INTRODUCTION TO RIEMANNIAN GEOMETRY

HÔNG VÂN LÊ

In this course we shall discuss the following notions: smooth manifolds, tangent spaces, fiber bundles, one parameter groups of transformations, topics on curvature and geodesics.

Recommended books:

- Sigmundur Gudmunsson, Riemannian Geometry,
http://www.matematik.lu.se/matematiklu/personal/sigma/index.html ,
- Juergen Jost, Riemannian Geometry and Geometric Analysis, Springer 2002.

Contents

1. Manifolds, differentiable maps, Lie groups

In this section we introduce the notion of differentiable manifolds, which are generalization of domains in Euclidean spaces with good enough properties to apply tools of differential calculus. Morphisms between differentiable manifolds are differentiable maps. Most important examples of differentiable manifolds are Lie groups.

1.1. **Topological manifolds and smooth manifolds.**

I suppose that you know what are curves and surfaces. Curves and surfaces are subjects of differential geometry at the very beginning of its development. In analytic geometry you already learn geometry of some curves and some surfaces. In differential geometry you shall use extensively the tool of differential calculus, taking derivatives, taking integrations, solving differential equations related with geometric objects.

Differential geometry has many applications in physics, economy, computer graphic, information theory, where we recognize pattern of differentiable manifolds and laws governed by differentiable mappings.

Let me now introduce you to geometric objects in differential geometry.

In classical analysis we are concerned with calculus on domains Ω of Euclidean spaces $\mathbb{R}^m$: we can differentiate a smooth function $f(x_1, \cdots, x_n)$ of n-variables on Ω, take integration of f etc. So any domain in $\mathbb{R}^n$ is an elementary geometric object in differential geometry. To enlarge the class of differential geometric objects we glue together domains of the same dimension. The obtained geometric objects look locally like open subsets in spaces $\mathbb{R}^n$.

Definition 1.1.1. Let M be a Hausdorff space [1] with a countable basis for its topology. We call M **an n-dimensional topological manifold**, if for each point $p \in M$ there is an open neighborhood U of p in M such that U is homeomorphic to an open set in $\mathbb{R}^n$ by some homeomorphism ϕ, which is also called *coordinate map of* U. Such a pair (U, ϕ) is called a **local coordinates systems** or a **chart** on M, and the number n is called *the dimension of* M^n.

Clearly every open set in $\mathbb{R}^n$ is a topological n-dimensional space. But there are plenty of closed subsets of $\mathbb{R}^n$ which are topological manifolds.

Example 1.1.2. The sphere $S^n = \{x \in \mathbb{R}^{n+1}, ||x|| = 1\}$ is a n-dimensional manifold. We write S^n as the union of two open sets $\{S^n \setminus N\} \cup \{S^n \setminus S\}$ where N and S are the north pole and the south pole of S^n. Clearly each point of S^n belongs to one of those two open sets. And these open sets are homeomorphic to $\mathbb{R}^n$ by stereographic projections π. The pair $(\{S^n \setminus N, \pi_N\}, \{S^n \setminus S, \pi_S\}$ is a chart on S^n.

We are now interested in the class of topological manifolds with a good gluing, that is a good agreement between different homeomorphisms ϕ_i and ϕ_j on the common domain $U_i \cap U_j$.

Definition 1.1.3. A (smooth) **atlas** on a topological manifold M is a collection $\mathcal{A} = \{(U_i, \phi_i)\}$ of charts such that U_i form an open covering of M and for each pair (U_i, ϕ_i) and (U_j, ϕ_j) in $\mathcal{A}$ the transition map

$$\Phi_{ij} = \phi_j \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$$

is a smooth map between the open sets in Euclidean space.

Two atlas are **equivalent**, if their union is also an atlas.

A **smooth structure** on a topological manifold is an equivalence class of smooth atlases, and a **smooth manifold** is a topological manifold with a specified smooth structure.

Exercise 1.1.4. (i) Prove that the chart on S^n in Example 1.1.2 is a smooth atlas.
(ii) Show that if M^m and N^n are topological manifolds, then the Cartesian product $M^m \times N^n$ is a topological manifolds. If $\{(U_i, \phi_i)\}$ is a smooth structure on M^m and $\{(V_j, \psi_j)\}$ is a smooth structure on N^n, then $\{(U_i \times U_j, \phi_i \times \psi_j)\}$ is a smooth structure on $M^m \times N^n$.

Hint. i) Show that $\pi_N(p_1, \cdots, p_{n+1}) = \frac{1}{1-p_1}(p_2, \cdots, p_{n+1}$ and $\pi_S(p_1, \cdots, p_{n+1}) = \frac{1}{1+p_1}(p_2, \cdots, p_{n+1}$.

It is known that there are topological manifolds which have no smooth structure and there are topological manifolds which have more than one smooth structure.

[1] an equivalent definition is the existence of unique limit for any sequence, a nice property of a Hausdorff space is that the compactness implies the closedness, http://planetmath.org/?op=getobj&from=objects&id=4203

Simon Donaldson (1983) proved that a positive definite intersection form of a simply connected smooth manifold of dimension 4 is diagonalisable to the identity matrix.

Michael Freedman (1982) had shown that any positive definite unimodular symmetric bilinear form is realized as the intersection form of some four-manifold; combining his and Donaldson's result, any non-diagonalizable intersection form gives rise to a four-dimensional topological manifold with no differentiable structure.

An exotic sphere is a differentiable manifold that is homeomorphic to the standard Euclidean n-sphere, but not diffeomorphic. That means that such a manifold M is a sphere from a topological point of view, but not from the point of view of its differential structure. The first exotic spheres were constructed by John Milnor (1956) in dimension n = 7 as S^3-bundles over S^4. He showed that the oriented exotic 7-spheres are the non-trivial elements of a cyclic group of order 28 under the operation of connected sum. In any dimension Milnor (1959) showed that the diffeomorphism classes of oriented exotic spheres form the non-trivial elements of an abelian monoid under connected sum, which is a finite abelian group if the dimension is not 4.

Remark 1.1.5. The concept of a differentiable manifold was implicitly introduced in the habilitation address of Bernhard Riemmann in Göttingen. The first clear formulation of this concept was given by Herman Weyl later in 1913.

1.2. **Differentiable maps.** Let M be a differentiable manifold. To study M we need to understand all social relations of M with other differentiable manifolds N, i.e. to study all maps from N to M and all maps from M to N. A mapping from M to $N = \mathbb{R}^n$ is also called a $\mathbb{R}^n$-valued function.

Definition 1.2.1. A continuous map $f : M_1 \to M_2$ is said to be **C^k-differentiable at a point** $p \in M_1$, if there is a chart (U, ϕ) on M_1 with $p \in U$ and a chart (V, ψ) on M_2 with $f(p) \in V$ such that the composition $\psi \circ f \circ \phi^{-1}$ is C^k-differentiable.

If f is C^k-differentiable at every point $p \in M_1$, then we say that f is a **C^k-map.**

Clearly the C^k-differentiability does not depend on the choice of local coordinates, since the composition of a smooth map (the change of local coordinates) with a map of class C^k is of class C^k.

We denote by $C^k(M_1, M_2)$ the set of all C^k-maps between M_1 and M_2. It follows immediately from the definition that the composition of two C^k-maps is also a C^k-map. In particular, for $\phi \in C^k(M_1, M_2)$

$$\phi^*(C^k(M_2)) \subset C^k(M_1), \tag{1.2.1}$$

where the induced map ϕ_* is defined as follows

$$\phi_*(f(x)) := f(\phi(x)).$$

In fact, we can take (1.2.1) for a definition of a C^k-map, i.e. a continuous map ϕ is said to be of class C^k, if (1.2.1) holds.

A smooth map f is called **diffeomorphism**, if it is bijective with a smooth inverse.

1.3. **Lie groups.** We now introduce main players in the study of differential geometry. These are Lie groups studied by Sophus Lie, Felix Klein, Wilhelm Killing, Elie Cartan. The theory of Lie group remains an active domain of research nowadays.

Definition 1.3.1. A Lie group is a smooth manifold G with a smooth multiplication and a smooth inversion.

Example 1.3.2. The set $GL_n(\mathbb{R}) = \{x \in M_n(\mathbb{R}) | \det(x) \neq 0\}$ is a smooth manifold, since it is an open set in the space $Mat_n(\mathbb{R}) = \mathbb{R}^{n^2}$. It is easy to check that the multiplication $\mu : GL_n(\mathbb{R}) \times GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ and the inverse $\tau : GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ is smooth.

Definition 1.3.3. A **homomorphism** between Lie groups G and H is a smooth map $\phi : G \to H$ which is also a homomorphism in the sense of group theory.

Example 1.3.4. The exceptional mapping $\exp : (\mathbb{R}, +) \to (\mathbb{R}^+, \times)$ is an isomorphism of Lie groups (i.e. it and its inverse are homomorphisms of Lie groups).

The idea of symmetry in differential geometry is expressed via the notion of an action of a Lie group on a (topological) differentiable manifold. For example we see that the round sphere S^2 is symmetric because we can rotate it along an axis, say $\vec{x}_3$. Equivalently, there is an action of S^1 on S^2, i.e. we have a map

$$S^1 \times S^2 \to S^2 : (\theta, r_3, r_2\phi) \mapsto (r_3, r_2, \phi + \theta)$$

In general, we say that a group G acts on a manifold M if there exists a differentiable map $\chi : G \times M \to M$ such that

$$\chi(g_1 \circ g_2, m) = \chi(g_1, \chi(g_2, m)).$$

So the action looks like an associative multiplication with value in M.

Exercise 1.3.5. Prove that the product of two rotations in $\mathbb{R}^3$ around $\vec{v}_i$ with an angle θ_i, $i = 1, 2$ is a rotation around some axis.

Hint. Prove that the product of two rotations leaves some vector unchanged, using the fact that the characteristic polynomial of a linear transformation on $\mathbb{R}^3$ has a real root.

2. Tangent bundle and tangent map

In this section, using the notion of a derivation of a function, we define the tangent bundle of a differentiable manifold M as a smooth manifold which is a Taylor expansion of M up to first order. We define the tangent map as a natural linear transformation of tangent space under a smooth map.

2.1. Tangent space and tangent map.

Denote by $C^\infty(M)$ the space of all smooth differentiable functions on a differentiable manifold M.

Definition 2.1.1. A tangent vector δ in a point $x_0 \in M$ is a $\mathbb{R}$-linear map $\delta : C^\infty(M) \to \mathbb{R}$ satisfying the Leibniz rule

$$\delta(f \cdot g) = f(x_0)\delta(g) + g(x_0)\delta(f) \text{ for all } f, g \in C^\infty(M).$$

Such a map is called **a derivation of** $C^\infty(M)$ **at** x_0.

It is easy to see that the space of tangent vectors in x_0 is a vector space. We denote this space by $T_{x_0}M$.

Example 2.1.2. i) The partial derivative ∂x_i defines a derivation at a point x by $\partial x_i(f)_x = (\partial x_i f)(x)$.
ii) Let $\gamma : \mathbb{R} \to M$ be a smooth map (curve) such that $\gamma(0) = p$. Then γ defines a derivation of $C^\infty(M)$ at point p by

$$D_{\dot\gamma}(f) := \frac{d}{dt}|_{t=0} f(\gamma(t))$$

for all $f \in C^\infty(M)$.

Remark 2.1.3 (Existence of the tangent map)**.** If $g : (M^m, x) \to (N^n, y)$ is a smooth map, then g induces a linear map $(Dg)_x : T_xM \to T_yN$ defined by

$$((Dg)_x\delta)\psi = \delta(\psi \circ g)$$

for any $\psi \in C^\infty(M)$.

Exercise 2.1.4. i) Show that if $g : (M, x) \to (N, y)$ and $f : (N, y) \to (P, z)$ are smooth map, then $(D(f \circ g))_x = (Df)_y \circ (Dg)_x$ (Chain rule).
ii) Show that $D_{\dot\gamma} = D\gamma(\partial t)$, where ∂t is the partial derivative on $\mathbb{R}$.

Corollary 2.1.5. *If $f : M \to N$ is a diffeomorphism in some neighborhood of $x \in M$ then $(Df_*)_x : T_xM \to T_{f(x)}M$ is an isomorphism.*

Corollary 2.1.5 says that the tangent space T_xM is determined by a coordinate neighborhood of x. Now we shall look at the tangent space of a point in $\mathbb{R}^n$.

Theorem 2.1.6. *If U is an open subset of $\mathbb{R}^n$ and D is a derivation at $p \in U$, then*

$$D = \sum_{1\le i\le n} D(x_i)\frac{\partial}{\partial x_i}_{|p}.$$

Proof. Write

$$f(x) - f(p) = \int_0^1 \frac{\partial f(p + t(x-p))}{\partial t}\, dt = \sum_{1\le i\le n}(x_i - p_i)\int_0^1 \frac{\partial f}{\partial x_i}(p + t(x-p))\, dt.$$

Integrating by parts we get

$$\int_0^1 \frac{\partial f}{\partial x_i}(p + t(x-p))\, dt =$$

$$= t\frac{\partial f}{\partial x_i}(p + t(x-p))|_0^1 - \int_0^1 \sum_{1\le j\le n}\frac{\partial^2 f}{\partial x_i \partial x_j}(p + t(x-p))(x_j - x_p)\, dt =$$

$$= \frac{\partial f}{\partial x_i}(x) + a_i(x)$$

where a_i are smooth and vanish at p. Now write

$$f(x) = f(p) + \sum_{1\le i\le n}(x_i - p_i)(\frac{\partial f}{\partial x_i}(p) + a_i(x)),$$

and applying D to both sides. We get

$$Df = \sum_{1\le \le n} D(x_i - p_i)(\frac{\partial f}{\partial x_i}(p) + a_i(x)) = \sum_{1\le i\le n} D(x_i)\frac{\partial f}{\partial x_i}(p).$$

□

Theorem 2.1.6 shows that the tangent space T_xM^n is spanned by $D_i = \frac{\partial}{\partial x_i}_{|p}$. Hence $\dim T_x\mathbb{R}^n = n$.

Exercise 2.1.7. Assume that $g = (g^1, \cdots, g^n)$ is a smooth map from $\mathbb{R}^m \to \mathbb{R}^n$. Prove that

$$(Dg)_p(D_i) = \sum_{1 \le j \le n} a_{ij} D_j$$

where $a_{ij} = (\partial g^j / \partial x_i)(p)$.

Hint. Use $(Dg)(\frac{\partial}{\partial x_i}(f(x)) = \frac{\partial}{\partial x_i}(f(g(x)) = \sum \frac{\partial f}{\partial x_j}\frac{\partial g^j}{\partial x_i}$.

2.2. **Tangent bundle and cotangent bundle.** We define the tangent bundle T_*M as the disjoint union $\cup_{x\in M} T_xM$. We shall provide T_*M with a topology and a smooth structure. Denote by π the projection $T_*M \to M$ sending the vector $v \in T_p(M)$ to the point p. Let U be a coordinate neighborhood of M. By Theorem 2.1.6 we have a set isomorphism $T_*U \overset{\tau}{=} U \times \mathbb{R}^n$ such that $\pi_1(\tau(v)) = \pi(v)$, where π_1 is the projection to the first factor. The isomorphism τ supplies T_*U with the product topology. Finally, the open sets on T_*M are generated by the open sets on T_*U_i, where $\{U_i\}$ is an open covering of M.

Proposition 2.2.1. *The space T_*M has a smooth manifold structure.*

Proof. By the above T_*M^m has an open covering $\{T_*U_i = U_i \times \mathbb{R}^m\}$, where $\{U_i\}$ is an open covering on M^m. We define a coordinate map $\tilde{\phi} : TU_i \to \mathbb{R}^m \times \mathbb{R}^m$ by

$$\tilde{\phi}_i(y, \delta) = (\phi_i(y), (D\phi_i)_y(\delta)),$$

where $\phi_i : U_i \to U \subset \mathbb{R}^n$ is a coordinate map. We have to show that the transition functions $\tilde{\phi}_{ij} = (\tilde{\phi}_j) \circ (\tilde{\phi}_i^{-1}) = (\phi_j \circ \phi_i^{-1}, D\phi_j \circ D\phi_i^{-1})$ are smooth. Using the chain rule in Exercise 2.1.4 we reduce the proof of the smoothness of $\tilde{\phi}_{ij}$ to the smoothness of the tangent map $D(\phi_j \circ \phi_i^{-1})$, which is obvious by the definition of the smooth atlas. □

We define the cotangent space $T^*_xM^m$ at $x \in M^m$ by $T^*_xM^m := Hom(T_*M^m, \mathbb{R})$. We define the cotangent bundle T^*M^m as the disjoint union $\cup_{x\in M^m} T^*_xM^m$. This space can be provided with a topology in the same way as we did for the tangent bundle TM^m, since we have $T^*U_i = U_i \times \mathbb{R}^m$.

Exercise 2.2.2. Prove that the cotangent bundle T^*M^m is a smooth manifold.

2.3. **Vector fields.** A vector field on a smooth manifold M is a smooth section s of the tangent bundle T_*M, i.e. a smooth map $M \to TM$ such that $\pi \circ s = Id$, where π is the canonical projection $TM \to M$.

Exercise 2.3.1. A section X of $T(M)$ is smooth, if and only $X(f)$ is smooth for all $f \in C^\infty(M)$.

Hint. Write X in local coordinates.

Since any tangent vector in $T_{x_0}M^m$ is a derivation of the algebra $C^\infty(M)$ at x_0, a vector field X defines a derivation $X : C^\infty(M) \to C^\infty(M)$, i.e. a $\mathbb{R}$-linear map satisfying

$$X(fg) = X(f)g + fX(g).$$

Denote the set of all derivations $C^\infty(M) \to C^\infty(M)$ by $Der(C^\infty(M))$. The bracket $[X,Y] := XY - YX$ turns $Der(C^\infty(M))$ into a Lie algebra, i.e. the following Jacobi identity holds

$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0.$$

Exercise 2.3.2. Every derivation of $C^\infty(M)$ arises from a smooth vector field.

Hint. Combine the definition of a tangent vector with Exercise 2.3.1.

Now we shall find a local normal form of a vector field.

Theorem 2.3.3 (Linearization of a vector field)**.** *Let X is a vector field on a smooth n-manifold M. Suppose that $X(p) \neq 0$. Then there is a local coordinate system (U, ϕ) around p such that $\phi(U) = [-\varepsilon, \varepsilon] \times [-\varepsilon, \varepsilon] \times \cdots [-\varepsilon, \varepsilon]$ and $\phi_*(X) = \partial/\partial x_1$.*

We prove this theorem by using the inverse function theorem, a very important theorem in analysis and in differential geometry.

Theorem 2.3.4 (Inverse function theorem)**.** *Let f be a C^k-map from an open domain $U \subset \mathbb{R}^n$ to $\mathbb{R}^n$. Suppose that the tangent map $Df_p : \mathbb{R}^n \to \mathbb{R}^n$ is invertible. Then there exists a neighborhood $U(p) \ni p$ in U such that the restriction f to $U(p)$ is bijective on the image $f(U(p)$, moreover, $f^{-1}_{|f(U(p)}$ is also a C^k-map.*

Proof of Theorem 2.3.3. Since this is a local result we may assume that M is an open set $U \subset \mathbb{R}^n$, $p = 0$, X never vanishes on U and after a change of coordinate $X(0) = \partial/\partial x_1$. We denote by $\mathbb{R}^{n-1}$ the vector space spanned on $(\partial/\partial x_2(0), \cdots, \partial/\partial x_n(0))$. Now let us consider the ODE system for $\sigma(t,x) : \mathbb{R}(t) \to \mathbb{R}^n$ with $x \in \mathbb{R}^{n-1}$

$$\frac{d\sigma(t,x)}{dt} = X(\sigma(t,x)) \tag{2.3.1}$$

with the initial condition

$$\sigma(0,x) = x. \tag{2.3.2}$$

We know that for any $x \in \mathbb{R}^{n-1}$ there is an $\varepsilon > 0$ such that this system has a unique smooth solution $\sigma : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$. Let $G : \mathbb{R}^n \supset U \to \mathbb{R}^n$ be a map defined by $G(t,x) := \sigma(t,x)$. The map G is smooth because the solution $\sigma(t,x)$ depends smoothly on x. We shall show that for a small neighborhood U of 0 the map G^{-1} is the required diffeomorphism whose tangent map DG^{-1} sends X to $\partial/\partial x_1$.

We observe that $(DG)_{0|\mathbb{R}^{n-1}} = Id$, since $G(0,x) = x$. Since $X(0) = \partial/\partial x_1$, the tangent map $(DG)_0$ is an invertible map. Hence G is a local diffeomorphism by inverse function theorem. By (2.3.1) $(DG)_{(t,x)}(\partial/\partial x_1) = X(G(t,x))$, hence $(DG^{-1})_y(X) = \partial/\partial x_1(G^{-1}(y))$. □

An **integral curve** for a smooth vector field X on a smooth manifold M is a parametrized curve $\sigma : ((-\varepsilon, \varepsilon), 0) \to (M, p)$ whose tangent at the point $\sigma(t)$ is the vector $X_{\sigma(t)}$:

$$X_{\sigma(t)} = \dot{\sigma}(t) = D\sigma(\frac{\partial}{\partial t}_{|t}).$$

Example 2.3.5. Let $M = S^3$. For each $S^3 \ni (z_0, z_1)$ we consider the vector field

$$X(z_0, z_1) = \frac{d \exp \sqrt{-1}t \cdot x}{dt}_{|t=0} = (\sqrt{-1}z_0, \sqrt{-1}z_1).$$

The integral curves of this vector field X are the fiber of the Hopf fibration.

As we have seen in the proof of Theorem 2.3.3, the local existence and uniqueness of an integral curve of a smooth vector field X through a point p where $X(p) \neq 0$ follows from the existence and uniqueness of the ODE (2.3.1) with the initial condition (2.3.2).

Now we shall examine the global existence of integral curves for a smooth vector field X.

Assume that $\sigma_j((a_j, b_j), c_j) \to (M, p)$, $j = 1, 2$, are two integral curves through p. After a translation of the parameter for σ_2 we may assume that $c_1 = c_2$. Then by the uniqueness of the solution of ODE we conclude that $\sigma_1 = \sigma_2$ on $(a_1, b_1) \cap (a_2, b_2)$. Hence we can extend σ_i on $(a_1, b_1) \cup (a_2, b_2)$. Applying Zorn's lemma we see that a maximal integral curve through a point p always exists. Up to a translation of parameters there are only four possible types of maximal integral curves with the following domains

$$(0, a),\ (0, \infty),\ (-\infty.0),\ (-\infty, \infty).$$

An integral curve is called **complete** if its domain is of the last type. A vector field is called complete if all of its integral curves are complete.

Proposition 2.3.6. *A vector field with compact support is complete.*

Proof. We consider the graph $Y = (X, \partial/\partial t)$ of X on $M \times \mathbb{R}$. The projection of the integral curve $\tau(t) := (\sigma(t), t)$ is an integral curve of X on M. Clearly $\tau(t)$ is complete if and only if $\sigma(t)$ is complete.

If the projection is not complete, then we can assume w.l.g. that it is of type (a, b) with $b < \infty$. We can assume further that X never vanishes on (a, b). Using the compactness of $sppt(X)$ we find a point $y \in M$ such that $\lim_{t_k \to \infty} \sigma(t_k) = y$. Let us consider point $(y, b) \in M \times \mathbb{R}$. Applying Theorem 2.3.3 we find a flow of Y through (y, b) in some time interval $(-\varepsilon, +\varepsilon)$. But then the integral curve $\tau(t)$ is defined on an interval $(a, b + \varepsilon)$, so b is not maximal. So we arrive at a contradiction. □

Exercise 2.3.7. Find integral curves of the following vector fields X on $\mathbb{R}^2$
a) $X = x\partial x + y\partial y$,
b) $X = y\partial x - x\partial y$.

Exercise 2.3.8. . i) Show that $[X, fY] = X(f)Y + f[X, Y]$.
ii) Compute $[x\partial x + y\partial y, y\partial x - x\partial y]$.

3. Submanifolds and fiber bundles

In this section we introduce the notion of submanifolds and fiber bundles. We show that these objects arise from special classes of smooth mappings between smooth manifolds.

3.1. **Submanifolds and a regular value of a smooth map.** Originally, manifolds were regarded as subsets of Euclidean space. In many case a nice subset can be situated very complicated in a smooth manifold. Let us consider the simplest case, how a curve can be situated in a manifold.

Let N be a subset in a smooth manifold M^m.

Definition 3.1.1. A subset N^n of M^m is called a smooth **n-dimensional submanifold** if for any point $p \in N^n$ there exists a chart $(U(p) \ni p, \phi_p)$ on M^m such that $\phi_p(U(p) \cap N^n) = \phi_p(U(p)) \cap (\mathbb{R}^n \subset \mathbb{R}^m)$ for some linear subspace $\mathbb{R}^n \subset \mathbb{R}^m$.

Example 3.1.2. Let $f : \mathbb{R}(x) \to S^1(\theta^1) \times S^1(\theta^2)$ be defined by

$$f(x) = (x\theta_1, \alpha \cdot x\theta_2),$$

where α is some constant. The image $f(x)$ is a submanifold in $S^1 \times S^2$, if and only if α is a rational number.

Remark 3.1.3. Let M^m be a smooth manifold and N^n a submanifold equipped with the induced topology. The restriction of the a chart (U_p, ϕ_p) to N^n provides a chart on N^n. Thus N is a topological manifold and the induced charts $(U(p) \cap N^n, \phi_{p|(U(p)\cap N^n)})$, $p \in N^n$ provide a smooth atlas for N.

How to describe a submanifold in a manifold? There are two ways to do it. In the first way we describe a submanifold explicitly (equivalently, parametrically) by giving some smooth map $f : N \to M$ and we verify that the image $f(N)$ is a smooth submanifold. In the second way we describe a submanifold implicitly, as a solution of a certain system of equations, more precisely as the preimage of a regular value of a smooth map f between smooth manifolds M^m and L^l, where $m \le l$.

A point $p \in M^m$ is called *a regular value* of a smooth map $f : M^m \to L^l$, $m \ge l$, if for any point $x \in f^{-1}(p)$ the differential (tangent map) $Df(x)$ has rank l.

A smooth map $f : N^n \to M^m$ whose tangent map is a linear embedding at every point $p \in N^n$ is said to be **an immersion**. An immersion is called **an embedding**, if it is 1-1 on its image. A map f is called *proper*, if the preimage $f^{-1}(S)$ of any compact set is a compact set.

The following theorem gives us a criterion for defining a submanifold parametrically.

Theorem 3.1.4. *Let M^m and N^n be smooth manifolds and let $f : N^n \to M^m$ be a proper embedding. Then $f(N^n)$ is a submanifold of M^m.*

Proof. We shall use the immersion condition to show that locally f looks like an embedding $\mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^{m-n}$. We shall use the properness and the embeddness to show the existence of a chart on M around a point $f(p)$, $p \in N^n$, satisfying the condition in the definition of a submanifold.

Choose $p \in N^n$ and set $q = f(p)$. We first show that there are coordinate systems (U, ϕ) and (V, ψ) around p and $f(p)$ respectively such that the composition $\psi \circ f \circ \phi^{-1}$ is a restriction of the coordinate inclusion $\mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^{m-n}$, i.e.

$$\pi \circ (\psi \circ f \circ \phi^{-1})_{|\phi(U)} = Id, \tag{3.1.1}$$

where π is the projection $\mathbb{R}^m \to \mathbb{R}^n$. This shall imply that $\psi(V \cap f(U)) = \psi(V) \cap \mathbb{R}^n$.

After translation we may assume that $\phi(p) = 0$ and $\psi(f(p)) = 0$. By assumption the derivative $(\psi \circ f \circ \phi^{-1})(0)$ is an embedding $\mathbb{R}^n \to \mathbb{R}^m$, so in suitable coordinates on $\mathbb{R}^n$ and $\mathbb{R}^m$ we can write

$$D(\psi \circ f \circ \phi^{-1})_0 = \begin{pmatrix} I_r \\ 0 \end{pmatrix}.$$

Let $D_0 := D(\psi \circ f \circ \phi^{-1})_0$. Then D_0 is an isomorphism from $\mathbb{R}^n$ to $D_0(\mathbb{R}^n) \subset \mathbb{R}^m$. Hence in the new chart $(U, D_0 \circ \phi)$ the composition $g = \psi \circ f \circ \phi^{-1} \circ D_0^{-1}$ is a map from $D_0(\phi(U)) \subset \mathbb{R}^n$ to $\mathbb{R}^m$ with $g(D_0(\phi(U))) \subset D_0(\mathbb{R}^n)$. Denote by π the projection from $\mathbb{R}^m$ to $D_0(\mathbb{R}^n)$. Then $\pi \circ g$ is a local diffeomorphism of $D_0(\phi_0(U))$. Hence, by the inverse function theorem, there exists a local right inverse $(\bar{g}) : D_0(\phi_0(U_1)) \to D_0(\phi_0(U_1))$ of $(\pi \circ g)$ for some smaller neighborhood $U_1 \subset U$ of p, i.e. $(\pi \circ g) \circ \bar{g} = Id$. Thus on U_1 with the coordinates $\tilde{\phi} := \bar{g}^{-1} \circ D_0 \circ \phi$ we get

$$\pi \circ (\psi \circ f \circ (\tilde{\phi})^{-1})_{|\tilde{\phi}(U_1)} = \pi \circ (g \circ \bar{g}) = Id.$$

Clearly $\tilde{\phi} := \pi \circ g \circ D_0 \circ \phi$ satisfies (3.1.1).

Now f is proper and an embedding, so we may shrink U_1 and V so that $f(U_1) = f(N^n) \cap V$, which implies that $\psi(V \cap f(N^n)) = \psi \circ f(U_1) = \psi(V) \cap \mathbb{R}^n$, was is required to prove. $\square$

Exercise 3.1.5. Let a point M moves with a constant speed on a ray ON which rotates around the origin O with a constant speed. Write the equation describing the trajectory of M. On which time interval $(t_0, t_1) \subset \mathbb{R}$ this equation defines a proper embedding from (t_0, t_1) to $\mathbb{R}^2$?

Now we turn to another very important class of submanifolds described implicitly.

Theorem 3.1.6 (Implicit function theorem)**.** *Let M^m and N^n be smooth manifold with $m \ge n$, and let q be a regular value of a smooth map $f : M^m \to N^n$. Then the set $f^{-1}(q)$ is a smooth submanifold of M^m.*

Proof. Let $p \in M_q := f^{-1}(q)$. We first show that there are coordinate charts (U, ϕ) and (V, ψ) around p and $q = f(p)$ such that $\phi(p) = 0$, $\psi(q) = 0$, and moreover the composition $\psi \circ f \circ \phi^{-1}$ is the restriction of the canonical projection

$$\pi : \mathbb{R}^m = \mathbb{R}^n \times \mathbb{R}^{m-n} \to \mathbb{R}^n \tag{3.1.2}$$

$$(x, y) \mapsto x. \tag{3.1.3}$$

First we can assume that $\psi(V) \subset \mathbb{R}^n \subset \mathbb{R}^m$. Furthermore, by translation, we can assume the equalities $\phi(p) = 0$ and $\psi(q) = 0$. Let us denote by D_0 the derivative $D(\psi \circ f \circ \phi^{-1})_0$. Denote by $\mathbb{R}^{m-n}$ the kernel of D_0. Clearly $\mathbb{R}^m = \mathbb{R}^{m-n} \oplus \mathbb{R}^n$. Applying a linear transformation B on $\mathbb{R}^n$ we can assume that D_0 is the canonical projection π.

Recall that $\phi^{-1}(U)$ is an open set in $\mathbb{R}^n \times \mathbb{R}^m$ with coordinates $(x_1, \cdots, x_n, y_1, \cdots, y_m)$. Let us define a map $g : \phi^{-1}(U) \to \psi(V) \times \mathbb{R}^{m-n}$ by

$$g(x, y) := (\psi \circ f \circ \phi^{-1}(x, y), y).$$

Then we have $Dg_{(0,0)} = Id$. So by inverse function theorem there is a local diffeomorphism J from $\psi(V) \times \mathbb{R}^{m-n} \to \phi^{-1}(U)$ such that $g \circ J = Id$. It follows that on the smaller open set $\phi^{-1}(U_1) \subset \phi^{-1}(U)$ we have

$$\psi \circ f \circ \phi^{-1} \circ J = \pi \circ g \circ J = \pi.$$

Clearly the map $\tilde{\phi} := J \circ \phi$.

Now let us assume that we have a covering U_α on M^m satisfying (3.1.2) and (3.1.3). Since $\psi \circ f \circ \phi^{-1}(\phi(U \cap f^{-1}(q)) = 0$, using (3.1.3) we get $\phi(U \cap f^{-1}(q)) \subset \mathbb{R}^{m-n}$. Hence $\phi((U \cap f^{-1}(q)) \subset \phi(U) \cap \mathbb{R}^{m-n}$. Using (3.1.3) again we get $\phi(U) \cap \mathbb{R}^{m-n} \subset \phi((U \cap f^{-1}(q))$. Hence $\phi((U \cap f^{-1}(q)) = \phi((U \cap f^{-1}(q))$. □

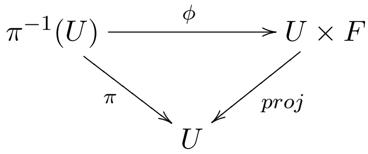
Example 3.1.7. Let $F : \mathbb{R}^n \to \mathbb{R}$ is given by $F(x) = \langle x, x \rangle$. Then $F'_x(v) = \langle 2x, v \rangle$ is an immersion, if $x \ne 0$. Hence the pre-image $F^{-1}(1) = S^n$ is a smooth manifold.

Exercise 3.1.8. Let us define $O_n(\mathbb{R}) = \{A \in Mat_n(\mathbb{R}) |\, AA^t = Id\}$. This is the group of orthogonal transformation on $\mathbb{R}^n$. Prove that $O_n(\mathbb{R})$ is a Lie group.

Hint. Denote by $Sym_n(\mathbb{R})$ the space of symmetric bilinear forms on $\mathbb{R}$. Show that $\sum_{i=1}^n (e^i)^*$ is a regular value of a map $f : Mat_n(\mathbb{R}) \to Sym_n(\mathbb{R})$, $f(A) = A \cdot A^t$.

3.2. **Fiber bundles.** Fiber bundles arise when we have a projection π from a differentiable manifold E^{m+n} onto a differentiable manifold M^m such that π is a differentiable map and all point $q \in M^m$ is a regular value of π. The implicit function theorem implies all the fiber $\pi^{-1}(p)$ is a smooth submanifold in E^{m+n}. Thus we can think of a fiber bundle as a family of smooth (sub)manifolds parametrized by the base space. When we require that the special structures on (sub)manifolds change smoothly we need to modify our definition of fiber bundles.

Definition 3.2.1. Let E be a topological space and $\pi : E \to B$ a continuous map. We call the quadruple $\xi = (E, F, \pi, B)$ a **fiber bundle** (or **fibration**), if for each point $b \in B$, there is an open set $U \subset B$ containing p such that $\pi^{-1}(U)$ is homeomorphic to $U \times F$ by a homeomorphism ϕ satisfying the following condition of commutativity diagram



π is called the projection, B is the base of the fibration, F is the fiber over B and E is the total space.

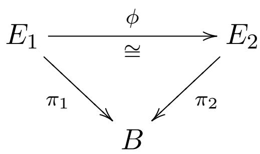
The pair (U, ϕ) is called a chart, or a local bundle coordinate system. The map ϕ is also called a trivialization of E over U.

Note that $\pi^{-1}(b)$, the fiber over b, denoted by F_b is homeomorphic to F for all $b \in B$.

If the base, fiber and the total space are smooth manifold, π and ϕ are smooth maps then we have a **smooth bundle**.

If the fiber is a vector space V and the transition bundle map $\phi_j \circ \phi_i^{-1} : (U_i \cap U_j) \times V \to (U_i \cap U_j) \times V$ restricted to each fiber $p \times V$ is a linear transformation on V, the bundle F is called a **vector bundle**.

Two bundles E_1 and E_2 over the same base B is called **isomorphic**, if there is a homeomorphism (diffeomophism) ϕ between the total spaces such that the following diagram commutes



The map ϕ is called a **bundle isomorphism**. If $E_1 = E_2$ then ϕ is called **a bundle automorphism.** The set of all bundle isomorphisms forms a group. This group is called **the gauge group** of E.

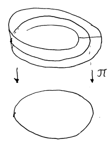
Example 3.2.2. - The simplest example of a bundle is the product bundle $E = B \times F$ with π being the projection on the first factor. Any bundle which is isomorphic to a product bundle is called **a trivial bundle**.

- Given two bundles (E_1, F_1, π_1, B_1) and (E_2, F_2, π_2, B_2) we shall define their product bundle by taking product of fibers and bases: $(E_1 \times E_2, F_1 \times F_2, \pi_1 \times \pi_2, B_1 \times B_2)$.

-Tangent bundles and cotangent bundles are vector bundles, since the transition bundle map restricted to each fiber p has the form $D(\phi_j^{-1} \circ \phi_i)$ or $D(\phi_j \circ \phi_i^{-1})^*$ which are linear maps.

- Given two vector bundles (E_1, V_1, π, M) and (E_2, V_2, π, M) with local trivialization ϕ_i^1 and ϕ_i^2 over a manifold M we construct the Whitney sum $E_1 \oplus E_2$ - a vector bundle over M with fiber $V_1 \oplus V_2$ by using the bundle transition functions $(\phi_j^1 \oplus \phi_j^2) \circ (\phi_i^1 \oplus \phi_i^2)^{-1}$. We also construct the Whitney product $E_1 \otimes E_2$ - a vector bundle over M with fiber $V_1 \otimes V_2$ using the bundle transition functions $(\phi_j^1 \otimes \phi_j^2) \circ (\phi_i^1 \otimes \phi_i^2)^{-1}$.

Example 3.2.3 (Möbius band)**.** If we twist a paper thin bank in and glue the ends together in the following way



then we get a Möbius band. It is a fiber bundle but it is not a trivial bundle, since if it were, then its boundary would consist of two components.

It is often important to know, if a given bundle is trivial or not. As examples let us consider the Hopf bundle and the canonical line bundle over a real projective space.

Example 3.2.4 (Hopf bundle)**.** Let us consider a map $\pi : S^3 \to S^2$ defined by

$$\pi(e^{i\theta_1} r_1, e^{i\theta_2} r_2) = (r_1, e^{i(\theta_2 - \theta_1)} r_2) \in \mathbb{R} \times \mathbb{C}.$$

It is is to see that π is a projection of S^3 onto $S^2$and the fibers π^{-1} are integral curves in example 2.3.5 which are circle. Clearly the Hopf bundle is not a trivial bundle, since S^3 is simply-connected.

Example 3.2.5. The space $\mathbb{R}P^n$ of all lines in $\mathbb{R}^{n+1}$ is a differentiable manifold. Let $[l]$ be a line in $\mathbb{R}^{n+1}$ through point $(x_0, \cdots, x_n)$, so we write $[l] = [x_0 : \cdots x_n]$. Clearly $\mathbb{R}P^n$ can be covered by $(n+1)$ open sets U_i defined by the condition $x_i \neq 0$. We define coordinate map $\phi_i : U_i \to \mathbb{R}^n$ by

$$\phi_i([x_0 : \cdots x_n]) = (\frac{x_0}{x_i}, \cdots, \frac{x_n}{x_i}).$$

It is easy to check that $\{U_i, \phi_i\}$ defines an atlas on $\mathbb{R}P^n$.

There is a canonical line bundle $l(n)$ over $\mathbb{R}P^n$ consists of all pair $([l], x)$ where x is a point in $[l]$. If $l(n)$ is trivial, then its sub-bundle $l^*(n) = \{([l], x|\, x \neq 0)\}$ were trivial, which implies that $l^*(n)$ has two connected component. On the other hand, it is easy to see that $l^*(n) = \mathbb{R}^3 \setminus \{0\}$ is connected. Hence $l(n)$ is not a trivial bundle.

Exercise 3.2.6. Denote by $\mathbb{C}P^n$ the space of all complex lines $[z]$ in $\mathbb{C}^{n+1}$. Prove that $\mathbb{C}P^n$ is a differentiable manifold. The canonical line bundle $L(n)$ over $\mathbb{C}P^n$ consists of all pair $([z], y)$ where y is a point in the complex line $[z]$. Prove that $L(n)$ is not a trivial vector bundle.

Hint. Using the argument in Example 3.2.5. Alternatively we can argue that if $L(n)$ is trivial, then the restriction of $L(n)$ to $\mathbb{R}P^n$ is trivial which implies that the canonical line bundle over $\mathbb{R}P^n$ is trivial.

4. Tensors and Riemannian metrics

In this section we consider natural vector bundles over a differentiable manifolds and their sections which are called tensors. A Riemannian metric is a special type of tensor, with which we can measure the length of curves on a manifold. We prove the existence of a Riemmanian metric on any differentiable manifold.

4.1. **Tensors and Riemannian metrics.** Recall that a vector field is a section of the tangent bundle TM. In general, a field on a manifold M is a section s of some vector bundle $\pi : E \to M$, i.e. a smooth map $M \to E$ such that $\pi \circ s = Id$. Clearly fields are generalization of the notion of functions.

Definition 4.1.1. A tensor ω of type (r, p) on a differentiable manifold M is a section of a natural vector bundle $TM \otimes_{r\,times} \cdots \otimes TM \otimes T^*M \otimes_{p\,times} \cdots \otimes T^*M$.

Clearly the space $T^{(r,p)}(M)$ of tensors of type (r, p) on M is a linear space. The ring $C^\infty(M)$ acts on $T^{(r,p)}(M)$ by

$$f(\omega)(x) := f(x) \cdot \omega(x).$$

This action provides $T^{(r,p)}(M)$ with a $C^\infty(M)$-module structure.

Let us look at a local expression of a tensor T of type (r, p). On a local coordinate neighborhood $U_i \subset \mathbb{R}^n$ the space $TU_i \otimes_{r\,times} \cdots \otimes TU_i \otimes T^*U_i \otimes_{p\,times} \cdots \otimes T^*U_i$ is a direct product $U_i \times (\mathbb{R}^n) \otimes_{r\,times} \cdots (\mathbb{R}^n) \otimes (\mathbb{R}^n)^* \otimes_{p\,times} \cdots \otimes (\mathbb{R}^n)^*$. Thus a tensor T at a point x has a value $\sum_{i_1,\cdots,i_r,j_1,\cdots,j_p} f_{i_1\cdots j_p} \partial x_{i_1} \otimes \cdots \otimes dx_{j_p}$.

Example 4.1.2. For any function f we define a tensor df of type $(0, 1)$ as follows: $df(X) = \partial_X(f)$.

We note that any tensor type ω type $(0, 1)$ defines a C^∞-linear map $\omega : Vect(M) \to C^\infty(M) : \omega(V)(x) := (\omega(x), V(x))$.

Exercise 4.1.3. Show that a linear function $\omega : Vect(M) \to C^\infty(M)$ is a tensor type $(0, 1)$ if and only if for all $f \in C^\infty(M)$ and $V \in Vect(M)$ we have $\omega(fV) = f\omega(V)$.

Definition 4.1.4. A Riemmannian metric g is a tensor of type $(1, 1)$ on M such that at each point $x \in M$ the value $g(x)$ is a symmetric positive definite bilinear form on T_xM.

Example 4.1.5. -Let M be a submanifold in $\mathbb{R}^n$. Then the restriction of the Euclidean metric to each tangent space T_xM is a positive definite bilinear form. This restriction defines a Riemannian metric on M.
- Let $f : N^n \to M^m$ be an immersion. If g is a Riemannian metric on M^m then f induces a Riemannian metric on N^n by

$$f^*(g)(X,Y)_x = g(Df(X), Df(Y)).$$

Exercise 4.1.6. Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ is defined by

$$f(x,y) = (x, y, z(x,y)).$$

Find the induced metric $f^*(g)$, where g is the Euclidean metric.

On a coordinate domain $U \subset \mathbb{R}^n$ a Riemannian metric g can be written as $g(x) = g_{ij}(x)dx^i dx^j$. On other coordinate domain $y(U)$ with $y : \mathbb{R}^n \to \mathbb{R}^n$ the same metric $g(y)$ has the following expression

$$g(y(x)) = \tilde{g}_{ij}(y(x))d\tilde{y}_i d\tilde{y}_j = \tilde{g}_{ij}(y(x))\frac{\partial y_i}{\partial x_k}dx^k \frac{\partial y_j}{\partial x_l}dx^l = g_{kl}(y(x))dx^k dx^l.$$

Hence

$$(\tilde{g}_{ij})(y(x)) = (\frac{\partial x_k}{\partial y_i})^*_x (g_{kl}(y(x)))(\frac{\partial x_l}{\partial y_j}).$$

Exercise 4.1.7. i) Compute the Riemmannian metric $dx^2 + dy^2$ in polar coordinates (r, ϕ) with $x = r\cos\phi$ and $y = r\sin\phi$.
ii) Compute the induced Riemmanian metric on the sphere in polar coordinate $z = \cos\theta, x = \sin\theta\cos\phi, y = \sin\theta\sin\phi$.

Remark 4.1.8. If $\gamma : [0,1] \to (M^m, g)$ is a curve on a Riemannian manifold, then we define the length $L([\gamma])$ by

$$L([\gamma]) := \int_0^1 |\dot{\gamma}(t)|\, dt.$$

Clearly the length of a curve γ does not depend on a parametrization of γ, i.e. if $s : [0,1] \to [0,1]$ is a monotone smooth function, then a new curve $\tilde{\gamma}(t) := \gamma(s(t))$ has the same length as $\gamma(t)$, since

$$\int_0^1 |\dot{\tilde{\gamma}}(t)|\, dt = \int_0^1 |\frac{d}{ds}\gamma||\dot{s}|\, dt = \int_0^1 |\dot{\gamma}|\, dt.$$

Exercise 4.1.9. i) Compute the length a a curve $\gamma(t) : [0,1] \to S^3$ defined by $\theta(t) = \frac{\pi}{4}, \phi(t) = \pi \cdot t$.

Exercise 4.1.10. Let (M, g) be a Riemannian manifold. Show that a linear function $V : Vect(M) \to C^\infty$ is a tensor type (1,0) if and only if for all $f \in C^\infty(M)$ and $W \in Vect(M)$ we have $g(V, fW) = fg(V, W)$.

4.2. **The existence of a Riemannian metric.**

Theorem 4.2.1. *On each differentiable manifold M^m there exists a Riemannian metric g.*

To prove this theorem we need the fact that every locally compact Hausdorff space M with countable basis is paracompact, i.e. any open covering on M possesses a locally finite refinement, see e.g. Kobayashi-Nomidzu, v.1. (In many textbooks ones requires that a differentiable manifold is a paracompact Hausdorff topological space). This fact implies the existence of partition of unity on a differentiable manifold. This fact is a fundamental Lemma in the theory of differentiable manifolds. Denote by *sppt* f *the support* of a function f on a differentiable manifold M, i.e. *sppt* $f := \{x \in M |\, f(x) \neq 0\}$.

Lemma 4.2.2 (Partition of unity). *Let M be a differentiable manifold, $(U_\alpha)_{|alpha\in A}$ an open covering. Then there exists a partition of unity, subordinate to (U_α). This means that there exists a locally finite refinement $(V_\beta)_{\beta\in B}$ of (U_α) and smooth functions $\phi_\beta : M \to \mathbb{R}$ with*
(i) sppt $\phi_\beta \subset V_\beta$ *for all* $\beta \in B$,
(ii) $0 \le \phi_\beta \le 1$ *for all* $x \in M, \beta \in B$,
(iii) $\sum_{\beta\in B} \phi_\beta(x) = 1$ *for all* $x \in M$.

Using a partition of unity we shall prove Theorem 4.2.1.

Proof. Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas on M. W.l.g. we can assume that (U_α) is locally finite. We define a Riemannian metric g on M by setting its value at a pair of tangent vectors $v, w \in T_pM$ as follows

$$\langle v, w\rangle = \sum_{\alpha\in A} \phi_\alpha(p) v^i_\alpha w^i_\alpha,$$

where $(v^1_\alpha, \cdots, v^n_\alpha)$ and $(w^1_\alpha \cdots, w^n_\alpha)$ are representation of $D\phi_\alpha(v)$ and $D\phi_\alpha(w)$ in $\mathbb{R}^n$. It is easy to check that g is well-defined. □

Remark 4.2.3. Let N^n be a submanifold in a Riemannian manifold (M^m, g). Then the restriction of g to the tangent bundle TN defines a Riemannian metric on N^n which we called **the induced metric**. Thus any smooth submanifold in $\mathbb{R}^n$ carries a Riemmanian metric which is induced from the Euclidean metric on $\mathbb{R}^n$. A celebrated theorem of Nash asserts that any a Riemannian metric on a manifold N^n is induced from the Eulidean metric on some space $\mathbb{R}^m$ via a smooth embedding $f : N^n \to \mathbb{R}^m$. A submanifold N^n in a Riemannian manifold (M^m, g) provided with the induced metric $\bar{g}$ is called **a Riemannian submanifold** of (M^m, g).

Exercise 4.2.4. Let N^n be a submanifold in a manifold M. Let ω be a tensor on N^n. Show that ω can be extended to a tensor on M^m.

Hint. Unse partition of unity.

5. Levi-Civita connection

In this section we introduce the notion of Levi-Civita connection on a Riemannian manifold which is a central notion in Riemannian geometry (a branch of differential geometry which studies Riemannian manifolds). The Levi-Civita connections give us tool to analyze global properties of a Riemmannian manifold M by using local invariants on M, since we can compare the tangent spaces at different points on M in a canonical way.

5.1. **Linear connections and metric connection.** We have observed in exercise 2.1.4(ii) that for a $p \in M$ the derivative $\partial_v f$ of function f is equal to the speed of the change of f along a curve $\gamma(t)$ at $t = 0$, if $\dot\gamma(0) = v$.

If $M = \mathbb{R}^n$ we can also take derivative $\partial_v W$ of any vector field W on $\mathbb{R}^n$ by setting

$$\partial_v W = \frac{d}{dt}_{t=0} W(\gamma(t)).$$

This formula is possible, since the tangent space $T_x\mathbb{R}^n$ at any point $x \in \mathbb{R}^n$ is canonically identified with $\mathbb{R}^n$.

On a general differentiable manifold there is a no canonical way to identify T_xM^m and T_yM^m. Thus we can imagine that there are many ways to define a derivative $\partial_v W$ of a vector field W on a differentiable manifold M for a tangent vector $v \in T_pM$. A linear connection on TM is a method to define such a derivative which we also require to satisfy some additional properties.

Denote by $\nabla_X V$ the (covariant) derivative of a vector field V at a tangent vector X depending on a connection ∇. Then $\nabla_X V$ must be linear in variable V

$$\nabla_X(V_1 + V_2) = \nabla_X(V_1) + \nabla_X(V_2). \tag{5.1.1}$$

This derivative must also satisfy the Leibniz rule

$$\nabla_X(f \cdot s) = df(X) \cdot s + f \cdot \nabla_X(s). \tag{5.1.2}$$

This derivative must be also linear on the variable X i.e.

$$\nabla_{X+Y}(s) = \nabla_X s + \nabla_Y s. \tag{5.1.3}$$

Now let (M, g) be a Riemannian manifold. We say that a connection ∇ is metric, if for all vector fields V, W and for all $X \in TM$ we have

$$\partial_X g(V, W) = g(\nabla_X V, W) + g(V, \nabla_X W). \tag{5.1.4}$$

We shall prove that a metric connection on a Riemannian manifold always exists. Moreover, this connection is unique, if we pose a certain condition on ∇.

Remark 5.1.1. The notion of connection ∇ can be defined for any vector bundle E over a smooth manifold M. We require that all conditions (5.1.1), (5.1.2), (5.1.3) holds for any tangent vector $X \in TM$ and any sections V, W of the vector bundle E.

5.2. **The existence and uniqueness of the Levi-Civita connection.** For a connection ∇ on TM we define its torsion $T(\nabla) \in C^\infty(TM \otimes T^*M)$ (a tensor of type $(0,2)$) by

$$T(V_p, W_p) := \nabla_V W - \nabla_W V - [V, W],$$

for any vector fields V on M with $V(p) = V_p$ and $W(p) = W_p$.

Exercise 5.2.1. Prove that T is well-defined, i.e. its value $T(V_p, W_p)$ does not depend on the choice of extensions V and W.

Hint. Compare with Exercise 4.1.3. Show that $T(fV, gW) = fgT(V, W)$ for all $f, g \in C^\infty(M)$. In particular we have that $T(V, W)_p = 0$ if $V_p = 0$ or $W_p = 0$.

Theorem 5.2.2. *On any Riemannian manifold there exists a unique metric torsion free connection.*

Proof. We set for $X, Y, Z \in Vect(M)$
(5.2.1)
$$\langle \nabla_X Y, Z\rangle := \frac{1}{2}\{X\langle Y, Z\rangle - Z\langle X, Y\rangle + Y\langle Z, X\rangle - \langle X, [Y, Z]\rangle + \langle Z, [X, Y]\rangle + \langle Y, [Z, X]\rangle\}$$

First we show the uniqueness of a torsion free metric connection, i.e. we have to show that any torsion free metric connection ∇ satisfies (5.2.1). Since ∇ is metric it should satisfy
$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle,$$
$$Y\langle Z, X\rangle = \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_Y X\rangle,$$
$$Z\langle X, Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle.$$
Using the torsion free condition we get
$$X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle$$
$$= 2\langle \nabla_X Y, Z\rangle - \langle [X, Y], Z\rangle + \langle Y, [X, Z]\rangle + \langle X, [Y, Z]\rangle$$
which yields (5.2.1).

Now we will show that (5.2.1) defines a torsion free metric connection, i.e. ∇ satisfies (5.1.1), (5.1.2), (5.1.3), 5.1.4) and $T(\nabla) = 0$. It will imply the existence of a torsion free metric connection. Then we shall show that any torsion free metric connection satisfies (5.2.1).

First we note that for any fixed $X, Y \in Vect(M)$ the value of $\nabla_X Y$ defined by RHS of (5.2.1) is a tensor field of type (1,0) on M. Clearly $\nabla_X Y$ defines a linear map $Vect(M) \to \mathbb{R}$. By Exercise 4.1.10 it suffices to show

(5.2.2)
$$\langle \nabla_X Y, (fZ)\rangle = f\langle \nabla_X, Y\rangle.$$

A straightforward calculation of the RHS of (5.2.1) gives
$$\langle \nabla_X Y, (fZ)\rangle = f\langle \nabla_X Y, Z\rangle + \frac{1}{2}[(Xf)\langle Y, Z\rangle + (Yf)\langle X, Z\rangle$$
$$-(Xf)\langle Y, Z\rangle - (Yf)\langle X, Z\rangle]$$
which is equal to the RHS of (5.2.2).

Thus ∇_X defines a linear map $Vect(M) \to Vect(M)$, so the first condition (5.1.1) holds. In the same manner we verify that properties (5.1.2) and (5.1.3) hold. So ∇ defines a linear connection. To check that ∇ is metric, we add the RHS of (5.2.1) associated to $\langle \nabla_X, Y\rangle$ and $\langle \nabla_X Z, Y\rangle$. Finally the torsion free condition
$$\langle \nabla_X Y, Z\rangle - \langle \nabla_Y X, Z\rangle = \langle [X, Y], Z\rangle$$
follows directly from (5.2.1). □

A torsion free metric connection on a Riemannian manifold M^m is called **the Levi-Civita connection**. Now let us look at the expression of Levi-Civita connection in local coordinate (x^i) with $g(x) = g_{ij}(x)dx^i dx^j$. Using the Leibniz rule, it suffices to compute $\nabla_{D_i} D_j$ for $i, j, = 1, m$. Now we define function $\Gamma^k_{ij}(x)$ as follows

$$\nabla_{D_i} D_j(x) = \Gamma^k_{ij}(x) D_k.$$

The functions Γ^k_{ij} are called **Chritoffel symbol**. By definition

$$\langle \nabla_{D_i} D_j, D_l \rangle = \langle \sum_{k=1}^m \Gamma^k_{ij} D_k, D_l \rangle = \sum \Gamma^k_{ij} g_{kl}.$$

Using (5.2.1) we get immediately

$$\langle \nabla_{D_i} D_j, D_l \rangle = \frac{1}{2}(D_i \langle D_j, D_l \rangle + D_j \langle D_l, D_i \rangle - D_l \langle D_i, D_j \rangle)$$

$$= \frac{1}{2}(\frac{\partial g_{jl}}{\partial x_i} + \frac{\partial g_{li}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_l}).$$

Thus, letting $g^{kl} := (g)^{-1}_{kl}$, we have

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \sum_{l=1}^m (\frac{\partial g_{jl}}{\partial x_i} + \frac{\partial g_{li}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_l}). \tag{5.2.3}$$

5.3. **Parallel transport and geodesics.** Let (M, g) be a Riemannian manifold, ∇ its Levi-Civita connection and $\gamma(t) : (-\varepsilon, \varepsilon) \to M$ be an embedded curve. Note that the restriction of TM to $\gamma(t)$ defines a vector bundle over submanifold $\gamma(t)|\, t \in (-\varepsilon, \varepsilon)$.

Lemma 5.3.1. *For a section $V(t) : \gamma(t) \to TM$ let $\tilde{V}$ be a extension of V to M. We define for any t a linear map $\nabla_{\dot\gamma(t)} : \Gamma(TM_{|\gamma(t)}) \to \Gamma(TM_{|\gamma(t)})$ as follows*

$$\nabla_{\dot\gamma(t)} V(t) := \nabla_{\dot\gamma(t)} \tilde{V}.$$

This linear map is well-defined. It is a connected on the vector bundle $TM_{|\gamma(t)}$.

From Lemma 5.3.1 we get immediately

Corollary 5.3.2. *The covariant derivative $\nabla_{\dot\gamma(t)} V$ for any vector field V on M depends only on the restriction of V to $\gamma(t)$.*

Proof of Lemma 5.3.1. It suffices to show that if $V(t)$ is a zero vector field, then $\nabla_{\dot\gamma(t)} \tilde{V} = 0$. Choose a local coordinate at $\gamma(t)$. W.l.g. we assume that $\dot\gamma(t) = \partial x_1$. Using the theorem on linearization of a vector field we can assume that $\gamma(t) = (t, 0, \cdots, 0)$ for a sufficient small t. Let $\tilde{V} = f_i(x) D_i$. Since $V_{|\gamma(t)} = 0$ we have $f_i(t, 0, \cdots, 0) = 0$. Now we compute

$$\nabla_{D_1} V(t)_0 = \nabla_{D_1} (f_i D_i)_0 = f_i (\nabla_{D_1} D_i)_0 + D_1(f_i) D_i = 0.$$

□

Exercise 5.3.3. Let $(N, \bar{g})$ is a Riemannian submanifold in a Riemannian manifold (M, g). Then the restriction of the tangent bundle TM to N is a direct sum of the tangent subbundle TN and the normal subbundle $TN^\perp$ consisting of vectors which is orthogonal to TN w.r.t. g. For any vector $V \in TM$ denote by V^T the tangential component of V. Now we define the following linear map $\tilde{\nabla}$ from $Vect(N) \times Vect(N) \to Vect(N)$

$$\nabla_X Y := (\nabla_X \tilde{Y})^T, \tag{5.3.1}$$

where $\tilde{Y}$ is any extension of Y from N^n to some vector field $\tilde{Y}$ on M. Prove that (5.3.1) defines the Levi-Civita connection on $(N, \bar{g})$.

Hint. Let $\tilde{Y}$ be extension of Y to M. Using Corollary 5.3.2 show that (5.3.1) does not depend on the extension $\tilde{Y}$. From here check that (5.3.1) defines a metric connection. To show that it is torsion free we use the identity $[\tilde{X}, \tilde{Y}](x) = [X, Y](x)$ for any $x \in N$, where $\tilde{X}$ is an extension of X.

Exercise 5.3.4. Prove that the normal bundle $(TS^2)^\perp$ of the sphere S^2 in $\mathbb{R}^3$ is a trivial vector bundle.

Definition 5.3.5. A family of vectors $v_t \in T_{\gamma(t)}M$ is said **parallel to** v_0 **along path** $\gamma(t)$ if

$$\nabla_{\dot{\gamma}(t)} v_t = 0.$$

A C^2-curve $\gamma \to M$ is called **geodesics**, if $\nabla_{\dot{\gamma}(t)}\dot{\gamma}(t) = 0$ for all t.

Theorem 5.3.6. *Let (M, g) be a Riemannian manifold and $I = (a, b)$ be an open interval on the real line $\mathbb{R}$. Further let $\gamma : I \to M$ be a smooth curve. Then for any $t_0 \in I$ and any given $X \in T_{\gamma(t)}M$ there exists a unique parallel vector field Y along $\gamma(t)$ such that $Y(t_0) = X$.*

Proof. W.l.g. we can assume that M is an open domain in $\mathbb{R}^n$. Thus

$$\dot{\gamma}(t) = \sum_i \dot{\gamma}^i(t)\partial x_i.$$

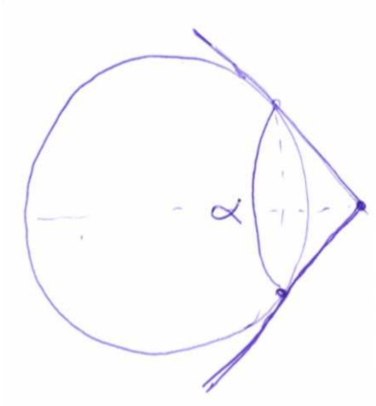
Now let $Y(t)$ be a vector field along $\gamma(t)$, $Y(t) = y^i(t)\partial x_i$. The condition that $Y(t)$ is a parallel vector field along $\gamma(t)$ is expressed in the following differential equation

$$\sum_{i,j} \dot{\gamma}^i(t)\nabla_i(y^j(t)\partial x_j) = 0. \tag{5.3.2}$$

Clearly (5.3.2) is a system of n OEDs. This system has a unique solution for any given initial value $Y(t_0)$ if the coefficients $\dot{\gamma}^i(t)$ is of class C^1. $\square$

Example 5.3.7. - On Euclidean space a vector field V is parallel along $\gamma(t)$, if and only if $V(\gamma(t))$ is a constant vector field. In particular a geodesic is a straight line and any straight line is a geodesic.

- Now we shall define a parallel transportation of a vector field V along a curve γ on



the sphere S^2 as in this figure. We regard γ as a curve in $\mathbb{R}^3$ as well as a curve in S^2 and in the cone $C\gamma$. By Exercise 5.3.3, the connection in the bundle $TC\gamma$ is obtained by the projection of the Euclidean connection to $TC\gamma$. Since the tangent bundle $TC\gamma$ of cone $C\gamma$ has the same restriction to γ as the restriction of $TS^2|\gamma$, it follows that we can consider the parallel transportation of V along γ as V belong to the tangent bundle $TC\gamma$. Now we pull out this cone isometrically to a planec $\mathbb{R}^2$ by cutting the cone along an edge C of it. The sliced cone does not cover the whole plane, there is a small angle θ inside the cone formed between the sliced edge C. Since the parallel transportation does not change under an isometric map, we can perform the parallel transportation of V on the pulled out cone. It is easy to see that V is moved to another vector V' parallel to it on $\mathbb{R}^2$, but if we close the sliced cone, then V' differs from V by the angle $\beta = 2\pi - \theta$. When γ is a great circle, then $\theta = 0$, so $V' = V$. If γ is very small then θ is very close to 0 and V' is close to V.

For the important geodesic equation we have the following local existence result.

Theorem 5.3.8. *Let (M, g) be a Riemannian manifold. For any $p \in M$ and $v \in T_pM$ there exists an open interval $I = (-\varepsilon, \varepsilon)$ and a unique geodesic $\gamma : I \to M$ such that $\gamma(0) = p$ and $\dot\gamma(0) = v$.*

Proof. It suffices to prove Theorem 5.3.8 for M being an open domain in $\mathbb{R}^m$. Let us write the equation for a geodesic $\gamma(t)$ in local coordinates

$$\nabla_{\dot\gamma(t)}(\dot\gamma(t)) = \sum_i \nabla_{\dot\gamma(t)}(\dot\gamma^i \partial x_i)$$

$$= \frac{d^2}{dt^2}(\gamma(t))\partial x_i + \dot\gamma^i(t)\gamma^j \Gamma^k_{ji}\partial x_k.$$

Thus the equation for a geodesic $\gamma(t)$ is a system of n second order ODE's. Hence follows the theorem. □

Exercise 5.3.9. - (i) Find all geodesics on the sphere S^n.
- (ii) Find all geodesics on the hyperbolic plane $H^2(x, y), y > 0$ with $g(X, Y)_{x,y} = \frac{1}{y^2}\langle X, Y\rangle$.

Hint. i) Using rotation, show that all geodesics are great circles.
ii) Using $\Gamma^1_{12} = \frac{1}{y} = -\Gamma^2_{11} = \Gamma^2_{22}$ show the geodesic equation

$$\ddot{x} = \frac{2\dot{x}\dot{y}}{y},\ \ddot{y} = \frac{(\dot{y})^2 - (\dot{x})^2}{y}.$$

Now use the trick

$$\frac{dy^2}{dx^2} = \frac{d}{dx}(\frac{\dot{y}}{\dot{x}}) = \frac{\ddot{y}\dot{x} - \ddot{x}\dot{y}}{(\dot{x})^3}$$

to show that the geodesics on H satisfy $(yy'') + (y')^2 = -1$ which are either line $\dot{x} = 0$ or circles $(x - C)^2 + y^2 = D^2$.

Remark 5.3.10. The existence of a geodesic on a Riemannian manifold allows us to define a map $T_pM \to M$ sending any vector v to point $\gamma(1)$ where $\gamma(t)$ is a geodesic through p and tangential to $\gamma(t)$. This map is called **the exponential map.** Since the tangent map of the exponential map at zero is identity, the exponential map provides a local diffeomorphism between an open neighborhood U_p of p and an open set in T_pM. It is known that in the chart $(U_p, \exp^{-1})$ the Riemannian metric has a nice form: $g_{ij}(0) = \delta_{ij}$, $\Gamma^i_{jk}(0) = 0$ for all i, j, k, (see J.Jost, Theorem 1.4.4.) This coordinate chart is called **a normal coordinate chart** on a Riemannian manifold (M, g).

5.4. **Geodesic and variation of the length of curves.** We will show that a geodesic $\gamma(t)$ joining two points p, q on a Riemannian manifold (M, g) has a locally minimal length among curves $\gamma'(t)$ joining the same points.

Let $\gamma : [a, b] \to M$ be a smooth curve. A **variation of** c is a differentiable map $F : [a, b]\times(-\varepsilon, \varepsilon) \to M$ with $F(t, 0) = \gamma(t)$ for all $t \in [a, b]$. The variation is called proper, if the endpoints stay fixed, i.e. $F(a, s) = \gamma(a)$ and $F(b, s) = \gamma(b)$ for all $s \in (-\varepsilon, \varepsilon)$. We set

$$\dot{F}(t, s) = \frac{d}{dt}F(t, s),\ F'(t, s) = \frac{d}{ds}F(t, s).$$

Now we compute the first variation

$$\frac{d}{ds}L(F(t, s)) = \frac{d}{ds}\int_a^b \langle \dot{F}(t, s), \dot{F}(t, s)\rangle^{1/2}\, dt$$

$$= \int_a^b \frac{\langle \nabla_{F'(t,s)}\dot{F}(t, s), \dot{F}(t, s)\rangle}{\langle \dot{F}(t, s), \dot{F}(t, s)\rangle^{1/2}},\ \text{since } \frac{d}{ds}\phi(F(t, s)) = \nabla_{F'(t,s)}\phi(F(t, s)) \text{ for any function } \phi,$$

$$= \int_a^b \frac{\langle \nabla_{\dot{F}(t,s)}F'(t, s), \dot{F}(t, s)\rangle}{\langle \dot{F}(t, s), \dot{F}(t, s)\rangle^{1/2}},\ \text{since } \nabla_{F'(t,s)}\dot{F}(t, s) - \nabla_{\dot{F}(t,s)}F'(t, s) = [F'(t, s), \dot{F}(t, s)] = 0,$$

$$(5.4.1) \qquad = \int_a^b [\frac{\frac{d}{dt}\langle F'(t, s), \dot{F}(t, s)\rangle}{\langle \dot{F}(t, s), \dot{F}(t, s)\rangle^{1/2}} - \langle\frac{F'(t, s), \nabla_{\dot{F}(t,s)}\dot{F}(t, s)\rangle}{\langle \dot{F}(t, s), \dot{F}(t, s)\rangle^{1/2}}]\, dt.$$

If $\gamma(t)$ is parametrized proportional to the arc-length, i.e. $||\dot{\gamma}(t, 0) = 0||$, (in this case $\gamma(t)$ is called **naturally parametrized**) then (5.4.1) becomes

$$(5.4.2) \quad \frac{d}{ds}_{s=0} L(F(t, s)) = \frac{1}{\langle\dot{\gamma}, \dot{\gamma}\rangle^{1/2}}\{\langle F'(t, 0), \dot{F}(t, 0)|^{t=b}_{t=a} - \int_a^b \langle F'(t, 0), \nabla_{\dot{\gamma}(t)}\dot{\gamma}(t)\, dt\}$$

Thus the equation (5.4.2) = 0 becomes $\nabla_{\dot\gamma(t)}\dot\gamma(t) = 0$, i.e. $\gamma(t)$ is a geodesic. Thus we have proved the following

Lemma 5.4.1. *Any geodesic $\gamma : [a,b] \to M$ is a critical point of the length functional w.r.t. to its proper variations. Any natural parametrized shortest curve $\gamma(t)$ joining two points p, q on M is a geodesic.*

Exercise 5.4.2. Let $H^2 = \{(x,y) \in \mathbb{R}^2, y > 0\}$ be the Poincare half plane with the metric $ds^2 = \frac{1}{y^2}(dx^2 + dy^2)$. Let $P = (\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and $Q = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. Compute the distance $d(P,Q)$.

5.5. **The second fundamental form and totally geodesic submanifolds.**

Definition 5.5.1. The second fundamental form of a Riemnnian submanifold $(N, \bar g, \bar\nabla)$ in a Riemannian manifold (M, g, ∇) at a point $x \in N$ is defined by

$$B : T_xN \times T_xM \to T_xN^\perp : (X,Y) \mapsto (\nabla_X Y)^\perp$$

Exercise 5.5.2. Prove that Definition 5.5.1 is well-defined. Prove that the second fundamental form is symmetric.

A submanifold $(N, \bar g) \subset (M, g)$ is called **totally geodesics**, if the second fundamental form B is vanishing everywhere.

Proposition 5.5.3. *Let $(N, \bar g)$ be a Riemannian submanifold in (M, g). Then the following conditions are equivalent:*
(i) N is totally geodesic in M,
(ii) Any geodesic $\gamma(t)$ in N is also a geodesic in M.

Proof. The result is a direct consequence of the following decomposition formula for a geodesic γ in N

$$\nabla_{\dot\gamma}\dot\gamma = \bar\nabla_{\dot\gamma}\dot\gamma + (\nabla_{\dot\gamma}\dot\gamma)^\perp = B(\dot\gamma, \dot\gamma).$$

□

Exercise 5.5.4. Prove that $S^k(1)$ is a totally geodesic submanifold in $S^n(1)$, if $k \le n$.

Exercise 5.5.5. Compute the second fundamental form of the submanifold $T^2 = \{|z_1| = 1 = |z_2|\}$ in the $S^3 = \{|z_1|^2 + |z_2|^2 = 1\}$.

5.6. **The Riemannian curvature tensor.** Since in a normal coordinate chart on a Riemmanian manifold (M, g) the metric g and its first derivatives coincide with the Euclidean metric on $\mathbb{R}^n$, there is no tensor depending on g and its first derivatives which is an invariant of (M, g). The curvature tensor of a Riemmanian manifold (M, g) is a tensor depending on the second derivatives of g which measures how far g differs from the standard Euclidean metric.

Theorem 5.6.1. *Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇. Then the curvature $R : Vect(M) \times Vect(M) \times Vect(M) \to Vect(M)$ defined by*

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z \tag{5.6.1}$$

is a tensor of type (3,1) on M.

Exercise 5.6.2. Prove this theorem.

Now let us compute the Riemannian curvatures R of a Riemannian manifold (M^m, g) in local coordinates. Let (U, x) be local coordinates on M^m. For $i, j, k, l = 1, ..., m$ put

$$R_{ijkl} = g(R(D_i, D_j)D_k, D_l.$$

Then

$$R_{ijkl} = \sum_{s=1}^{m} g_{sl}(\frac{\partial \Gamma^s_{jk}}{\partial x_i} - \frac{\partial \Gamma^s_{ik}}{\partial x_j} + \sum_{r=1}^{m}(\Gamma^r_{jk}\Gamma^s_{ir} - \Gamma^r_{ik}\Gamma^s_{jr})), \tag{5.6.2}$$

where Γ^k_{ij} are the Christoffel symbols of the Levi-Civita connection. The formula (5.6.2) is obtained by applying (5.6.1) to vector fields D_i.

It is easy to see that the Riemannian curvature on $\mathbb{R}^n$ with the Euclidean metric is equal to zero. Thus if a Riemmannian manifold (M, g) has non-zero curvature tensor R it cannot be isometric to an open domain in $\mathbb{R}^n$, i.e. there is no diffeomorphism $f \to U$, U is an open domain in $\mathbb{R}^n$ such that f induces the Euclidean metric from $\mathbb{R}^n$ to M^n.

Proposition 5.6.3. *Let (M, g) be a smooth Riemannian manifold. For vector fields X, Y, Z, W on M the following identities hold*
(i) $R(X, Y)Z = -R(Y, X)Z$,
(ii) $R(X, Y)Z + R(Z, X)Y + R(Y, Z)X = 0$,
(iii)$g(R(X, Y)Z, W) = -g(R(X, Y)W, Z)$,
(iv) $g(R(X, Y)Z, W) = g(R(Z, W)X, Y)$.

Proof. See J.Jost, proof of Lemma 3.3.1. □

Now we introduce the notion of sectional curvature. For this we need the following technical result.

Lemma 5.6.4. *Let (M, g) be a Riemannian manifold and X, Y, Z, W be tangent vectors at p such that the two 2-planes $X \wedge Y$ and $Z \wedge W$ are identical. Then*

$$\frac{g(R(X, Y)Y, X)}{|X \wedge Y|^2} = \frac{g(R(Z, W)W, Z)}{|Z \wedge W|^2}.$$

Proof. First we consider the case $Z = X$ and $W = Y + \lambda X$ with $\lambda \in \mathbb{R}$. In this case the identity is a consequence of $R(X, X)Y = 0$ for all X, Y (see Proposition (i)). Now assume that $W = aY + bX$, $a \neq 0$. Then by the above $R(X, W) = aR(X, Y)$. Since $vol(Z \wedge W) = vol(X \wedge Y) = a^{-1}vol(X \wedge W)$ we have $vol(Z \wedge W) = a^{-1}vol(X \wedge W)$. Consequently $Z = a^{-1}X + bW$. So $R(Z, W) = a^{-1}R(X, W) = R(X, Y)$. □

Using Lemma we define **the sectional curvature** $K(X \wedge Y)$ of plane $X \wedge Y \subset T_pM$ as the value defined in the Lemma. In particular, on any two-dimensional Riemannian surface (M^2, g) the sectional curvature K_g is a function $M \to \mathbb{R}$.

Exercise 5.6.5. Compute the section curvature of $S^n(1)$ and of the Poincare half plan $(x, y)|, y > 0$ with $g(X, Y) = \frac{1}{y^2}\langle X, Y\rangle$.

The global behaviour of the sectional curvature on (M^2, g) gives us information about topology of M^2.

Theorem 5.6.6 (The Gauss-Bonnet Theorem)**.** *Let M^2 be an orientable closed surface and g be a Riemmannian metric on M^2. Then*

$$\int_{M^2} K_g dA = \chi(M^2)$$

where dA is element of area of M^2 and $\chi(M)$ is the Euler number of M^2.

We need to explain how to take the integration of a continuous function K_g over a surface (M^2, g). Using partition of unity, it suffices to take integration over coordinate domains (U_k, g_{ij}) so that

$$\int_{M^2} f\, dA = \sum_k \int_{U_k} \alpha_k \cdot f\, dA,$$

where the number of covering U_k is finite, and α_k is a partition of unity subordinate to U_k. Furthermore we can assume that $U = [-\varepsilon, \varepsilon] \times [-\varepsilon, \varepsilon]$ and $g = g_{ij} dx^i dx^j$. Here we define

$$\int_U f dA := \int_{-\varepsilon}^{\varepsilon} \int_{-\varepsilon}^{\varepsilon} f \cdot (\det(g_{ij}))^{1/2} dx dy.$$

It is easy to see that the above expression does not depend on the choice of coordinates (x, y) on U.

The Gauss-Bonnet Theorem is obtained from the Gauss-Bonnet formula for a simply connected domain D on M^2. Let D be a bounded domain on M^2 such that its boundary ∂D is a piece-wise differentiable curve C consisting of m differentiable curves. Then

$$\int_D K\, dA + \int_C k_g\, ds + \sum_{i=1}^m (\pi - \alpha_i) = 2\pi.$$

Here k_g is the geodesic curvature of C (the value $|\nabla_{\dot{C}} \dot{C}|$, if C is naturally parametrized), and α_i are inner angles of C where C s not differentiable.

For a proof of the Gauss-Bonnet formula we refer to the book by Millman, Elements of Differential Geometry,1977.

Exercise 5.6.7. Let $S \subset \mathbb{R}^3$ be an embedded surface such that $K \le 0$. Prove that there are no two geodesics γ_1 and γ_2 starting from the same point p and meet again in a point q enclosing a simple region.

Lemma 5.6.8. *The value of Riemannian curvature tensor R at any point p of a Riemannian (M^m, g) is defined by the value of its sectional curvature K on all tangent planes $X \wedge Y \subset T_pM$.*

Proof. Denoting $K(X, Y) := K(X \wedge Y)|X \wedge Y|^2$, by Proposition 5.6 we have

$$\begin{aligned}\langle R(X, Y)Z, W\rangle = K(X + W, Y + Z) - K(X + W, Y) - K(X + W, Z)\\ -K(X, Y + Z) - K(W, Y + Z) + K(X, Z) + K(W, Y)\\ -K(Y + W, X + Z) + K(Y + W, X) + K(Y + W, Z)\\ +K(Y, X + Z) + K(W, X + Z) - K(Y, Z) - K(W, X).\end{aligned}$$

□

A Theorem of Shur asserts that if the dimension of M is at least 3 and the sectional curvature is constant at each point, i.e

$$K(X \wedge Y) = f(x) \text{ for } X, Y \in T_x M$$

then $f(x) = const$. In this case we say that (M, g) is a **space form**.

Example 5.6.9. The space S^n with its canonical metric is a space form, since for any point $p \in S^n$ and any pair of tangent planes $X \wedge Y$ and $Z \wedge W$ in $T_p S^n$ there exists a rotation of $\mathbb{R}^n$ which preserves p and sends $X \wedge Y$ to $Z \wedge W$. The projective space $\mathbb{R}P^n$ carries a metric of constant positive sectional curvature, since it is the quotient of the group $\mathbb{Z}_2$ acting on the sphere $S^n : x \mapsto -x$. This action preserves the metric on the sphere, therefore it descends to a metric on $\mathbb{R}P^n$. In local coordinates the metrics on S^n and $\mathbb{R}P^n$ are identical.

It is known that any simply connected Riemannian manifold of positive constant curvature is isometric to some sphere $S^n(r)$. Any simply connected Riemannian manifold of zero constant curvature (also called flat space) is isometric to Euclidean space $\mathbb{R}^n, g_0 = \sum dx_i^2$. Any simply connected Riemmanian manifold of constant negative sectional curvature is isometric to a hyperbolic space $H^n(-a^2) = (R_1^+ \times \mathbb{R}^{n-1}, g(x_1, \cdots, x_n) = \frac{1}{a^2 x_1^2}(\sum_{i=1}^n dx_i^2)$.

If dimension of M is at least 3, we can consider the average of sectional curvature $K(X \wedge Y)$. This curvature is called the Ricci curvature. It is defined as follows

$$Ric(X, Y)_p := \sum_i R(X, e_i)e_i, Y$$

where e_i is any orthonormal frame in $T_p M$.

Likewise, if the Ricci curvature is constant at each point, i.e. $Ric(X, X) = c(x)g(X, X)$, then $c(X)$ is constant. In this case (M, g) is an Einstein manifold.