Some constructive bounds on Ramsey numbers

Alexandr Kostochka*, Pavel Pudlák†, and Vojtech Rödl‡

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Abstract

We present explicit constructions of three families of graphs that yield the following lower bounds on Ramsey numbers: $R(4,m) \geq \Omega(m^{8/5}),\ R(5,m) \geq \Omega(m^{5/3}),\ R(6,m) \geq \Omega(m^2)$.

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1 Introduction

For two positive integers s and m, the Ramsey number $R(s,m)$ is the least integer R such that every graph on R vertices contains either a clique of size s or an independent set of size m. The best lower bounds on Ramsey numbers are proven by probabilistic methods which do not give explicit constructions of graphs without a clique of size s and an independent set of size m. All known constructions give weaker lower bounds.

In this note we show constructions that give nontrivial lower bounds on the Ramsey numbers $R(s,m)$ for $s = 4, 5, 6$. We describe three families of explicitly defined graphs that give the following bounds:

$$R(4,m) \geq \Omega(m^{8/5}),\ R(5,m) \geq \Omega(m^{5/3}),\ R(6,m) \geq \Omega(m^2).$$

These graphs are defined similarly to the graphs introduced in [6], which give the constructive lower bound

$$R(3,m) \geq \Omega(m^{3/2}). \tag{1}$$

In [6] the lower bound was proved using an algebraic argument. For the sake of completeness, we repeat this construction and give a combinatorial proof of the

*Department of Mathematics, University of Illinois, Urbana, IL 61801, USA and Sobolev Institute of Mathematics, Novosibirsk-90, 630090, Russia. E-mail: kostochk@math.uiuc.edu. Research of this author was supported by NSF grant DMS-0650784 and grant 08-01-00673 of the Russian Foundation for Fundamental Research.

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‡Emory University, Atlanta, GA, USA, partially supported by NSF grant DMS 0800070.

lower bound. The first constructive lower bound of the form (1) was found by Alon [1].

A different generalization of the construction of [6] was presented in [2]. That construction gives polynomial lower bounds for $R(s, m)$ with the exponent of the polynomials increasing with s. However, for $s = 3, 4, 5, 6$, the construction does not give anything interesting.

To the best of our knowledge, the above bounds are the best known constructive bounds. However, they are far from nonconstructive lower bounds [7, 10]:

$$R(3, m) \geq \Omega(m^2/\log m),\ R(4, m) \geq \Omega((m/\log m)^{2.5}),$$
$$R(5, m) \geq \Omega((m/\log m)^3),\ R(6, m) \geq \Omega((m/\log m)^{3.5}).$$

2 Preliminaries

Let G be a graph. *The superline graph*, H_G, of G is constructed as follows. The vertices of H_G are the edges of G, and ef is an edge in H_G if e and f are disjoint edges of G and there exists an edge g of G that connects an end of e with an end of f (i.e., if the edges e, g and f form a path in G).

Let $\alpha(G)$ denote the *independence number* of G, i.e., the size of a largest independent set in G.

Lemma 2.1 *For every triangle-free graph G and its superline graph H_G,*

$$\alpha(H_G) \leq \alpha(G).$$

Proof. Let A be an independent set in H_G. Let B be the set of vertices in G of the edges in A. Then the subgraph $G[B]$ of G induced by B has no triangles and does not contain paths of length 3. So, the components of $G[B]$ are stars, hence $G[B]$ has an independent set of size $|A|$. ■

We also need the following modification of the superline graph construction. Let G be a bipartite graph with bipartition of vertices (U, V) and let $\prec$ be a linear ordering of the edges of G. We denote by $H_G^{\prec}$ the graph whose vertices are the edges of G and a pair $\{uv, u'v'\}$, with $u \neq u' \in U$ and $v \neq v' \in V$ is an edge in $H_G^{\prec}$ if either $uv \prec u'v'$ and uv' is an edge in G or $u'v' \prec uv$ and $u'v$ is an edge in G. (In particular, $H_G^{\prec}$ is a subgraph of H_G.)

Lemma 2.2 *For every bipartite graph G on n vertices, every bipartition of G and every ordering $\prec$ of its edges,*

$$\alpha(H_G^{\prec}) < n.$$

Proof. Let A be a set of n edges of G. Then A contains a cycle. Let $(u_0, v_0, u_1, v_1, , \ldots, , u_{k-1}, v_{k-1}, u_0)$ be a cycle formed by some edges in A. Then for some $0 \leq i < k$, $u_i v_i \prec u_{i+1} v_{i+1}$ (where we count $i + 1$ modulo k). Hence $\{u_i v_i, u_{i+1} v_{i+1}\}$ is an edge in $H_G^{\prec}$, which proves that A is not an independent set. ■

We also need the following concepts. A bipartite graph is called a *generalized k-gon,* if it is regular and has diameter k and girth $2k$.

Let G be a bipartite graph with bipartition (U, V). A mapping $\pi : U \to V$ is called a *polarity,* if it is a bijection such that for every $u_1, u_2 \in U$

$$u_1\pi(u_2) \in E(G) \text{ iff } u_2\pi(u_1) \in E(G).$$

If a polarity exists, then the *polarity graph* G^π is the graph with vertex set U, and $u_1u_2 \in E(G^\pi)$ iff $u_1 \neq u_2$ and $u_1\pi(u_2) \in E(G)$.

We also define the *polarity graph with loops* $G^{\pi,o}$. It is obtained from the polarity graph G^π by adding loops uu for every u such that $u\pi(u)$ is an edge in G. The reason for using these graphs is that this construction preserves regularity; namely, if G is d-regular, then so is $G^{\pi,o}$. Our convention is that by adding a loop to a vertex v its degree increases by 1 (note that some authors postulate that the increase of the degree is 2). We use graphs $G^{\pi,o}$ to estimate the independence numbers of graphs G^π. Otherwise all graphs in this paper will be loop-free.

Recall the well known inequality

$$\alpha(G) \leq n\frac{\lambda}{d}, \tag{2}$$

where G is d-regular graph without loops of order n, λ is the second largest absolute value of an eigenvalue and $\alpha(G)$ denotes the cardinality of the largest independent set in G. For graphs with loops the corresponding inequality is

$$\alpha(G) \leq n\frac{\lambda+1}{d}, \tag{3}$$

where we allow loops to be present in independent sets. Both inequalities are immediate consequences of the well-known inequality

$$\left| e(A, B) - \frac{d}{n}|A| \cdot |B| \right| \leq \lambda\sqrt{|A| \cdot |B|}, \tag{4}$$

where $e(A, B)$ denotes the number of edges (u, v) such that $u \in A$ and $v \in B$ (including loops when A and B are not disjoint); this inequality is Corollary 2.5, Chapter 9 of [3].

Let G be a bipartite graph with bipartition (U, V). Let M be the matrix of G such that the rows of M are indexed by vertices in U, the columns of M are indexed by vertices in V, M_{uv} equals 1 if (u, v) is an edge, and equals 0 otherwise. We use the eigenvalues of MM^T for generalized k-gons as they are presented in Tanner [11]. Observe that G has a polarity if and only if one can choose an ordering of vertices of G so that the matrix M is symmetric. In such a case, $MM^T = M^2$ and M is also the adjacency matrix of the polarity graph with loops $G^{\pi,o}$. Since the eigenvalues of M^2 are squares of the eigenvalues of M, one can determine the absolute values of the eigenvalues of $G^{\pi,o}$ from the eigenvalues of MM^T.

3 Constructive bound $R(3, m) \geq \Omega(m^{3/2})$

Let P_q be the incidence graph of the classical projective plane $PG(2, q)$ with $q^2 + q + 1$ points and $q^2 + q + 1$ lines, where q is a prime power. P_q is a regular graph of degree $q + 1$. Let $\prec$ be an arbitrary ordering of the edges of P_q. We use the graphs $H^{\prec}_{P_q}$. The following properties can be easily verified:

1. $H^{\prec}_{P_q}$ has $(q + 1)(q^2 + q + 1)$ vertices.
2. $H^{\prec}_{P_q}$ is triangle-free. *Proof:* If $p_1l_1 \prec p_2l_2 \prec p_3l_3$ would form a triangle, then (p_1, l_3, p_2, l_2) would be a 4-cycle in P_q. ■
3. The largest independent set in $H^{\prec}_{P_q}$ has size $\leq 2(q^2+q+1)$, by Lemma 2.2.

4 Constructive bound $R(5, m) \geq \Omega(m^{5/3})$

The construction and the proof is the same as above, except that instead of graphs without $K_{2,2}$ we use denser graphs that do not contain $K_{3,3}$. Such graphs were constructed in [5, 4].

We use the bipartite version of Brown's construction. Let U and V be two copies of $\mathbb{F}_p^3$, where p is a prime and $\mathbb{F}_p$ is the finite field with p elements. Connect two vertices $(u_1, u_2, u_3) \in U$ and $(v_1, v_2, v_3) \in V$ by an edge if they are different and $(u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2 = 0 \pmod p$. The resulting graph B_p

1. has $2p^3$ vertices,
2. is regular of degree $p^2 - p$, hence it has $p^5 - p^4$ edges,
3. does not contain $K_{3,3}$.

Consider the graph $H^{\prec}_{B_p}$, where $\prec$ is an arbitrary linear order. Let $n = p^5 - p^4$ be the number of vertices of $H^{\prec}_{B_p}$. The maximal independent set has size $m \leq 2p^3$, according to Lemma 2.2, hence $n = \Omega(m^{5/3})$. The proof that $H^{\prec}_{B_p}$ does not contain K_5 is also similar. By way of contradiction, suppose that $u_1v_1 \prec \ldots \prec u_5v_5$ are vertices of K_5 in $H^{\prec}_{B_p}$. Then vertices $u_1, u_2, u_3, v_3, v_4, v_5$ induce a $K_{3,3}$ in B_p.

5 Constructive bound $R(4, m) \geq \Omega(m^{8/5})$

Constructions of generalized 4-gons are known for every prime power q, but those with polarities exist if and only if q is an odd power of 2. For an explicit presentation of such a polarity, see [8]. Let Q^{π}_q and $Q^{\pi,o}_q$ denote the polarity graph and the polarity graph with loops of a generalized 4-gon for q an odd power of 2. We need the following properties:

1. $Q_q^{\pi,o}$ is a $(q+1)$-regular graph of order q^3+q^2+q+1.

2. Q_q^{π} has at least $(q^4+q^3+q^2+q)/2$ edges.

3. Q_q^{π} does not contain cycles C_3, C_4, C_6 (but it does contain C_5).

4. The second largest absolute value of an eigenvalue of $Q_q^{\pi,o}$ is $\sqrt{2q}$.

5. The size of any independent set in Q_q^{π} is at most $\frac{(q^3+q^2+q+1)(\sqrt{2q}+1)}{q+1}$.

Properties 1 and 3 were proved in [8]. Property 2 immediately follows from Property 1. According to [11], the eigenvalues of the square of the adjacency matrix of $Q_q^{\pi,o}$ are $(q+1)^2, 2q, 0$, which implies Property 4. Property 5 follows from Property 4 and inequality (3).

Consider the superline graph $H_{Q_q^{\pi}}$. Let m be its independence number and n be its order. Then $m = O(q^{5/2})$ and $n = \Theta(q^4)$, whence $n = \Omega(m^{8/5})$. It remains to prove that it does not contain K_4. To this end we use Property 3 of Q_q.

Lemma 5.1 *If G is a graph such that K_4 is a subgraph of H_G, then G contains a cycle C_3, or C_4, or C_6.*

Proof. Suppose that the edges $11', \ldots, 44'$ of G are vertices of a K_4 in H_G. We consider two cases depending on how the first edge is connected to the three others, see Figures 1,2.

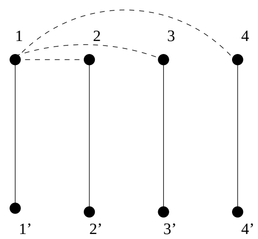

Figure 1: Case 1.

Case 1. The sets $\{2,2'\}, \{3,3'\}$, and $\{4,4'\}$ have to be adjacent to each other. Since Q_q is C_3- and C_4-free, they can be connected only by edges $i'j'$, $2 \le i < j \le 4$. But then we would have triangle $(2',3',4')$.

Case 2. We consider the possibilities how sets $\{2,2'\}, \{3,3'\}, \{4,4'\}$ are connected to each other without creating cycles of lengths 3,4 or 6. For $22'$ and $33'$ there is the unique possibility $2'3'$.

One can check that each of the edges $2'4, 3'4'$ would produce a C_6 and each of the edges $24', 34'$ would produce a C_4. Hence we must have (24 or $2'4'$) and

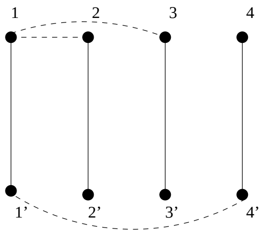

Figure 2: Case 2.

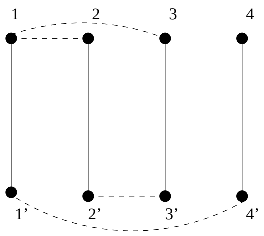

Figure 3: Case 2, continued.

(34 or $3'4$). Thus we have 4 possibilities; each of them produces C_4 or C_6 :

$$\begin{array}{ll} 24, 34 & \mapsto (1,2,4,3) \\ 24, 3'4 & \mapsto (2,2',3'4) \\ 2'4', 34 & \mapsto (1,2,2',4',4,3) \\ 2'4', 3'4 & \mapsto (2',4',4,3') \end{array}$$

■

6 Constructive bound $R(6,m) \geq \Omega(m^2)$

Let P_q^π be the polarity graph of the classical projective plane $PG(2,q)$, where q is a prime power. This graph has a simple explicit definition. The vertices of P_q^π are the *points* of the projective plane P_q. Given two distinct points with homogeneous coordinates (a,b,c) and (a',b',c'), $a,b,c,a',b',c' \in GF_q$, they form an edge in P_q^π, if $aa'+bb'+cc'=0$. P_q^π has $n=q^2+q+1$ vertices; $q+1$ vertices have degree q, and q^2 vertices have degree $q+1$, hence the number of edges is $\frac{1}{2}q(q+1)^2$. This graph is also known as the *Erdős-Rényi graph*; for more information on its independence number, see [9].

The second largest absolute value of an eigenvalue of the polarity graph with loops $P_q^{\pi,o}$ is $\lambda = \sqrt{q}$. This fact also follows from the results of [11], since P_q is a generalized 3-gon.

Whence by (3),

$$\alpha(P_q^\pi) = \alpha(P_q^{\pi,o}) \le n\frac{\sqrt{q}+1}{q+1} = O(n^{3/4}).$$

Our explicit lower bound on $R(6,m)$ is based on $H_{P_q^\pi}$. The number of vertices of this graph is $\Omega(n^{3/2})$ and $\alpha(H_{P_q^\pi}) \le \alpha(P_q^\pi) = O(n^{3/4}) = O(\sqrt{n^{3/2}})$. Hence it remains to prove that $H_{P_q^\pi}$ does not contain K_6. Again, the only property we use is that P_q^π does not contain C_4.

Lemma 6.1 *If G is a graph such that K_6 is a subgraph of H_G, then C_4 is a subgraph of G.*

Proof. Suppose that $11', \ldots, 66'$ are disjoint edges of a graph G that form a K_6 in H_G.

Case 1: One end of $66'$ is adjacent to ends of at least four other edges, say, 6 is adjacent to $1,2,3,4$. Assuming there is no C_4 in G, we cannot have edges ij', for $1 \le i \ne j \le 4$. Furthermore we cannot have two non-disjoint edges on $\{1,2,3,4\}$, or five edges on $\{1',2',3',4'\}$. Thus we must have two disjoint edges on $\{1,2,3,4\}$, say, 12 and 34, see Figure 4. This forces the remaining four edges connecting $11', \ldots, 44'$ to be $1'3', 1'4', 2'3', 2'4'$. Due to symmetry, it suffices to consider how $11'$ is connected with $33'$. The edge 13 would produce the 4-cycle $(1,3,4,6)$; $13'$ would produce the cycle $(1,3',3,6)$; $1'3$ would produce the cycle $(1',3,6,1)$. But if we have edges $1'3', 1'4', 2'3', 2'4'$, then we get the 4-cycle $(1',3',2',4')$.

Hence Case 1 leads to a contradiction.

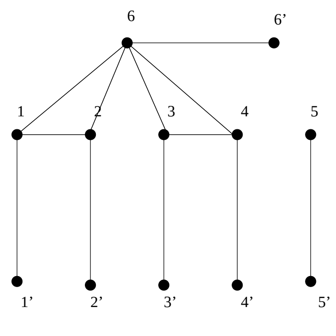

Figure 4: Case 1.

Case 2: One end of $66'$ is adjacent to ends of two edges and the other is adjacent to ends of three edges, say, as shown in Figure 5.

Assuming G does not contain C_4, there are no edges of the form ij', $i \ne j$ either for $1 \le i,j \le 2$ or for $3 \le i,j \le 5$. Furthermore, there can be only one edge of the form ij for $3 \le i,j, \le 5$. Thus we have at least two edges on $\{3',4',5'\}$, say, $3'4', 4'5'$.

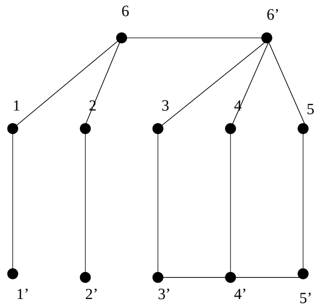

Figure 5: Case 2.

In the following two claims we assume that C_4 is not a subgraph of G.

Claim 1. There is no edge connecting the sets $\{1, 2\}$ and $\{3, 4, 5\}$.
Proof. Such an edge would form C_4 with $6, 6'$. ■

Claim 2. There is no edge connecting the sets $\{1', 2'\}$ and $\{3', 4', 5'\}$.
Proof. Suppose $i'j'$ is such an edge $i < j$. W.l.o.g. assume $i = 2$ and $j \in \{3, 4\}$. Let k be the other element of $\{3, 4\}$, thus jk is an edge. Then the only way $\{i, i'\}$ can be connected with $\{k, k'\}$ is by the edge $i'k'$. Indeed, $2k'$ would form the 4-cycle $(2, k', j', 2')$ and $2'k$ would form the cycle $(2', k, k', j')$. Thus we have the situation shown in Figure 6.

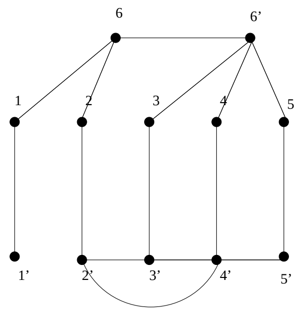

Figure 6: Case 2, continued.

Now consider how $\{2, 2'\}$ is connected to $\{5, 5'\}$. The edge $2'5'$ would produce 4-cycle $(2', 3', 4', 5')$; $2'5$ would produce 4-cycle $(2', 5, 5', 4')$; $25'$ would produce 4-cycle $(2, 5', 5, 4')$. This contradiction finishes the proof of Claim 2.

According to Claims 1 and 2, there are only cross edges connecting pairs $(1, 1'), (2, 2')$ with pairs $(3, 3'), (4, 4'), (5, 5')$. Since we need 6 edges to connect these sets, one of the vertices $1', 2', 3', 4', 5'$ must be incident with two such edges. Suppose for instance that it is $1'$ and it is adjacent to 3 and 4. Then $(1', 3, 6', 4)$ is a 4-cycle. The other cases are similar. ■

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