Kähler manifolds and the Hodge conjecture

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Kähler manifolds form a bridge between algebraic geometry and complex differential geometry. In this course we shall discuss the following concepts: complex manifolds, Kähler metrics, sheaf cohomology, Sobolev spaces, pseudo-differential operators, elliptic differential operators. We shall prove the Hodge decomposition theorem, the Lefschetz decomposition theorem as well as the Kodaira embedding theorem. We end the course with the Hodge conjecture.

Recommended literature: - C. Voisin, "Hodge theory and complex algebraic geometry" , 1, Cambrige University Press, 2002,

- R. O. Wells, "Differential analysis on complex manifolds", Prentice-Hall, 1973.

Contents

1 Complex manifolds and almost complex manifolds

1.1 Manifolds and complex manifolds

I suggest you to look at my lecture on differential geometry
(http://www.math.cas.cz/∼hvle/DG2008.pdf)

I would like to repeat quickly a definition of a differentiable manifold. (There are many equivalent definitions).

1.1.1. Definition Let M be a Hausdorff space with a countable basis for its topology. We call M **an n-dimensional topological manifold**, if for each point $p \in M$ there is an open neighborhood U of p in M such that U is homeomorphic to an open set in $\mathbb{R}^n$ by some homeomorphism ϕ. Such a pair (U, ϕ) is called a **local coordinates systems** or a **chart** on M.

1.1.2. Example The sphere $S^n = \{x \in \mathbb{R}^{n+1}, ||x|| = 1\}$ is a n-dimensional manifold. We write S^n as the union of two open sets $\{S^n \setminus N\} \cup \{S^n \setminus S\}$ where N and S are the north pole and the south pole of S^n. Clearly each point of S^n belongs to one of those two open sets. And these open sets are homeomorphic to $\mathbb{R}^n$ by stereographic projections π. The pair $\{S^n \setminus N, \pi\}, (\{S^n \setminus S, \pi\})$ is a chart on S^n.

We are now interested in the class of topological manifolds with a good gluing, that is a good agreement between different homeomorphisms ϕ_i and ϕ_j on the common domain $U_i \cap U_j$.

1.1.3. Definition. A (smooth) **atlas** on a topological manifold M is a collection $\mathcal{A} = \{(U_i, \phi_i)\}$ of charts such that $\{U_i\}$ form an open covering of M and for each pair (U, ϕ_i) and (V, ϕ_j) in $\mathcal{A}$ the transition map

$$\Phi_{ij} = \phi_j \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$$

is a smooth map between open sets in Euclidean space.

Two atlas are **equivalent**, if their union is also an atlas.

A **smooth structure** on a topological manifold is an equivalence class of atlases, and a **smooth manifold** is a topological manifold with a specified smooth structure.

Transition functions define the structure of a differentiable manifold completely.

1.1.4. Definition Let M^{2n} be a differentiable manifold of dimension 2n. We say that M^{2n} is equipped with a complex structure if there is an atlas $\{U_i, \phi_i\}$ on M^{2n} such that U_i is diffeomorphic to an open set in C^n and the transition function Φ_{ij} is **holomorphic** for all i, j.

1.1.5. Examples of complex manifolds For $n = 2$ the sphere S^2 is a complex manifold. We cover S^2 by $\phi_0^{-1}(\mathbb{C}_0) \cup \phi_1^{-1}(\mathbb{C}_1)$ where $\mathbb{C}_0 = \{z_0 \in \mathbb{C}\}$ is given by stereographic projection ϕ_0 from the north pole and $\mathbb{C}_1 = \{z_1 \in \mathbb{C}\}$ is given by stereographic projection ϕ_1 from the south pole. We assume that the diameter of S^2 is 1. Now it is easy to check that the transition function

$$\phi_0(\phi_0^{-1}(\mathbb{C}_0) \cap \phi_1^{-1}(\mathbb{C}_1)) = (\mathbb{C}_0 \setminus \{0\}) \to (\mathbb{C}_1 \setminus \{0\}) = \phi_1(\phi_0^{-1}(\mathbb{C}_0) \cap \phi_1^{-1}(\mathbb{C}_1))$$

is given by

$$\phi_{01}(z_0) = (z_0)^{-1}.$$

This transition function is holomorphic. Moreover we observe that this transition function coincides with the transition function on $\mathbb{C}P^1 = D_0 \cup D_1$ with $D_i = \{[z_0 : z_1], | \ z_i \neq 0\}$ with a coordinate function $f_0([z_0 : z_1]) = z_1/z_0$ and $f_1([z_0 : z_1]) = z_0/z_1$.

1.1.6. Exercise Show that the space $\mathbb{C}P^n$ is a complex manifold.

The notion of differentiable maps (or functions) can be defined easily in the category of differentiable manifolds using atlas on differentiable manifolds. In the same way we can define the notion of holomorphic maps (or functions) in the category of complex manifolds.

1.1.7. Exercise Prove that any holomorphic function F on a compact complex manifold is a constant.

Hint Use the Cauchy formula for holomorphic function f of one variable

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{\xi - z} d\xi$$

to investigate the maximum of F.

1.1.8. Definition A compact subset N in a complex manifold M^n is called **a complex submanifold**, if for each point $x \in N$ there is a chart (U, ϕ) around x in M such that $\phi(U \cap N) = \phi(U) \cap \mathbb{C}^k$ for some subspace $\mathbb{C}^k \subset \mathbb{C}^n \supset \phi(U)$.

For example $\mathbb{C}P^k$ is a complex submanifold in $\mathbb{C}P^n$ for any $1 \leq k < n$.

1.2 Vector bundles

1.2.1. Definition A real (complex) topological vector bundle of rank m over a topological space X is a topological space E equipped with a map $E \to X$ such that for an open cover $\{U_i\}$ of X we have local trivialisation homeomorphism

$$\tau_i : \pi^{-1}(U_i) \cong U_i \times \mathbb{R}^m (resp. U_i \times \mathbb{C}^m)$$

such that
(i) $pr_1 \circ \tau_i = \pi$.
(ii)The transition morphisms

$$\tau_{ij} := \tau_j \circ \tau_i^{-1} : \tau_i(\pi^{-1}(U_i \cap U_j)) \to \tau_J(\pi^{-1}(U_i \cap U_j))$$

are $\mathbb{R}$-linear (resp. $\mathbb{C}$-linear) on each fiber $u \times \mathbb{R}^m$ (resp. $u \times \mathbb{C}^m$).

The transition functions $\tau_{ij}(x)$ take values in the space of real (resp. complex) matrices of size $m \times m$.

If all the transition functions τ_{ij} of a complex vector bundle V over a complex manifold M are holomorphic, then V is called **holomorphic vector bundle**. Clearly in this case V is a complex manifold.

A **section** of a vector bundle $E \xrightarrow{\pi} X$ is a map $\sigma : X \to E$ such that $\pi \circ \sigma = Id_X$.

For any real (resp. complex) manifold M^n we can associate a very important vector bundle of dimension n over M^n, the **tangent bundle** of M^n. Let $\{U_i, \phi_i)$ be an atlas on M^n. Then the tangent space TM^n is covered by open sets $U_i \times \mathbb{R}^n$ and the transition morphisms between $(U_i \cap U_j) \times \mathbb{R}^n \subset U_i \times \mathbb{R}^n$ and $(U_i \cap U_j) \times \mathbb{R}^n \subset U_j \times \mathbb{R}^n$ are given by

$$(x, v) \mapsto (u, \phi_{ij}^*(v)).$$

Here ϕ_{ij} is the transition function between the open set $\phi_i(U_i \cap U_j)$ and $\phi_j(U_i \cap U_j)$ of $\mathbb{R}^n$ and ϕ_{ij} is its Jacobian matrix at the point x.

A section of the tangent bundle is called a **vector field**.

The points of the tangent bundle can be identified with equivalence classes of differentiable maps $\gamma : [-\varepsilon, \varepsilon] \to M^n$ for the equivalence relation

$$\gamma \cong \gamma' \Longrightarrow \gamma(0) = \gamma'(0), \ \frac{d}{dt}_{t=0}\gamma = \frac{d}{dt}_{t=0}\gamma'.$$

1.2.2. Exercise Let us denote by $C^k(M^n)$ the space of k-differentiable functions on M^n. Show that any tangent vector $[\gamma](x)$ defines a linear map $D_{[\gamma]} : C^k(M^n) \to \mathbb{R}$ by $D_{[\gamma]}(f) = \frac{d}{dt}_{t=0} f(\gamma(t))$.

Using 1.2.2 we map the space of vector fields to the space of all derivations $Der(C^\infty(M^n))$. It turns out that this map is an isomorphism. Thus we shall define the **Lie bracket of two vector fields** on M^n by putting

$$[X, Y](f) := (XY - YX)f$$

for all $f \in C^\infty(M)$.

The dual vector bundle $T^*M := Hom(TM, M \times \mathbb{R})$ is called the **cotangent bundle** of M^n.

1.2.3. A differential form ω of degree k is a section of $\Lambda^k(T^*M^n)$. If the section is holomorphic, then ω is called holomorphic differential form.

We define the differential operator $d : \Omega^p(M^n) \to \Omega^{p-1}(M^n)$ by

$$d\omega(X_0, X_1, \cdots, X_r) = \frac{1}{r+1}\sum_{i=0}^{r} X_i(\omega(X_0, \cdots, \hat{X}_i, \cdots, , X_r))$$

$$+\frac{1}{r+1}\sum_{0\le i<j\le r}(-1)^{i+j}\omega([X_i, X_j], X_0, \cdots, \hat{X}_i, \cdots \hat{X}_j, \cdots, X_r)).$$

1.2.4. Exercise Prove that $d^2 = 0$.

1.2.5. Tangent bundle of a complex manifold Let M^{2n} be a complex manifold of real dimension $2n$ and let (U_i, ϕ_i) be holomorphic local charts. Then the tangent bundle $T_(U_i)$ can be identified with $U_i \times \mathbb{C}^n$. Moreover the transition function ϕ_{ij} is holomorphic by hypothesis. Equivalently the Jacobian $\phi_{ij}(x)_* : \mathbb{C}^n \to \mathbb{C}^n$ is a $\mathbb{C}$-linear map for all $x \in U_i$. This tangent bundle is also called **the holomorphic tangent bundle** of M^{2n}.

Now we define linear operators

$$J_i : TU_i \to TU_i$$

identified with $Id \times \sqrt{-1}$ acting on $U_i \times \mathbb{C}^n$. By the remark above these linear operators define a global endomorphism, written J. Clearly $J^2 = -Id$.

1.3 Almost complex manifolds and ∂-operator

We can characterize complex manifolds among a class of differentiable manifolds with special structures using 1.2.5.

1.3.1. Definition An almost complex structure on a differentiable manifold M^{2n} is an endomorphism J of TM^{2n} such that $J^2 = -Id$. Equivalently J defines a structure of a complex vector bundle. An almost complex structure is called **integrable**, if there is a complex structure on M^{2n} which induces I.

Rephrasing in terms of G-structure, we can say that a almost complex structure is a $GL(n, \mathbb{C})$-structure. (Having an almost complex structure is having the group $Gl(n, \mathbb{C})$ acting at every fiber $Gl(2n, R)$ of the principal bundle - frame bundle $F(M) \to M$, that is having a section of the associated bundle $Gl(n, 2R)/Gl(n, \mathbb{C})$.)

It is a problem of algebraic topology (so it is a soft problem) to define if a manifold admits an almost complex structure or not. The existence of a complex structure is more subtle problem. For example we know that S^6 admits almost complex structure but we do not know if it admits complex structure. But if we fix an almost complex structure and ask if it comes from a complex structure then we can answer certainly by the Newlander-Nirenberg theorem.

First we shall need to look more careful at the structure of the tangent bundle of an almost complex manifold.

Let V be a real vector space with a complex structure J and $V \otimes_{\mathbb{R}} \mathbb{C}$ the complexification of V. Putting

$$J(v \times \alpha) = J(v) \otimes \alpha$$

we extend the $\mathbb{R}$-linear transformation J to a $\mathbb{C}$-linear transformation also denoted by J of the space $V \otimes_R \mathbb{C}$. This extended transformation also satisfies $J^2 = -Id$, hence J has two eigenvalues $\pm\sqrt{-1}$. We denote by $V_{1,0}$ the eigenspace corresponding to the eigenvalue $\sqrt{-1}$ and by $V_{0,1}$ the eigenspace corresponding to the eigenvalue $-\sqrt{-1}$. Then

$$V \otimes_{\mathbb{R}} \mathbb{C} = V_{1,0} \oplus V_{0,1}.$$

It is easy to see that our complex space (V, J) is $\mathbb{C}$-linearly isomorphic to $V_{1,0}$.

If M^{2n} is almost complex manifold then we have the decomposition of its complexified tangent bundle $TM^{2n}_{\mathbb{C}} = (T_{1,0}M^{2n}) \oplus (T_{0,1}M^{2n})$. Now we note that J also acts on the cotangent bundle by

$$(Jv^*, w) := (v, Jw).$$

According to this induced action we can decompose the complexified cotangent bundle $T^*M_{\mathbb{C}}$ as $(T^{1,0}M^{2n}) \oplus (T^{0,1}M^{2n})$ and therefore we have

$$\Lambda^k(TM^{2n}_{\mathbb{C}}) = \sum_{p+q=k} \Lambda^p(T^*M^{2n})^{1,0} \wedge \Lambda^q(T^*M^{2n})^{0,1}. \tag{1.3.2}$$

A section of the bundle $(T_{1,0}M^{2n})$ (resp. $(T_{0,1}M^{2n})$) is called a vector field of type $(1,0)$ (resp. $(0,1)$). The Lie bracket on the space of vector fields on M^{2n} extends $\mathbb{R}$-linearly to the Lie bracket on the space of complex vector fields.

1.3.3. Newlander-Nirenberg Theorem *The almost complex structure is integrable, iff the bracket of two vector fields of type* $(1,0)$ *is a vector field of type* $(1,0)$.

We shall not prove the Newlander-Nirenberg Theorem here. We observe that this theorem, like the Frobenius theorem, can be stated in dual terms using the differential forms. For the statement we need to introduce the notions of the operator ∂ and operator $\bar{p}$.

Let $\omega \in \Omega^{p,q}(M^{2n})$ where M^{2n} is an almost complex manifold. We define $\partial\omega$ to be the $(p+1,p)$-component of $d\omega$ and $\bar{\omega}$ to be the $(p,q+1)$-component of $d\omega$. By (1.3.3) we get

$$d\Omega^{p,q} \subset \Omega^{p+1,q}(M^{2n}) \oplus \Omega^{p,q+1}(M^{2n}),$$

hence $d = \partial + \bar{\partial}$.

1.3.4. Exercise. 1. Let α be a differential form on $\mathbb{C}^3 = \mathbb{R}^6(x_1, y_1, x_2, y_2, x_3, y_3)$. Compute $\partial\alpha$ for $\alpha = (x_2 y_2) x_3^{-1} dx_1 \wedge dx_2$.
2. Show that on any complex manifold we have $\partial^2 = 0 = \bar{\partial}^2$ and $\partial\bar{\partial} + \bar{\partial}\partial = 0$.
3. Prove that the Newlander-Nirenberg Theorem is equivalent to the following statement. For any form $\omega \in \Omega^1_{\mathbb{C}}(M^{2n})$ we have $d\omega = \partial\omega + \bar{\partial}\omega$.

Hint i) Show that $\partial f = \sum_i \partial z_i f dz^i$, where $\partial z_i = (1/2)(\partial x_i - \sqrt{-1}\partial y_i)$ and $dz_i = dx_i + \sqrt{-1} dy_i$. For (ii) use (iii). For (iii) use the formula for $d\omega$ in 1.2.3.

2 Sheaf cohomology

Sheaves are used to keep track of the relationship between local and global data.

2.1 Sheaves

Given a topological space X, a sheaf F on X associated to each open set U a group $\mathcal{F}(U)$, called the section of F over U and to each pair $U \subset V$ an open sets of map $r_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$ called the restriction map, satisfying

2.1.1. For any triple $U \subset V \subset W$ of open sets

$$r_{W,U} = r_{V,U} \cdot r_{W,V}.$$

By virtue of this relation we may write $\sigma_{|u}$ for $r_{V,U}(\sigma)$ without loss of information.

2.1.2. For any pair of open set $U, V \subset M$ and sections $\sigma \in \mathcal{F}(U), \tau \in \mathcal{F}(V)$ such that

$$\sigma_{U \cap V} = \tau_{U \cap V}$$

there exists a section $\rho \in \mathcal{F}(U \cup V)$ with

$$\rho_{|U} = \sigma, \ \rho_{|V} = \tau_{|V} = \tau.$$

2.1.3. If $\sigma \in \mathcal{F}(U \cup V)$ and

$$\sigma_{|U} = \sigma_{|V} = 0$$

then $\sigma = 0$.

2.1.4. Examples a- On any differentiable manifold M the following groups are sheaves: $C^\infty(U)$ -the set of smooth functions on U, $\Omega^p(U)$- the set of smooth p-forms on U.

b- On any complex manifold M the following sets are sheaves: $\mathcal{O}(U)$ - the set of holomorphic functions on U, $\mathcal{O}^*(U)$ - the set of nonzero holomorphic functions on U, Ω_h^p- the set of holomorphic p-forms on U.

c- If $M \subset N$ is a subspace, $\mathcal{F}$ a sheaf on M then we can **extend $\mathcal{F}$ by zero** to obtain a sheaf $\tilde{\mathcal{F}}$ on N, settings

$$\tilde{\mathcal{F}}(U) = \mathcal{F}(U \cap M).$$

2.1.5. A map of sheaves $\mathcal{F} \stackrel{\alpha}{\to} \mathcal{G}$ on M is given by a collection of homomorphisms $\{\alpha_U : \mathcal{F}(U) \to \mathcal{G}(U) | \, U \subset M\}$ such that for $U \subset V \subset M$, α_U and α_V commute with the restriction map. The **kernel** of the map $\alpha : \mathcal{F} \to \mathcal{G}$ is just a sheaf $Ker(\alpha)$ given by $Ker\,(\alpha)(U) = Ker(\alpha_U : \mathcal{F}(U) \to \mathcal{G}(U))$. It is easy to check that this assignment defines a sheaf. The **cokernel** of α is defined as follows. A section s of the cokernel sheaf $Coker\,(\alpha)$ over U is given by an open cover $\{U_\alpha\}$ of U together with sections $s_\alpha \in \mathcal{G}(U_\alpha)$ such that for all α, β

$$s_{\alpha|U_\alpha \cap U_\beta} - s_{\beta|U_\alpha \cap U_\beta} \in \alpha_{U_\alpha \cap U_\beta}(\mathcal{F}(U_\alpha \cap U_\beta)).$$

We identify two such collections $\{(U_\alpha, s_\alpha)\}$ and $\{(U'_\alpha, s'_\alpha)\}$ if for all $p \in U$ and U_α, U'_β, there exists V with $p \in V \subset (U_\alpha \cap U'_\beta)$ such that $s'_{\alpha|V} - s'_{\beta|V} \in \alpha_V(\mathcal{F}(V))$.

2.1.6. Exercise a. Show that $Ker\,\alpha$ is a sheaf and $im(\alpha) := \cup_i(im\,\alpha_{U_i}(F(U_i))$ is not always a sheaf. Use the exponential map $\exp : \mathcal{O} \to \mathcal{O}^*$ (see 2.1.7) for the second statement.

Hint Condition 2.1.2 does not hold for contractible domains U_1 and U_2 with $U_1 \cup U_2 = \mathbb{C}^*$ and for $f_1 \in \mathcal{O}^*(U_1) = f_2 = (z \mapsto z) \in im(\exp(U_i)(\mathcal{O}(U_i)))$. The function $(z \mapsto \ln z)$ is not single-valued in $\mathbb{C}^*$.)

b. Check that the definition $Coker\,(\alpha)(U) = \mathcal{G}(U)/\alpha_U(\mathcal{F}(U))$ does not satisfy the condition of a sheaf.

We say that a sequence of sheaf maps

$$0 \to \mathcal{E} \stackrel{\alpha}{\to} \mathcal{F} \stackrel{\beta}{\to} \mathcal{G} \to 0$$

is exact if $\mathcal{E} = ker(\beta)$ and $\mathcal{G} = coker(\alpha)$. In this case we also say that $\mathcal{E}$ is the subsheaf of $\mathcal{F}$ and $\mathcal{G}$ is the quotient sheaf of $\mathcal{F}$ by $\mathcal{E}$ written $\mathcal{F}/\mathcal{E}$.

2.1.7. Example On any complex manifold the sequence

$$0 \to \mathbb{Z} \stackrel{i}{\to} \mathcal{O} \stackrel{\exp}{\to} \mathcal{O}^* \to 0$$

is exact, where i is the obvious inclusion and exp the exponential map $\exp(f) = \exp(2\pi\sqrt{-1}f)$. This fundamental sequence is called the exponential sequence.

2.2 Cohomology of sheaves

Let $\mathcal{F}$ be a sheaf on M and $\underline{U} = \{U_\alpha\}$ a locally finite cover. We define

$$C^0(U, \mathcal{F}) = \Pi_\alpha \mathcal{F}(U_\alpha),$$

$$C^p(U, \mathcal{F}) = \Pi_{\alpha_0 \neq \alpha_1 \neq \cdots \neq \alpha_p} \mathcal{F}(U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}),$$

for all $1 \leq p < \infty$. The above notations mean that an element $\sigma \in C^p(U, \mathcal{F})$, if it has form $\sigma = \{\sigma_{\alpha_0 \cdots \alpha_p} \in \mathcal{F}(U_{\alpha_0} \cap \cdots \cap U_{\alpha_p})\}$ and $\alpha_0 \cdots \alpha_p$ run *all over* the set of $\alpha_0 \neq \cdots \neq \alpha_p$. An element $\sigma \in C^p(\underline{U}, \mathcal{F})$ is called a **p-cochain** of $\mathcal{F}$. We define a coboundary operator

$$\delta : C^p(\underline{U}, \mathcal{F}) \to C^{p+1}(\underline{U}, \mathcal{F})$$

by the formula

$$(\delta\sigma)_{i_0, \cdots, i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \sigma_{i_0, \cdots, \hat{i_j}, \cdots, i_{p+1} | U_{i_0} \cap \cdots \cap U_{i_p}}.$$

A p-cochain σ is called a **cocycle**, if $\delta\sigma = 0$.

2.2.1. Exercise Show that any cocycle σ must satisfy the skew-symmetry condition $\sigma_{i_0, \cdots, i_p} = -\sigma_{i_0, \cdots, i_{q-1}, i_{q+1}, i_q, i_{q+2}, \cdots i_p}$.

A p-cochain σ is called **coboundary** if $\sigma = \delta\tau$ for some $\tau \in C^p(\underline{U}, \mathcal{F})$. It is easy to see that $\delta^2 = 0$ and we set

$$Z^p(\underline{U}, \mathcal{F}) = \ker \delta \subset C^p(\underline{U}, \mathcal{F}),$$

$$H^p(\underline{U}, \mathcal{F}) = \frac{Z^p(U, \mathcal{F})}{\delta C^{p-1}(\underline{U}, \mathcal{F})}.$$

Now given two coverings $\underline{U} = \{U_\alpha\}_{\alpha \in I}$ and $\underline{U}' = \{U'_\beta\}_{\beta \in I'}$ of M we say that $\underline{U}'$ is a refinement of $\underline{U}$ if for every $\beta \in I'$ there exists $\alpha \in I$ such that $U'_\beta \subset U_\alpha$, we write $\underline{U}' < \underline{U}$. For $\underline{U}' < \underline{U}$ we can choose a map $\phi : I' \to I$ such that $U_\beta \subset U_{\phi\beta}$ for all β. Then we have a natural map

$$\rho_\phi : C^p(\underline{U}, \mathcal{F}) \to C^p(\underline{U}', \mathcal{F})$$

given by

$$(\rho_\phi \sigma)_{\cdots} = \sigma_{\phi(\cdots)}.$$

Evidently $\delta \circ \rho_\phi = \rho_\phi \circ \delta$ therefore ρ_ϕ induces a homomorphism of the corresponding cohomology groups.

2.2.2. Exercise Prove that the induced homomorphism on cohomology groups does not depend on the choice of ϕ.

Hint Check that the chain maps associated to two inclusion associations ϕ and ψ are chain homotopic and thus induces the same map on cohomology.

2.2.3. Definition The p^{th}-Čech cohomology group of $\mathcal{F}$ on M to be the direct limit of $H^p(\underline{U}, \mathcal{F})$ as $\underline{U}$ becomes finer and finer:

$$H^p(M, \mathcal{F}) := \overset{\lim}{\longrightarrow}_{\underline{U}} H^p(\underline{U}, \mathcal{F}).$$

We recall that the inductive limit of $H^i(U, \mathcal{F})$ is the quotient of the union $\cup_i H^i(U_i)$ by the following equivalent relation $\mathcal{R}$. Two elements $g_1 \in H^i(U_1)$ and $g_2 \in H^i(U_2)$ are equivalent, if there exists an element $g_3 \in H^i(U_3)$ such that $g_3 = \rho(g_1) = \rho(g_2)$.

2.2.4. Exercise Show that $H^0(M, \mathcal{F}) = \mathcal{F}(M)$.

Now we shall compute the Čech cohomology for a fine sheaf $\mathcal{F}$ that admits partition of unity. More precisely for any $U = \cup U_\alpha$ we can find a family of homomorphisms $\eta_\alpha : \mathcal{F}(U_\alpha) \to \mathcal{F}(U)$ with sum 1 such that $sppt\, \eta_\alpha \subset U_\alpha$. Examples of such a sheaf is the sheaf of smooth functions or the sheaf of smooth differential forms on a smooth manifold.

2.2.5. Lemma *Suppose that $\mathcal{F}$ is a fine sheaf over M. Then $H^p(M, \mathcal{F}) = 0$ for all $p \geq 1$.*

Proof Given $\sigma \in Z^p(U, \mathcal{F})$ we define $\tau \in C^{p-1}(\underline{U}, \mathcal{F})$ by setting

$$\tau_{\alpha_0 \cdots \alpha_{p-1}} = \sum_{\beta \in I} \eta_\beta \sigma_{\beta, \alpha_0 \cdots \alpha_{p-1}}.$$

Now it is easy to check that $\delta\tau = \sigma$. □

2.2.6. Exact cohomology sequence. *Let $0 \to \mathcal{E} \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{G} \to 0$ be a short exact sequence of sheaves on M. Then*

$$0 \to H^0(M, \mathcal{E}) \to H^0(M, \mathcal{F}) \to H^0(M, \mathcal{G})$$

$$\to H^1(M, \mathcal{E}) \to H^1(M, \mathcal{F}) \to H^1(M, \mathcal{G})$$

$\ldots$

$$\to H^p(M, \mathcal{E}) \to H^p(M, \mathcal{F}) \to H^p(M, \mathcal{G})$$

is a (long) exact sequence.

2.2.7. Remark A sequence $0 \to \mathcal{E} \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{G} \to 0$ is exact at $\mathcal{F}$, if $Im\, \alpha = \ker \beta$. Since α is injective, we get $Im\, \alpha = im\, \alpha$. We observe that, the injectivity of α is equivalent to the injectivity of $\alpha_x : \mathcal{E}_x \to \mathcal{F}_x$ for all x. Furthermore the exactness at $\mathcal{F}$ is equivalent to the exactness at $\mathcal{F}_x$ for all x. The exactness at $\mathcal{G}$ means that, $Im\,\beta(U) = \mathcal{G}(U)$ for all U. By definition of stack this identity is equivalent to the exactness at $\mathcal{G}_x$. Thus the exactness of the sequence $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ is equivalent to the exactness of the sequences

$$0 \to \mathcal{E}_x \to \mathcal{F}_x \to \mathcal{G}_x \to 0$$

for all x.

2.2.8. Example of an exact sequence of sheaves The following sequence is exact

$$0 \to \mathbb{Z} \stackrel{2\pi i\cdot}{\to} \mathcal{O} \stackrel{\exp}{\to} \mathcal{O}^* \to 0,$$

$$0 \to \mathcal{O} \to \mathcal{M} \to \mathcal{P} \to 0,$$

where $\mathcal{P}$ is the quotient sheaf $\mathcal{M}/\mathcal{O}$:

$$\mathcal{P}(U) = \{p_n\} \subset U \text{ and } f_n \in \mathcal{M}_p/\mathcal{O}_p.$$

2.2.9. Definition of homomorphism in the long exact sequence. Since α und β

$$C^p(U,\mathcal{E}) \stackrel{\alpha}{\to} C^p(U,\mathcal{F}) \stackrel{\beta}{\to} C^p(U,\mathcal{G}),$$

commute with δ, we get the induced homomorphisms

$$H^p(U,\mathcal{E}) \stackrel{\alpha^*}{\to} H^p(U,\mathcal{F}) \stackrel{\beta^*}{\to} H^p(U,\mathcal{G}).$$

Since $H^p(M,\mathcal{F})$ is the direct limit of $H^p(U,\mathcal{F})$, we obtain

$$H^p(M,\mathcal{E}) \to H^p(M,\mathcal{F}) \to H^p(M,\mathcal{G}).$$

Now we define the co-boundary operator $\delta : H^p(M,\mathcal{G}) \to H^{p+1}(M,\mathcal{E})$ as follows. Let $\sigma \in H^p(M,\mathcal{G})$. Then there exists an open set $U \subset M$ and an element $\sigma^r \in C^p(U,\mathcal{G})$ such that

$$\delta^*(\sigma^r) = 0,\ [\sigma^r] = \sigma.$$

We consider the following commutative diagram

$$\begin{array}{ccccc} C^p(U,\mathcal{E}) & \stackrel{\alpha}{- - - \to} & C^p(U,\mathcal{F}) & \stackrel{\beta}{- - - \to} & C^p(U,\mathcal{G}) \\ \downarrow \delta & & \downarrow \delta & & \downarrow \delta \\ C^{p+1}(U,\mathcal{E}) & \stackrel{\alpha}{- - - \to} & C^{p+1}(U,\mathcal{F}) & \stackrel{\beta}{- - - \to} & C^{p+1}(U,\mathcal{G}) \end{array}$$

Since β is surjective, there exists $\mu \in C^p(U, \mathcal{F})$ such that

$$\beta(\mu) = i \circ \sigma.$$

Clearly $\delta\mu \in \ker \beta$. Therefore there exists $\tau \in C^{p+1}(U, \mathcal{E})$ such that

$$\alpha(\tau) = \delta\beta.$$

We set : $\delta^*([\sigma]) = [\tau]$.

2.2.10. Exercise Show that this definition does not depend on the choice of μ and τ.

For most of exact sequences $0 \to \mathcal{E} \to \mathcal{F} \to 0$ we can find a covering U such that for each open set $U = U_{i_0} \cap \cdots \cap U_{i_p}$ the sequence

$$0 \to \mathcal{E}(U) \to \mathcal{F}(U) \to \mathcal{G}(U) \to 0$$

is exact. In this case we see easily that the sequence

$$H^p(U, \mathcal{E}) \to H^p(U, \mathcal{F}) \to H^p(U, \mathcal{G})$$

ist exact.

2.2.11. Exercise Show that in this case the sequence

$$H^p(U, \mathcal{F}) \xrightarrow{\beta^*} H^p(U, \mathcal{G}) \xrightarrow{\delta^*} H^{p+1}(U, \mathcal{E})$$

ist exact.

Hint. We have to show that $im\beta^* = \ker \delta^*$.

3 Complex line bundles and holomorphic line bundles

3.1 Chern class of a complex line bundle

Let $\mathcal{A}$ and $\mathcal{A}^*$ denote the sheaves of C^∞-functions and nonzero C^∞-functions, respectively. The transition functions of a complex line bundle L over M satisfy the condition

$$g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = 1$$

hence they give a Čech cocycle

$$\{g_{\alpha\beta}\} \in C^1(M, \mathcal{A})$$

3.1.l. Lemma *The complex line bundle L is determined up to C^∞-isomorphism by the cohomology class* $[\{g_{\alpha\beta}\}] \in H^1(M, \mathcal{A}^*)$.

Proof We need to show that two cocycles $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ define the same complex line bundle if and only if their difference is a Čech coboundary. Now let us remember that the cocycles $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ are defined by trivializations $\{\phi_\alpha\}$ and $\{\phi'_\alpha\}$ of L respectively. These trivializations are related by the relation

$$\phi'_\alpha = f_\alpha \cdot \phi_\alpha \tag{3.1.2}$$

for some $f_\alpha \in C^\infty(U_\alpha)$. From (3.1.2) we get

$$g'_{\alpha\beta} = \frac{f_\alpha}{f_\beta} \cdot g_{\alpha\beta}. \tag{3.1.3}$$

Now (3.1.2) implies that the difference of the two cocycles is a coboundary. The inverse statement can be deduced easily. □

Now we consider the following exact sequence

$$0 \to \mathbb{Z} \to \mathcal{A} \to \mathcal{A}^* \to 0.$$

This sequence gives the following exact sequence

$$H^1(M, \mathcal{A}) \to H^1(M, \mathcal{A}^*) \xrightarrow{\delta'} H^2(M, \mathbb{Z}).$$

The image of the cocycle of the transition functions of L via δ' is called **the Chern class** of L. Since $\mathcal{A}$ is a fine sheaf we get $H^1(M, \mathcal{A}) = 0 = H^2(M, \mathcal{A})$. Thus δ' is an isomorphism. Hence we get

3.1.2. Proposition *A complex line bundle is defined up to C^∞ isomorphism by its Chern class.*

3.2 Holomorphic line bundles and divisors

Now we shall look carefully at a class of complex line bundles, the class of holomorphic line bundles. Using the same argument as in the previous subsection we obtain the following

3.2.1. Lemma *A holomorphic line bundle L over a complex is defined up to isomorphism by the cohomology class $H^1(M, \mathcal{O}^*)$ associated with its transition functions.*

The set $H^1(M, \mathcal{O}^*)$ is called **the Picard group** of M, denoted $Pic(M)$.

How to construct holomorphic line bundles? Any complex line bundle can be obtained from a universal complex line bundle on the classifying space (namely the complex Grassmannian) but we do not have (till now) classifying space for holomorphic line bundles. But there is a way to construct a holomorphic line bundle from a complex codimension 1 analytical subvariety, called also an analytical hypersurface, in a complex manifold M. We say that an analytical hypersurface V is **irreducible**, if V cannot written as the union of two analytical varieties different from V.

3.2.2. A divisor D on M is a locally finite formal linear combination

$$D = \sum a_i V_i$$

of irreducible hypersurfaces of M.

Locally finite means that for any $p \in M$ there exists a neighborhood of p meeting only a finite number of V_i appearing in D.

Divisors can be described in sheaf theoretic terms as follows. Let $\mathcal{M}^*$ denote the multiplicative sheaf of meromorphic functions on M.

3.2.3. Lemma *A divisor is a global section of the quotient sheaf* $\mathcal{M}^*/\mathcal{O}^*$.

Proof First we shall introduce the notion of the order of a meromorphic function along an irreducible analytical hypersuface. For an irreducible analytical hypersurface $V \ni p$ and any holomorphic function g defined near p we define the order $ord_{V,p}(g)$ as the largest number a such that

$$g = f^a \cdot h$$

in a local neighborhood U_p of p and $f, h \in \mathcal{O}(U_p)$.

3.2.4. Exercise Show that $ord_{V,p} g$ is independent of p.

Hint Since V is connected we need to show only the local independence.

Exercise 3.2.4 shows that we can define just $ord_V(g)$. Now let f be a meromorphic function on M. We define

$$ord_V(f) = ord_V(g) - ord_V(h),$$

where f can be written locally as the quotient g/h. Clearly this definition does not depend on the choice of the local representation of f.

Now we shall prove a first statement of Lemma 3.2.3. A global section $\{f\}$ of $\mathcal{M}^*/\mathcal{O}^*$ is given by an open cover $\{U_\alpha\}$ of M and nonzero meromorphic functions f_α in U_α with

$$\frac{f_\alpha}{f_\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta). \tag{3.2.5}$$

Then we associate to $\{f\}$ the divisor

$$D = \sum_V ord_V(f_\alpha) \cdot V.$$

(In this sum V enters only if $V \cap U_\alpha \neq 0$.) This association is well-defined, since $ord_V(f_\alpha) = ord_V f_\beta$ by (3.2.5). Moreover this map is additive w.r.t. the group structure on domain and target. Its kernel is clearly zero. Thus to prove that this map is an isomorphism we need to show that it is epimorphism. Given

$$D = \sum_{V_i} a_i V_i$$

we can define an open cover $\{U_\alpha\}$ of M such that in each U_α every V_i appearing in D ha a local defining function $g_{i\alpha} \in \mathcal{O}(U_\alpha)$. Now we set

$$f_\alpha = \Pi_i g_{i\alpha}^{\alpha_i} \in \mathcal{M}^*(U_\alpha). \tag{3.2.6}$$

Then $\{f_\alpha\}$ is a global section of $\mathcal{M}^*/\mathcal{O}^*$. This completes the proof of Lemma 3.2.3. □

Now we shall consider the exact sequence

$$0 \to \mathcal{O}^* \xrightarrow{i} \mathcal{M}^* \xrightarrow{j} \mathcal{M}^*/\mathcal{O}^*$$

and the related cohomology exact sequence

$$H^0(M, \mathcal{M}^*) \xrightarrow{j_*} H^0(M, \mathcal{M}^*/\mathcal{O}^*) \xrightarrow{\delta} H^1(M, \mathcal{O}^*).$$

The map δ is a homomorphism from the group of divisors to the group of holomorphic line bundles over M. We denote by $[D]$ the image of a divisor D in the class of holomorphic line bundle. By the exact sequence the line bundle $L = [D]$ is trivial if and only if D is a divisor of a meromorphic function.

3.2.7. Exercise i) Let D be a smooth analytic hypersurface in M. Show that there is a section $s \in H^0(M, [D])$ such that $s^{-1}(0) = D$.
ii) Show that the following sheaf sequence is exact

$$0 \to \mathcal{O}_M(L \otimes [-D]) \xrightarrow{\otimes s_0} \mathcal{O}_M(L) \xrightarrow{r} \mathcal{O}_D(L_{|D}) \to 0$$

where s_0 is any holomorphic section in $H^0(M, \mathcal{O}([D])$ such that $s^{-1} = D$ and r is the restriction map.

Hint Let D is given by local data $f_\alpha \in \mathcal{M}(U_\alpha)$ (see also (3.2.6)). Then f_α gives a holomorphic sections_f of $[D]$ with $s^{-1}(0) = D$.

4 Kähler manifolds

4.1 Hermitian metrics and Kähler metrics

Let M^n be a complex manifold of complex dimension n. Denote by J the complex structure on TM considered as a real vector bundle. A **Hermitian metric** g on M^n is a metric (i.e. a section of the real vector bundle $S^2_{pos}T^*M$ whose fiber consists of positive definite quadratic forms over T_xM) which satisfies the following invariant condition

$$g(JX, JY) = g(X, Y).$$

It is easy to check that the bilinear form

$$\omega_g(X, Y) := g(JX, Y),$$

is skew-symmetric (and hence a differential 2-form) and non-degenerate. The form ω_g is called **the Kähler form of** g**.**

4.1.1. Exercise i) Prove the existence of a Hermitian metric on any complex manifold.
ii) Show that the form $h(X, Y) = g(X, Y) + \sqrt{-1}\omega(X, Y)$ is a $\mathbb{C}$-valued Hermitian bilinear form on TM considered as complex vector bundle (i.e. $J = \sqrt{-1}$).

Hint (i) Use partition of unity to glue Hermitian metrics on each local coordinate neighborhood.

4.1.2. A Kähler metric is a Hermitian metric whose Kähler form is closed.

4.2 Examples of Kähler manifolds

4.2.1 Any Hermitian metric on a Riemannian surface Σ_g is a Kähler metric.

4.2.2 Let Λ be a lattice in $\mathbb{C}^n$, i.e. Λ is a discrete subgroup of $\mathbb{C}^n$. Then $\mathbb{C}^n/\Lambda$ is called **a complex torus**. Clearly any complex torus carries a Kähler metric which is induced from the constant metric on $\mathbb{C}^n$.

4.2.3. The Fubini-Study-metric on $\mathbb{C}P^n$. We define the following 2-form on $\mathbb{C}P^n$

$$\omega_{F-S} = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\ln\frac{||z||}{||z_i||}^2$$

where $z = [z_0 : z_1, \cdots z_n] \in \mathbb{C}P^n$ with $z_i \neq 0$.

4.2.4. Exercise 1. Show that this definition does not depend on the choice of i.

2. Show that ω_{F-S} is a closed and non-degenerate 2-form.

Hint To show that ω_{F-S} is closed we use 1.3.4.1 to get $\omega = \frac{1}{4\pi}(\partial + \bar{\partial})(\bar{\partial} - \partial) \ln \frac{||z||}{||z_i||}^2$.

To show that ω_{F-S} is non-degenerated we write it explicitly. Let $t^k = z^k/z_i$. Then

$$g = 4\frac{(1 + \sum_k t^k \bar{t}^k)(\sum_k dt^k d\bar{t}^k) - (\sum_k \bar{t}^k)(\sum_k t^k d\bar{t}^k)}{(1 + \sum_k t^k \bar{t}^k)^2}.$$

4.2.5 A complex submanifold N in a complex manifold M is a submanifold whose tangent bundle is also complex w.r.t to the complex structure on M. If N is a complex submanifold of a Kähler manifold M then the restriction of the Kähler form on M to N is the Kähler form of the restriction of the Kähler metric from M to N. Hence N is also a Kähler manifold.

4.2.6. Exercise Show that the Kähler form of any complex submanifold M^n in $\mathbb{C}P^N$ represents a non-trivial element in $H^2(M, \mathbb{Z})$.

Hint It suffices to prove that the Fubini-Study form ω_{F-S} is integral. Since Since $H^*(\mathbb{C}P^n, \mathbb{Z}) = \mathbb{Z}$ it implies that there exists a number λ_n such that $\lambda_n[\omega_{F-S}] = E$, where $E \in H^2(\mathbb{C}P^n)$ is the generator of the ring $H^*(\mathbb{C}P^n, \mathbb{Z})$. Prove that $\lambda_n = 1$ by showing that it agrees with the volume form on $\mathbb{C}P^1$.

4.3 The Kähler-Hodge identities

First we note that any complex manifold M^{2n} has a canonical orientation defined by J, namely frames $(v_1, Jv_1, \cdots, v_n, Jv_n)$ and $(w_1, Jw_1, \cdots, w_n, Jw_n)$ have the same orientation: the determinant of a matrix preserving a complex structure J is always positive. Thus given a Hermitian metric g we can define the volume form vol - a $2n$-differential form on M^{2n} such that $vol(e_1, \cdots, e_{2n}) = 1$ if $(e_1, \cdots, e_{2n})$ is an oriented orthonormal basis.

4.3.1. Hodge star operator. The presence of a Hermitian g metric on a complex manifold M^n defines a linear algebraic operator, the Hodge star operator

$$* : T_x^{p,q}(M^n) \to T_x^{n-p,n-q}(M^n)$$

by requiring

$$< \psi, \phi >_{\tilde{g}} = < vol_x, \psi \wedge *\phi >_{\tilde{g}} . \tag{4.3.1.1}$$

Here $\tilde{g}$ is the induced Hermitian metric on the space $(\wedge^k T_x^*(M^n))_{\mathbb{C}}$ considered as a real vector space, i.e. we consider the space $(\wedge^k T_x^*(M^n))_{\mathbb{C}} \otimes \mathbb{R}$ obtained by extending the scalar product on $\wedge^k T_x^*(M)$ first to a Hermitian bilinear form $(,)$ on its complexification and then take the real part of this Hermitian form (cf.

4.1.1.ii). So $\tilde{g}$ is a Hermitian metric on $T^{p,q}_x(M)$ with respect to the canonical complex structure $Jv = \sqrt{-1} \cdot v$.

4.3.1.2. Exercise i) Show that the definition (4.3.1.1) is equivalent to the definition

$$(\psi, \phi) vol_x = \psi \wedge *\phi.$$

ii) Show that $(*)^2 \phi = (-1)^{p+q} \phi$ for $\phi \in T^{p,q}_x M^{2n}$.

Hint To prove i) express the LHS of the equation as $< \psi, \phi > vol_x + < \psi, \phi > \sqrt{-1} vol_x$. To prove ii) show that for $\eta = \eta_{I\bar{J}} \phi_I \wedge \bar{\phi}_J$ we have

$$*\eta = 2^{p+q-n} \varepsilon_{I\bar{J}} \bar{\eta}_{I\bar{J}} \phi_{I^0} \bar{\phi}_{J^0}$$

where $I^0 = \{1, \cdots, n\} \setminus I$ and $\varepsilon_{I\bar{J}}$ is the sign of the permutation

$$(1, \cdots n, 1', \cdots, n') \mapsto (i_1, \cdots, i_p, j_1 \cdots j_q, i^0_1, \cdots, i^0_{n-p}, j^0_1, \cdots, j^0_{n-q}).$$

Now we set

$$\bar{\partial}^* = - * \bar{\partial} *, \tag{4.3.2}$$

$$\partial^* = - * \partial * . \tag{4.3.3}$$

4.3.4. Lemma *Suppose that M is compact. For each $\psi \in \Omega^{p,q-1}(M^n)$ and $\phi \in \Omega^{p,q}(M^n)$ we have*

$$\int_M (\bar{\partial}\psi, \phi) = \int_M (\psi, \bar{\partial}^* \phi). \tag{4.3.4.1}$$

Proof We have

$$\int_M (\bar{\partial}\psi, \phi) = \int_M \bar{\partial}\psi \wedge *\phi = (-1)^{p+q} \int_M \psi \wedge \bar{\partial} * \phi + \int_M \bar{\partial}(\psi \wedge *\phi). \tag{4.3.5}$$

Since $\partial = d$ on the forms of type $(n, n-1)$ the second term on the RHS of (4.3.5) is

$$\int_M d(\psi \wedge *\phi) = 0.$$

Using formula (4.3.1.2) for $*\bar{\partial}^* = -(*)^2 \bar{\partial} * = (-1)^{p+q} \bar{\partial} *$ we deduce Lemma 4.3.4 from (4.3.5) immediately. □

4.3.6. Exercise Prove an analog of (4.3.4.1) for ∂^*.

Hint Change the complex structure on M.

4.3.7. The Λ operator is defined on Hermitian manifolds as follows. First we note that the Kähler form ω is a $(1,1)$-form. Let

$$L : T_x^{p,q}(M) \to T_x^{p+1,q+1}(M)$$

be defined by

$$L(\eta) = \eta \wedge \omega.$$

Then we define Λ to be the adjoint $L^* : T_x^{p,q}(M) \to T_x^{p-1,q-1}(M)$

$$< L^*\psi, \phi >_x = < \psi, L\phi >_x \tag{4.3.7.1}$$

for all $z \in M$.

It is easy to see that if L is $\mathbb{C}$-linear then L^* must be anti-$\mathbb{C}$-linear. Hence (4.3.7.1) is equivalent to

$$(L\phi, \psi)_x = (\phi, L^*\psi)_x. \tag{4.3.7.2}$$

This implies for $\phi \in T_x^{p,q}M$ and $\psi \in T_x^{p+1,q+1}M$

$$L\phi \wedge *\psi = \phi \wedge L * \psi = \phi \wedge *\Lambda\psi$$

Hence (cf. with (4.3.4))

$$\Lambda = (-1)^{p+q} * L * . \tag{4.3.7.3}$$

4.3.8 The Kähler-Hodge identities On any Kähler manifold we have

$$[\Lambda, \bar\partial] = -\sqrt{-1}\partial^*, \tag{4.3.8.1}$$

$$[\Lambda, \partial] = \sqrt{-1}\bar\partial^*. \tag{4.3.8.2}$$

We refer the recommended books for different proofs of these identities. We remark that recently Verbitsky extend these identities on a large class of almost Hermitian manifolds with $d\omega \in \Lambda^{3,0}(M) \oplus \Lambda^{0,3}(M)$ (i.e. $\partial\omega = 0 = \bar\partial\omega$. (arxiv:math/0510618).

4.3.9. Remark The Hodge star operator $*$ can be defined on an oriented Riemannian manifold in the same way by requiring (4.3.1.1) holds.

5 The Hodge theory on compact complex manifolds

In this section we demonstrate application of elliptic methods to the study of topolopgy of comapct complex manifolds and compact Kähler manifolds

5.1 The Dolbeault cohomology group

In 1.3.4.1 we derived the equality $\partial^2 = 0 = \bar{\partial}^2$ for any complex manifold M^n. Thus the complex $(\Omega^{p,q}, \bar{\partial})$ is a differential complex, whose cohomology group

$$H^{p,q}_{\bar{\partial}}(M^n) = \frac{\ker \bar{\partial} \subset \Omega^{p,q}(M^n)}{\bar{\partial}(\Omega^{p,q-1}(M^n))}$$

is called **the Dolbeault cohomology group**.

Thus any element $\alpha \in H^{p,q}_{\bar{\partial}}(M^n)$ can be represented by a $\bar{\partial}$-closed (p,q)-form. Which representative of α is the best? If M is compact and provided with a Hermitian metric we can define the L_2-norm on the space $\Omega^{p,q}(M^n)$ by using the inner product

$$<\psi, \phi>:= \int_{M^n} <\psi, \phi> vol.$$

5.1.1. Lemma *A $\bar{\partial}$-closed form $\psi \in Z^{p,q}_{\bar{p}}(M^n) := \ker \bar{\partial} \subset \Omega^{p,q}(M^n)$ is of minimal L_2-norm in the space $\psi + \bar{\partial}\Omega^{p,q-1}(M^n)$, iff $\bar{\partial}^*\psi = 0$.*

Proof If $\bar{\partial}^*\psi = 0$ then for any $\eta \in \Omega^{p,q-1}$

$$\begin{aligned} ||\psi + \bar{\partial}\eta||^2_{L^2} &=< \psi + \bar{\partial}\eta, \psi + \bar{\partial}\eta >= \\ &= ||\psi||^2_{L^2} + ||\bar{\partial}\eta||^2_{L^2} + 2 < \psi, \bar{\partial}\eta >= \\ &= ||\psi||^2_{L^2} + ||\bar{\partial}\eta||^2_{L^2} + 2 < \bar{\partial}^*\psi, \eta >= \\ &= ||\psi||^2_{L^2} + ||\bar{\eta}||^2_{L^2} \ge ||\psi||^2_{L^2}. \end{aligned}$$

Conversely if ψ has the smallest norm, then for any $\eta \in \Omega^{p,q-1}(M)$ we have

(5.1.2) $$\frac{d}{dt}_{t=0} ||\psi + t\bar{\partial}\eta||^2 = 0.$$

Expanding LHS of (5.1.2) we get

(5.1.3) $$\frac{d}{dt}_{t=0} < \psi + t\bar{\partial}\eta, \psi + t\bar{\partial}\eta >=< \psi, \bar{\partial}\eta >= 0.$$

Hence $\bar{\partial}^*\psi = 0$. □

5.1.4. Exercise Show that the two first oder equations

$$\bar{\partial}\psi = 0, \qquad \bar{\partial}^*\psi = 0$$

can be replaced by th single second oder equation

$$\triangle_{\bar{\partial}}\psi = (\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial})\psi = 0$$

Hint Look at the equation $(\triangle_{\bar{\partial}}\psi, \psi) = 0$.

5.2 The Hodge theorem

5.1.5 motivates us to look at the space of solution of the **Laplacian equation** $\triangle_{\bar\partial}\psi = 0$. Differential forms ψ on M satisfying this equation are called **harmonic forms**. The space of harmonic (p,q)-forms is denoted by $\mathcal{K}^{p,q}(M)$. The Hodge theorem allows us to identify the Dolbault cohomology groups as well as the de Rham coomolgy groups with the space of harmonic forms (Corollary 5.2.5)

5.2.1. The Hodge Theorem on compact complex manifolds
1. $\dim \mathcal{K}^{p,q}(M) < \infty$.
2. *The orthogonal projection* $K : \Omega^{p,q}(M) \to \mathcal{K}^{p,q}(M)$ *is well defined, and there exists a unique operator, the Green's operator*

$$G : \Omega^{p,q}(M) \to \Omega^{p,q}(M)$$

with

$$Id = K + \triangle_{\bar\partial} G \tag{5.2.2}$$

on $A^{p,q}(M)$.

A proof of Hodge Theorem 5.2.1 shall be given in 8.2.2 based on the general theory on elliptic differential operators on vector bundles over a compact Riemannian manifolds which we shall explain in next three sections. Examples for elliptic differential operators is the Laplacian $\triangle_{\bar\partial}$ acting on $\Omega^{p,q}(M)$ as well as the usual Laplacian

$$\triangle_d : \Omega^p(M^n) \to \Omega^p(M)$$

acting on the space $\Omega^p(M^n)$ of p-forms on a compact Riemannian manifold M^n by

$$\triangle_d := dd^* + d^*d$$

with

$$d^*(\omega^p) = (-1)^p *^{-1} d * . \tag{5.2.2.a}$$

(See also (4.3.9) for the star operator. In the Riemannian case we can verify that $*^2 = (-1)^{k(n-k)}$ on $T^k_x M^n$.)

5.2.2.b. Remark. If n is even, then $*^2 = (-1)^p$ on $\Omega^p(M)$. If we extend operator $*$ $\mathbb{C}$-linearly to operator $\tilde{*}$ on the space of complex valued forms (some peoples use $*$ for $\tilde{*}$ and some other peoples denote the Hodge operator defined in (4.3.1.1) by $\bar{*}$) then we have

$$(\alpha, \beta) vol = \alpha \wedge \tilde{*}\bar\beta.$$

That implies

$$(5.2.2.c) \qquad \tilde{*}\beta = \overline{*\beta}.$$

On the other hand (5.2.2.a) implies for a complex form ω^p

$$(5.2.2.d) \qquad d^*_A\omega^p = -d\tilde{*}d\tilde{*}.$$

It is easy to see that (5.2.2.c) and (5.2.2.d) imply formula (4.3.2) for adjoint of ∂ and $\bar{\partial}$. Conversely (4.3.2) also implies (5.2.2.a) for complex manifolds.

5.2.3. Exercise Show that d^* is a formal adjoint of d in the sense that for all $\psi \in \Omega^p(M)$ and $\phi \in \omega^{p-1}(M)$ we have

$$< d^*\psi, \phi >_{L_2} = < \psi, d\phi > .$$

A p-form ω on M^n is called harmonic, if $\triangle_d\omega = 0$.

5.2.4. Example. A Kähler form ω on M^n is harmonic because $d\omega = 0$ and $*\omega = \omega^{n-1}$, hence $*d*\omega = 0$. Since $d = \partial + \bar{\partial}$, we also have $\triangle_{\bar{\partial}}\omega = 0$.

Denote by $\mathcal{H}^p$ the space of p-harmonic forms on M^n. The following theorem is an analog of theorem 5.2.1.

5.2.5. The Hodge theorem on compact Riemannian manifolds *We have the following direct decomposition (w.r.t. the L_2-metric)*

$$\Omega^p(M^n) = \mathcal{H}^p(M^n) \oplus \triangle_d(\Omega^p(M^n)).$$

5.2.6. Corollary *i) If M^n is a compact manifold then the cohomology group $H^k(M^n, \mathbb{R})$ is isomorphic to the space of harmonic k-forms, and hence it is finite dimensional.*
ii) If M^n is a compact complex manifold then the Dolbeault cohomology group $H^{p,q}_{\bar{\partial}}(M^n)$ is isomorphic to the space of harmonic (p,q)-forms and hence finite dimensional.

Proof. The first statement follows from the fact the the decomposition of a p-form ω into

$$(5.2.6) \qquad \omega = \omega_h + \omega_d + \omega_{d^*}$$

is unique. To prove this we assume that $\omega = 0$ and apply the RHS of (5.2.6) with d and d^* seperately Since $dd^*z = 0$ if and only $d^*z = 0$, and $d^*dz = 0$ if and only if $dz = 0$, we conclude that $\omega_d = 0 = \omega_{d^*}$. Hence $\omega_h = 0$.

The second statement follows in the same way. □

5.3 The Hodge decomposition theorem on compact Kähler manifolds

Now let M^{2n} be a compact Kähler manifold of real dimension 2n. Thank to the Kähler-Hodge identities (4.3.8) we have the following special relation between the Dolbeault Laplacian $\triangle_{\bar\partial}$ and $\triangle_d$. Let $\triangle_\partial := \partial\partial^* + \partial^*\partial$, where $\partial^* = - * \partial *$.

5.3.1. Lemma *On a Kähler manifold we have*

$$\triangle_d = 2\triangle_\partial = 2\triangle_{\bar\partial}.$$

Proof We have

$$(5.3.1.1) \qquad \triangle_d = (\partial + \bar\partial)(\partial^* + \bar\partial^*) + (\partial^* + \bar\partial^*)(\partial + \bar\partial).$$

Using the Kähler-Hodge identities we expand the RHS of (5.3.2)

$$(\partial + \bar\partial)(\partial^* - \sqrt{-1}[\Lambda, \partial]) + (\partial^* - \sqrt{-1}[\Lambda, \partial])(\partial + \bar\partial) =$$

$$= \partial\partial^* + \bar\partial\partial^* + \sqrt{-1}\bar\partial\partial\Lambda - \sqrt{-1}\bar\partial\Lambda\partial + \partial^*\partial + \partial^*\bar\partial - \sqrt{-1}\Lambda\partial\bar\partial + \sqrt{-1}\partial\Lambda\bar\partial.$$

Writing $\partial^* = \sqrt{-1}[\Lambda, \bar\partial]$ we obtain

$$\partial^*\bar\partial = -\sqrt{-1}\bar\partial\Lambda\bar\partial = -\bar\partial\partial^*.$$

Thus

$$\triangle_d = \triangle_\partial + \sqrt{-1}[\Lambda, \bar\partial]\partial + \sqrt{-1}\partial[\Lambda, \bar\partial] = 2\triangle_{\bar\partial}.$$

The other equality is obtained by changing the complex structure J to $-J$. □

5.3.2. Theorem *We have the following decomposition as a direct sum*

$$(5.3.2.1) \qquad \mathcal{H}^k(M) = \oplus_{p+q=k}\mathcal{K}^{p,q}(M).$$

Proof First we have the inclusion

$$\mathcal{K}^{p,q}(M) \subset \mathcal{H}^{p+q}(M)$$

since any $\triangle_{\bar\partial}$-harmonic form is also a $\triangle_d$-harmonic form by Lemma 5.3.1. Thus it suffices to show the other inclusion. In its turns the inclusion of LHS of (5.3.2.1) in the RHS of (5.3.2.1) is a consequence of the following fact F which we shall prove. If α is a $\triangle_d$-harmonic form then its (p, q)-component is also $\triangle_d$ harmonic, and therefore belongs to $\mathcal{K}^{p,q}(M)$ by Lemma 5.3.1. To prove the fact F its suffices to show that

$$(5.3.3). \qquad \triangle_d(\Omega^{p,q}(M)) \subset \Omega^{p,q}(M).$$

But $\triangle_d = 2\triangle_{\bar\partial}$ and $\triangle_{\bar\partial}\Omega^{p,q}(M) \subset \Omega^{p,q}(M)$ which implies (5.3.3). □

Now let us recall that the De Rham cohomology group of a differentiable manifold M is defined as follows

$$H^q_{DR}(M, \mathbb{R}) = \frac{\ker(d : \Omega^q(M) \to \Omega^{q+1}(M))}{Im\,(d : \Omega^{q-1}(M) \to \Omega^q(M))}.$$

We can also consider the group $H^q_{DR}(M, \mathbb{C})$ by replacing the complex $\Omega^q(M)$ by its complexification $\Omega^q_{\mathbb{C}}(M)$.

5.3.3. Corollary *We have the induced decomposition.*

$$H^k_{DR}(M, \mathbb{C}) = \oplus_{p+q=k} H^{p,q}(M).$$

Moreover this decomposition does not depend on the choice of Kähler metric.

Proof The first statement follows from Theorem 5.3.2. To see that this decomposition does not depend on the choice of Kähler metric we recall that the Dolbeault cohomology groups does not depend on the choice of Kähler metric. □

It is known (exercise ???) that $H^*_{sing}(M, \mathbb{R}) = H^*_{DR}(M, \mathbb{R})$.
Elements of $H^{2p}(M, \mathbb{R}) \subset H^{2p}(M, \mathbb{C})$ which lies in the group $H^{p,p}(M, \mathbb{C})$ is called a (p,p)-class.

5.3.4. Examples of (p,p)-classes i) The class $[\omega^p] \in H^{2p}(M, \mathbb{R})$ is a (p,p)-class on a Kähler manifold (M^{2n}, J, ω).
ii) If N^{n-p} is a complex submanifold of complex codimension p in a compact Kähler manifold M^n, then its Poincare dual class $PD[N^{n-p}]$ considered as an element of $H^{2p}(M^n, \mathbb{R})$ is a (p,p)-class. Indeed it suffices to show that for all $k \neq p$ we have

(5.3.4.1) $$\int_{M^n} PD[N^{n-p}] \wedge \alpha^{n-k,n+k-2p} = \int_{N^{n-p}} \alpha^{n-k,n+k-2p} = 0$$

for any (harmonic) $(n-k, n+k-2p)$-form α. Here we also represent $PD[N^{n-p}]$ by a closed differential form. Since N^{n-p} is complex, its tangent space $T_z N^{n-p}$ at any point $z \in N^{n-p}$ is a vector of form $(n-p, n-p)$. Thus (5.3.4.1) holds.

5.3.5. Exercise. Compute the group $H^{p,q}(T^n)$ of a complex torus T^n and the group $H^{p,q}(\mathbb{C}P^n)$.

Hint All the 1-forms dz_i and $d\bar z_i$ on T^n and their wedge products are solutions of $\triangle_{\bar\partial}\theta = 0 : \bar\partial^* dz_i = \pm \bar{*}\bar\partial\bar{*} dz_i = 0 = \bar\partial dz_i = 0$. So $H^{p.q} = \binom{n}{p}\binom{n}{q}$. For the second statement use the fact the $H^*(\mathbb{C}P^n, \mathbb{Z}) = (\mathbb{Z}[x], x^{n+1} = 0)$ and x is represented by the Kähler form.

6 Sobolev spaces and differential operators

In the previous section we considered Laplacian operators acting on the spaces of smooth sections of certain vector bundles over Riemannian manifolds or Hermitian manifolds. These vector bundles are also equipped with a natural metric, hence the space of the smooth sections of these vector bundles is also equipped with the induced inner product. Its completion is a Hilbert space. In order to apply the technique of Hilbert spaces (or more generally of Banach spaces) to the theory of differential operators it is important to have other inner products (resp. norms) on the space of smooth sections. The most important norms are the Sobolev norms. We shall consider here only a special class of Sobolev norms which are called $W^{p,2}$-norm, also often denoted by L^2_p-norm.

6.1 Sobolev spaces

Locally any section on a real vector bundle V^k over a Riemannian manifold M^n can be considered as a function from a bounded domain $\Omega \subset \mathbb{R}^n$ to $\mathbb{R}^k$. We assume that the domain $\mathbb{R}^n$ is equipped with a Riemannian metric (which is not necessary Euclidean) and hence with a measure (the volume form), and target space $\mathbb{R}^k$ is equipped with an inner product $<,>$. Let $(x_1, \cdots, x_n)$ be coordinates on $\mathbb{R}^n$ and $D_i = \frac{\partial}{\partial x_i}$. Then for an n-tuple $\alpha = (i_1, \cdots, i_n)$ we shall denote by D^α the differential operator $D_1^{i_1} \cdots D_n^{i_n}$. Set $|\alpha| := \sum_{k+1}^n i_k$.

6.1.1. Definition Let $f, g \in C^\infty(\Omega, \mathbb{R}^k)$. Then we define

$$< f, g >_{W^{p,2}(\Omega)} := \int_\Omega < f, g > + \sum_{|\alpha| \le p} \int_\Omega < D^\alpha f, D^\alpha g >$$

This inner product induces a $W^{p,2}(\Omega)$-norm by

$$||u||_{W^{p,2}(\Omega)} = < u, u >^{\frac{1}{2}}_{W^{1,2}(\Omega)} .$$

Let E be a vector bundle equipped with a (Euclidean or Hermitian) metric over a compact Riemannian manifold M^n. To define a $W^{p,2}$-norm on the space $\Gamma^\infty(E^k)$ of smooth sections of E we use partition of unity on M and a local trivialization of E over the chosen covering on M. Thus any smooth section on E can be written as a sum of $\mathbb{R}^k$-valued function on a domain $U \subset \mathbb{R}^n$. Now the inner products on the fixed local trivialization over the fixed covering extends to the inner product on the whole space $\Gamma^\infty(E)$. This inner product induces then the $W^{p,2}$-norm on $\Gamma^\infty(E)$. Now we denote by $W^{p,2}(E)$ the completion of $\Gamma^\infty(E)$. This space is called the Sobolev space of sections over E equipped with a $W^{p,2}$-norm.

It is an easy fact that different $W^{p,2}$-norms on the same Sobolev space are equivalent, i.e. the choice of covering as well as of trivialization of E does not effect on the equivalence class of any $W^{p,2}$-norm on E. Thus as a topological space $W^{p,2}(E)$ is uniquely defined and we can work with any local trivialization on E.

The following two theorems are fundamental in the theory of Sobolev spaces. Denote by $\Gamma^k(E)$ the Banach space of all C^k-sections of E.

6.1.2. Sobolev imbedding theorem *Let $n = \dim M^n$. Then for $s > [n/2] + k + 1$ we have the embedding*

$$W^{s,2}(E) \subset \Gamma^k(E).$$

6.1.3. Rellich imbedding theorem *The natural inclusion*

$$j : W^{s,2}(E) \to W^{t,2}(E)$$

for $t < s$ is a compact linear operator.

We recall that a linear operator L is **compact**, if the image of any closed ball is compact.

We shall not prove these theorems and refer to the book by Wells for a partial proof and the the book by Adams on the Sobolev spaces for a complete proof.

These theorems are very important in proving the existence and smoothness of solutions of elliptic differentail equations. Using the Rellich theorem we can usually show the existence of a generalized solution - an element in an appropriate Sobolev space $W^{p,2}$, p is given, by considering a Cauchy sequence of approximated generalized sequence of bounded norm in another Sobolev space. Sometime we can use tools of functional analysis to show the existence of a generalized solution to a given linear elliptic equation. To show a generalized solution is smooth according to the Sobolev theorem we need to show that the gen higher norms $W^{q,2}$, for $q = p+1$ and iterating this process. This estimate is usually achieved by considering the equation which poses a constrain to the derivatives of its solutions.

Now we shall define the notion of an adjoint operator and a formal adjoint operator. Suppose that E and F are vector bundles over a compact Riemannian manifold M^n. Let $L : \Gamma^\infty(E) \to \Gamma^\infty(F)$ be a $\mathbb{R}$-linear operator. Then operator

$$L^* : \Gamma^\infty(F) \to \Gamma^\infty(E)$$

is called the **adjoint** of L if for all $f \in \Gamma^\infty(E)$ and $g \in \Gamma^\infty(F)$ we have

$$< Lf, g >_{L^2} = < f, L^* g >_{L^2} . \qquad (6.1.4)$$

This definition can be extended to non-compact manifold M^n but we shall take for defining (6.1.4) only sections f, g with compact support. The operator L^* shall be called the **formal adjoint** of L.

6.1.5. Exercise Prove that $(L_1 \circ L_2)^* = L_2^* \circ L_1^*$.

6.2 Differential operators

In this section we shall consider only linear differential operators, since they are necessary and sufficient for their applications in the Hodge theory.

Now let E^p and F^q be vector bundles over a differentiable manifold M^n. A linear operator $L : \Gamma^\infty(E) \to \Gamma^\infty(F)$ is called a (linear) **differential operator**, if on any local trivialization of E^p over a local coordinate $(x_1, \cdots, x_n)$ for any section f of E^p over U

$$f(x) = (f_1(x), \cdots, f_p(x))$$

we have

$$L(f)_i(x) = \sum_{j=1}^{p} \sum_{|\alpha|\le k} a^{ij}_\alpha(x) D^\alpha f_j(x),\ i = 1, \cdots, q. \tag{6.2.1}$$

We say that L has order k, if in its local representation (6.2.1) there is no partial differentiation of order larger than k.

The following proposition explains why we can use the Sobolev space theory to study the differential operators. Denote by $Diff_k(E, F)$ the vector space of all linear differential operators of order k on E with values in F.

6.2.2. Proposition *1. For each $L \in Diff_k(E, F)$ there is extension*

$$\bar{L}_s : W^{s,2}(E) \to W^{s-k,2}(F)$$

for all s.
2. For each $L \in Diff(E, F)$ there exists its adjoint $L^* : \Gamma^\infty(F) \to \Gamma^\infty(E)$ and its adjoint

$$\overline{(L^*)}_s : W^{s,2}(F) \to W^{s-k,2}(E)$$

with $\overline{(L^*)}_s = \overline{L_s}^*$.

6.2.3. Exercise Prove 6.2.2.

Hint Use the Fourier transformation

$$\hat{f}(y) = \frac{1}{(2\pi)^n} \int e^{-i<x,y>} f(x)\, dx$$

and notice that

$$||f||^2_{L^2_s(\mathbb{R}^n)} = \int |\hat{f}(y)|^2 (1 + |y|^2)^s\, dy$$

because

$$\widehat{D^\alpha f}(y) = i^{|\alpha|} y^\alpha \hat{f}(y).$$

Here $y^\alpha = y^{\alpha_1} \cdots y^{\alpha_n}$ for $\alpha = (\alpha_1, \cdots \alpha_n)$.

6.3 Symbol of differential operators

The main characteristic of a linear differential operator L acting on E and taking values in F is its symbol, a linear operator acting on a finite dimensional space π^*E and taking values in a finite dimensional space π^*F. Here we define π^*E and π^*F as follows. Let $p : E \to M$ and $p : F \to M$ be vector bundle over a manifold M. Let $T'M = T^*M \setminus \{0\}$, i.e. T'_xM is a subset consisting of all non-zero vectors of T^*_xM. Let $\pi : T'M \to M$ be the projection. Then π^*E (resp. π^*F) is the induced vector bundle ($\pi^*E = \{(x, v) \in T'M \times E | \, \pi(x) = p(E)\}$. Now we put

$$Smbl_k(E, F) = \{\sigma \in Hom(\pi^*E, \pi^*F)|$$

$$\sigma(x, \rho v) = \rho^k \sigma(x, v) \text{ for all } (x, v) \in T'M \text{ and all } \rho > 0\}.$$

Now we shall define a linear map

$$\sigma_k : Diff_k(E, F) \to Smbl_k(E, F),$$

(6.3.1)
$$\sigma_k(L)(x, v)e = L(\frac{1}{k!}[g - g(x)]^k \cdot f)(x) \in F_x,$$

where
- $g \in C^\infty(M)$ with $dg_x = v$, and
- $f \in \Gamma^\infty(E)$ with $f(x) = e$.

6.3.2. Exercise Show that (6.3.1) does not depend on the choice of g and f.

Hint Let $D_i g = \xi_i$, so $v = (\xi_1, \cdots, \xi_n)$. For $L = \sum_{|\nu| \le k} A_\nu D^\nu$ as in (6.2.1) show that

$$\sigma_k(L)(x, v)e = \sum_{|\nu| = k} A_\nu \xi_1^{\nu_1} \cdots \xi_n^{\nu_n}.$$

6.3.3. Definition The image $\sigma_k(L)$ is called **the k-symbol** of L.

6.3.4. Example: Symbol of the d-operator Let us consider the de Rham complex

$$\Omega^0(M^n) \xrightarrow{d} \Omega^1(M^n) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M^n) \to 0.$$

It generates the corresponding complex

$$\Lambda^0 T^*_x \overset{\sigma_1(d)(x,v)}{\to} \Lambda^1 T^*_x \overset{\sigma_1(d)(x,v)}{\to} \cdots$$

with

$$\sigma_1(d)(x, v)e = v \wedge e$$

for all $e \in \Lambda^p T^*_x$.

6.3.5. Exercise Compute the symbol of the operator ∂ and $\bar{\partial}$.

Hint. Let $v\wedge : \Omega^p(M) \to \Omega^{p+1}(M)$. Show that $(v\wedge)^*\omega = (-1)^{deg\,\omega} * v \wedge *$.

6.3.6. Exercise. i) Prove that $\sigma_{r+s}(P\cdot Q) = \sigma_r(P)\cdot\sigma_s(Q)$ for $P \in Diff_r(E,F)$ and $Q \in Diff_s(F,G)$. Use this compute $\sigma(\triangle_d)$ and $\sigma(\triangle_{\bar{\partial}})$.

ii) Prove that $\sigma_k(L^*) = (-1)^k(\sigma_k(L))^*$. Compute $\sigma(d^*)$, $\sigma(\partial^*)$ and $\sigma(\bar{\partial}^*)$.

Hint (i) Use (6.3.1).

(ii) Use the partition of unity and the fact that $(D^\nu)^* = (-1)^{|\nu|}D^\nu$. Then use the formula in 6.3.2 and 6.3.6.i.

7 Pseudo differential operators

The class of differential operators is too small in order to contain the inverse for a very important class of differential operators - elliptic differential operators, whose symbols are invertible. Thus we need to investigate a larger class containing differential operators - class of pseudo differential operators. This class is very natural if we look at differential operators in views of Fourier transformations. Suppose that U is an open domain in $\mathbb{R}^n$ and $p(x,\xi)$ polynomial of degree m in $\xi \in \mathbb{R}^n$ whose coefficients are smooth functions in $x \in U$. We shall associate $p(x,\xi)$ with a differential operator $D(p)$ by setting

$$D(p) := p(x, D)$$

where $D = (\sqrt{-1}D_1, \cdots, \sqrt{-1}D_n)$ is a formal substitute for $\xi = (\xi_1, \cdots x_n) \in \mathbb{R}^n$ with $D_j = \partial/\partial x_j$. Using the inverse Fourier transformation we get the following formula for all $u \in C_0^\infty(U)$ ($C_0^\infty(U)$ is the subset in $C^\infty(U)$ consisting of all functions with compact support):

$$[D(p)u]x = [p(x,D)u]x = \int_{\mathbb{R}^n} p(x,\xi)\hat{u}(\xi)e^{\sqrt{-1}<x,\xi>}d\xi,$$

where

$$\hat{u}(\xi) = \frac{1}{2\pi^n}\int u(x)e^{-\sqrt{-1}<x,\xi>}\,dx$$

the Fourier transformation of u.

7.1 Pseudo differential operators

Let U be an open set in $\mathbb{R}^n$ and m an arbitrary integer. We denote by $\tilde{S}^m(U)$ the subset on $C^\infty(U \times \mathbb{R}^n)$ consisting of functions $p(x,\xi)$ satisfying the following condition. For each compact subset $K \subset U$ and for any multi-indices α, β exists a constant $C_{\alpha,\beta,K}(p) \in \mathbb{R}$ such that

$$|[D_x^\beta D_\xi^\alpha q](x,\xi)| \le C_{\alpha,\beta,K}(1+|\xi|)^{m-|\alpha|} \text{ for all } x \in K, \xi \in \mathbb{R}^n, \tag{7.1.1}$$

Now we shall consider the subset $S^m(U) \subset \tilde{S}^m(U)$ defined by the following conditions (7.1.2) and (7.1.3)

$$\lim_{\lambda\to\infty} \frac{p(x,\lambda\xi)}{\lambda^m} \text{ exists for } \xi \neq 0, \tag{7.1.2}$$

$$p(x,\xi) - \psi(\xi)\sigma_m(p)(x,\xi) \in \tilde{S}^{m-1}(U), \tag{7.1.3}$$

where $\sigma_m(p)(x,\xi)$ denotes the LHS of (7.1.2) and $\psi(\xi)$ is some cut off function on $\mathbb{R}^n$ with support in the unit ball in $\mathbb{R}^n$.

Next we define subset $\tilde{S}_0^m(U) \subset \tilde{S}^m(U)$ as follows. A function $p \in \tilde{S}^m(U)$ belongs to $\tilde{S}_0^m(U)$ if there exists a compact $K \subset U$ such that for each fixed ξ the function $p(x,\xi)$ in variable $x \in U$ has compact support in K.

Finally we set $S_0^m(U) = S^m(U) \cap \tilde{S}_0^m(U)$.

Set for $p \in \tilde{S}^m(U)$ and $u \in C_0^\infty(U)$:

$$(L(p)u)x := \int p(x,\xi)\hat{u}(\xi)e^{\sqrt{-1}<x,\xi>}\, d\xi, \tag{7.1.4}$$

Operator $L(p)$ defined by (7.1.4) is called **a linear pseudo-differential operator of order** m.

This definition is easily extended to the case of a linear operator $L : \Gamma_0^\infty(E^l) \to \Gamma^\infty(F^k)$ for vector bundles E^l of dimension l and F^k of dimension k over a differentiable manifold M^n by using any trivialization of E^l over a given covering $\{U_i\}$ of M^n. Namely L is called a linear pseudo differential operator of order m, if for any given covering $\{U_i\}$ on M operator L can be locally expressed by a matrix $L(p_j^i), i = \overline{1,l}, j = \overline{1,k}$ of $l \times k$ pseudo differential operators $L(p_j^i) : \Gamma_0^\infty(U_i) \to \Gamma^\infty(U_i)$ of order m as in (7.1.4).

Denote the set of linear pseudo differential operators of order m from E to F by $PDiff_m(E,F)$. The most important properties of pseudo differential operators are summarized in the following Theorem.

7.1.5. Theorem *1) If $L \in PDiff_m(E,F)$ then L can be extended to a continuous linear operator $\bar{L}_s : W^s(E) \to W^{s-m}(F)$ for all s.*
2) There exists a canonical linear map

$$\sigma_m : PDiff_m(E,F) \to Smbl_m(E,F)$$

which on a trivialization of E and F over U is defined the formula

$$\sigma_m(L_U)(x,\xi) = [\sigma_m(p^i_j)(x,\xi)].$$

Furthermore this linear map σ_m is surjective.

We refer to the book by Wells for a proof of this Theorem.

7.1.6. Exercise(compare with 6.3.6.) Let M^n be a compact manifold, E, F and G vector bundles over M^n. Show that

i) If $P \in PDiff_s(F,G)$ and $Q \in PDiff_r(E,F)$ then $P \circ Q \in PDiff_{r+s}(E,G)$ with $\sigma_{r+s}(P \circ Q) = \sigma_s(P) \circ \sigma_r(Q)$.

ii) If $P \in PDiff_m(E,F)$ then there exists its adjoint operator $P^* \in PDiff_m(F,E)$, moreover $\sigma_m(P^*) = (-1)^m[\sigma_m(P)]^*$.

7.2 Parametrics for elliptic pseudo differential operators

We shall show (Theorem 7.2.4) that a pseudo-differential operator is invertible, if its symbol is invertible.

7.2.1. Definition Let $L \in PDiff_k(E,F)$ over a manifold M^n. We say that L is **elliptic**, if $\sigma_k(L)(x,\xi) : E_x \to F_x$ is isomorphism for all (x,ξ). We also say in this case that $\sigma_k(L)$ is elliptic.

We say that a linear operator $L : \Gamma^\infty(E) \to \Gamma^\infty(F)$ is a **smoothing operator**, if there is a smooth extension $\bar{L}_s : W^{s,2}(E) \to W^{s+1,2}(F)$ for all s. We also say that L is an operator of order -1 and we denote the set of all linear operators of order (-1) from E to F by $OP_{-1}(E,F)$.

7.2.2. Exercise Show that if $L \in OP_{-1}(E,F)$ over a compact manifold M^n then $Id_{s+1,s} \circ L$ is a compact operator, i.e. for any s the image $L(B_s)$ of a unit ball w.r.t. the norm $W^{s,2}$ is compact. Here $Id_{s+1,s}$ denotes the embedding $W^{s+1,2}(F) \to W^{s,2}(F)$.

Hint Use the Rellich theorem.

7.2.3. Definition Let $L \in PDiff(E,F)$. Operator $\tilde{L} \in PDiff(F,E)$ is called **parametrix** (or pseudo inverse) for L, if it satisfies the following conditions

$$L \circ \tilde{L} - Id_F \in OP_{-1}(F),$$

$$\tilde{L} \circ L - Id_E \in OP_{-1}(E).$$

7.2.4. Theorem *Let k be an arbitrary integer and $L \in PDiff_k(E,F)$ elliptic operator. Then there exists a parametrix for L.*

Proof Since $L \in PDiff_k(E,F)$ its symbol $\sigma_k(L)$ is invertible, i.e. $\sigma_k(L)^{-1}(x,\xi) : F_x \to E_x$ is an element in $Smbl_k(F,E)$. According to Theorem 7.1.5.2 there exists $\tilde{L} \in PDiff_k(F,E)$ such that $\sigma_{-k}(\tilde{L}) = \sigma_k(L)^{-1}$. Now

$$\sigma_0(L \circ \tilde{L} - Id_F) = \sigma_k(L) \circ \sigma_{-k}(\tilde{L}) - \sigma_0(Id_F) = 0.$$

Hence now applying 7.2.2. (ii) we get

$$L \circ \tilde{L} - Id_F \in OP_{-1}(F,F).$$

In the same way we prove the $\tilde{L} \circ L - Id_E \in OP_{-1}(E,E)$. □

8 Decomposition theorems for self-adjoint elliptic differential operators

We shall apply results in the previous section to study the space of solutions of linear self-adjoint elliptic differential operators on a compact manifold M^n. For $L \in Diff(E,F)$ we set

$$\mathcal{H}_L = \{\xi \in \Gamma^\infty(E) : \ L\xi = 0\}.$$

8.1 Finiteness theorem for elliptic differential operators

We recall that a bounded linear operator T on a Banach space is called **a Fredholm operator**, if its range is closed and T has finite dimensional kernel and cokernel.

8.1.1. Lemma *Let $L \in PDiff(E,F)$ be elliptic differential operator. Then there exists a parametric P for L such that $L \circ P$ and $P \circ L$ are extended continuously to Fredholm operators mapping $W^{s,2}(F) \to W^{s,2}(F)$ and $W^{s,2}(E) \to W^{s,2}(E)$ correspondingly for all s.*

Proof Recall that (7.2.2) $L \circ P = Id - S$, where S is a smoothing compact operator. Hence $L \circ P$ extends to operator acting on $W^{s,2}(F)$. Its kernel consists of all x such that $Sx = x$, hence the kernel is compact because S is compact and therefore $\dim \ker L < \infty$. Next we note that $Im(L \circ P)$ is orthogonal to $\ker I - S^*$ and therefore the range of $L \circ P$ is closed. Since S is a smoothing operator its adjoint S^* is also a smoothing operator. Hence $\dim coker\,(L \circ P) < \infty$. □

8.1.2. Theorem *Let $L \in Diff_k(E,F)$ be an elliptic operator and $\mathcal{H}_{L_s} = \ker\{L_s : W^{2,2}(E) \to W^{s-k,2}(F)\}$. Then*

a) $\mathcal{H}_{L_s} \subset \Gamma^\infty(X, E)$ *for all* s
b) $\dim \mathcal{H}_{L_s} = \dim \mathcal{H}_L < \infty$, $\dim W^{s-k}(F)/L_s(W^s(E)) < \infty$.

Proof First we show that $\dim \mathcal{H}_{L_s} < \infty$ for all s. Let P be a parametrix for L. Then by Lemma 8.1.1 we get

$$(P \circ L)_s : W^{s,2}(E) \to W^{s,2}(F)$$

has a finite dimensional kernel. Since $\ker L_s \subset \ker(P \circ L)_s$, the operator L_s has also a finite dimensional kernel. So to prove (a) it suffices $\mathcal{H}_{L_s}$ contains only a smooth sections (regularity of the solution to L). Since $S := P \circ L - Id$ is a smoothing operator we get for any $\xi \ker L_s \subset W^{s,2}(E)$

$$\xi = (P \circ L - S)\xi = -S\xi \in W^{s+1,2}(E)$$

so in fact $\xi \in W^{l,2}(E)$ for all l. By the Sobolev embedding theorem $\xi \in \Gamma^\infty(E)$. This proves a) and first part of b). The second part of b) follows from the inclusion $Im\,(L_s \circ P_{s-k}) \subset Im\,L_s \subset W^{s,2}(F)$ and the Fredholm property of $L_s \circ P_{s-k}$ (see 8.1.1). □

8.2 Proof of the Hodge decomposition theorem

First we prove a generalization of the Hodge theorem 5.2.1.

8.2.1 Theorem *Let* $L \in Diff_m(E)$ *be a linear self-adjoint elliptic operator, i.e.* $L = L^*$. *Then exists a linear map* $\hat{L} : \Gamma^\infty(E) \to \Gamma^\infty(E)$ *and* $G_L : \Gamma^\infty(E) \to \Gamma^\infty(E)$ *such that:*
a) $\hat{L}(\gamma^\infty(E)) = \mathcal{H}_L(E)$,
b) $L \circ G_L + \hat{L} = G_L \circ L + \hat{L} = Id_E$,
c) $\Gamma^\infty(E) = \mathcal{H}_L(E) \oplus G_L \circ L(\Gamma^\infty(E)) = \mathcal{H}_L(E) \oplus L \circ G_L(\Gamma(E))$.

Proof We take $\hat{L}$ to be the orthogonal projection in $L^2(E)$ on the (finite dimensional) space $\mathcal{H}_L(E)$. Taking into account (8.1.2.b) this proves (a).

Now we shall construct G_L for (b), (c). Let $\mathcal{H}_L(E)^\perp$ be the orthogonal complement of $\mathcal{H}_L(E)$ in $W^{0,2}(E)$. Now denote by L_m the continuous extension of L from $W^{m,2}(E) \to W^{0,2}(E)$. Clearly

$$L_m(W^{m,2}(E)) \subset \mathcal{H}_L(E)^\perp$$

since L is self-adjoint. The restriction

$$\bar{L}_m : W^{m,2}(E) \cap \mathcal{H}_L(E)^\perp) \to \mathcal{H}_L(E)^\perp$$

is injective, since L is linear, and any element in the kernel of L_m is in $\mathcal{H}_L(E)$. Since L is self-adjoint and L_m is Fredholm, for each $\tau \in \mathcal{H}_L(E)^\perp$ there exists $\xi \in$

$W^{m,2}(E)$ such that $L_m\xi = \tau$. This proves that $\bar{L}_m$ is onto on $W^{0,2}(E) \cap \mathcal{H}_L(E)^\perp$. So $\bar{L}_m$ is a bijection.

By Banach theorem on the inverse map there exists an continuous linear inverse map $\bar{G}_0$ of $\bar{L}_m$. Denote by G_0 its linear extension to the whole space $W^{0,2}(E)$ by setting $G_{0|\mathcal{H}_L(E)} = 0$. Now set $G_L := G_{|\Gamma^\infty(E) \subset W^{0,2}(E)}$. Note that statement (b) follows from the identity

$$L_m \circ G_L + \hat{L} = Id_E = G_L \circ L_m + \hat{L}, \tag{8.2.2}$$

which holds because $L_m \circ G_L = Id_{|\mathcal{H}_L(E)^\perp} = G_L \circ L_m$. The last statement (c) also follows from (8.2.2). □

Clearly the Hodge Theorems 5.2.1 and 5.2.4 follow from 8.2.1 by setting $E = \Gamma^{p,q}(M^n)$, and $L = \mathcal{L}$ in the first case and $E = \Omega^p(M^n)$, $L = \triangle_d$ in the second case.

8.2.3. Exercise. Let G be the Green operator for the Laplacian $\triangle_d$ as in Theorem 8.2.1. Show that the pseudo differential operator $dGd^* : \Omega^1(M) \to \Omega^1(M)$ is the orthogonal projection to the subspace $d(C^\infty(M))$.

9 The Lefschetz decomposition

Besides the Hodge decomposition theorem 5.3.2 it is also very useful to consider another decomposition of the cohomology group of a compact Kähler manifold. The idea which leads to the Lefschetz decomposition is the fact that the space of harmonic forms is invariant under the multiplication by ω. Moreover this operation is an isomorphism on appropriated spaces.

We recall that L is operator acting on differential form by ω and Λ is its adjoint: $\Lambda = *^{-1}L*$.

9.1. Lemma *We have the commutation relation for all $x \in M^{2n}$*

$$[L, \Lambda] = (k - n)Id \text{ on } \Lambda^k(T^*_x(M^{2n})).$$

Proof The Lemma can be obtained by straightforward calculation in linear algebra using explicit formula for L and Λ (s. e.g. Voisin for explicit calculations).

Since ω is closed we have an induced operator also denoted by L

$$L : H^k(M^{2n}, \mathbb{R}) \to H^{k+2}(M^{2n}, \mathbb{R})$$

We say that an element $\alpha \in H^{k-2r}$ is primitive, if $L^{n-k+2r-1}\alpha = 0 \in H^{2n-k+2r+2}(M^{2n}, \mathbb{R})$.

9.2. Theorem (The Lefschetz decomposition) *i) If X is a compact Kähler manifold of dimension n, then for every $k \le n$*

$$L^{n-k} : H^k(X, \mathbb{R}) \to H^{2n-k}(X, \mathbb{R})$$

is an isomorphism.
ii) Each cohomology class $\alpha \in H^k(M^{2n}, \mathbb{R})$ admits a unique decomposition

$$\alpha = \sum_r L^r \alpha_r$$

We shall first show that the analogous decomposition theorem holds on the level of differential forms.

9.3. Lemma *The morphism of vector bundles*

$$L^{n-k} : \Lambda^k(T_x^* M^{2n}) \to \Lambda^{2n-k}(T_x^* M^{2n})$$

is an isomorphism.

Proof of Lemma 9.3 Since L is linear and $\dim \Lambda^k(T_x^* M^{2n}) = \dim \Lambda^{2n-k}(T_x^* M^{2n})$ its suffices to show that L^{n-k} is injective. Using Lemma 9.1 we get

$$[L^r, \Lambda] = L[L^{r-1}, \Lambda] + [L, \Lambda]L^{r-1}.$$

Using induction we get for all $r \ge 1$

$$[L^r, \Lambda] = [r(k-n) + r(r-1)]L^{r-1}. \tag{9.4}$$

Now we shall prove Lemma 9.3 by induction on k. Clearly L^{n-0} is an isomorphism. Assume that L^{n-r} is not injective and $\alpha \in \ker L^{n-r} \subset \Lambda^k(T_x^* M^{2n})$. Then using (9.4) we get

$$L^r \Lambda \alpha - [r(k-n) + r(r-1)]L^{r-1}\alpha = L^{r-1}(L\Lambda - (r(k-n) + r(r-1))Id)\alpha = 0. \tag{9.5}$$

By induction step (9.5) implies that

$$(L\Lambda - r[(k-n) + (r-1)]Id)\alpha = 0. \tag{9.6}$$

Since $k - n + r - 1 \ne 0$ we get from (9.6) that $\alpha = L\beta$, where $\beta \in \Lambda^{k-2}(T_x^* M^{2n})$ and combining Lemma 9.1 with (9.6) we get $L^{r+1}\beta = 0$. Using the induction step we get $\beta = 0$, hence $\alpha = 0$. □

Proof of Theorem 9.2. Clearly the spaces of harmonic forms are invariant under the action of L since ω is a harmonic form. Then the first statement follows from the Hodge theory (Corollary 5.2.5) that the space of harmonic forms is isomorphic to the de Rham cohomology groups and from Lemma 9.3 which leads to isomorphism of spaces of harmonic forms of the corresponding degrees.

Using Lemma 9.3 we reduce (ii) to the case of $k \le n$. First we shall prove the uniqueness of the decomposition. Suppose that the decomposition is not unique. Then there exists α_r primitive such that

$$\sum_{r\ge 0} L^r \alpha_r = 0. \tag{9.7}$$

Suppose that in this sum $\alpha_0 = 0$. Then

$$L(\sum_{r\ge 0} L^{r-1}\alpha_r)) = 0$$

and by Lemma 9.3 we get $\sum_{r\ge 0} L^{r-1}\alpha_r = 0$ since $L^{r-1}\alpha_r$ is a k-form with $k \ge 1$.

Now suppose that $\alpha_0 \ne 0$ so α_0 is a form of degree k. Since $L^{n-k+1}\alpha_0 = 0$, applying L^{n-k} to RHS of (9.7) to get

$$L^{n-k+1}(\sum_{r\ge 1} L^r \alpha_r) = 0 = L^{n-k+2}(\sum_{r\ge 1} L^{r-1}\alpha_r).$$

Lemma 9.3 implies again that $\sum_{r\ge 1} L^{r-1}\alpha_r = 0$. By induction step we get $\alpha = 0$. This proves the uniqueness of the decomposition.

Now we shall show the existence of the decomposition. It suffices to assume that $k \le n$. Lemma 9.3 shows that there exits $\beta \in \Lambda^{k-2}(T^*_x M^{2n})$ such that $L^{n-k+2}\beta = L^{n-k+1}\alpha$. Thus $\alpha_0 = \alpha - L\beta$ is primitive, and $\alpha = L\beta + \alpha_0$. The induction step gives us the desired decomposition. □

9.10. Remark Using the the Lefschetz decomposition theorem we can define a nondegenerated bilinear form Q on the space $H^r(M, \mathbb{C})$ by

$$Q(\xi, \eta) = \sum_{s\ge (r-n)^+} (-1)^{\frac{r(r+1)}{2}+s} \int_M L^{n-r+2s}(\xi_s \wedge \eta_s),$$

where $\xi + \sum L^s \xi_s$, $\eta + \sum L^s \eta_s$ are the decomposition into primitive forms. This bilinear from is called the Hodge-Riemannian bilinear relation. It plays important properties in theory of Kähler manifolds.

10 The Kodaira embedding theorem

It is important to know if a compact complex manifold M^n is a projective algebraic manifold i.e. it can be realized as a complex submanifold of $\mathbb{C}P^N$ for some large

N. As we have seen in 4.2.5 and 4.2.6 any projective algebraic manifold is Kähler, moreover its Kähler form represents a non-trivial element in the group $H^2(M^n, \mathbb{Z})$, it is natural to ask if a compact complex manifold M^n admitting a Kähler form in $H^2(M^n, \mathbb{Z})$ is also a projective algebraic manifold. This question was posed by Hodge and solved by Kodaira.

10.1 Hodge manifolds, positive line bundles and the Kodaira theorem

10.1.1. A Hodge manifold is a compact complex manifold M^n admitting a Kähler metric h whose associated Kähler form Ω_h is integral, i.e. $[\Omega_h] \in H^2(M^n, \mathbb{Z})$.

10.1.2. The Kodaira embedding theorem *Let M^n be a Hodge manifold. Then M^n is projective algebraic.*

It is important to find another feature which also characterizes Hodge manifolds.

10.1.3. Proposition *If M^n is a Hodge manifold with a Kähler form Ω_h then there exists a holomorphic line bundle L on M^n such that $c_1(L) = [\Omega_h]$.*

We shall not prove this theorem which is a consequence of the Lefschetz $(1,1)$-form theorem which states that if ω is an integral $(1,1)$-form, then there exits a holomorphic line bundle L over M such that $c_1(L) = [\omega]$.

Now we shall say that a holomorphic line bundle L over M^{2n} is **positive**, if L satisfies the condition in Proposition 10.1.3 for some Kähler form Ω_h. A holomorphic line bundle L is called semi-positive, if $c_1(L) = [\omega_h]$ where ω_h is a closed $(1,1)$ such that the bilinear form $h = \omega_h(., J.)$ is semi-positive.

10.1.4. Exercise Show that
i) If L_1 and L_2 are positive then $L_1 \otimes L_2$ is positive.
ii) For any positive line bundle L and for any give line bundle K there exists n such that $L^n \otimes K$ is positive.
iii) If L is positive on M and $f : N \to M$ is a holomorphic map, then $f^*(L)$ is semi positive on N.

Now we state the most important theorem on positive line bundles. We refer to the book of Wells for an exposition of th proof of this theorem which is based on the Hodge theory as well as the theory of elliptic differential operators of orders 2.

10.1.5 Kodaira-Nakano vanishing theorem *Let $L \to M^n$ be a positive line bundle. Then*

$$H^q(M, \Omega^p(L)) = 0 \text{ for } p + q > n.$$

Here $\Omega^p(L)$ denotes the sheaf of holomomorphic p-forms valued in L

10.2 Line bundles and maps to projective spaces

Let L be a holomorphic line bundle, and $\mathcal{O}(L)$ the sheaf of holomorphic sections of L. The group $H^0(M, \mathcal{O}(L))$ is the set of all global holomorphic sections of L. This space is of finite dimension. Let $s_0, \cdots, s_N$ be a basis of $H^0(M, \mathcal{O}(L))$. Considering s_i as generalized functions we want to parametrize M by values of $s_i(x)$. Let U_α be an open neighborhood on M. Then the value $[s_0^\alpha(x), \cdots, s_N^\alpha(x)] \in \mathbb{C}^N$ depends on the choice of trivialization of L, but the value $\tau_L(x) := [s_0^\alpha(x) : \cdots : s_N^\alpha(x)] \in \mathbb{C}P^N$ does not depend on the choice of a trivialization, since two trivializations differ by multiplication with a scalar $\lambda \in \mathbb{C}^*$. Thus our parametrization τ_L of M by s_i is a holomorphic map to $\mathbb{C}P^N$. Of course here we need not to forget the condition that this map is defined at any point $x \in M$, i.e. there exists some k such that $s_k(x) \neq 0$. If so the restriction map

$$\mathbb{C}^{N+1} = H^0(M, \mathcal{O}(L)) \stackrel{r_x}{\to} L_x = \mathbb{C}. \tag{10.2.1}$$

is a non-trivial map and r_x is surjective.

If (10.2.1) holds we want to know if τ_L is an embedding, i.e. it is injective map and the differential $D\tau_L(x)$ is also injective for all $x \in M$. The injectivity of τ_L is equivalent to the fact that $\lambda(s_0(x), \cdots, s_N(x)) \neq (s_0(y), \cdots, s_N(y))$ for all $\lambda \in \mathbb{C}^*$. Equivalently $\ker r_x \neq \ker r_y$. That last condition holds if and only if $\ker r_{x,y} = \ker r_x \cap \ker r_y$ has codimension 2, or equivalently

$$\mathbb{C}^{N+1} = H^0(M, \mathcal{O}(L)) \stackrel{r_{x,y}}{\to} L_x \otimes L_y = \mathbb{C}^2$$

is surjective.

Finally the differential $D\tau_L(x) : T_xM \to T_{\tau_L(x)}\mathbb{C}P^n$ is injective x, if and only if in a coordinate neighborhood $U_\alpha \ni x$ with $s_0(x) \neq 0$ and $D\tau_L(x) = \{ds_1(x), \cdots, ds_N(x)\} : T_xM \to \mathbb{C}^N\}$ is an embedding. By changing "coordinates s_i by adding λs_0 to $s_i(x)$ we can assume that the image $\tau_L(x) = (1, 0, \cdots, 0)$. Denote by $\mathcal{J}_x(L)$ the sheaf of all section in $H^0(M, \mathcal{O}(L))$ vanishing at x, then $\mathcal{J}_x(L)$ is generated by $\{s_1, \cdots, s_N\}$. Thus the condition that $D\tau_L(x)$ is injective is equivalent to the fact that

$$\bigcap_{i \geq 1} ds_i(x) \neq 0.$$

Equivalently, for all $v \in T_x M$ there exists $s_i, i \geq 1$ such that $ds_i(V) \neq 0$. After a linear transformation, it is equivalent to the fact, that for any $w^* \in T_x^* M$ there is $s \in \mathcal{J}_x(L)$ such that $ds(x) = v$. To write this condition independent of the choice of a neighborhood U_α we note that the change of U_α to U_β leads to the change of corresponding differential ds, $s \in \mathcal{J}_x(L)$ as follows

$$ds^\alpha(x) = ds^\beta(x) \cdot g_{\alpha\beta},$$

where $g_{\alpha\beta}$ is the transition function of L under the change of coordinates and trivializations ϕ_α, ϕ_β of L (so $s^\alpha = \phi_\alpha^* s$, $s^\beta = \phi_\beta^* s$.) Thus in fact we have defined a map

$$d_x : \mathcal{J}_x(L) \to T_x^* \otimes L_x$$

and by the argument above, d_x is surjective, iff $D\tau_L(x)$ is injective.

11 Proof of the Kodaira embedding theorem

11.1. Remark Why we need to blow up M, if $\dim M \geq 2$?

11.1 Blow-up of a Kähler manifold

Let M is a complex manifold. A **blow-up of** M **at a point** $x \in M$ is a manifold $\tilde{M}$ together with a holomorphic projection $\pi : \tilde{M} \to M$ with the following property. The preimage $E := \pi^{-1}(x)$ is a divisor in M which is biholomorphic to $\mathbb{C}P^{n-1}$. The map $\pi : \tilde{M} \setminus E \to M \setminus \{x\}$ is a diffeomorphism. Since this operation is local we shall first construct a blow up of the origin in the disk $D \subset \mathbb{C}^n$. Let $z = (z_1, \cdots, z_n)$ be complex coordinates in D and $l = [l_1 : \cdots l_n]$ homogeneous coordinates in $\mathbb{C}P^{n-1}$. We denote by

$$\tilde{D} := \{(z, l) \in D \times \mathbb{C}P^{n-1} | \, z_i l_j = z_j l_i \text{ for all } i, j\}.$$

If we consider $l \in \mathbb{C}P^{n-1}$ is a line in $\mathbb{C}^n$ the above condition means that $z \in l$. Now $\pi : \tilde{D} \to D$ is the projection to the first factor: $(z, l) \mapsto z$. Clearly π is ismorphism if $z \neq 0$ and $E := \pi^{-1}(0) = \mathbb{C}P^{n-1}$.

Now let M be a complex manifold of dimension n, $x \in M$ and $U \to D$ is a complex neighborhood a round x. Let $p : \tilde{D} \setminus E \to U - \{x\} \subset M$ is the blow up of U according to the above recipe. Then we define a blowup of M at x to be

$$\tilde{M}_x = M \setminus \{x\} \cup_p \tilde{D}.$$

11.1.1. Exercise Let L be a positive line bundle on M. Prove that there exists a number k_0 such that the line bundle $\pi^*(L)^k \otimes [-E]$ is positive on $\tilde{M}$, for $k \geq k_0$.

Hint.

11.1.2. Exercise Let us denote by K_{M^n} the holomorphic line bundle $\Lambda^n T^* M$. Show that

$$K_{\tilde{M}} = \pi^* K_M \otimes (n-1)[E].$$

Hint .

11.1.3. Exercise Let E be a holomorphic vector bundle over a complex manifold M^n. Prove that $\mathcal{O}(E) = \Omega^n(E \otimes K_M^*)$.

Hint. Associate each $f = f(z) \cdot e(z) \in \mathcal{O}(E)$ to $f(x) \cdot dz_1 \wedge \cdots \wedge dz_n e(z) \cdot d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n \in \Omega^n(E \otimes K_M^*)$.

11.2 Proof of the Kodaira theorem

Let L be a positive line bundle on the compact complex manifold M. We shall show that there is a k_0 such that

11.3.1. Lemma *The restriction map*

$$H^0(M, \mathcal{O}(L^k)) \stackrel{r_{x,y}}{\to} L_x^k \oplus L_y^k$$

is surjective for all $x \neq y$, $k \leq k_0$.

11.3.2. Lemma *The differential map*

$$H^0(M, \mathcal{J}_x(L^k)) \stackrel{d_x}{\to} {T^*}'_x \oplus L_x^k$$

is surjective for all $x \in M$, $k \leq k_0$.

By the discussion in the previous section these Lemmas imply the Kodaira embedding theorem.

Proof of Lemma 11.3.1 Suppose that $\dim_{\mathbb{C}} M \geq 2$. Let $\tilde{M} \stackrel{\pi}{\to} M$ denote the blow-up of M at both x and y, $E_x = \pi^{-1}(x)$, and $E_y = \pi^{-1}(y)$ the exceptional divisors of the blow-up. Denote by E the divisor $E_x + E_y$ and $\tilde{L} = \pi^* L$. If $\dim_{\mathbb{C}} M = 1$, then we let $\tilde{M} = M$ and $\pi = Id$.

Now we consider the pullback map on sections

$$\pi^* : H^0(M, \mathcal{O}_M(L^k)) \to H^0(\tilde{M}, \mathcal{O}_{\tilde{M}}(\tilde{L}^k)).$$

For any global section $\tilde{\sigma}$ of $\tilde{L}^k$, the section of L^k given by σ over $M \setminus \{x, y\}$ extends by Hartogs' theorem to a global section $\sigma \in H^0(M, \mathcal{O}(L))$ and so we see that π^* is an isomorphism. Furthermore, by definition $\tilde{L}^k$ is trivial along E_x and E_x, hence

$$H^0(E, \mathcal{O}_E(\tilde{L}^k)) = L_x^k \oplus L_y^k.$$

Now let us consider the following commutative diagram

$$\begin{array}{ccc} H^0(\tilde{M}, \mathcal{O}_{\tilde{M}}(\tilde{L}^k)) & \xrightarrow{r_E} & H^0(E, \mathcal{O}_E(\tilde{L}^k)) \\ \uparrow & & \| \\ H^0(M, \mathcal{O}(L^k)) & \xrightarrow{r_{x,y}} & L^k_x \oplus L^k_y \end{array}$$

where r_E denotes the restriction to E. Thus to prove 11.3.1 it suffices to show that r_E is surjective.

Note that we have the following exact sheaf sequence (see 3.2.7.ii)

$$0 \to \mathcal{O}_{\tilde{M}}(\tilde{L}^k \otimes [-E]) \to \mathcal{O}_{\tilde{M}}(\tilde{L}^k) \stackrel{r_E}{\to} \mathcal{O}_E(\tilde{L}^k_{|E}) \to 0.$$

Choose k_1 such that $L^{k_1} \otimes K^*_M$ is positive on M. Using 10.1.4.ii and 11.1.1 we can choose k_2 such that $\tilde{L}^k \otimes [-E]^n$ is positive for $k \geq k_2$. By Lemma 11.1.3 we have

$$K_{\tilde{M}} = \tilde{K}_M \otimes [E]^{n-1}, \tag{11.3.3}$$

where $tildeK_M = \pi^* K_M$. Thus for $k \geq k_0 = k_1 + k_2$ we have

$$\mathcal{O}_{\tilde{M}}(\tilde{L}^k \otimes [-E]) = \Omega^n_{\tilde{M}}(\tilde{L}^k \otimes [-E] \otimes K^*_{\tilde{M}}) =$$

$$\stackrel{11.3.3}{=} \Omega^n_{\tilde{M}}((\tilde{L}^{k_1} \otimes \tilde{K}^*_M) \otimes (\tilde{L}^{k'} \otimes [-E]^n)$$

with $k' \geq k_2$.

By our choice $\tilde{L}^{k'} \otimes [E]^n$ is positive on $\tilde{M}$, and $L^{k_1} \otimes K^*_M$ is positive on M so $\tilde{L}^{k_1} \otimes \tilde{K}^*_M)$ is semipositive on $\tilde{M}$. Thus $(\tilde{L}^{k_1} \otimes \tilde{K}^*_M) \otimes (\tilde{L}^{k'} \otimes [-E]^n)$ is positive on $\tilde{M}$. By Kodaira vanishing theorem we get for $k \geq k_0$

$$H^1(\tilde{M}, \mathcal{O}_{\tilde{M}}(\tilde{L}^k \otimes [-E]) = H^1(\tilde{M}, \Omega^n_{\tilde{M}}(\tilde{L}^{k_1} \otimes \tilde{K}^*_M) \otimes (\tilde{L}^{k'} \otimes [-E]^n) = 0$$

Hence the map $r_E H^0(\tilde{M}, \mathcal{O}_{\tilde{M}}(\tilde{L}^k)) \to H^0(E, \mathcal{O}_E(\pi^* L^k)$ is surjective for $k \geq k_0$. Since M is compact we can choose k_0 independent of choice of x. □

Proof of Lemma 11.3.2 We use the same notations as in the proof of the previous Lemma except we do not take any y so $E = E_x$. As before

$$\pi^* : H^0(M, \mathcal{O}_M(L^k)) \to H^0(\tilde{M}, \mathcal{O}_{\tilde{M}}(\tilde{L}^k))$$

is an isomorphism. Further, if $\sigma \in H^0(M, \mathcal{O}_M(L^k))$ then $\sigma(x) = 0$ if and only if $\tilde{\sigma} = \pi^* \sigma$ vanishes on E, thus π^* restricts to give an isomorphism

$$\pi^* : H^0(M, \mathcal{J}_x(L^k) \to H^o(\tilde{M}, \mathcal{O}_{\tilde{M}}(\tilde{L}^k \otimes [-E]).$$

As in (??) we can identify

$$H^0(E, \mathcal{O}_E(\tilde{L}^k \otimes [-E]) = L_x^k \otimes H^0(E, \mathcal{O}_E([-E])) \cong L_x^k \otimes T_x^{*\prime}.$$

Using the following commutative diagram

$$\begin{array}{ccc} H^0(\tilde{M}, \mathcal{O}_{\tilde{M}}(\tilde{L}^k \otimes [-E])) & \xrightarrow{r_E} & H^0(E, \mathcal{O}_E(\tilde{L}^k \otimes [-E])) \\ \pi^* \uparrow \cong & & \| \\ H^0(M, \mathcal{J}_x(L^k)) & \xrightarrow{d_x} & {T^*}'_x \oplus L_y^k \end{array}$$

It suffices to show that r_E is surjective for $k >> 0$. Consider the following exact sequence (see 3.2.7.ii)

$$0 \to \mathcal{O}_{\tilde{M}}(\tilde{L}^k \otimes [-E]) \to \mathcal{O}_{\tilde{M}}(\tilde{L}^k \otimes [-E]) \overset{r_E}{\to} \mathcal{O}_E(\tilde{L}^k \otimes [-E]) \to 0.$$

Again choose k_1 such that $L^{k_1} \otimes K_M^*$ is positive on M and k_2 such that $\tilde{L}^{k'} \otimes [-E]^{n+1}$ is positive on $\tilde{M}$ for $k' \geq k_2$. For $k \geq k_0 = k_1 + k_2$ we have

$$\mathcal{O}_{\tilde{M}}(\tilde{L}^k \otimes [-E]^2) = \Omega_{\tilde{M}}^n((\tilde{L}^{k_1} \oplus (\tilde{L}^{k'} \otimes [-E]^{n+1}))$$

with $k' \geq k_2$. It follows by Kodaira vanishing theorem that

$$H^1(\tilde{M}, \mathcal{O}_{\tilde{M}}(\tilde{L}^k \otimes [-E]^2)) = 0$$

for $k \geq k_0$. Hence r_E is surjective. Since M is compact we can choose k_0 independent of x. □

12 The Hodge conjecture

The Hodge conjecture characterizes algebraic cycles in a projective algebraic manifold M via the Hodge structure on the cohomology group of M.

12.1 Algebraic cycles

Let M^{2n} be a projective algebraic manifold. An **algebraic set** $Z \subset M^n$ is locally given as the zero set of holomorphic functions on M^{2n}. If Z is compact we can find a neighborhood V of Z in M^{2n} and a finitely generated ideal $I(Z)$ of holomorphic functions on V such that Z is the zero set of $I(Z)$. Furthermore a point $z \in Z$ is called **regular** or **smooth**, if the rank of the set $\{df(z), | \, f \in I(Z)\}$ is maximal.

It is easy to see that the set of regular points $Z_{smooth} \subset Z$ is a smooth submanifold in M^n. Its dimension $d(Z) := \dim Z_{smooth}$ is called the dimension of Z. We also write Z^k if $k = \dim Z$.

12.1.1. Proposition *Algebraic set $Z^k \subset M^n$ defines an element $[Z] \in H^{2n-2k}(M, \mathbb{Q})$ as follows*

$$[Z]^{2n-2k}([Y] \in H_{2n-2k}(M, \mathbb{Q})) = [Z_{smooth} \cap [Y]] \in \mathbb{Q}.$$

Proof Since the set of singularity is of codimension 2, this intersection number does not depend on the choice of representative of $[Y]$. □

12.1.2. Remark In fact we can define the image of $[Z] \in H^{2n-2k}(M, \mathbb{Z}) \subset H^{2n-2k}(M, \mathbb{Z})$ by using the fact that $H^l(M^n - S^{n-r}, \mathbb{Z}) = H^l(M, Z)$ if $l \leq 2r$ and using the Thom isomorphism $H^k(X^n, X^n \setminus Y^{n-k}, \mathbb{Z}) = H^0(Y, \mathbb{Z})$, (see [Voisin] for more detail.).

An **algebraic cycle** of dimension $2k$ in M^n is a linear combination $Z = \sum n_i Z_i$ where Z_i are closed algebraic sets in M^n.

12.1.3. Exercise Let M be a compact Kähler manifold and $Z6k$ an algebraic cycle. Show that the image $[Z^k] \in H^{2n-2k}(M, \mathbb{Q}) \subset H^{2n-2k}(M, \mathbb{C})$ is an element of degree $(n-k, n-k)$.

12.2 The Hodge conjecture

A class $[Y] \in H^{2n-2k}(M, \mathbb{Q})$ is called a **Hodge class**, if $[Y] \in H^{n-k,n-k}(M, \mathbb{C})$. The Hodge conjecture states that any Hodge class is a multiple (with a coefficient in $\mathbb{Q}$) of an algebraic class.

12.2.2. Exercise. Prove the Hodge conjecture for complex torus CP^n.