) is satisfied when

$$0 < a < M(p, b) := \sup_{\alpha\in[1,\infty]} \frac{K(p\,\alpha^*, p, [0, 2\pi])}{\|(1 + b\,\cos(t))_+\|_{\alpha,[0,2\pi]}}.$$

Figure 3.7 was obtained by means of the software system Mathematica. It gives the graph of function $M(p, .)$ for several fixed values of p. Let us recall that $M(2, 0) = 1/4$ and $M(2, 1) \doteq 0.16448$.

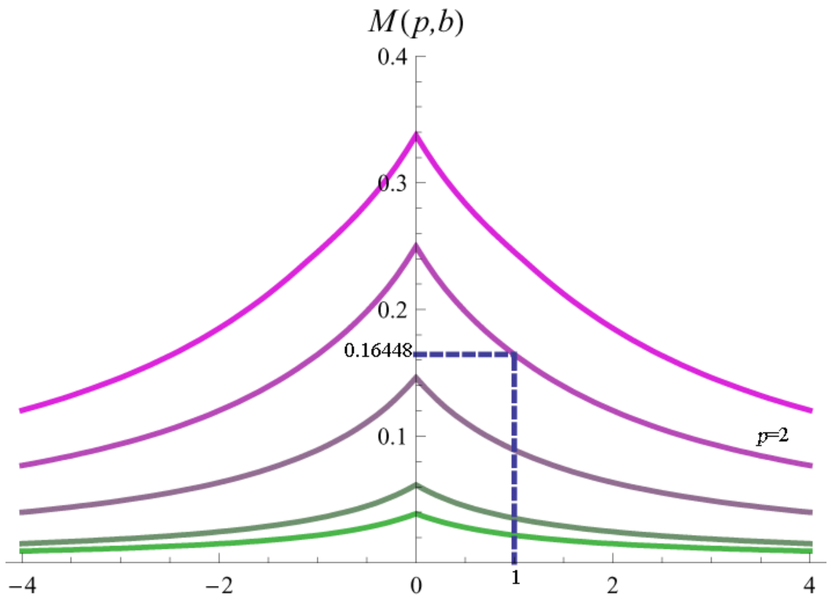

Figure 1: Graph of $M(p,b)$ for $p = 1.5,\ 2,\ 2.7,\ 3.7,\ 4.2$ (from above to below)

3.8. Remark. Theorem 3.5 generalizes [4, Theorem 3.2] where the authors proved that problem (1.3) fulfills the strong anti–maximum principle whenever $p = 2$, $\bar{\mu} > 0$ and $\|\mu_+\|_{\alpha,J} < K(2\,\alpha^*, 2, J)$ for some α, $1 \le \alpha \le \infty$. Actually, in [4, Theorem 3.2] it is proved that under the above conditions problem (1.3) has a unique solution given by the expression

$$u(t) = \int_0^T G(t,s)\,h(s)\,\mathrm{d}s, \quad t \in J, \tag{3.11}$$

where G is the corresponding Green's function and G is positive on $J \times J$. Moreover, one can see in [18] that problem (1.3) with $p = 2$ fulfils the anti–maximum principle if and only if G is nonnegative on $J \times J$.

On the other hand, the particular choice of the constant potential $\mu = (\pi/|J|)^2$ (for which $\|\mu_+\|_{\infty,J} = (\pi/|J|)^2 = K(2,2,J)$) shows that it is not possible to guarantee the strict positivity of G on $J \times J$ when the equality is attained in the norm estimate (see [3, Lemma 2.5]). In general for $p = 2$, it follows from (3.11) that if problem (1.3) fulfills the strong anti–maximum principle then, for any $t \in J$, the function $G(t,\cdot)$ vanish on, at most, a set of Lebesgue measure zero. Since $u'' + \mu(t)\,u$ coupled with the periodic boundary value conditions generates a self–adjoint operator, the Green's function G is symmetric and, as consequence, the zeroes of $G(\cdot,s)$ form a set of Lebesgue measure zero, as well.

4. Applications to singular periodic problems

Consider singular periodic problems of the form

$$(\phi_p(u'))' = f(t,u), \quad u(0) = u(T), \quad u'(0) = u'(T), \tag{4.1}$$

where $0 < T < \infty$, $1 < p < \infty$, and $f \in Car([0,T] \times (0,\infty))$.

The following existence principle which relies on the comparison of the given problem (4.1) with a related quasilinear problem fulfilling the antimaximum principle has been proved in [7, Theorems 4.2 and 4.3] (see also [14, Theorems 8.28 and 8.29]).

4.1. Theorem. *Let $r > 0$, $A \geq r$, $B \geq A$, μ, $\beta \in L_1[0,T]$ be such that $\mu \geq 0$ a.e. on $[0,T]$, $\bar{\mu} > 0$, $\bar{\beta} \leq 0$ (with $\overline{\beta} < 0$ if $1 < p < 2$), (3.2) fulfils the antimaximum principle,*

$$f(t,x) \leq \beta(t) \quad \text{for a.e. } t \in [0,T] \text{ and all } x \in [A,B], \tag{4.2}$$

$$f(t,x) + \mu(t)\,\phi_p(x-r) \geq 0 \quad \text{for a.e. } t \in [0,T] \text{ and all } x \in [r,B] \tag{4.3}$$

and

$$B - A \geq \frac{T}{2}\,\|m\|_1^{p^*-1}, \tag{4.4}$$

where

$$m(t) = \max\big\{\sup\{f(t,x) \colon x \in [r,A]\}, \beta(t), 0\big\} \quad \text{for a.e. } t \in [0,T]. \tag{4.5}$$

Then problem (4.1) has a solution u such that

$$r \leq u \leq B \quad \text{on } [0,T] \quad \text{and} \quad \|u'\|_\infty < \phi_p^{-1}(\|m\|_1). \tag{4.6}$$

Taking into account that, if $r > 0$, $B \geq A$, $1 \leq \alpha \leq \infty$, $\mu \in L_\alpha[0,T]$, $\bar{\mu} \geq 0$ and relations (3.3) and (4.3) are true, then the same assumptions are satisfied also with μ_+ in place of μ and, moreover $\overline{\mu_+} > 0$, we can see that Theorem 4.1 may be reformulated as follows.

4.2. Theorem. *Let $r > 0$, $A \geq r$, $B \geq A$. Furthermore, let $\beta \in L_1[0,T]$ be such that $\bar{\beta} \leq 0$ (with $\overline{\beta} < 0$ if $1 < p < 2$) and let μ verify the assumptions of Theorem 3.5 for some α, $1 \leq \alpha \leq \infty$, and let the relations (4.2)–(4.5) be satisfied.*
Then problem (4.1) has a solution u such that (4.6) is true.

In what follows we will consider the periodic problem for the Duffing type equation

$$(\phi_p(u'))' + a(t)\,\phi_p(u) = g(u) + e(t), \quad u(0) = u(T), \quad u'(0) = u'(T), \tag{4.7}$$

where

$$1 < p < \infty, \quad g \in C(0,\infty), \quad e \in L_1[0,T] \quad \text{and} \quad a \in L_\infty[0,T]. \tag{4.8}$$

Furthermore, we denote $g_* = \inf \operatorname{ess}_{x\,\in\,(0,\infty)}\, g(x)$,

$$c\,(p) = \begin{cases} 2^{p-2} & \text{if } \ 1 < p < 2, \\ 1 & \text{if } \ 2 \le p < \infty \end{cases} \quad \text{and} \quad d(p) = \begin{cases} 1 & \text{if } \ 1 < p < 2, \\ 2^{p-2} & \text{if } \ 2 \le p < \infty. \end{cases}$$

It is well known (cf. e.g. [13, Section VIII.4.2] that

$$c\,(p)\,(x^{p-1} + y^{p-1}) \le (x+y)^{p-1} \le d(p)\,(x^{p-1} + y^{p-1}) \tag{4.9}$$

hold for all $x,\ y \in [0,\infty)$.

Next assertion is an immediate consequence of Theorem 4.2.

4.3. Corollary. *Assume* (4.8) *and*

$$\bar{e} + \limsup_{x\to\infty} \big(g(x) - a_*\, x^{p-1}\big) < 0. \tag{4.10}$$

Furthermore, let there be $\mu \in L_\infty[0,T]$ *fulfilling the assumptions of Theorem* 3.5 *for some* $\alpha,\ 1 \le \alpha \le \infty,$ *and such that the relation*

$$e_* + \inf_{x>0} \left(g(x) + \left(\frac{\mu_{+\,*}}{d(p)} - a^* \right) x^{p-1} \right) > 0 \tag{4.11}$$

is true.

Then problem (4.7) *has a positive solution* u.

Proof. Put $f(t,x) = e(t) + g(x) - a(t)\, x^{p-1}$ for $x \in (0,\infty)$ and a.e. $t \in [0,T]$.

Step 1. By (4.9), we have

$$\begin{aligned} f(t,x) + \mu_+(t)\,(x-r)^{p-1} &= e(t) + g(x) - a(t)\,x^{p-1} + \mu_+(t)\,(x-r)^{p-1} \\ &\ge e_* + g(x) + \left(\frac{\mu_+(t)}{d(p)} - a(t) \right) x^{p-1} - \mu_+(t)\, r^{p-1} \\ &\ge e_* + \inf_{x>0} \left(g(x) + \left(\frac{\mu_{+\,*}}{d(p)} - a^* \right) x^{p-1} \right) - \mu_+^*\, r^{p-1} \end{aligned}$$

for a.e. $t \in [0,T]$, all $r \in (0,\infty)$ and all $x \in [r,\infty)$. Therefore, having in mind (4.11), we can see that f satisfies (4.3) with

$$r = \left(\frac{1}{\mu_+^*} \left(e_* + \inf_{x>0} \left(g(x) + \left(\frac{\mu_{+\,*}}{d(p)} - a^* \right) x^{p-1} \right) \right) \right)^{\frac{1}{p-1}}$$

and $B > r$ arbitrarily large.

Step 2. If $\limsup_{x\to\infty}\big(g(x)-a_*\,x^{p-1}\big)=-\infty$, we choose $A\ge r$ so that

$$g(x)-a_*\,x^{p-1}<-\bar e-1\quad\text{for}\ \ x\ge A$$

and define $\beta(t):=e(t)-\bar e-1$. Then $\bar\beta=-1<0$ and (4.2) holds with $B>A$ arbitrarily large.

If $\limsup_{x\to\infty}\big(g(x)-a_*\,x^{p-1}\big)>-\infty$, then, due to (4.10), we can find $A\ge r$ such that

$$\bar e+g(x)-a_*\,x^{p-1}<\frac{1}{2}\left(\bar e+\limsup_{x\to\infty}\big(g(x)-a_*\,x^{p-1}\big)\right)<0\quad\text{for}\ \ x\in[A,\infty).$$

Consequently,

$$\begin{aligned}f(t,x)&=\big(\bar e+g(x)-a_*\,x^{p-1}\big)+(a_*-a(t))\,x^{p-1}+(e(t)-\bar e)\\&<\frac{1}{2}\left(\bar e+\limsup_{x\to\infty}(g(x)-a_*\,x^{p-1})\right)+(e(t)-\bar e)\end{aligned}$$

holds for a.e. $t\in[0,T]$ and all $x\in[A,\infty)$. Therefore, (4.2) is satisfied with

$$\beta(t):=\tfrac{1}{2}\left(\bar e+\limsup_{x\to\infty}(g(x)-a_*\,x^{p-1})\right)+(e(t)-\bar e)$$

and $B>A$ arbitrarily large.

Step 3. Theorem 4.2 yields the existence of a solution u to (4.7) such that $u\ge r$ on $[0,T]$. □

4.4. Remark. Let $g(x)=\gamma\,x^{-\lambda}$ with $\lambda,\,\gamma\in(0,\infty)$ and let $\mu\in L_\infty[0,T]$ fulfil the assumptions of Theorem 3.5 for some α, $1\le\alpha\le\infty$. (Notice that this will be certainly true if $\mu^*\le\big(\frac{\pi_p}{T}\big)^p$.) Furthermore, one can see that (4.10) can be satisfied only if $a_*>0$ or $a_*=0$ and $\bar e<0$. Define

$$h(x):=g(x)+\left(\frac{\mu_{+*}}{d(p)}-a^*\right)\,x^{p-1}\quad\text{for}\ \ x\in(0,\infty).$$

Thus, condition (4.11) reduces to $e_*+\inf_{x\in(0,\infty)}h(x)>0$. If $\mu_{+*}<d(p)\,a^*$, then $\lim_{x\to\infty}h(x)=-\infty$. Hence, in such a case, assumption (4.11) can not be satisfied. On the other hand, it is easy to verify that assumption (4.11) will be satisfied whenever

$$\mu_{+*}>d(p)\,a^*\quad\text{and}\quad e_*+h(x_0)>0,\quad\text{where}\ \ x_0=\left(\frac{\gamma\,\lambda}{(p-1)\,\left(\frac{\mu_{+*}}{d(p)}-a^*\right)}\right)^{\frac{1}{\lambda+p-1}}$$

or

$\mu_{+*} = d(p)\, a^* \quad \text{and} \quad e_* > 0.$

Recall that for the particular choices $\gamma = 1$ and

$$p = 2,\ \alpha = +\infty,\ \mu(t) \equiv (\pi/T)^2 \quad \text{and} \quad a(t) \equiv k \in (0, (\pi/T)^2)$$

or

$$\alpha = +\infty,\ \mu(t) \equiv (\pi_p/T)^p \quad \text{and} \quad a(t) \equiv k \in (0, (\pi_p/T)^p),$$

more detailed results can be found also in [7, Corollary 4.5] or [16, Corollary 3.7], respectively.

5. Final comments

1.- Monotone method. For the linear case ($p = 2$), besides the existence result expressed in the previous theorem, it is possible to perform a monotone iteration, as in [19, Theorem 3.2], starting at the lower (upper) solution and converging to the minimal (maximal) solution. However for the nonlinear case, $p \neq 2$, Theorem 3.5 is not strong enough to develop the monotone method. In order to do this, as one can see in [5] for a Neumann boundary value problem, we would need the following strong version

$$\left.\begin{array}{c} (\phi_p(u'))' + \mu(t)\,\phi_p(u) \geq (\phi_p(v'))' + \mu(t)\,\phi_p(v), \\ u(a) - u(b) = 0 = v(a) - v(b), \\ u'(a) - u'(b) \geq 0 \geq v'(a) - v'(b). \end{array}\right\} \Longrightarrow u \geq v.$$

2.- Nonhomogeneous problems. It would be interesting to give an anti-maximum principle for the equation

$$(\phi_p(u'(t)))' + \mu(t)\,\phi_q(u(t)) = h(t), \quad t \in [a,b], \quad 1 \leq p, q \leq \infty, \tag{5.1}$$

together with different kinds of boundary conditions. The set of eigenvalues and eigenfunctions for the equation

$$\phi_p(u'(t))' + \mu\,\phi_q(u(t)) = 0,$$

with constant $\mu \in \mathbb{R}$ and Dirichlet, Neumann or periodic boundary conditions, has been described in [10]. However as far as the authors are aware, only anti-maximum principles has been studied for equation (5.1) with Neumann boundary conditions, $1 \leq p \leq \infty$ and $q = 2$ in [5].

3.- Sign-changing potential. In Theorem 3.5 we deal with an indefinite potential $\mu(t)$ (i.e., $\mu(t)$ can change sign) such that $\bar{\mu} > 0$. Anti-maximum principles for the p-Laplacian under Neumann or Dirichlet boundary conditions with indefinite potential has been investigated in the past decades (see [12] and references therein).

6. Appendix

6.1. Lemma. *The relations (see (2.5))*

$$K(1,p,J) = \frac{\varkappa(1,p)}{|J|^{2p-1}} \quad \textit{and} \quad K(\infty,p,J) = \frac{2^p}{|J|^{p-1}}$$

hold for each p, $1 < p < \infty$, *and each bounded interval* $J \subset \mathbb{R}$.

Proof. Recall that, by [17], $K(\beta,p,J)$ is given by (2.4) if $1 < \beta < \infty$.

<u>*Case* $\beta = 1$</u>. Letting $\beta \to 1$ in the relation

$$K(\beta,p,J)\,\|u\|^p_{\beta,J} \le \|u'\|^p_{p,J} \quad \text{for all } u \in W^{1,p}_0(J), \tag{6.1}$$

we obtain

$$\frac{\varkappa(1,p)}{|J|^{2p-1}}\,\|u\|^p_{1,J} \le \|u'\|^p_{p,J} \quad \text{for all } u \in W^{1,p}_0(J). \tag{6.2}$$

Moreover, as noticed in [24, Remark 2.2 (i)], the equality in (6.2) is achieved by some $u \in W^{1,p}_0(J)$. This proves the equality $K(1,p,J) = \dfrac{\varkappa(1,p)}{|J|^{2p-1}}$.

<u>*Case* $\beta = \infty$</u>. First, recall that $\lim_{\beta\to\infty} \|u\|^p_{\beta,J} = \|u\|^p_{\infty,J}$ holds for all $u \in W^{1,p}_0(J)$ and all $p \in (1,\infty)$ (cf. e.g. [20, Theorem I.3.1]). Therefore, letting $\beta \to \infty$ in (6.1), we get

$$\frac{2^p}{|J|^{p-1}}\,\|u\|^p_{\infty,J} \le \|u'\|^p_{p,J} \quad \text{for all } u \in W^{1,p}_0(J).$$

On the other hand, let $C > 0$ be an arbitrary constant such that

$$C\,\|u\|^p_{\infty,J} \le \|u'\|^p_{p,J} \quad \text{for all } u \in W^{1,p}_0(J).$$

Since $\|u\|^p_{\beta,J} \le \|u\|^p_{\infty,J}\,|J|^{p/\beta}$, we have

$$\frac{C}{|J|^{p/\beta}}\,\|u\|^p_{\beta,J} \le C\,\|u\|^p_{\infty,J} \le \|u'\|^p_{p,J} \quad \text{for all } \ u \in W^{1,p}_0(J),$$

and, by the definition of $K(\beta,p,J)$, it follows that

$$\frac{C}{|J|^{p/\beta}} \le K(\beta,p,J).$$

Thus, letting $\beta \to \infty$, we get

$$C \le \frac{2^p}{|J|^{p-1}},$$

wherefrom the equality $K(\infty,p,J) = \dfrac{2^p}{|J|^{p-1}}$ immediately follows. □

The following lemma is essentially [11, Proposition 3.2], where the assumption on the positivity of the potential μ was needed only to show the uniqueness of the obtained solution. We include the proof for the convenience of the reader.

6.2. Lemma. *Suppose that* $\mu \in L_{1,loc}(\mathbb{R})$*. Then for each* $t_0 \in \mathbb{R}$ *and* $\widetilde{x}, \widetilde{y} \in \mathbb{R}$*, the initial value problem*

$$(\phi_p(u'))' + \mu(t)\,\phi_p(u) = 0, \quad u(t_0) = \widetilde{x},\ u'(t_0) = \widetilde{y} \tag{6.3}$$

possesses at least one solution defined on the whole $\mathbb{R}$*.*

Proof. The existence of a local solution of (6.3) defined on the interval $(t_0 - \delta, t_0 + \delta)$ for some $\delta > 0$ is a direct consequence of the classical Carathéodory theory for ordinary differential equations.

Assume that u is a solution to (6.3) on (a_1, b_1) and $t_0 \in (a_1, b_1)$ To prove that this solution can be extended to the whole $\mathbb{R}$, it suffices to show that there are constants $M_0,\ M_1 \in (0, \infty)$ such that the relations

$$|u(t)| \le M_0 \quad \text{and} \quad |u'(t)| \le M_1 \tag{6.4}$$

are true for all $t \in (a_1, b_1)$. To this aim, assume that an arbitrary $t \in [t_0, b_1)$ is given. Integrating the differential equation in (6.3) over $[t_0, t]$, we obtain

$$|u'(t)|^{p-1} \le |\widetilde{y}|^{p-1} + v(t) \tag{6.5}$$

and

$$|u(t)| \le |\widetilde{x}| + \int_{t_0}^{t} \left(|\widetilde{y}|^{p-1} + v(s)\right)^{\frac{1}{p-1}} \mathrm{d}s \tag{6.6}$$

where

$$v(t) = \int_{t_0}^{t} |\mu(s)|\, |u(s)|^{p-1}\, \mathrm{d}s. \tag{6.7}$$

Next, using (4.9), we get

$$\begin{aligned}
|u(t)|^{p-1} &\le \left(|\widetilde{x}| + \int_{t_0}^{t} \left(|\widetilde{y}|^{p-1} + v(s)\right)^{\frac{1}{p-1}} \mathrm{d}s\right)^{p-1} \\
&\le \left(|\widetilde{x}| + d(p^*) \int_{t_0}^{t} \left(|\widetilde{y}| + v(s)^{\frac{1}{p-1}}\right) \mathrm{d}s\right)^{p-1} \\
&\le d(p) \left((|\widetilde{x}| + (b_1 - a_1)\, d(p^*)\, |\widetilde{y}|)^{p-1} + ((b_1 - a_1)\, d(p^*))^{p-1}\ v(t)\right) \le C\,(1 + v(t))
\end{aligned}$$

where

$$C = \max\{d(p)\,(|\widetilde{x}| + (b_1 - a_1)\, d(p^*)\, |\widetilde{y}|)^{p-1}, d(p)\ ((b_1 - a_1)\, d(p^*))^{p-1}\}.$$

To summarize, we have

$$|u(t)|^{p-1} \le C\,(1 + v(t)) \quad \text{for all} \ \ t \in [t_0, b_1). \tag{6.8}$$

Therefore, differentiating (6.7), we find that the inequality

$$v'(t) \le C\,|\mu(t)|\,(1 + v(t))$$

holds for a.e. $t \in [t_0, b_1)$. Hence, by Gronwall's inequality, there is a constant $\widetilde{C} \in (0, \infty)$ such that $v(t) \le \widetilde{C}$ for all $t \in [t_0, b_1)$. Now, by (6.8) and (6.5), we conclude that

$$|u(t)| \le (C\,(1+\widetilde{C}))^{\frac{1}{p-1}} \quad \text{and} \quad |u'(t)| \le (|\widetilde{y}|^{p-1} + \widetilde{C})^{\frac{1}{p-1}} \quad \text{for all} \quad t \in [t_0, b_1),$$

i.e. (6.4) is true for $t \in [t_0, b_1)$. Analogously, we would show that (6.4) is true also for $t \in (a_1, t_0]$. □

References

[1] V. Bevc., J. L. Palmer and C. Süsskind, On the design of the transition region of axi-symmetric magnetically focused beam valves, *J. British Inst. Radio Engineers*, **18** (1958), 696708.

[2] D.W. Boyd, Best constants in a class of integral inequalities, *Pacific J. Math.*, **30** (1969), 367–383.

[3] A. Cabada, The method of lower and upper solutions for second, third, fourth, and higher order boundary value problems, *J. Math. Anal. Appl.*, **185** (1994), 2, 302–320.

[4] A. Cabada and J. A. Cid, On the sign of the Green's function associated to Hill's equation with an indefinite potential, *Appl. Math. Comput.* **205** (2008) 303-308.

[5] A. Cabada, P. Habets and R.L. Pouso, Optimal existence conditions for ϕ–Laplacian equations with upper and lower solutions in the reversed order, *J. Differential Equations*, **166** (2000), 385–401.

[6] A. Cabada and S. Lois, Existence results for nonlinear problems with separated boundary conditions, *Nonlinear Anal.*, **35** (1999), no. 4, 449–456.

[7] A. Cabada, A. Lomtatidze and M. Tvrdý, Periodic problem involving quasilinear differential operator and weak singularity, *Adv. Nonlinear Stud.*, **7** (2007), no. 4, 629–649.

[8] J. Campos, J. Mawhin and R. Ortega, Maximum principles around an eigenvalue with constant eigenfunctions, *to appear in Comunications in Contemporary Mathematics.*

[9] M. Cherpion, C. De Coster and P. Habets, Monotone iterative methods for boundary value problems, *Differential Integral Equations*, **12** (1999), no. 3, 309–338.

[10] P. Drábek and R. Manásevich, On the closed solution to some nonhomogeneous eigenvalue problems with p-Laplacian, *Differential Integral Equations*, **12** (1999), no. 6, 773–788.

[11] M. García-Huidobro, R. Manásevich and M. Ótani, Existence results for p-Laplacian-like systems of ODE's *Funkcialaj Ekvacioj*, **46** (2003), no. 2, 253–285.

[12] T. Godoy, J.P. Gossez and S. Paczka, On the antimaximum principle for the p-Laplacian with indefinite weight, *Nonlinear Anal. T.M.A.*, **51** (2002), no. 3, 449–467.

[13] T.H. Hildebrandt, *Introduction to the Theory of Integration*, Academic Press, New York - London, 1963.

[14] I. Rachůnková, S. Staněk and M. Tvrdý, *Solvability of Nonlinear Singular problems for Ordinary Differential Equations*, Contemporary Mathematics and Its Applications, Vol. 5, Hindawi Publishing Corporation, in print.

[15] I. Rachůnková and M. Tvrdý, Periodic problems with $\phi-$Laplacian involving non-ordered lower and upper functions, *Fixed Point Theory*, **6** (2005), 99–112.

[16] I. Rachůnková, M. Tvrdý and I. Vrkoč, Existence of nonnegative and nonpositive solutions for second order periodic boundary value problems, *J. Differential Equations*, **176** (2001), 445–469.

[17] G. Talenti, Best constant in Sobolev inequality, *Ann. Mat. Pura Appl.*, **(4)** (1976), no. 110, 353–372.

[18] P. Torres, Existence of one-signed periodic solutions of some second-order differential equations via a Krasnoselskii fixed point theorem, *J. Differential Equations*, **190** (2003), 643–662.

[19] P.J. Torres and M. Zhang, A monotone iterative scheme for a nonlinear second order equation based on a generalized anti-maximum principle, *Math. Nachr.*, **251** (2003), 101–107.

[20] K. Yosida, *Functional Analysis*, Springer Verlag, Berlin - Heidelberg - New York, 1978.

[21] M. Zhang, A relationship between the periodic and the Dirichlet BVPs of singular differential equations, *Proc. Roy. Soc. Edinburgh Sect. A*, **128** (1998) 1099–1114.

[22] M. Zhang, Nonuniform Nonresonance of Semilinear Differential Equations, *J. Differential Equations*, **166** (2000), 33–50.

[23] M. Zhang, The rotation number approach to eigenvalues of the one-dimensional p-Laplacian with periodic potentials, *J. London Math. Soc. (2)*, **64** (2001), no. 1, 125–143.

[24] M. Zhang, Certain classes of potentials for p-Laplacian to be non-degenerate, *Math. Nachr.*, **278** (2005), no. 15, 1823–1836.

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