Regulated functions and the Perron-Stieltjes integral

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Dedicated to Professor Otakar Borůvka on the occasion of his ninetieth birthday

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Abstract. Properties of the Perron-Stieltjes integral with respect to regulated functions are investigated. It is shown that linear continuous functionals on the space $\mathbb{G}_{\mathrm{L}}(a,b)$ of functions regulated on $[a,b]$ and left-continuous on (a,b) may be represented in the form

$$F(x) = qx(a) + \int_a^b p(t)\mathrm{d}[x(t)],$$

where $p \in \mathbb{R}$ and $q(t)$ is a function of bounded variation on $[a,b]$. Some basic theorems (e.g. integration-by-parts formula, substitution theorem) known for the Perron-Stieltjes integral with respect to functions of bounded variation are established.
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0 . Introduction

This paper deals with the space $\mathbb{G}(a,b)$ of regulated functions on a compact interval $[a,b]$. It is known that when equipped with the supremal norm $\mathbb{G}(a,b)$ becomes a Banach space, and linear bounded functionals on its subspace $\mathbb{G}_{\mathrm{L}}(a,b)$ of functions regulated on $[a,b]$ and left-continuous on (a,b) can be represented by means of the Dushnik-Stieltjes (interior) integral. This result is due to H. S. Kaltenborn [7], cf. also Ch. S. Hönig [5, Theorem 5.1]. Together with the known relationship between the Dushnik-Stieltjes integral, the σ-Young-Stieltjes integral and the Perron-Stieltjes integral (cf. Ch. S. Hönig [6] and Š. Schwabik [11],[12]) this enables us to see that F

is a linear bounded functional on $\mathbb{G}_{\mathrm{L}}(a,b)$ if and only if there exists a real number q and a function $p(t)$ of bounded variation on $[a,b]$ such that

$$F(x) = qx(a) + \int_a^b p(t)\mathrm{d}[x(t)] \quad \text{for any} \quad x \in \mathbb{G}_{\mathrm{L}}(a,b),$$

where the integral is the Perron-Stieltjes integral. We will give here the proof of this fact based only on the properties of the Perron-Stieltjes integral. To this aim, the proof of the existence of the integral

$$\int_a^b f(t)\mathrm{d}[g(t)]$$

for any function f of bounded variation on $[a,b]$ and any function g regulated on $[a,b]$ is crucial. Furthermore, we will prove extensions of some theorems (e.g. integration-by-parts and substitution theorems) needed for dealing with generalized differential equations and Volterra-Stieltjes integral equations in the space $\mathbb{G}(a,b)$.

1 . Preliminaries

Throughout the paper $\mathbb{R}^n$ denotes the space of real n-vectors, $\mathbb{R}^1 = \mathbb{R}$. Given $x \in \mathbb{R}^n$, its arguments are denoted by $x_1, x_2, \dots, x_n$ $(x = (x_1, x_2, \dots, x_n))$. $\mathbb{N}$ stands for the set of all natural numbers ($\mathbb{N} = \{1, 2, \dots\}$). Given $M \subset \mathbb{R}$, χ_M denotes its characteristic function ($\chi_M(t) = 1$ if $t \in M$ and $\chi_M(t) = 0$ if $t \notin M$.)

Let $-\infty < a < b < \infty$. The sets $d = \{t_0, t_1, \dots, t_m\}$ of points in the closed interval $[a,b]$ such that $a = t_0 < t_1 < \dots < t_m = b$ are called *divisions of* $[a,b]$. Given a division d of $[a,b]$, its elements are usually denoted by $t_0, t_1, \dots, t_m$. The couples $D = (d, \xi)$, where $d = \{t_0, t_1, \dots, t_m\}$ is a division of $[a,b]$ and $\xi = (\xi_1, \xi_2, \dots, \xi_m) \in \mathbb{R}^m$ is such that

$$t_{j-1} \le \xi_j \le t_j \quad \text{for all} \quad j = 1, 2, \dots, m$$

are called *partitions of* $[a,b]$.

A function $f : [a,b] \mapsto \mathbb{R}$ which possesses finite limits

$$f(t+) = \lim_{\tau \to t+} f(\tau) \quad \text{and} \quad f(s-) = \lim_{\tau \to s-} f(\tau)$$

for all $t \in [a,b)$ and all $s \in (a,b]$ is said to be *regulated on* $[a,b]$. The set of all regulated functions on $[a,b]$ is denoted by $\mathbb{G}(a,b)$. Given $f \in \mathbb{G}(a,b)$, we define $f(a-) = f(a)$, $f(b+) = f(b)$,

$$\Delta^+ f(t) = f(t+) - f(t), \ \Delta^- f(t) = f(t) - f(t-)$$

and

$$\Delta f(t) = f(t+) - f(t-).$$

for any $t \in [a,b]$. (In particular, we have $\Delta^- f(a) = \Delta^+ f(b) = 0$, $\Delta f(a) = \Delta^+ f(a)$ and $\Delta f(b) = \Delta^- f(b)$.)

It is known (cf. [5, Corollary 3.2a]) that if $f \in \mathbb{G}(a,b)$, then for any $\varepsilon > 0$ the set of points $t \in [a,b]$ such that

$$|\Delta^+ f(t)| > \varepsilon \quad \text{or} \quad |\Delta^- f(t)| > \varepsilon$$

is finite. Consequently, for any $f \in \mathbb{G}(a,b)$ the set of its discontinuities in $[a,b]$ is countable. The subset of $\mathbb{G}(a,b)$ consisting of all functions regulated on $[a,b]$ and left-continuous on (a,b) will be denoted by $\mathbb{G}_{\rm L}(a,b)$.

A function $f : [a,b] \mapsto \mathbb{R}$ is called a *finite step function on* $[a,b]$ if there exists a division $\{t_0, t_1, \dots, t_m\}$ of $[a,b]$ such that f is constant on every open interval (t_{j-1}, t_j), $j = 1, 2, \dots, m$. The set of all finite step functions on $[a,b]$ is denoted by $\mathbb{S}(a,b)$. A function $f : [a,b] \mapsto \mathbb{R}$ is called a *break function on* $[a,b]$ if there exist sequences

$$\{t_k\}_{k=1}^\infty \subset [a,b], \quad \{c_k^-\}_{k=1}^\infty \subset \mathbb{R} \quad \text{and} \quad \{c_k^+\}_{k=1}^\infty \subset \mathbb{R}$$

such that $t_k \neq t_j$ for $k \neq j$, $c_k^- = 0$ if $t_k = a$, $c_k^+ = 0$ if $t_k = b$,

$$\sum_{k=1}^\infty \Big(|c_k^-| + |c_k^+| \Big) < \infty$$

and

$$f(t) = \sum_{t_k \le t} c_k^- + \sum_{t_k < t} c_k^+ \qquad \text{for} \quad t \in [a,b] \tag{1.1}$$

or equivalently

$$f(t) = \sum_{k=1}^\infty c_k^- \, \chi_{[t_k,b]}(t) + c_k^+ \, \chi_{(t_k,b]}(t) \quad \text{for} \quad t \in [a,b].$$

Clearly, if f is given by (1.1), then $\Delta^+ f(t_k) = c_k^+$ and $\Delta^- f(t_k) = c_k^-$ for any $k \in \mathbb{N}$ and $f(t-) = f(t) = f(t+)$ if $t \in [a,b] \setminus \{t_k\}_{k=1}^\infty$. Furthermore, we have $f(a) = 0$ and

$$\operatorname{var}_a^b f = \sum_{k=1}^\infty \Big(|c_k^-| + |c_k^+| \Big)$$

for any such function. The set of all break functions on $[a,b]$ is denoted by $\mathbb{B}(a,b)$.

$\mathbb{BV}(a,b)$ denotes the set of all functions with bounded variation on $[a,b]$ and

$$\|f\|_{\mathbb{BV}} = |f(a)| + \operatorname{var}_a^b f \quad \text{for} \quad f \in \mathbb{BV}(a,b).$$

It is known that for any $f \in \mathbb{BV}(a,b)$ there exist uniquely determined functions $f^{\mathrm{C}} \in \mathbb{BV}(a,b)$ and $f^{\mathrm{B}} \in \mathbb{BV}(a,b)$ such that f^{C} is continuous on $[a,b]$, f^{B} is a break function on $[a,b]$ and $f(t) = f^{\mathrm{C}}(t) + f^{\mathrm{B}}(t)$ on $[a,b]$ (the *Jordan decomposition of* $f \in \mathbb{BV}(a,b)$). In particular, if $W = \{w_k\}_{k\in\mathbb{N}}$ is the set of discontinuities of f in $[a,b]$, then

$$f^{\mathrm{B}}(t) = \sum_{k=1}^{\infty} \Big(\Delta^- f(w_k)\, \chi_{[w_k,b]}(t) + \Delta^+ f(w_k)\, \chi_{(w_k,b]}(t)\Big) \in [a,b]. \tag{1.2}$$

Moreover, if we put

$$f_n^{\mathrm{B}}(t) = \sum_{k=1}^{n} \Big(\Delta^- f(w_k)\, \chi_{[w_k,b]}(t) + \Delta^+ f(w_k)\, \chi_{(w_k,b]}(t)\Big) \quad \text{on} \quad [a,b] \tag{1.3}$$

for any $n \in \mathbb{N}$, then

$$\lim_{n\to\infty} \|f_n^{\mathrm{B}} - f^{\mathrm{B}}\|_{\mathbb{BV}} = 0 \tag{1.4}$$

(cf. e.g. [14, the proof of Lemma I.4.23]). Obviously,

$$\mathbb{S}(a,b) \subset \mathbb{B}(a,b) \subset \mathbb{BV}(a,b) \subset \mathbb{G}(a,b).$$

Given $f \in \mathbb{G}(a,b)$, we define

$$\|F\| = \sup_{t\in[a,b]} |f(t)|.$$

Clearly, $\|f\| < \infty$ for any $f \in \mathbb{G}(a,b)$ and when endowed with this norm, $\mathbb{G}(a,b)$ becomes a Banach space (cf. [5, Theorem 3.6]). It is known that $\mathbb{S}(a,b)$ is dense in $\mathbb{G}(a,b)$ (cf. [5, Theorem 3.1]). It means that $f : [a,b] \mapsto \mathbb{R}$ is regulated if and only if it is a uniform limit on $[a,b]$ of a sequence of finite step functions. Obviously, $\mathbb{G}_{\mathrm{L}}(a,b)$ is closed in $\mathbb{G}(a,b)$ and hence it is also a Banach space. (Neither $\mathbb{S}(a,b)$ nor $\mathbb{B}(a,b)$ are closed in $\mathbb{G}(a,b)$, of course.)

For some more details concerning regulated functions see the monographs by Ch. S. Hönig [5] and by G. Aumann [1] and the papers by D. Fraňková [2] and [3].

The integrals which occur in this paper are the Perron-Stieltjes integrals. We will work with the following definition which is a special case of the definition due to J. Kurzweil [8].

Let $-\infty < a < b < \infty$. An arbitrary positive valued function $\delta : [a,b] \mapsto (0,\infty)$ is called a *gauge on* $[a,b]$. Given a gauge δ on $[a,b]$, the partition (d,ξ) of $[a,b]$ is said to be δ-fine if

$$[t_{j-1}, t_j] \subset (\xi_j - \delta(\xi_j), \xi_j + \delta(\xi_j)) \quad \text{for any} \quad j = 1,2,\dots,m.$$

Given functions $f,\, g:\, [a,b] \mapsto \mathbb{R}$ and a partition $D=(d,\xi)$ of $[a,b]$, let us define

$$S_D(f\,\Delta g) = \sum_{j=1}^{m} f(\xi_j)[g(t_j)-g(t_{j-1})].$$

We say that $I \in \mathbb{R}$ is the *Kurzweil integral* of f with respect to g from a to b and denote

$$I = \int_a^b f(t)\,\mathrm{d}[g(t)] \quad \text{or} \quad I = \int_a^b f\,\mathrm{d}g$$

if for any $\varepsilon > 0$ there exists a gauge δ on $[a,b]$ such that

$$\left| I - S_D(f\,\Delta g)\right| < \varepsilon$$

for all δ-fine partitions D of $[a,b]$.

The Perron-Stieltjes integral with respect to a function not necessarily of bounded variation was defined by A. J. Ward [15] (cf. also S. Saks [10, Chapter VI]). It can be shown that the Kurzweil integral is equivalent to the Perron-Stieltjes integral (cf. [11, Theorem 2.1], where the assumption $g \in \mathbb{BV}(a,b)$ is not used in the proof and may be omitted). Consequently, the Riemann-Stieltjes integral (both of the norm type and of the σ-type, cf. [4]) is its special case. The relationship between the Kurzweil integral, the σ-Young-Stieltjes integral and the Perron-Stieltjes integral was described by Š. Schwabik (cf. [11] and [12]).

Since we will make use of some of the properties of the σ-Riemann-Stieltjes integral, let us indicate here the proof that this integral is included in the Kurzweil integral. (For the definition of the σ-Riemann-Stieltjes integral, see e.g. [4, Sec. II.9].)

1.1. Proposition. *Let $f, g : [a,b] \mapsto \mathbb{R}$ and $I \in \mathbb{R}$ be such that the σ-Riemann-Stieltjes integral $\sigma \int_a^b f\,\mathrm{d}g$ exists and equals I. Then the Perron-Stieltjes integral $\int_a^b f\,\mathrm{d}g$ exists and equals I, as well.*

Proof. Let

$$\sigma \int_a^b f\,\mathrm{d}g = I \in \mathbb{R},$$

i.e. for any $\varepsilon > 0$ there is a division $d_0 = \{s_0, s_1 \ldots, s_{m_0}\}$ of $[a,b]$ such that for any division $d = \{t_0, t_1, \ldots, t_m\}$ which is its refinement ($d_0 \subset d$) and any $\xi \in \mathbb{R}^m$ such that $D=(d,\xi)$ is a partition of $[a,b]$ the inequality

$$\Big| S_D(f\,\Delta g) - I \Big| < \varepsilon$$

is satisfied. Let us define

$$\delta_\varepsilon(t) = \begin{cases} \frac{1}{2}\min\{|t-s_j|; j=0,1,\ldots,m_0\} & \text{if } t \notin d_0, \\ \varepsilon & \text{if } t \in d_0. \end{cases}$$

Then a partition $D=(d,\xi)$ of $[a,b]$ is δ_ε-fine only if for any $j=1,2,\ldots,m_0$ there is an index i_j such that $s_j=\xi_{i_j}$. Furthermore,

$$S_D(f\,\Delta g) = \sum_{j=1}^{m}\Big[f(\xi_j)[g(t_j)-g(\xi_j)] + f(\xi_j)[g(\xi_j)-g(t_{j-1})]\Big]$$

for any partition $D=(d,\xi)$ of $[a,b]$. Consequently, for any δ_ε-fine partition $D=(d,\xi)$ of $[a,b]$ the corresponding integral sum $S_D(f\,\Delta g)$ equals the integral sum $S_{D'}(f\,\Delta g)$ corresponding to a partition $D'=(d',\xi')$, where d' is a division of $[a,b]$ such that $d_0 \subset d'$, and hence

$$\Big|S_{D'}(f\,\Delta g) - I\Big| < \varepsilon.$$

This means that the Kurzweil integral $\int_a^b f\,\mathrm{d}g$ exists and

$$\int_a^b f\,\mathrm{d}g = \sigma\int_a^b f\,\mathrm{d}g = I$$

holds. □

To prove the existence of the Perron-Stieltjes integral $\int_a^b f\,\mathrm{d}g$ for any $f \in \mathbb{BV}(a,b)$ and any $g \in \mathbb{G}(a,b)$ in Theorem 2.8 the following assertion is helpful.

1.2. Proposition. *Let $f \in \mathbb{BV}(a,b)$ be continuous on $[a,b]$ and let $g \in \mathbb{G}(a,b)$, then both the σ-Riemann-Stieltjes integrals*

$$\sigma\int_a^b f\,\mathrm{d}g \quad \textit{and} \quad \sigma\int_a^b g\,\mathrm{d}f$$

exist.

Proof. Let $f \in \mathbb{BV}(a,b)$ which is continuous on $[a,b]$ and $g \in \mathbb{G}(a,b)$ be given .According to the integration-by-parts formula [4, II.11.7] for σ-Riemann-Stieltjes integrals to prove the lemma it is sufficient to show that the integral $\sigma\int_a^b g\,\mathrm{d}f$ exists.

First, let us assume that an arbitrary $\tau \in [a,b]$ is given and $g = \chi_{[a,\tau]}$. Let us put

$$d_0 = \begin{cases} \{a,b\} & \text{if } \tau = a \text{ or } \tau = b, \\ \{a,\tau,b\} & \text{if } \tau \in (a,b). \end{cases}$$

It is easy to see that then for any partition $D=(d,\xi)$ such that

$$d_0 \subset d = \{t_0, t_1, \dots, t_m\}$$

we have $\tau = t_k$ for some $k \in \{0,1,\dots,m\}$ and

$$S_D(g\,\Delta f) = \begin{cases} f(\tau) - f(a) & \text{if } \ \xi_{k+1} > \tau, \\ f(t_{k+1}) - f(a) & \text{if } \ \xi_{k+1} = \tau. \end{cases}$$

Since f is assumed to be continuous, it is easy to show that for a given $\varepsilon > 0$, there exists a division d_* of $[a,b]$ such that $d_0 \subset d_*$ and

$$\big|S_D(g\,\Delta f) - [f(\tau) - f(a)]\big| < \varepsilon$$

holds for any partition $D=(d,\xi)$ of $[a,b]$ with $d_* \subset d$, i.e.

$$\sigma\int_a^b \chi_{[a,\tau]}\mathrm{d}f = f(\tau) - f(a) \quad \text{for all} \quad \tau \in [a,b].$$

By a similar argument we could show the following relations:

$$\sigma\int_a^b \chi_{[a,\tau)}\mathrm{d}f = f(\tau) - f(a) \quad \text{for all} \quad \tau \in (a,b],$$

$$\sigma\int_a^b \chi_{[\tau,b]}\mathrm{d}f = f(b) - f(\tau) \quad \text{for all} \quad \tau \in [a,b],$$

and

$$\sigma\int_a^b \chi_{(\tau,b]}\mathrm{d}f = f(b) - f(\tau) \quad \text{for all} \quad \tau \in [a,b),$$

as well.

It follows that the integral

$$\sigma\int_a^b g\mathrm{d}f$$

exists for any $f \in \mathbb{BV}(a,b)$ continuous on $[a,b]$ and any $g \in \mathbb{S}(a,b)$ (cf. Remark 2.2).

Now, if $g \in \mathbb{G}(a,b)$ is arbitrary, then there exists a sequence $\{g_n\}_{k=1}^{\infty} \subset \mathbb{S}(a,b)$ such that

$$\lim_{n\to\infty} \|g_n - g\| = 0.$$

Since by the preceding part of the proof of the lemma all the integrals $\sigma\int_a^b g_n\mathrm{d}f$ have a finite value, by means of the convergence theorem [4, Theorem II.15.1] valid for σ-Riemann-Stieltjes integrals we obtain that the integral $\sigma\int_a^b g\,\mathrm{d}f$ exists and the relation

$$\lim_{n\to\infty}\sigma\int_a^b g_n\,\mathrm{d}f=\sigma\int_a^b g\,\mathrm{d}f\in\mathbb{R}$$

holds. This completes the proof. □

A direct corollary of Proposition 1.2 and of [4, Theorem II.13.17] is the following assertion which will be helpful for the proof of the integration-by-parts formula Theorem 2.15. (Of course, we could prove it by an argument similar to that used in the proof of Proposition 1.2, as well.)

1.3. Corollary. *Let $f\in\mathbb{BV}(a,b)$ and $g\in\mathbb{G}(a,b)$. Let*

$$\Delta^+f(t)\Delta^+g(t)=\Delta^-f(t)\Delta^-g(t)=0\quad\textit{for all}\quad t\in(a,b).$$

Then both the σ-Riemann-Stieltjes integrals

$$\sigma\int_a^b f\,\mathrm{d}g\quad\textit{and}\quad\sigma\int_a^b g\,\mathrm{d}f$$

exist. □

It is well known (cf. e.g. [14, Theorems I.4.17, I.4.19 and Corollary I.4.27] that if $f\in\mathbb{G}(a,b)$ and $g\in\mathbb{BV}(a,b)$, then the integral $\int_a^b f\,\mathrm{d}g$ exists and the inequality

$$\Big|\int_a^b f\,\mathrm{d}g\Big|\le\|f\|\big(\mathrm{var}_a^b g\big)$$

holds. The Kurzweil integral is an additive function of intervals and possesses the usual linearity properties. For the proofs of these assertions and some more details concerning the Kurzweil integral with respect to functions of bounded variation see e.g. [8], [9], [13] and [14].

2 . Perron-Stieltjes integral with respect to regulated functions

In this section we deal with the integrals

$$\int_a^b f(t)\,\mathrm{d}[g(t)]\quad\text{and}\quad\int_a^b g(t)\,\mathrm{d}[f(t)],$$

where $f \in \mathbb{BV}(a,b)$ and $g \in \mathbb{G}(a,b)$. we prove some basic theorems (integration - by - parts formula, convergence theorems, substitution theorem and unsymmetric Fubini theorem) needed in the theory of Stieltjes integral equations in the space $\mathbb{G}(a,b)$. However, our first task is the proof of existence of the integral $\int_a^b f\,\mathrm{d}g$ for any $f \in \mathbb{BV}(a,b)$ and any $g \in \mathbb{G}(a,b)$. First, we will consider some simple special cases.

2.1. Proposition. *Let $g \in \mathbb{G}(a,b)$ be arbitrary. Then for any $\tau \in [a,b]$ we have*

$$\int_a^b \chi_{[a,\tau]}\mathrm{d}g = g(\tau+) - g(a), \tag{2.1}$$

$$\int_a^b \chi_{[a,\tau)}\mathrm{d}g = g(\tau-) - g(a), \tag{2.2}$$

$$\int_a^b \chi_{[\tau,b]}\mathrm{d}g = g(b) - g(\tau-), \tag{2.3}$$

$$\int_a^b \chi_{(\tau,b]}\mathrm{d}g = g(b) - g(\tau+) \tag{2.4}$$

and

$$\int_a^b \chi_{[\tau]}\mathrm{d}g = g(\tau+) - g(\tau-), \tag{2.5}$$

where $\chi_{[a)}(t) \equiv \chi_{(b]}(t) \equiv 0$ and the convention $g(a-) = g(a)$, $g(b+) = g(b)$ is used.

Proof. Let $g \in \mathbb{G}(a,b)$ and $\tau \in [a,b]$ be given.

a) Let $f = \chi_{[a,\tau]}$. It follows immediately from the definition that

$$\int_a^\tau f\,\mathrm{d}g = g(\tau) - g(a).$$

In particular, 2.1 holds in the case $\tau = b$. Let $\tau \in [a,b)$, let $\varepsilon > 0$ be given and let

$$\delta_\varepsilon(t) = \begin{cases} \frac{1}{2}|\tau - t| & \text{if } \tau < t \le b, \\ \varepsilon & \text{if } t = \tau. \end{cases}$$

It is easy to see that any δ_ε-fine partition $D = (d,\xi)$ of $[\tau,b]$ must satisfy

$$\xi_1 = t_0 = \tau, \quad t_1 < \tau + \varepsilon \quad \text{and} \quad S_D(f\,\Delta g) = g(t_1) - g(\tau).$$

Consequently,

$$\int_\tau^b f\,\mathrm{d}g = g(\tau+) - g(\tau)$$

and

$$\int_a^b f\,\mathrm{d}g = \int_a^\tau f\,\mathrm{d}g + \int_\tau^b f\,\mathrm{d}g$$
$$= g(\tau) - g(a) + g(\tau+) - g(\tau) = g(\tau+) - g(a),$$

i.e. the relation (2.1) is true for every $\tau \in [a, b]$.

b) Let $f = \chi_{[a,\tau)}$. If $\tau = a$, then $f \equiv 0$, $g(\tau-) - g(a) = 0$ and (2.2) is trivial. Let $\tau \in (a, b]$. For a given $\varepsilon > 0$, let us define a gauge δ_ε on $[a, \tau]$ by

$$\delta_\varepsilon(t) = \begin{cases} \frac{1}{2}|\tau - t| & \text{if } a \le t < \tau, \\ \varepsilon & \text{if } t = \tau. \end{cases}$$

Then for any δ_ε-fine partition $D = (d, \xi)$ of $[a, \tau]$ we have

$$t_m = \xi_m = \tau, \quad t_{m-1} < \tau - \varepsilon \quad \text{and} \quad S_D(f\,\Delta g) = g(t_{m-1}) - g(a).$$

It follows immediately that

$$\int_a^\tau f\,\mathrm{d}g = g(\tau-) - g(a)$$

and in view of the obvious identity

$$\int_\tau^b f\,\mathrm{d}g = 0,$$

this implies (2.2).

c) The remaining relations follow from 2.1, 2.2 and the equalities

$$\chi_{[\tau,b]} = \chi_{[a,b]} - \chi_{[a,\tau)}, \quad \chi_{(\tau,b]} = \chi_{[a,b]} - \chi_{[a,\tau]}$$

and

$$\chi_{[\tau]} = \chi_{[a,\tau]} - \chi_{[a,\tau)}.$$

□

2.2. Remark. Since any finite step function is a linear combination of functions $\chi_{[\tau,b]}$ $(a \le \tau \le b)$ and $\chi_{(\tau,b]}$ $(a \le \tau < b)$, it follows immediately from Proposition 2.1 that the integral $\int_a^b f\,\mathrm{d}g$ exists for any $f \in \mathbb{S}(a, b)$ and any $g \in \mathbb{G}(a, b)$.

Other simple cases are covered by

2.3. Proposition. *Let $\tau \in [a,b]$. Then for any function $f : [a,b] \mapsto \mathbb{R}$ the following relations are true*

$$\int_a^b f\,\mathrm{d}\chi_{[a,\tau]} = \begin{cases} -f(\tau) & \text{if } \tau < b, \\ 0 & \text{if } \tau = b, \end{cases} \tag{2.6}$$

$$\int_a^b f\,\mathrm{d}\chi_{[a,\tau)} = \begin{cases} -f(\tau) & \text{if } \tau > a, \\ 0 & \text{if } \tau = a, \end{cases} \tag{2.7}$$

$$\int_a^b f\,\mathrm{d}\chi_{[\tau,b]} = \begin{cases} f(\tau) & \text{if } \tau > a, \\ 0 & \text{if } \tau = a, \end{cases} \tag{2.8}$$

$$\int_a^b f\,\mathrm{d}\chi_{(\tau,b]} = \begin{cases} f(\tau) & \text{if } \tau < b, \\ 0 & \text{if } \tau = b \end{cases} \tag{2.9}$$

and

$$\int_a^b f\,\mathrm{d}\chi_{[\tau]} = \begin{cases} -f(a) & \text{if } \tau = a, \\ 0 & \text{if } a < \tau < b, \\ f(b) & \text{if } \tau = b, \end{cases} \tag{2.10}$$

where $\chi_{[a)}(t) \equiv \chi_{(b]}(t) \equiv 0$ and the convention $g(a-) = g(a)$, $g(b+) = g(b)$ is used.

For the *proof* see [14, I.4.21 and I.4.22]. □

2.4. Corollary. *Let $W = \{w_1, w_2, \dots, w_n\} \subset [a,b]$, $c \in \mathbb{R}$ and $h : [a,b] \mapsto \mathbb{R}$ be such that*

$$h(t) = c \quad \text{for all} \quad t \in [a,b] \setminus W. \tag{2.11}$$

Then

$$\int_a^b f\,\mathrm{d}h = f(b)[h(b) - c] - f(a)[h(a) - c] \tag{2.12}$$

holds for any function $f : [a,b] \mapsto \mathbb{R}$.

Proof. A function $h : [a,b] \mapsto \mathbb{R}$ fulfils (2.11) if and only if

$$h(t) = c + \sum_{k=1}^{n} [h(w_j) - c]\chi_{[w_j]}(t) \quad \text{on} \quad [a,b].$$

Thus the formula (2.12) follows from (2.6) (with $\tau = b$) and from (2.10) in Proposition 2.3. □

2.5. Remark. It is well known (cf. [14, I.4.17] or [13, Theorem 1.22]) that if $g \in \mathbb{BV}(a,b)$, $h : [a,b] \mapsto \mathbb{R}$ and $h_n : [a,b] \mapsto \mathbb{R}$, $n \in \mathbb{N}$, are such that $\int_a^b h_n \mathrm{d}g$ exist for any $n \in \mathbb{N}$ and $\lim_{n\to\infty} \|h_n - h\| = 0$, then $\int_a^b h \mathrm{d}g$ exists and

$$\lim_{n\to\infty} \int_a^b h_n \mathrm{d}g = \int_a^b h \mathrm{d}g \tag{2.13}$$

holds. To prove an analogous assertion for the case $g \in \mathbb{G}(a,b)$ we need the following auxiliary assertion.

2.6. Lemma. *Let $f \in \mathbb{BV}(a,b)$ and $g \in \mathbb{G}(a,b)$. The the inequality*

$$|S_D(f\, \Delta g)| \le \big(|f(a)| + |f(b)| + \mathrm{var}_a^b f\big)\|g\| \tag{2.14}$$

holds for an arbitrary partition D of $[a,b]$.

Proof. For an arbitrary partition $D = (d,\xi)$ of $[a,b]$ we have (putting $\xi_0 = a$ and $\xi_{m+1} = b$)

$$\begin{aligned} |S_D(f\, \Delta g)| &= |f(b)g(b) - f(a)g(a) - \sum_{j=1}^{m+1} [f(\xi_j) - f(\xi_{j-1})] g(t_{j-1})| \\ &\le \Big(|f(b)| + |f(a)| + \sum_{j=1}^{m+1} |f(\xi_j) - f(\xi_{j-1})|\Big)\|g\| \\ &\le \big(|f(a)| + |f(b)| + \mathrm{var}_a^b f\big)\|g\|. \end{aligned}$$

□

2.7. Theorem. *Let $g \in \mathbb{G}(a,b)$ and let $h_n, h : [a,b] \mapsto \mathbb{R}$ be such that*

$$\int_a^b h_n \mathrm{d}g \quad \textit{exists for any} \quad n \in \mathbb{N} \quad \textit{and} \quad \lim_{n\to\infty} \|h_n - h\|_{\mathbb{BV}} = 0.$$

Then $\int_a^b h \mathrm{d}g$ exists and (2.13) holds.

Proof. Since

$$|f(b)| \le |f(a)| + |f(b) - f(a)| \le |f(a)| + \mathrm{var}_a^b f,$$

we have by (2.14)

$$|S_D((h_m - h_k)\Delta g)| \le 2\|h_m - h_k\|_{\mathbb{BV}}\, \|g\|$$

for all $m, k \in \mathbb{N}$ and all partitions D of $[a, b]$. Consequently,

$$\Big|\int_a^b (h_m - h_k)\,\mathrm{d}g\Big| \le 2\|h_m - h_k\|_{\mathbb{BV}}\,\|g\|$$

holds for all $m, k \in \mathbb{N}$. This immediately implies that there is a $q \in \mathbb{R}$ such that

$$\lim_{n\to\infty}\int_a^b h_n\,\mathrm{d}g = q.$$

It remains to show that

$$q = \int_a^b h\,\mathrm{d}g. \tag{2.15}$$

For a given $\varepsilon > 0$, let $n_0 \in \mathbb{N}$ be such that

$$\Big|\int_a^b h_{n_0}\,\mathrm{d}g - q\Big| < \varepsilon \quad \text{and} \quad \|h_{n_0} - h\|_{\mathbb{BV}} < \varepsilon, \tag{2.16}$$

and let δ_ε be such a gauge on $[a, b]$ that

$$\Big|S_D(h_{n_0}\,\Delta g) - \int_a^b h_{n_0}\,\mathrm{d}g\Big| < \varepsilon \tag{2.17}$$

for all δ_ε-fine partitions D of $[a, b]$. Given an arbitrary δ_ε-fine partition D of $[a, b]$, we have by (2.16), (2.17) and Lemma 2.6

$$\begin{aligned}
&|q - S_D(h\,\Delta g)| \\
&\quad \le \Big|q - \int_a^b h_{n_0}\,\mathrm{d}g\Big| + \Big|\int_a^b h_{n_0}\,\mathrm{d}g - S_D(h_{n_0}\,\Delta g)\Big| \\
&\qquad + \Big|S_D(h_{n_0}\,\Delta g) - S_D(h\,\Delta g)\Big| \\
&\quad \le 2\varepsilon + |S_D([h_{n_0} - h]\,\Delta g)| \le 2\varepsilon + 2\|h_{n_0} - h\|_{\mathbb{BV}}\,\|g\| \le 2\varepsilon\,(1 + \|g\|)
\end{aligned}$$

wherefrom the relation (2.15) immediately follows. This completes the proof of the theorem. □

Now we can prove the following

2.8. Theorem. *Let $f \in \mathbb{BV}(a, b)$ and $g \in \mathbb{G}(a, b)$. Then the integral*

$$\int_a^b f\,\mathrm{d}g$$

exists and the inequality

$$\Big|\int_a^b f\,\mathrm{d}g\Big| \le \big(|f(a)| + |f(b)| + \operatorname{var}_a^b f\big)\|g\| \tag{2.18}$$

holds.

Proof. Let $f \in \mathbb{BV}(a,b)$ and $g \in \mathbb{G}(a,b)$ be given. Let $W = \{w_k\}_{k\in\mathbb{N}}$ be the set of discontinuities of f in $[a,b]$ and let $f = f^{\mathrm{C}} + f^{\mathrm{B}}$ be the Jordan decomposition of f (i.e. f^{C} is continuous on $[a,b]$ and f^{B} is given by (1.2)). We have

$$\lim_{n\to\infty} \|f_n^{\mathrm{B}} - f^{\mathrm{B}}\|_{\mathbb{BV}} = 0$$

for f_n^{B}, $n \in \mathbb{N}$, given by (1.3). By (2.3) and (2.4),

$$\int_a^b f_n^{\mathrm{B}}\mathrm{d}g = \sum_{k=1}^n \big[\Delta^+ f(w_k)(g(b) - g(w_k+)) + \Delta^- f(w_k)(g(b) - g(w_k-))\big] \tag{2.19}$$

holds for any $n \in \mathbb{N}$. Thus according to Theorem 2.7 the integral $\int_a^b f^{\mathrm{B}}\mathrm{d}g$ exists and

$$\int_a^b f^{\mathrm{B}}\mathrm{d}g = \lim_{n\to\infty}\int_a^b f_n^{\mathrm{B}}\mathrm{d}g. \tag{2.20}$$

The integral $\int_a^b f^{\mathrm{C}}\mathrm{d}g$ exists as the σ-Riemann-Stieltjes integral by Proposition 1.2. This means that $\int_a^b f\,\mathrm{d}g$ exists and

$$\int_a^b f\,\mathrm{d}g = \int_a^b f^{\mathrm{C}}\mathrm{d}g + \int_a^b f^{\mathrm{B}}\mathrm{d}g = \int_a^b f^{\mathrm{C}}\mathrm{d}g + \lim_{n\to\infty}\int_a^b f_n^{\mathrm{B}}\mathrm{d}g.$$

The inequality (2.18) follows immediately from Lemma 2.6. □

2.9. Remark. Since

$$\sum_{k=1}^\infty \big|\big[\Delta^+ f(w_k)(g(b) - g(w_k+)) + \Delta^- f(w_k)(g(b) - g(w_k-))\big]\big|$$
$$\le 2\|g\|\sum_{k=1}^\infty \Big(|\Delta^+ f(w_k)| + |\Delta^- f(w_k)|\Big) \le 2\|g\|(\operatorname{var}_a^b f) < \infty,$$

we have in virtue of (2.19) and (2.20)

$$\int_a^b f^{\mathrm{B}} \mathrm{d}g = \sum_{k=1}^{\infty} \big[\Delta^+ f(w_k)(g(b) - g(w_k+)) + \Delta^- f(w_k)(g(b) - g(w_k-))\big]. \tag{2.21}$$

As a direct consequence of Theorem 2.8 we obtain

2.10. Corollary. *Let* $h_n \in \mathbb{G}(a,b)$, $n \in \mathbb{N}$, *and* $h \in \mathbb{G}(a,b)$ *be such that*

$$\lim_{n\to\infty} \|h_n - h\| = 0.$$

Then for any $f \in \mathbb{BV}(a,b)$ *the integrals*

$$\int_a^b f\,\mathrm{d}h \quad \text{and} \quad \int_a^b f\,\mathrm{d}h_n, \ n \in \mathbb{N}, \quad \text{exist}$$

and

$$\lim_{n\to\infty} \int_a^b f\,\mathrm{d}h_n = \int_a^b f\,\mathrm{d}h.$$

2.11. Lemma. *Let* $h : [a,b] \to \mathbb{R}$, $c \in \mathbb{R}$ *and* $W = \{w_k\}_{k\in\mathbb{N}} \subset [a,b]$ *be such that* (2.11) *and*

$$\sum_{k=1}^{\infty} |h(w_k) - c| < \infty \tag{2.22}$$

hold. Given $n \in \mathbb{N}$, *let us put* $W_n = \{w_1, w_2, \dots, w_n\}$ *and*

$$h_n(t) = \begin{cases} c & \text{if } t \in [a,b] \setminus W_n, \\ h(t) & \text{if } t \in W_n. \end{cases} \tag{2.23}$$

Then $h_n \in \mathbb{BV}(a,b)$ *for any* $n \in \mathbb{N}$, $h \in \mathbb{BV}(a,b)$ *and*

$$\lim_{n\to\infty} \|h_n - h\|_{\mathbb{BV}} = 0 \tag{2.24}$$

Proof. The functions h_n, $n \in \mathbb{N}$, and h evidently have a bounded variation on $[a,b]$. For a given $n \in \mathbb{N}$, we have

$$h_n(t) - h(t) = \begin{cases} 0 & \text{if } t \in W_n \text{ or } t \in [a,b] \setminus W \\ c - h_n(t) & \text{if } t = w_k \text{ for some } k > n. \end{cases}$$

Thus,

$$\lim_{n\to\infty} h_n(t) = h(t) \quad \text{on} \quad [a,b] \tag{2.25}$$

and, moreover,

$$\sum_{j=1}^{m} \left|\big(h_n(t_j) - h(t_j)\big) - \big(h_n(t_{j-1}) - h(t_{j-1})\big)\right| \le 2 \sum \sum_{k=n+1}^{\infty} |h(w_k) - c|$$

holds for any $n \in \mathbb{N}$ and any division $\{t_0, t_1, \dots, t_m\}$ of $[a,b]$. Consequently,

$$\operatorname{var}_a^b (h_n - h) \le 2 \sum_{k=n+1}^{\infty} |h(w_k) - c| \tag{2.26}$$

holds for any $n \in \mathbb{N}$. In virtue of the assumption (2.22) the right-hand side of (2.26) tends to 0 as $n \to \infty$. Hence (2.24) follows from (2.25) and (2.26). □

2.12. Proposition. *Let $h : [a,b] \mapsto \mathbb{R}$, $c \in \mathbb{R}$ and $W = \{w_k\}_{k\in\mathbb{N}}$ be such that* (2.11) *and* (2.22) *hold. Then*

$$\int_a^b h \, \mathrm{d}g = \sum_{k=1}^{\infty} [h(w_k) - c] \Delta g(w_k) + c\, [g(b) - g(a)]$$

holds for any $g \in \mathbb{G}(a,b)$.

Proof. Let $g \in \mathbb{G}(a,b)$ be given. Let $W_n = \{w_1, w_2, \dots, w_n\}$ for $n \in \mathbb{N}$ and let the functions h_n, $n \in \mathbb{N}$, be given by (2.23). Given an arbitrary $n \in \mathbb{N}$, then (2.1) (with $\tau = b$) and (2.5) from Proposition 2.1 imply

$$\int_a^b h_n \, \mathrm{d}g = \sum_{k=1}^{n} [h(w_k) - c] \Delta g(w_k) + c[g(b) - g(a)].$$

Since (2.22) yields

$$\sum_{k=1}^{n} \big|[h(w_k) - c] \Delta g(w_k)\big| \le 2\, \|g\| \sum_{k=1}^{\infty} |h(w_k) - c| < \infty$$

and Lemma 2.11 implies

$$\lim_{n\to\infty} \|h_n - h\|_{\mathbb{BV}} = 0,$$

we can use Theorem 2.7 to prove that

$$\int_a^b h \, \mathrm{d}g = \lim_{n\to\infty} \int_a^b h_n \, \mathrm{d}g = \sum_{k=1}^{\infty} [h(w_k) - c] \Delta g(w_k) + c\, [g(b) - g(a)].$$

□

2.13. Proposition. *Let* $h : [a,b] \mapsto \mathbb{R}$, $c \in \mathbb{R}$ *and* $W = \{w_k\}_{k\in\mathbb{N}}$ *fulfil* (2.11). *Then*

$$\int_a^b f \, \mathrm{d}h = f(b)[h(b)-c] - f(a)[h(a)-c] \tag{2.27}$$

holds for any $f \in \mathbb{BV}(a,b)$.

Proof. Let $f \in \mathbb{BV}(a,b)$. For a given $n \in \mathbb{N}$, let $W_n = \{w_1, w_2, \dots, w_n\}$ and let h_n be given by (2.23). Then

$$\lim_{n\to\infty} \|h_n - h\| = 0. \tag{2.28}$$

Indeed, let $\varepsilon > 0$ be given and let $n_0 \in \mathbb{N}$ be such that $k \ge n_0$ implies

$$|h(w_k) - c| < \varepsilon. \tag{2.29}$$

(Such an n_0 exists since $|h(w_k) - c| = |\Delta^- h(w_k)| = |\Delta^+ h(w_k)|$ for any $k \in \mathbb{N}$ and the set of those $k \in \mathbb{N}$ for which the inequality (2.29) does not hold may be only finite.) Now, for any $n \ge n_0$ and any $t \in [a,b]$ such that $t = w_k$ for some $k > n$ ($t \in W \setminus W_n$) we have

$$|h_n(t) - h(t)| = |h_n(w_k) - h(w_k)| = |c - h(w_k)| < \varepsilon.$$

Since $h_n(t) = h(t)$ for all the other $t \in [a,b]$ ($t \in \big([a,b] \setminus W\big) \cup W_n$), it follows that $|h_n(t) - h(t)| < \varepsilon$ on $[a,b]$, i.e.

$$\|h_n - h\| < \varepsilon.$$

This proves the relation (2.28).

By Corollary 2.4 we have for any $n \in \mathbb{N}$

$$\int_a^b f \, \mathrm{d}h_n = f(b)[h(b)-c] - f(a)[h(a)-c].$$

Making use of (2.28) and Corollary 2.10 we obtain we obtain

$$\int_a^b f \, \mathrm{d}h = \lim_{n\to\infty} \int_a^b f \, \mathrm{d}h_n = f(b)[h(b)-c] - f(a)[h(a)-c].$$

□

2.14. Corollary. *Let* $h \in \mathbb{BV}(a,b)$, $c \in \mathbb{R}$ *and* $W = \{w_k\}_{k\in\mathbb{N}}$ *fulfil* (2.11). *Then* (2.27) *holds for any* $f \in \mathbb{G}(a,b)$.

Proof. By Proposition 2.12, (2.27) holds for any $f \in \mathbb{BV}(a,b)$. Making use of the density of $\mathbb{S}(a,b) \subset \mathbb{BV}(a,b)$ in $\mathbb{G}(a,b)$ and of the convergence theorem mentioned in Remark 2.5 we complete the proof of our assumption. □

2.15. Theorem. (Integration-by-parts) *If* $f \in \mathbb{BV}(a,b)$ *and* $g \in \mathbb{G}(a,b)$, *then both the integrals* $\int_a^b f\,\mathrm{d}g$ *and* $\int_a^b g\,\mathrm{d}f$ *exist and*

$$\int_a^b f\,\mathrm{d}g + \int_a^b g\,\mathrm{d}f = f(b)g(b) - f(a)g(a) + \sum_{t\in[a,b]} \big[\Delta^- f(t)\Delta^- g(t) - \Delta^+ f(t)\Delta^+ g(t)\big]. \tag{2.30}$$

Proof. The existence of the integral $\int_a^b g\,\mathrm{d}f$ is well known, while the existence of $\int_a^b g\,\mathrm{d}f$ is guaranteed by Theorem 2.8. Furthermore,

$$\begin{aligned}\int_a^b f\,\mathrm{d}g &+ \int_a^b g\,\mathrm{d}f \\ &= \int_a^b f(t)\,\mathrm{d}[g(t) + \Delta^+ g(t)] + \int_a^b g(t)\,\mathrm{d}[f(t) - \Delta^- f(t)] \\ &\quad - \int_a^b f(t)\,\mathrm{d}[\Delta^+ g(t)] + \int_a^b g(t)\,\mathrm{d}[\Delta^- f(t)].\end{aligned}$$

It is easy to verify that the function $h(t) = \Delta^+ g(t)$ fulfils the relation (2.11) with $c = 0$ and $h(b) = 0$. Consequently, Proposition 2.13 yields

$$\int_a^b f(t)\,\mathrm{d}[\Delta^+ g(t)] = -f(a)\Delta^+ g(a).$$

Similarly, by Corollary 2.14 we have

$$\int_a^b g(t)\,\mathrm{d}[\Delta^- f(t)] = \Delta^- f(b)g(b).$$

Hence

$$\int_a^b f\,\mathrm{d}g + \int_a^b g(t)\,\mathrm{d}f = \int_a^b f(t)\,\mathrm{d}[g(t+)] + \int_a^b g(t)\,\mathrm{d}[f(t-)] + f(a)\Delta^+ g(a) + \Delta^- f(b)g(b). \tag{2.31}$$

The first integral on the right-hand side may be modified in the following way:

$$\int_a^b f(t)\,\mathrm{d}[g(t+)] = \int_a^b f(t-)\,\mathrm{d}[g(t+)] + \int_a^b \Delta^- f(t)\,\mathrm{d}[g(t+)]. \tag{2.32}$$

Making use of Proposition 2.12 and taking into account that $\Delta g_1(t) = \Delta g(t)$ on $[a,b]$ for the function g_1 defined by $g_1(t) = g(t+)$ on $[a,b]$, we further obtain

$$\int_a^b \Delta^- f(t)\,\mathrm{d}[g(t+)] = \sum_{t\in[a,b]} \Delta^- f(t)\Delta g(t). \tag{2.33}$$

Similarly,

$$\begin{aligned}\int_a^b g(t)\,\mathrm{d}[f(t-)] &= \int_a^b g(t+)\,\mathrm{d}f(t-) - \int_a^b \Delta^+ g(t)\,\mathrm{d}[f(t-)]\\ &= \int_a^b g(t+)\,\mathrm{d}[f(t-)] - \sum_{t\in[a,b]} \Delta^+ g(t)\Delta f(t).\end{aligned} \tag{2.34}$$

The function $f(t-)$ is left-continuous on $[a,b]$, while $g(t+)$ is right-continuous on $[a,b)$. It means that both the integrals

$$\int_a^b f(t-)\,\mathrm{d}[g(t+)] \quad \text{and} \quad \int_a^b g(t+)\,\mathrm{d}[f(t-)]$$

exist as the σ-Riemann-Stieltjes integrals (cf. Corollary 1.3), and making use of the integration-by-parts theorem for these integrals (cf. [4, Theorem II.11.7]) we get

$$\int_a^b f(t-)\,\mathrm{d}[g(t+)] + \int_a^b g(t+)\,\mathrm{d}[f(t-)] = f(b-)g(b) - f(a)g(a+). \tag{2.35}$$

Inserting (2.32) - (2.35) into (2.31) we get

$$\begin{aligned}\int_a^b f\,\mathrm{d}g + \int_a^b g\,\mathrm{d}f &= f(b-)g(b) - f(a)g(a+)\\ &\quad + \sum_{t\in[a,b]} \Delta^- f(t)[\Delta^- g(t) + \Delta^+ g(t)]\\ &\quad - \sum_{t\in[a,b]} [\Delta^- f(t) + \Delta^+ f(t)]\Delta^+ g(t)\\ &\quad + f(a)\Delta^+ g(a) + \Delta^- f(b)g(b)\\ &= f(b)g(b) - f(a)g(a)\\ &\quad + \sum_{t\in[a,b]} \big[\Delta^- f(t)\Delta^- g(t) - \Delta^+ f(t)\Delta^+ g(t)\big]\end{aligned}$$

and this completes the proof □

The following proposition describes some properties of indefinite Perron-Stieltjes integrals.

2.16. Proposition. *Let $f : [a,b] \mapsto \mathbb{R}$ and $g : [a,b] \mapsto \mathbb{R}$ be such that $\int_a^b f\,\mathrm{d}g$ exists. Then the function*

$$h(t) = \int_a^t f\,\mathrm{d}g$$

is defined on $[a,b]$ and

(i) if $g \in \mathbb{G}(a,b)$, then $h \in \mathbb{G}(a,b)$ and

$$\Delta^+ h(t) = f(t)\Delta^+ g(t), \quad \Delta^- h(t) = f(t)\Delta^- g(t) \quad \text{on} \quad [a,b]; \tag{2.36}$$

(ii) if $g \in \mathbb{BV}(a,b)$ and f is bounded on $[a,b]$, then $h \in \mathbb{BV}(a,b)$.

Proof. The former assertion follows from [8, Theorem 1.3.5]. The latter follows immediately from the inequality

$$\sum_{j=1}^m \Big| \int_{t_{j-1}}^{t_j} f\,\mathrm{d}g \Big| \leq \sum_{j=1}^m \big[\|f\| \, (\mathrm{var}_{t_{j-1}}^{t_j} g) \big] = \|f\| (\mathrm{var}_a^b g)$$

which is valid for any division $\{t_0, t_1, \dots, t_m\}$ of $[a,b]$. □

In the theory of generalized differential equations the substitution formula

$$\int_a^b h(t)\,\mathrm{d}\Big[\int_a^t f(s)\,\mathrm{d}[g(s)] \Big] = \int_a^b h(t) f(t)\,\mathrm{d}[g(t)] \tag{2.37}$$

is often needed. In [4, II.19.3.7] this formula is proved for the σ-Young-Stieltjes integral under the assumption that $g \in \mathbb{G}(a,b)$, h is bounded on $[a,b]$ and the integral $\int_a^b f\,\mathrm{d}g$ as well as one of the integrals in (2.37) exists. In [14, Theorem I.4.25] this assertion was proved for the Kurzweil integral. Though it was assumed there that $g \in \mathbb{BV}(a,b)$, this assumption was not used in the proof. We will give here a slightly different proof based on the Saks-Henstock lemma (cf. e.g. [13, Lemma 1.11]).

2.17. Lemma. (Saks-Henstock) *Let $f, g : [a,b] \mapsto \mathbb{R}$ be such that the integral $\int_a^b f\,\mathrm{d}g$ exists. Let $\varepsilon > 0$ be given and let δ be a gauge on $[a,b]$ such that*

$$\Big| S_D(f\,\Delta g) - \int_a^b f\,\mathrm{d}g \Big| < \varepsilon$$

holds for any δ-fine partition D of [a, b]. Then for an arbitrary system $\{([\beta_i,\gamma_i],\sigma_i),\ i=1,2,\dots,k\}$ *of intervals and points such that*

$$a \le \beta_1 \le \sigma_1 \le \gamma_1 \le \beta_2 \le \dots \le \beta_k \le \sigma_k \le \gamma_k \le b \tag{2.38}$$

and

$$[\beta_i,\gamma_i] \subset [\sigma_i - \delta(\sigma_i), \sigma_i + \delta(\sigma_i)], \quad i = 1,2,\dots,k,$$

the inequality

$$\Big| \sum_{i=1}^{k} \Big[f(\sigma_i)[g(\gamma_i) - g(\beta_i)] - \int_{\beta_i}^{\gamma_i} f\,\mathrm{d}g \Big] \Big| < \varepsilon \tag{2.39}$$

holds. □

Making use of Lemma 2.17 we can prove the following useful assertion

2.18. Lemma. *If* $f:[a,b]\mapsto\mathbb{R}$ *and* $g:[a,b]\mapsto\mathbb{R}$ *are such that* $\int_a^b f\,\mathrm{d}g$ *exists, then for any* $\varepsilon > 0$ *there exists a gauge* δ *on* $[a,b]$ *such that*

$$\sum_{j=1}^{m} \Big| f(\xi_j)[g(t_j) - g(t_{j-1})] - \int_{t_{j-1}}^{t_j} f\,\mathrm{d}g \Big| < \varepsilon \tag{2.40}$$

holds for any δ-fine partition (d,ξ) *of* $[a,b]$.

Proof. Let $\delta:[a,b]\mapsto(0,\infty)$ be such that

$$\Big| S_D(f\,\Delta g) - \int_a^b f\,\mathrm{d}g \Big| = \Big| \sum_{j=1}^{m} f(\xi_j)[g(t_j) - g(t_{j-1})] - \int_{t_{j-1}}^{t_j} f\,\mathrm{d}g \Big| < \tfrac{\varepsilon}{2}$$

for all δ-fine partitions $D=(d,\xi)$ of $[a,b]$. Let us choose an arbitrary δ-fine partition $D=(d,\xi)$ of $[a,b]$. Let $\gamma_i = t_{p_i}$ and $\beta_i = t_{p_i-1}$, $i=1,2,\dots,k$, be all the points of the division d such that

$$f(\xi_{p_i})[g(\gamma_i) - g(\beta_i)] - \int_{\beta_i}^{\gamma_i} f\,\mathrm{d}g \ge 0.$$

Then the system $\{([\beta_i,\gamma_i],\sigma_i),\ i=1,2,\dots,k\}$, where $\sigma_i = \xi_{p_i}$, fulfils (2.38) and (2.39) and hence we can use Lemma 2.17 to prove that the inequality

$$\sum_{i=1}^{k} \Big| f(\xi_{p_i})[g(\gamma_i) - g(\beta_i)] - \int_{\beta_i}^{\gamma_i} f\,\mathrm{d}g \Big| < \tfrac{\varepsilon}{2}$$

is true. Similarly, if $\omega_i = t_{q_i}$ and $\theta_i = t_{q_i-1}$, $i=1,2,\dots,r$ are all points of the division d such that

$$f(\xi_{q_i})[g(\omega_i) - g(\theta_i)] - \int_{\theta_i}^{\omega_i} f\,\mathrm{d}g \le 0,$$

the the inequality

$$\sum_{i=1}^{r} \left| f(\xi_{q_i})[g(\omega_i) - g(\theta_i)] - \int_{\theta_i}^{\omega_i} f\,\mathrm{d}g \right| < \tfrac{\varepsilon}{2}$$

follows from Lemma 2.17, as well. Summarizing, we conclude that

$$\begin{aligned}
&\sum_{j=1}^{m} \left| f(\xi_j)[g(t_j) - g(t_{j-1})] - \int_{t_{j-1}}^{t_j} f\,\mathrm{d}g \right| \\
&\quad = \sum_{i=1}^{k} \left| f(\xi_{p_i})[g(\gamma_i) - g(\beta_i)] - \int_{\beta_i}^{\gamma_i} f\,\mathrm{d}g \right| \\
&\qquad + \sum_{i=1}^{r} \left| f(\xi_{q_i})[g(\omega_i) - g(\theta_i)] - \int_{\theta_i}^{\omega_i} f\,\mathrm{d}g \right| \\
&\quad < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

This completes the proof. □

2.19. Theorem. *Let $h : [a,b] \mapsto \mathbb{R}$ be bounded on $[a,b]$ and let $f, g : [a,b] \mapsto \mathbb{R}$ be such that the integral $\int_a^b f\,\mathrm{d}g$ exists. Then the integral*

$$\int_a^b h(t) f(t)\,\mathrm{d}[g(t)]$$

exists if and only if the integral

$$\int_a^b h(t)\,\mathrm{d}\Big[\int_a^t f(s)\,\mathrm{d}[g(s)] \Big]$$

exists, and in this case the relation (2.37) *holds.*

Proof. Let $|h(t)| \le C < \infty$ on $[a,b]$. Let us assume that the integral

$$\int_a^b h(t) f(t)\,\mathrm{d}[g(t)]$$

exists and let $\varepsilon > 0$ be given. There exists a gauge δ_1 on $[a,b]$ such that

$$\Big| \sum_{j=1}^{m} h(\xi_k) f(\xi_j)[g(t_j) - g(t_{j-1})] - \int_a^b h(t) f(t)\,\mathrm{d}[g(t)] \Big| < \tfrac{\varepsilon}{2}$$

is satisfied for any δ_1-fine partition (d,ξ) of $[a,b]$. By Lemma 2.18 there exists a gauge δ on $[a,b]$ such that $\delta(t)\le\delta_1(t)$ on $[a,b]$ and

$$\sum_{j=1}^m \Big| f(\xi_j)[g(t_j)-g(t_{j-1})] - \int_{t_{j-1}}^{t_j} f\,\mathrm{d}g\Big| < \frac{\varepsilon}{2C}$$

holds for any δ-fine partition (d,ξ) of $[a,b]$. Let us denote

$$k(t)=\int_a^t f\,\mathrm{d}g \quad\text{for}\quad t\in[a,b].$$

Then for any δ-fine partition $D=(d,\xi)$ of $[a,b]$ we have

$$\begin{aligned}
&\Big|S_D(h\,\Delta k) - \int_a^b h(t)f(t)\,\mathrm{d}[g(t)]\Big|\\
&\quad=\Big|\sum_{j=1}^m h(\xi_j)\int_{t_{j-1}}^{t_j} f\,\mathrm{d}g - \sum_{j=1}^m h(\xi_j)f(\xi_j)[g(t_j)-g(t_{j-1})]\\
&\qquad+\sum_{j=1}^m h(\xi_j)f(\xi_j)[g(t_j)-g(t_{j-1})] - \int_a^b h(t)f(t)\,\mathrm{d}[g(t)]\Big|\\
&\quad\le\Big|\sum_{j=1}^m h(\xi_j)\Big[\int_{t_{j-1}}^{t_j} f\,\mathrm{d}g - f(\xi_j)[g(t_j)-g(t_{j-1})]\Big]\Big|\\
&\qquad+\Big|\sum_{j=1}^m h(\xi_j)f(\xi_j)[g(t_j)-g(t_{j-1})] - \int_a^b hf\,\mathrm{d}g\Big| < \varepsilon
\end{aligned}$$

This implies the existence of the integral

$$\int_a^b h\,\mathrm{d}k$$

and the relation (2.37). The second implication can be proved in an analogous way. □

The convergence result 2.10 enables us to extend the known theorems on the change of integration order in iterated integrals

$$\text{(2.41)}\qquad \int_c^d g(t)\,\mathrm{d}\Big[\int_a^b h(t,s)\,\mathrm{d}[f(s)]\Big],\quad \int_a^b \Big(\int_c^d g(t)\,\mathrm{d}_t[h(t,s)]\Big)\,\mathrm{d}[f(s)],$$

where $-\infty<c<d<\infty$ and h is of strongly bounded variation on $[c,d]\times[a,b]$ (cf. [14, Theorem I.6.20]). In what follows $\mathrm{v}(h)$ denotes the *Vitali variation* of the function h on $[c,d]\times[a,b]$ (cf. [4, Definition III.4.1] or [14, I.6.1]). For a given $t\in[c,d]$, $\operatorname{var}_a^b h(t,.)$ denotes the variation of the function $s\in[a,b]\mapsto h(t,s)\in\mathbb{R}$ on $[a,b]$. Similarly, for $s\in[a,b]$ fixed, $\operatorname{var}_c^d h(.,s)$ stands for the variation of the function $t\in[c,d]\mapsto h(t,s)\in\mathbb{R}$ on $[c,d]$.

2.20. Theorem. *Let $h:[c,d]\times[a,b]\mapsto\mathbb{R}$ be such that*

$$\mathrm{v}(h)+\operatorname{var}_a^b h(c,.)+\operatorname{var}_c^d h(.,a)<\infty.$$

Then for any $f\in\mathbb{BV}(a,b)$ and any $g\in\mathbb{G}(c,d)$ both the integrals (2.41) *exist and*

$$\int_c^d g(t)\,\mathrm{d}\Big[\int_a^b h(t,s)\,\mathrm{d}[f(s)]\Big]=\int_a^b\Big(\int_c^d g(t)\,\mathrm{d}_t[h(t,s)]\Big)\mathrm{d}[f(s)]. \tag{2.42}$$

Proof. Let us notice that by [14, Theorem I.6.20] our assertion is true if g is also supposed to be of bounded variation. in the general case of $g\in\mathbb{G}(a,b)$ there exists a sequence $\{g_n\}_{n=1}^{\infty}\subset\mathbb{S}(a,b)$ such that $\lim_{n\to\infty}\|g-g_n\|=0$. Then, since the function

$$v(t)=\int_a^b h(t,s)\,\mathrm{d}[f(s)]$$

is of bounded variation on $[c,d]$ (cf. the first part of the proof of [14, Theorem I.6.20]), the integral on left-hand side of (2.42) exists and by Corollary 2.10 and [14, Theorem I.6.20] we have

$$\begin{aligned}\int_c^d g(t)\,\mathrm{d}\Big[\int_a^b h(t,s)\,\mathrm{d}[f(s)]\Big]&=\lim_{n\to\infty}\int_c^d g_n(t)\,\mathrm{d}\Big[\int_a^b h(t,s)\,\mathrm{d}[f(s)]\Big]\\&=\lim_{n\to\infty}\int_a^b\Big(\int_c^d g_n(t)\,\mathrm{d}_t[h(t,s)]\Big)\mathrm{d}[f(s)].\end{aligned} \tag{2.43}$$

Let us denote

$$w_n(t)=\int_c^d g_n(t)\,\mathrm{d}_t[h(t,s)]\quad\text{for}\quad s\in[a,b]\quad\text{and}\quad n\in\mathbb{N}.$$

Then $w_n\in\mathbb{BV}(a,b)$ for any $n\in\mathbb{N}$ (cf. [14, Theorem I.6.18]) and by [14, Theorem I.4.17] mentioned here in Remark 2.5 we obtain

$$\lim_{n\to\infty}w_n(s)=\int_c^d g_n(t)\,\mathrm{d}_t[h(t,s)]:=w(s)\ \text{ on }\ [a,b].$$

As

$$|w_n(s)-w(s)|\le\|g_n-g\|\big(\operatorname{var}_c^d h(.,s)\big)\le\|g_n-g\|\big(\mathrm{v}(h)+\operatorname{var}_c^d h(.,a)\big)$$

for any $s \in [a,b]$ (cf. [14, Lemma I.6.6]), we have

$$\lim_{n\to\infty} \|w_n - w\| = 0.$$

It means that $w \in \mathbb{G}(a,b)$ and by Theorem 2.8 the integral

$$\int_a^b w(s)\,\mathrm{d}[f(s)] = \int_a^b \Big(\int_c^d g(t)\,\mathrm{d}_t[h(t,s)]\Big)\,\mathrm{d}[f(s)]$$

exists as well. Since obviously

$$\lim_{n\to\infty} \int_a^b \Big(\int_c^d g_n(t)\,\mathrm{d}_t[h(t,s)]\Big)\,\mathrm{d}[f(s)] = \lim_{n\to\infty} \int_a^b w_n(s)\,\mathrm{d}[f(s)]$$
$$= \int_a^b w(s)\,\mathrm{d}[f(s)] = \int_a^b \Big(\int_c^d g(t)\,\mathrm{d}_t[h(t,s)]\Big)\,\mathrm{d}[f(s)],$$

the relation (2.42) follows from (2.43). □

3 . Linear bounded functionals on $\mathbf{G}_L$(a,b)

By Theorem 2.8 the expression

$$F_\eta(x) = q\,x(a) + \int_a^b p\,\mathrm{d}q \tag{3.1}$$

is defined for any $x \in \mathbb{G}(a,b)$ and any $\eta = (p,q) \in \mathbb{BV}(a,b) \times \mathbb{R}$. Moreover, for any $\eta \in \mathbb{BV}(a,b) \times \mathbb{R}$, the relation (3.1) defines a linear bounded functional on $\mathbb{G}_{\mathrm{L}}(a,b)$.

Proposition 2.3 immediately implies

3.1. Lemma. *Let $\eta = (p,q) \in \mathbb{BV}(a,b) \times \mathbb{R}$ be given. Then*

$$\begin{aligned} F_\eta(\chi_{[a,b]}) &= q, \\ F_\eta(\chi_{(\tau,b]}) &= p(\tau) \quad \textit{for any} \quad \tau \in [a,b), \\ F_\eta(\chi_{[b]}) &= p(b). \end{aligned} \tag{3.2}$$

□

3.2. Corollary. *If $\eta = (p,q) \in \mathbb{BV}(a,b) \times \mathbb{R}$ and $F_\eta(x) = 0$ for all $x \in \mathbb{S}(a,b)$ which are left-continuous on (a,b), then $p(t) \equiv 0$ on $[a,b]$ and $q = 0$.* □

3.3. Lemma. *Let $x \in \mathbb{G}(a,b)$ be given. Then for a given $\eta = (p,q) \in \mathbb{BV}(a,b) \times \mathbb{R}$,*

$$\begin{aligned} F_\eta(x) &= x(a) \quad \text{if } p \equiv 0 \text{ on } [a,b] \text{ and } q = 1, \\ F_\eta(x) &= x(b) \quad \text{if } p \equiv 1 \text{ on } [a,b] \text{ and } q = 1, \\ F_\eta(x) &= x(\tau-) \text{ if } p = \chi_{[a,\tau)} \text{ on } [a,b],\ \tau \in (a,b] \text{ and } q = 1, \\ F_\eta(x) &= x(\tau+) \text{ if } p = \chi_{[a,\tau]} \text{ on } [a,b],\ \tau \in [a,b) \text{ and } q = 1. \end{aligned} \tag{3.3}$$

Proof follows from Proposition 2.1. □

3.4. Corollary. *If $x \in \mathbb{G}(a,b)$ and $F_\eta(x) = 0$ for all $\eta = (p,q) \in \mathbb{BV}(a,b) \times \mathbb{R}$, then*

$$x(a) = x(a+) = x(\tau-) = x(\tau+) = x(b-) = x(b) \tag{3.4}$$

holds for any $\tau \in (a,b)$. In particular, if $x \in \mathbb{G}_L(a,b)$ (x is left-continuous on (a,b)) and $F_\eta(x) = 0$ for all $\eta = (p,q) \in \mathbb{BV}(a,b) \times \mathbb{R}$, then $x(t) \equiv 0$ on $[a,b]$. □

3.5. Remark. The space $\mathbb{BV}(a,b) \times \mathbb{R}$ is supposed to be equipped with the usual norm ($\|\eta\|_{\mathbb{BV}\times\mathbb{R}} = |q| + \|p\|_{\mathbb{BV}}$ for $\eta = (p,q) \in \mathbb{BV}(a,b) \times \mathbb{R}$). Obviously, it is a Banach space with respect to this norm.

3.6. Proposition. *The spaces $\mathbb{G}_L(a,b)$ and $\mathbb{BV}(a,b) \times \mathbb{R}$ form a dual pair with respect to the bilinear form*

$$x \in \mathbb{G}_L(a,b), \eta \in \mathbb{BV}(a,b) \times \mathbb{R} \mapsto F_\eta(x). \tag{3.5}$$

Proof follows from Corollaries 3.2 and 3.4. □

On the other hand, we have

3.7. Lemma. *If F is a linear bounded functional on $\mathbb{G}_L(a,b)$ and*

$$p(t) = \begin{cases} F(\chi_{(t,b]}) & \text{if } t \in [a,b), \\ F(\chi_{[b]}) & \text{if } t = b, \end{cases} \tag{3.6}$$

then $p \in \mathbb{BV}(a,b)$ and

$$|p(a)| + |p(b)| + \operatorname{var}_a^b p \le 2\|F\|, \tag{3.7}$$

where

$$\|F\| = \sup_{x \in \mathbb{G}_L(a,b), \|x\| \le 1} |F(x)|.$$

Proof is analogous to that of part c (i) of [5, Theorem 5.1]. Indeed, for an arbitrary division $\{t_0, t_1, \dots, t_m\}$ of $[a,b]$ we have

$$\begin{aligned}
&\sup_{|c_j|\le 1, c_j\in\mathbb{R}} \Big| p(a)c_0 + p(b)c_{m+1} + \sum_{j=1}^{m} [p(t_j) - p(t_{j-1})]c_j \Big| \\
&= \sup_{|c_j|\le 1, c_j\in\mathbb{R}} \Big| F\big(c_0\,\chi_{(a,b]} + c_{m+1}\,\chi_{[b]} - \sum_{j=1}^{m-1} c_j\,\chi_{(t_{j-1},t_j]} + c_m\,\chi_{(t_{m-1},b)}\big)\Big| \\
&\le \sup_{\|h\|\le 2, h\in\mathbb{G}_{\mathrm{L}}(a,b)} |F(h)| = 2\|F\|.
\end{aligned}$$

In particular, for $c_0 = \operatorname{sgn} p(a)$, $c_{m+1} = \operatorname{sgn} p(b)$ and $c_j = \operatorname{sgn}(p(t_j) - p(t_{j-1}))$, $j = 1, 2, \dots, m$, we get

$$|p(a)| + |p(b)| + \sum_{j=1}^{m} |p(t_j) - p(t_{j-1})| \le 2\|F\|,$$

and the inequality (3.7) immediately follows. □

Using the ideas from the proof of [5, Theorem 5.1] we may now prove the following representation theorem.

3.8. Theorem. *F is a linear bounded functional on $\mathbb{G}_L(a,b)$ ($F \in \mathbb{G}_L^*(a,b)$) if and only if there is an $\eta = (p,q) \in \mathbb{BV}(a,b) \times \mathbb{R}$ such that*

$$(3.8) \qquad F(x) = F_\eta(x)\Big(:= q x(a) + \int_a^b p\,\mathrm{d}x\Big) \quad \textit{for any} \;\; x \in \mathbb{G}_L(a,b).$$

The mapping

$$\Phi : \eta \in \mathbb{BV}(a,b) \times \mathbb{R} \mapsto F_\eta \in \mathbb{G}_L^*(a,b)$$

is an isomorphism.

Proof. Let a linear bounded functional F on $\mathbb{G}_{\mathrm{L}}(a,b)$ be given and let us put

$$(3.9) \qquad q = F(\chi_{[a,b]}) \quad \text{and} \quad p(t) = \begin{cases} F(\chi_{(t,b]}) & \text{if } \; t \in [a,b), \\ F(\chi_{[b]}) & \text{if } \; t = b. \end{cases}$$

Then Lemma 2.6 implies $\eta = (p,q) \in \mathbb{BV}(a,b) \times \mathbb{R}$ and by Lemma 3.1 we have

$$\begin{aligned}
&F(\chi_{[a,b]}) = F_\eta(\chi_{[a,b]}), \\
&F(\chi_{(t,b]}) = F_\eta(\chi_{(t,b]}) \quad \text{for any} \;\; t \in [a,b)
\end{aligned}$$

and

$$F(\chi_{[b]}) \;= F_\eta(\chi_{[b]}).$$

Since all functions from $\mathbb{S}(a,b)\cap\mathbb{G}_{\rm L}(a,b)$ obviously are finite linear combinations of the functions

$$\chi_{[a,b]},\ \chi_{(\tau,b]},\ \tau\in[a,b),\ \chi_{[b]},$$

it follows that $F(x)=F_\eta(x)$ holds for any $x\in\mathbb{S}(a,b)\cap\mathbb{G}_{\rm L}(a,b)$. Now, the density of $\mathbb{S}(a,b)\cap\mathbb{G}_{\rm L}(a,b)$ in $\mathbb{G}_{\rm L}(a,b)$ implies that

$$F(x)=F_\eta(x) \quad \text{for all} \quad x\in\mathbb{G}_{\rm L}(a,b).$$

This completes the proof of the first assertion of the theorem.

Given an $x\in\mathbb{G}_{\rm L}(a,b)$, then Lemma 2.6 yields

$$|F_\eta(x)|\le\big(|p(a)|+|p(b)|+\operatorname{var}_a^b p+|q|\big)\|x\|$$

and consequently,

$$\|F_\eta\|\le|p(a)|+|p(b)|+\operatorname{var}_a^b p+|q|\le 2\big(\|p\|_{\mathbb{BV}}+|q|\big)=2\|\eta\|_{\mathbb{BV}\times\mathbb{R}}.$$

On the other hand, according to Lemma 3.7 we have

$$\|p\|_{\mathbb{BV}}\le\big(|p(a)|+|p(b)|+\operatorname{var}_a^b p\big)\le 2\|F\|.$$

Furthermore, in virtue of (3.9) we have $|q|\le\|F\|$ and hence

$$\|\eta\|_{\mathbb{BV}\times\mathbb{R}}=\|p\|_{\mathbb{BV}}+|q|\le 2\|F\|.$$

It means that

$$\tfrac{1}{2}\|F\|\le\|\eta\|_{\mathbb{BV}\times\mathbb{R}}\le 3\|F\|$$

and this completes the proof of the theorem. □

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