APPLICATION OF INTEGRAL GEOMETRY TO MINIMAL SURFACES

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1. Introduction

The theory of higher dimensional minimal surfaces, especially its main branch—the Plateau problem, has been intensively developed since the sixties when E. R. Reifenberg, H. Federer, W. H. Fleming, E. De Giorgi and F. Almgren proved the existence and almost regularity theorems for solutions of the higher dimensional Plateau problem (or simply speaking, globally minimal surfaces) in different contexts of geometric measure theory. After that, the other part of the theory, namely, construction, classification and study of geometry of globally minimal surfaces has been developed rapidly. The first non-trivial example of globally minimal surfaces was obtained by H. Federer by showing that every Kähler submanifold is globally (homologically) minimal in its ambient Kähler manifold [7]. His method of employing exterior powers of the Kähler form in Kähler manifolds has been generalized for other Riemannian manifolds in the works of M. Berger, H. B. Lawson, Dao Trong Thi, R. Harvey and H. B. Lawson ([1], [17], [3], [13]). Now, this method is called the calibration method and it has various applications in the study of geometry of globally minimal surfaces as well as of (locally) minimal surfaces ([5], [6], [11], [18], [19],...). Other interesting examples of globally minimal surfaces were obtained by A. T. Fomenko [9], [22] by using an estimate from below for the volume of globally minimal surfaces in Riemannian manifolds. His idea came from Griffiths' idea of using exhaustion functions on algebraic manifolds in the Nevanlinna theory. His method allows us to construct homological

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minimal submanifolds when the coefficient group of homologies may be finite ($\mathbb{Z}_p$) or infinite ($\mathbb{Z}$). Note that the calibration method works only for homology groups with coefficients in $\mathbb{R}$. But Fomenko's method which depends on an estimate involving only the injective radius, Riemannian curvature of ambient manifolds and dimension of submanifolds, cannot give us so many examples of globally minimal surfaces. To our knowledge, up to now, all non-trivial examples of globally minimal surfaces are obtained by using the above-mentioned methods with the exception of some globally minimal hypersurfaces with large symmetry groups where one can reduce the problem of higher dimension to dimension 2 which can be completely analysed. This reduction method was invented by W-Y. Hsiang and H. B. Lawson [14] and [16].

This paper is an attempt to fill the gap between the calibration method and the Fomenko method. This new method may also be called an analog of the calibration method for discrete coefficients of homology groups (of Riemannian manifolds). The idea is simple; it also comes from complex geometry. Let us recall the Grofton-type formula (which originates in probability theory [23]).

Theorem. [2, p. 146] *Let $f : M \to \mathbb{C}P^n$ be a compact holomorphic curve with or without boundary. Then*

$$\int_{\mathbb{C}P^n} \#(f(M) \cap \gamma)\, d\gamma = \mathrm{Area}(M), \tag{1.1}$$

where γ is a (complex) hyperplane of $\mathbb{C}P^n$, and the space of these hyperplanes is identified with $\mathbb{C}P^n$ equipped with the invariant measure, and $\#(X)$ denotes the number of points in X.

A more detailed analysis shows that if we replace a holomorphic curve M by any (real) two-dimensional surface M', then the equality (1.1) becomes an inequality, where the right-hand side is greater than the left one (see Proposition 2.11 and Proposition 3.10 which we call *Integral Wirtinger Inequality*). So, this strengthened Crofton-type formula gives us a new proof of homological minimality of $\mathbb{C}P^1$, and moreover, an estimate on the measure of all (complex) hyperplanes meeting a fixed holomorphic curve k times (see Equidistribution Theorem [2, p. 146] and Theorem 4.1). In fact, some authors have used similar integral formulae in order to estimate the volume of 2-dimensional analytical sets in $\mathbb{C}^n$, but their formulae concern only the simplest case of real dimension 1 (cf. [15] and references in that paper). Our idea is a natural generalization of the Crofton-type formula. Namely, we want to estimate the volume of a submanifold $N \subset M$ by its intersection number $\#(N \cap N_\lambda^*)$, where N_λ^* is a family of submanifolds in M. Since the algebraic intersection number is a homology invariant we hope to get an estimate from below for the volume of a submanifold realizing a given cycle. The use of intersection number as a homology invariant explains the analogy between this method and the calibration method, which essentially employs another homology invariant—the Stokes formula. But in view of the Federer stability theorem [8] the relation between these methods proves to be more intimate; in many cases, the effectiveness of one method leads to the effectiveness of the other one (see Sec. 4). Applying this intersection method we obtain some old and new examples of globally minimal submanifolds in Grassmannian spaces. In a few cases this gives us a classification theorem for globally minimal submanifolds in a certain class (see Sec. 3 and Sec. 4) and their new properties such as equidistribution in measure of globally minimal surfaces. Other applications of integral geometry to minimal surfaces will appear in our next paper. The present note is based on a revised form of the author's preprint [20].

2. General Construction and Examples

Let us begin with a simple example.

Example 2.1. Let M^m be a Riemannian manifold and TM its tangent bundle. Let the Riemannian metric on M be naturally lifted on TM. Then M^m realizes a nontrivial cycle in the homology group $H_m(TM, \mathbf{Z}_2)$ and moreover it has the minimal volume in its homology class $[M]$. In fact, if M' is another submanifold in TM and realizing the cycle $[M] \in H_m(TM, \mathbf{Z}_2)$, then M' must meet every fiber π_x, $x \in M$. Consequently, the projection $\pi : M' \to M$ is surjective. It is easy to see that the projection π decreases the volume element (in any dimension not exceeding $\dim M = m$). Hence we get the assertion. This example is interesting because if M is not orientable then $H_m(TM, \mathbf{Z}) = 0$ and the classical calibration method is not applicable!

Now let us give a *general construction*, which generally does not depend on fibrations (such simple fibrations as the above example occur very rarely). Let us consider a Riemannian manifold M^m. Suppose we have a family $(M)^*$ of n-dimensional submanifolds $N_y \subset M$, $y \in (M)^*$. Suppose further that $(M)^*$ is a smooth manifold with a volume element $\mu_y = \mathrm{vol}_{m^*}$, where m^* is the dimension of $(M)^*$. For every $X \subset M$ denote by $S_X \subset (M)^*$ the set of all submanifolds N_y passing through the set X. Now we fix a point $x \in M$ and a $(m - n)$-dimensional subspace $V^{m-n} \subset T_xM$. Denote by $B(x, V^{m-n}, r)$ the geodesic ball of radius r in M with its center at x and its tangent space at x equal to V^{m-n}. Let us consider the following limit

$$cd(x, V^{m-n}) = \lim_{r \to 0} \frac{\mathrm{vol}_{m^*}(S_{B(x, V^{m-n}, r)})}{\mathrm{vol}(B(x, V^{m-n}, r))}. \tag{2.1}$$

Suppose for every $x \in M$ the set S_x is a compact smooth submanifold in $(M)^*$. Then the limit in (2.1) exists. To compute this limit we fix a submanifold S_x and a small normal neighborhood of S_x in $(M)^*$. Obviously, there exists a fiber bundle F over S_x in this neighborhood such that S_x is embedded into it as a zero section of generic position. For instance, in order to construct F we can use the exponential map from the normal bundle over S_x to $(M)^*$. For every $y \in S_x$, with the help of F, we can construct a map F_y from a neighborhood of $x \in M$ to the fixed neighborhood of S_x as follows: $M \ni x' \mapsto S_{x'} \cap p^{-1}y$, where $p^{-1}y$ is the fiber over $y \in S_x$. Since S_x meets fibers transversally, the map F_y is well defined in a sufficiently small neighborhood of x, that is, $p^{-1}y$

meets $S_{x'}$ only at one point. Then we have

$$cd(x, V^{m-n}) = \int_{S_x} \mathrm{vol}(\overline{T_y S_x} \wedge dF_y(\overline{V^{m-n}})).$$

Here for any linear subspace L we denote by $\bar{L}$ the unit polyvector associated with L. We call the limit in (2.1) *a deformation coefficient* $cd(x, V^{m-n})$. Put

$$\sigma(M)^*_{m-n} = \max\{cd(x, V^{m-n}) | x \in M, V^{m-n} \subset T_x M\}.$$

Suppose that $\sigma(M)^*_{m-n} > 0$. The following theorem is related to integral geometry on Riemannian manifolds.

Theorem 2.1. *Let W be a compact $(m-n)$-dimensional submanifold in M. Then its volume can be estimated from below*:

$$\mathrm{vol}(W) \geq (\sigma(M)^*_{m-n})^{-1} \int_{(M)^*} \#(W \cap N_y)\mu_y. \tag{2.2}$$

Proof. It is easy to find a finite triangulation W_i^ε of W by simplices of diameter less than ε, that is, $W = \bigcup_i W_i^\varepsilon$ and $\mathrm{vol}_{m-n}(W_i^\varepsilon \cap W_j^\varepsilon) = 0$ if $i \neq j$, such that for every i the number of connected components of the intersection of W_i^ε with any submanifold N_y is at most one. So we have:

$$\mathrm{vol}(W) = \sum_i \mathrm{vol}(W_i^\varepsilon), \tag{2.3}$$

$$\int_{(M)^*} \#(W \cap N_y)\, dy = \sum_i \int_{(M)^*} \#(W_i^\varepsilon \cap N_y)\, dy. \tag{2.4}$$

With the help of (2.3) and (2.4) Theorem 2.1 can be proved if we show (2.2) for W_i^ε instead of W. Hence, in view of our assumption it suffices to prove:

$$\mathrm{vol}(W^\varepsilon) \geq (\sigma(M)^*_{m-n})^{-1} \int_{S_{W^\varepsilon}} \mu_y. \tag{2.2.ε}$$

Letting $\varepsilon \to 0$ we get the infinitesimal version of (2.2.ε):

$$\lim_{\varepsilon \to 0} \frac{\mathrm{vol}(W^\varepsilon)}{\mathrm{vol}(S_{W^\varepsilon})} = cd(x, T_x W^\varepsilon)^{-1} \geq (\sigma(M)^*_{m-n})^{-1}. \tag{2.2.0}$$

Obviously, (2.2.0) follows from (2.1). By integrating we obtain (2.2.ε). This completes the proof.

In Example 2.1, if we exhaust TM^m by compact bundles TM_R of tangent vectors of length R over M, then we can also get the deformation coefficient $\sigma(TM_R)^*_m = 1$. Here the set $(TM_R)^*$ consisting of m-dimensional tangent balls of radius R is diffeomorphic to M.

Corollary 2.2. *Lower bound of the volume of nontrivial cycles in Riemannian manifolds. Suppose $N \subset M$ is a k-dimensional submanifold realizing a nontrivial cycle $[N] \in H_k(M^{n+k}, G)$, $G = \mathbf{Z}$ or $\mathbf{Z}_2$. Let $(M)^*$ be a family of n-dimensional submanifolds N^*_χ realizing a nontrivial cycle $[N^*] \in H_n(M^{n+k}, G)$. Let χ be the (algebraic) intersection number of $[N]$ and $[N^*]$. Then we get:*

$$\mathrm{vol}(N) \geq \chi \cdot (\sigma(M)^*_k)^{-1} \cdot \mathrm{vol}(M)^*.$$

We note that Theorem 2.1 is still valid for a compact k-dimensional set W almost everywhere smooth except singularities of codimension 1. On the other hand, it is well-known that homological volume-minimizing cycles are such sets [7]. So Corollary 2.2 yields the following criterion for global minimality.

Corollary 2.3. *Let $N \subset M$ be a k-cycle almost everywhere smooth except singularities of codimension 1. Suppose that the inequality in Corollary 2.2 is an equality for N. Then N is a globally minimal cycle.*

Example 2.2. Consider the group U_n equipped with the standard bi-invariant metric, that is, on the tangent space $T_eU_n = \mathfrak{u}_n$, this metric is defined as follows: $\langle \xi, \eta \rangle = -\mathrm{tr}(\xi\eta)$. Applying Corollary 2.3 we will show that the subgroup S^1 of all diagonal scalar elements is a homological minimal submanifold. Indeed, U_n is a fibred space over $S^1 : g \mapsto \det(g)$, whose fibers are congruent with the subgroup SU_n. First, we note that SU_n meets S^1 at exactly n points $x_k = \mathrm{diag}\left(\exp\dfrac{2ki\pi}{n}\right)$; $k = 0, \ldots, n-1$. Therefore, any fibre $a \cdot SU_n$, $a \in S^1$, meets S^1 exactly in n points $a \cdot x_k$. Clearly, at every intersection point $y = a \cdot x_k$ the tangent spaces T_yS^1 and $T_y(a \cdot SU_n)$ are perpendicular. Further, we observe that the algebraic intersection number between S^1 and SU_n equals n since S^1 is homologous to n times of the circle U_1 which generates the homology group $H_1(U_n, \mathbf{Z})$. Now, it is easy to see that if we set $(M)^*$ to be the space of cosets of the subgroup SU_n in U_n, then $\sigma(M)^*_1 = 1$, and by Corollary 2.3, S^1 has the minimal length in its homology class of $H_1(U_n, \mathbf{Z})$).

In most of our applications we are interested in cycles of compact homogeneous Riemannian spaces. We shall denote $(\cdot)$ the group multiplication or the action of a group on homogeneous spaces. Sometimes we omit this notation $(\cdot)$ if no confusion arises. Let $M = G/H$, where H is a compact subgroup in a compact group G. Let K be another compact subgroup of G. Denote L the intersection of H and K. We consider the space $(M)^*$ of all submanifolds $g \cdot K/L \subset G/H$ which are obtained from K/L by the left shift g, $g \in G$. Obviously, G acts transitively on $(M)^*$. Let us denote $I(K)$ its isotropy group at the point $e \cdot K/L \in (M)^*$.

Lemma 2.4. *The isotropy group $I(K)$ coincides with the subgroup $K\cdot(H\cap N(K))$, where $N(K)$ is the normalizer of the subgroup K in G.*

Proof. Clearly, the subgroup $I(K)$ consists of all elements $g\in G$ such that $g\cdot K\subset K\cdot H$. So we have

$$I(K)=\bigcap_{k\in K}(K\cdot H\cdot k)=\bigcup_{h\in H}\left\{\bigcap_{k\in K}(K\cdot h\cdot k)\right\}.$$

Let $h\in H$ be an element such that the intersection $\bigcap_{k\in K}(K\cdot h\cdot k)$ is not empty. We easily verify that the last condition is equivalent to h being an element of the normalizer $N(K)$. Hence the lemma follows immediately.

The condition under which submanifold $y\cdot K/L\subset M$ contains a point $x=(g\cdot H)/H\in M$ is the relation $y\in g\cdot H\cdot K$. So we have the following lemma.

Lemma 2.5. *Let $x=\{gH\}\in M=G/H$. Then the set $S_x\subset(M)^*=G/I(K)$ is the submanifold gH/L', where $L'=H\cap I(K)$.*

Our purpose now is to compute the deformation coefficient $cd(x,V)$ for $x\in M$. Without loss of generality we can assume that $x=\{eH\}$, and then $V\subset T_{\{eH\}}M$. Denote by $\mathfrak{g}$ the Lie algebra of G. Let us consider the map $\pi_*:\mathfrak{g}\to T_{\{eH\}}M$ which is induced by the natural projection $\pi:G\to G/H=M$. Let $\mathfrak{h}^{\mathfrak{g}}$ be the orthogonal complement (with respect to some Ad_G-invariant metric on $\mathfrak{g}$) to the subalgebra $\mathfrak{h}$ in $\mathfrak{g}$. Then we identify $T_{\{eH\}}M$ with $\mathfrak{h}^{\mathfrak{g}}$ by the map π_*. This isomorphism π_* is an isomorphism of Ad_H-modules. From now on we consider the metric on $\mathfrak{h}^{\mathfrak{g}}$ which is induced by the isomorphism π_*^{-1}.

Proposition 2.6. *Let $k=\operatorname{codim}(K/L)$. Then the k-dimensional deformation coefficient $cd(\{eH\},V^k)$ depends only on the H-action orbit passing through the k-dimensional subspace V^k on the space $\bigwedge^k(\mathfrak{h}^{\mathfrak{g}})$.*

Proof. Let us denote by exp the exponential map from Lie algebra onto Lie group. We note that we can replace the family of exhausting geodesic balls $B(\{eH\},V,r)$ and the corresponding set $S_{B(\{eH\},V,r)}$ in the formula (2.1) by any family of exhausting submanifolds $B'(\{eH\},V,r)$ and $S_{B'(\{eH\},V,r)}$ such that: $T_{\{eH\}}B'(\{eH\},V,r)=V$, $B'(\{eH\},V,r_1)\subset B'(\{eH\},V,r_2)$ if $r_1\le r_2$, and $B'(\{eH\},V,r)\to\{eH\}$ when $r\to 0$. We choose $B'(\{eH\},V,r)=\{\exp V(r)\cdot H\}/H$, where $V(r)$ denotes the ball of radius r in the tangent space $V\subset\mathfrak{h}^{\mathfrak{g}}\subset\mathfrak{g}$. Hence, according to Lemma 2.5 we get $S_{B'(\{eH,V,r)}=\exp V(r)\cdot H/L'$. Therefore we obtain

$$cd(\{eH\},V)=\lim_{r\to 0}\frac{\operatorname{vol}(\exp V(r)\cdot H/L')}{\operatorname{vol}(\exp V(r)\cdot e/H)}. \tag{2.5}$$

We choose an orthonormal basis of vectors $\{v_i\}$ in V. Fix a point $x=\{\tilde{x}L'\}\in H/L'\subset G/I(K)$, where $\tilde{x}\in H\subset G$. The tangent space to $\exp V(r)\cdot H/L'$ at the point x is the sum of the tangent spaces $T_x(H/L')$ and $T_x(\exp V(r)\cdot x)$. Consider the map

$$\rho : V(r) \to \exp V(r)\cdot\{\tilde{x}L'\}; \qquad v \mapsto \exp v\cdot\{\tilde{x}L\}'.$$

Its differential $d\rho$ sends the vector v_i to the projection of the vector

$$\frac{d}{dt}\exp tv_i\cdot\tilde{x}|_{t=0} \in T_xG$$

on the tangent space $T_x(G/I(K))$ since $G/I(K)$ is the quotient space of the right $I(K)$-action on G. Denote $\hat{v}_i(x)$ the resulting vector $d\rho(v_i) \in T_x(G/I(K))$. Then we have $T_x(\exp V(r)\cdot x) = \operatorname{span}\{\hat{v}_i, i = 1, \ldots, n\}$. So (2.5) can be rewritten as follows:

$$cd(\{eH\}, V) = \int_{H/L'} \operatorname{vol}(\overline{T_x(H/L')} \wedge \hat{V}_x)\mu_x, \tag{2.6}$$

where $\overline{T_x(H/L')}$ denotes the unit polyvector associated with $T_x(H/L')$, and $\hat{V}_x = \hat{v}_1(x) \wedge \cdots \wedge \hat{v}_k(x)$. First, we note that $\operatorname{vol}(\overline{T_x(H/L')} \wedge \hat{V}_x) = |\langle\hat{V}_x, \overline{W_x}\rangle|$, where the associated subspace W_x is the orthogonal complement to $T_x(H/L')$ in $T_x(G/I(K))$. Secondly, we observe that for each $h \in H$ we have

$$\widehat{Ad_h v_i}(x) = h_*\hat{v}_i(h^{-1}\cdot x).$$

Therefore we obtain

$$cd(\{eH\}, Ad_h V) = \int_{H/L'} |\langle h\cdot\hat{V}_{h^{-1}\cdot x}, \overline{W_x}\rangle|\,\mu_x. \tag{2.7}$$

Now Proposition 2.6 immediately follows from (2.6), (2.7) and the G-invariance of the metric on $G/I(K)$.

Let us consider the case when the invariant metrics on G/H and $G/I(K)$ are canonical (i.e. they are obtained from a bi-invariant metric on G factorized by the action of its subgroups H and $I(K)$ respectively.) In this case the formula for $cd(\{eH\}, V)$ has a very simple expression. Denote by $\mathfrak{h}$ and $\mathfrak{k}$ the Lie algebras of the subgroups H and K respectively. Let W be the orthogonal complement to the span of these subalgebras in $\mathfrak{g}$, that is,

$$\mathfrak{g} = W \oplus (\mathfrak{h} + \mathfrak{k}).$$

Then we obtain the following lemma.

Lemma 2.7. *Under the above assumptions we have*

$$cd(\{eH\}, V) = \int_{H/L'} |\langle\overline{V}, Ad_{\tilde{x}}(\overline{W})\rangle|\,dx. \tag{2.8}$$

Proof. Denote by pr($\mathfrak{h}$) the orthogonal projection of $\mathfrak{h}$ onto the orthogonal complement to $\mathfrak{k}$ in ($\mathfrak{h} + \mathfrak{k}$). We have the following orthogonal decomposition

$$T_{\{eI(K)\}}G/I(K) = W \oplus \operatorname{span}\{z \in \operatorname{pr}(\mathfrak{h}) | \langle z, \mathfrak{h} \cap \mathfrak{n}(\mathfrak{k})\rangle = 0\},$$

and

$$T_{\{eL'\}}H/L' = \operatorname{span}\{z \in \operatorname{pr}(\mathfrak{h}) | \langle z, \mathfrak{h} \cap \mathfrak{n}(\mathfrak{k})\rangle = 0\}.$$

Therefore, the normal fiber $W_{\{eL'\}}$ coincides with W. Since $\tilde{x} \in H$ the shift $L_{\tilde{x}}$ preserves the normal bundle of H/L' in $G/I(K)$. Hence, $W_x = \tilde{x}_* W$.

Our next aim is to compute $\hat{v}_i(x)$. Let us choose an orthonormal basis $f_1, \ldots, f_N$ of the space $lI(K)^G = T_{\{eI(K)\}}G/I(K)$. The shift $L_{\tilde{x}}: G/I(K) \to G/I(K)$, $\{gI(K)\} \mapsto \tilde{x} \cdot \{gI(K)\}$, sends the vector f_i to the vector $f_i^{\tilde{x}}(x)$. Obviously, $f_i^{\tilde{x}}(x)$ is an orthogonal basis of the tangent space $T_x(G/I(K))$. Straightforward calculation shows that

$$\langle \hat{v}_i(x), f_j^{\tilde{x}}(x)\rangle = \langle v_i, Ad_{\tilde{x}} f_j\rangle,$$

where $\langle \ , \ \rangle$ in the right-hand side of the above formula denotes the restriction of the bi-invariant metric on G to the algebra $\mathfrak{g}$.

Now, taking into account (2.7) (with $h = e$) we immediately get the formula (2.8). Clearly, the space W is invariant under the action $Ad_{L'}$. Therefore, the integrand on the right-hand side of (2.8) depends only on x. This completes the proof of Lemma 2.7.

Example 2.3. Let $M = S^n = SO_{n+1}/SO_n$, and $(M)^* = SO_{n+1}/S(O_{k+1} \times O_{n-k})$ be the set of great (totally geodesic) k-dimensional spheres in S^n. Here $H = SO_n$ acts on the Grassmannian $G_{n-k}(T_e M) \cong SO_n/S(O_k \times O_{n-k})$ transitively. This means that $cd(x, V)$ is a constant ζ_{n-k}. Taking into account (2.2.ε), (2.2.0) (which become equalities in this case) and (2.3), (2.4) we get:

Proposition 2.8. [23] *Let N^{n-k} be a submanifold in S^n. Then its volume can be computed from the following formula*:

$$\operatorname{vol}(N^{n-k}) = \zeta_{n-k} \cdot \int_{SO_{n+1}/S(O_{k+1} \times O_{n-k})} \#(N^{n-k} \cap S^k(x))\mu_x,$$

where $\zeta_{n-k} = 1/2 \operatorname{vol}(S^{n-k}) \cdot \operatorname{vol}(SO_{n+1}/S(O_{k+1} \times O_{n-k}))^{-1}$.

The same formula holds for a submanifold $N^{n-k} \subset \mathbf{R}P^n$, but we should replace S^k by $\mathbf{R}P^k$. Further, we note that any projective space $\mathbf{R}P^k$ meets almost all projective spaces of complementary dimension at one point (cf. Proposition 3.6). Hence in view of Corollary 2.3 we obtain:

Proposition 2.9. *The projective space $\mathbf{R}P^k$ has the minimal volume in its homology class $[\mathbf{R}P^k] \in H_k(\mathbf{R}P^n, \mathbf{Z}_2) = \mathbf{Z}_2$.*

This proposition was obtained by Fomenko [9] using a different method of geodesic defects.

Example 2.4. Let $M = \mathbf{C}P^n = U_{n+1}/(U_n \times U_1)$. Then $T_e\mathbf{C}P^n = \mathbf{C}^n = \mathbf{R}^{2n}$, and $H = U_n \times U_1$ does not act on $G_k(\mathbf{R}^{2n})$ transitively. But H acts on the complex Grassmannian $G_k(\mathbf{C}^n)$ transitively, and H also acts on the Lagrangian Grassmannian $GL(\mathbf{C}^n) = U_n/O_n$ transitively. Considering the family $(M)_1^* = U_{n+1}/(U_{n-k+1} \times U_k)$ of all canonically embedded complex projective spaces of dimension $(n-k)$ in M, and the family $(M)_2^* = U_{n+1}/O_{n+1}$ of all canonically embedded real projective spaces of dimension n in M, we get:

Proposition 2.10. a) *Crofton type formula. Let N^{2k} be a complex manifold in $\mathbf{C}P^n$. Then its volume can be computed from the following formula:*

$$\operatorname{vol}(N^{2k}) = \zeta_k^C \cdot \int_{U_{n+1}/(U_{n-k+1}\times U_k)} \#(N^{2k} \cap \mathbf{C}P^{n-k}(x))\mu_x,$$

where the constant ζ_k^C does not depend on N^{2k}.

b) *Let N^n be a Lagrangian manifold in $\mathbf{C}P^n$. Then its volume can be computed from the following formula:*

$$\operatorname{vol}(N^n) = \zeta_n^L \cdot \int_{U_{n+1}/O_{n+1}} \#(N^n \cap \mathbf{R}P^n(x))\mu_x,$$

where the constant ζ_n^L does not depend on N^n and U_{n+1}/O_{n+1} is the space of all real projective spaces of dimension n in $\mathbf{C}P^n$.

When $k = 1$ we have the following inequality.

Proposition 2.11. *Integral Wirtinger Inequality. Let N^2 be a real surface in $\mathbf{C}P^n$. Then the following inequality holds*

$$\int_{\mathbf{C}P^n} \#(N^2 \cap \gamma)\, d\gamma \le \operatorname{Area}(N^2),$$

where γ is a (complex) hyperplane of $\mathbf{C}P^n$, and the space of these hyperplanes is identified with $\mathbf{C}P^n$ equipped with the invariant measure. Moreover, the inequality becomes an equality if and only if N^2 is a complex curve.

Proof. We consider the family $(\mathbf{C}P^n)^*$ of complex hyperplanes in $\mathbf{C}P^n$. According to Theorem 2.1 it suffices to show that the associated deformation coefficient $cd(x, V^2)$ attains its maximal value if and only if V^2 is a complex line. Using the above notations we have $H = U_n \times U_1$, $K = U_1 \times U_n$, $L = L' = U_1 \times U_{n-1} \times U_1$, and then $H/L' = \mathbf{C}P^{n-1}$. With the help of (2.8) we get

$$cd(\{eH\}, V^2) = \int_{\mathbf{C}P^{n-1}} |\langle \overline{V^2}, Ad_{\tilde{x}}(\overline{W})\rangle|\, dx.$$

Let $L'' = \{1\} \times U_{n-1} \times U_1$. Then $S^{2n-1} = H/L''$ is also considered as the unit sphere in the orthogonal complement $(\mathfrak{l}'')^H$ to $\mathfrak{l}''$ in $\mathfrak{h}$. We consider the Hopf fibration $S^{2n-1} \to \mathbf{C}P^{n-1}$. It is well-known that the Hopf fibres are the U_1 orbits on S^{2n-1}, and the invariant Riemannian metric on $\mathbf{C}P^n$ is obtained from the one on S^{2n-1} factorized by the U_1 action. Therefore we get

$$cd(\{eH\}, V^2) = \mathrm{vol}(U_1)^{-1} \int_{S^{2n-1}} |\langle \overline{V^2}, Ad_{\tilde{x}}(\overline{W}) \rangle| \, dx.$$

Now we apply the normal form theorem of Harvey and Lawson to V^2.

Lemma 2.12. [13, Lemma 6.13] *There exists a unitary basis v_i, Jv_i in $\mathbf{C}^n = T_{\{eH\}}\mathbf{C}P^n$ such that $V^2 = \cos\tau \cdot v_1 \wedge Jv_1 + \sin\tau \cdot v_1 \wedge v_2$.*

Taking into account the equality $Ad_{\tilde{x}}(\overline{W}) = x \wedge Jx$ for $x \in S^{2n-1} \subset (\mathfrak{l}'')^{\mathfrak{h}}$ we obtain

$$cd(\{eH\}, V^2) = \mathrm{vol}(U_1)^{-1} \int_{S^{2n-1}} |\langle \cos\tau \cdot v_1 \wedge Jv_1 + \sin\tau \cdot v_1 \wedge v_2, x \wedge Jx \rangle| \, dx. \quad (2.9)$$

Let $a_i(x) = \langle x, v_i \rangle$ and $b_i(x) = \langle x, Jv_i \rangle$. From (2.9) we get

$$cd(\{eH\}, V^2) = \mathrm{vol}(U_1)^{-1} \int_{S^{2n-1}} |(a_1^2(x) + b_1^2(x))\cos\tau + (-a_1(x)b_2(x) + a_2(x)b_1(x))\sin\tau| \, dx. \quad (2.10)$$

Since the integrand in (2.10) is homogeneous of degree 2 on $\mathbf{R}^{2n}$, we observe that our calculation can be reduced to the one on sphere S^3. Namely, there exists a constant χ_n such that

$$cd(\{eH\}, V^2) = \chi_n \int_{S^3} |(a_1^2(x) + b_1^2(x))\cos\tau + \sin\tau(-a_1(x)b_2(x) + a_2(x)b_1(x))\sin\tau| \, dx.$$

Hence we obtain

$$cd(\{eH\}, V^2) \le \chi_n \left(\int_{S^3} |a_1^2(x)\cos\tau - a_1(x)b_2(x)\sin\tau| + |b_1^2(x)\cos\tau + a_2(x)b_1(x)\sin\tau| \, dx \right) \quad (2.11)$$

We choose the torus coordinates on S^3. Namely we put

$$a_1(x) = \sin\beta(x)\cos\alpha(x), \qquad a_2(x) = \sin\beta(x)\sin\alpha(x),$$

$$b_1(x) = \cos\beta(x)\cos\gamma(x), \qquad b_2(x) = \cos\beta(x)\sin\gamma(x),$$

where $\beta \in [0, \pi]$, $\alpha \in [0, 2\pi]$, $\gamma \in [0, 2\pi]$. So, the action of the group $S^1 \times S^1$ on S^3 given by: $\alpha(x) \to \alpha(x) + \theta_1$, $\gamma(x) \to \gamma(x) + \theta_2$ preserves the invariant measure on S^3. In these coordinates (2.11) becomes the following inequality

$$cd(\{eH\}, V^2) \leq \chi_n \left(\int_{S^3} |\sin^2 \beta \cos \alpha \cos(\alpha + \tau)| \mu(\alpha, \beta, \gamma) \right.$$
$$\left. + \int_{S^3} |\cos^2 \beta \cos \gamma \cos(\gamma - \tau)| \mu(\alpha, \beta, \gamma) \right), \qquad (2.11')$$

where μ is the invariant measure on S^3. Applying the Schwarz inequality for integrals to the right-hand side of (2.11′) we get

$$cd(\{eH\}, V^2) \leq \chi_n \left\{ \left(\int_{S^3} |\sin^2 \beta \cos^2 \alpha| \mu \right)^{1/2} \cdot \left(\int_{S^3} |\sin^2 \beta \cos^2(\alpha + \tau)| \mu \right)^{1/2} \right.$$
$$\left. + \left(\int_{S^3} |\cos^2 \beta \cos^2 \gamma| \mu \right)^{1/2} \cdot \left(\int_{S^3} |\cos^2 \beta \cos^2(\gamma - \tau)| \mu \right)^{1/2} \right\}.$$

As it was mentioned above the transformation $g(\tau): \alpha \to \alpha + \tau, \gamma \to \gamma - \tau$ preserves the invariant measure μ. Therefore we get

$$cd(\{eH\}, V^2) \leq \chi_n \int_{S^3} |\sin^2 \beta \cos^2 \alpha + \cos^2 \beta \cos^2 \gamma| \mu. \qquad (2.12)$$

The inequality (2.11) becomes an equality if and only if $\tau = 0$. Observe that the right-hand side of (2.12) equals $cd(\{eH\}, v_1 \wedge Jv_1)$. This means that the deformation coefficient $cd(\{eH\}, V^2)$ attains its maximal value only at complex lines. Our proof is completed.

Remark. From the above proof we immediately deduce a dual proposition which replaces a two-dimensional surface $N^2 \subset \mathbf{C}P^n$ by a surface of codimension 2. A proof for the case of an arbitrary k will be given in Sec. 3 (see Proposition 3.10).

3. Minimal Cycles in Grassmannian Manifolds

We denote by $G_k(\mathbf{R}^n)$ the Grassmannian of unoriented k-planes through the origin in $\mathbf{R}^n$ and its 2-sheeted covering by $G_k^+(\mathbf{R}^n)$. We denote by $G_k(\mathbf{C}^n)$ and $G_k(\mathbf{H}^n)$ the complex Grassmannian and the quaternionic Grassmannian respectively. The question of finding and classifying globally minimal cycles in Grassmannian manifolds has attracted attention of many mathematicians. The first non-trivial result was obtained by A. T. Fomenko in 1972 using his method of geodesic defects [9], [22] and by M. Berger in the same year using calibration method [1]. In particular, Fomenko proved that the canonically embedded real projective space $\mathbf{R}P^l \to \mathbf{R}P^n$, $l \leq n$, is globally minimal, and Berger proved that $\mathbf{H}P^k$ is homologically volume-minimizing in $\mathbf{H}P^n$ if $k \leq n$. Recently, employing Euler forms and their "adjusted powers" as calibration H. Gluck, F. Morgan and W. Ziller proved that *if* $k = even \geq 4$, *then each*

$$G_1^+(\mathbf{R}^{k+1}) \subset G_2^+(\mathbf{R}^{k+2}) \subset \cdots \subset G_l^+(\mathbf{R}^{k+l})$$

is uniquely volume minimizing in its homology class [11]. H. Tasaki showed that the same proof implies that $G_k(\mathbf{H}^{m+k})$ is uniquely volume minimizing in its homology class in $G_n(\mathbf{H}^{m+n})$ for all m, even and odd [25]. In this section using our method we prove:

Theorem 3.1. *The canonically embedded real Grassmannian submanifold* $G_k(\mathbf{R}^{k+m})$ *in* $G_l(\mathbf{R}^{l+m})$, $k \leq l$, *has the minimal volume in its homology class with coefficients in* $\mathbf{Z}$ *or* $\mathbf{Z}_2$.

We will show in Sec. 4 that this theorem implies the G-M-Z Theorem mentioned above. But the G-M-Z Theorem implies our Theorem only in the case when m is even and $G = \mathbf{Z}$, because when m is odd, each $G_k^+(\mathbf{R}^{k+m})$ bounds over the reals in $G_l^+(\mathbf{R}^{l+m})$.

Theorem 3.1′. *Classification Theorem. Let* M *be a volume-minimizing cycle of the non-trivial homology class* $[G_k(\mathbf{R}^{m+k})] \in H_*(G_l(\mathbf{R}^{m+l}), G)$, *where* $G = \mathbf{Z}$ *or* $\mathbf{Z}_2$. *Then* M *must be one of these sub-Grassmannians.*

Theorem 3.2. *The canonically embedded complex Grassmannian submanifold* $G_k(\mathbf{C}^{k+m})$ *in* $G_l(\mathbf{C}^{l+m})$, $k \leq l$, *has the minimal volume in its homology class with coefficients in* $\mathbf{Z}_2$.

Theorem 3.3. *The canonically embedded quaternionic Grassmannian submanifold* $G_k(\mathbf{H}^{k+m})$ *in* $G_l(\mathbf{H}^{l+m})$, $k \leq l$, *has the minimal volume in its homology class with coefficients in* $\mathbf{Z}_2$.

Remark. Of course, we can also prove these theorems with respect to integral homologies (and then real homologies) by the same method.

Proof of Theorem 3.1. We apply results of Sec. 2 to $G = SO_{l+m}$, $H = S(O_l \times O_m)$, $I(K) = K = S(O_k \times O_{l-k+m})$, $L = L' = S(O_k \times O_{l-k} \times O_m)$. We consider the family $(M)^* = SO_{l+m}/S(O_k \times O_{l-k+m})$ of homogeneous subspaces obtained from $G_{l-k}(\mathbf{R}^{l-k+m})$ by the action of the group $SO(\mathbf{R}^{l+m})$ (see Sec. 2). Let V be a km-dimensional subspace of $T_e G_l(\mathbf{R}^{l+m})$, where $e = \{eH\}$. According to Lemma 2.7 we get:

$$cd(e, V) = \int_{S(O_l \times O_m)/S(O_k \times O_{l-k} \times O_m)} |\langle \overline{V}, Ad_{\tilde{x}} \overline{W} \rangle| \, dx$$

$$= \int_{SO_l/S(O_k \times O_{l-k})} |\langle \overline{V}, Ad_{\tilde{x}} \overline{W} \rangle| \, dx, \tag{3.1}$$

where W denotes the tangent space $T_e G_k(\mathbf{R}^{k+m})$.

Clearly, the group SO_l acts on the tangent space $T_e G_l(\mathbf{R}^{l+m}) = \mathbf{R}^l \otimes \mathbf{R}^m$ as the sum of m irreducible representations π_1 of dimension l. Namely, in the matrix representation of $T_e G_l(\mathbf{R}^{l+m}) \to so_{l+m}$ these irreducible spaces can be chosen as m columns $\mathbf{R}^l_i$. Let us denote by I the canonical operator of the decomposition $T_e G_l(\mathbf{R}^{l+m}) = \bigoplus \mathbf{R}^l_i$ with respect to the adjoint action of SO_l, that is, $I \cdot Ad = Ad \cdot I$ and $I(\mathbf{R}^l_i) = \mathbf{R}^l_{i+1}$. Obviously, we have $W = W_1 \oplus I(W_1) \oplus \cdots \oplus I^{m-1}(W_1)$, where $W_1 = W \cap \mathbf{R}^l_1$. So we get $Ad_g W = Ad_g W_1 \oplus I(Ad_g W_1) \oplus \cdots \oplus I^{m-1}(Ad_g W_1)$. Now we consider the following fibration $j: SO_l/SO_{l-k} \to SO_l/S(O_k \times O_{l-k})$, where the total space is considered as the Stiefel manifold of frames of k orthonormal vectors in R^l_1, and the base is the Grassmannian of unit simple k-vectors in $\mathbf{R}^l$, which is identified with the set of all $Ad_{\tilde{x}} W$. Thus, if x is a frame of k orthonormal vectors $(v_1, \ldots, v_k)$, then $j(x) = v_1 \wedge \cdots \wedge v_k$. Let the metrics on the above spaces be the standard ones. Since the volume of each fibre O_k is a constant $\lambda_{k,l}$, we can rewrite integral (3.1) as follows

$$cd(e, V) = \lambda_{k,l} \int_{SO_l/SO_{l-k}} |\langle \bar{V}, j(x) \wedge I(j(x)) \wedge \cdots \wedge I^{m-1}(j(x))\rangle| \, dx. \tag{3.2}$$

We consider the fibration $SO_l/SO_{l-k} \to SO_l/SO_{l-k+1}$ with fibre S^{l-k}; it maps a k-frame $x = (v_1, \ldots, v_k)$ to a $(k-1)$-frame $x' = (v_1, \ldots, v_{k-1})$. Denote by $\mathbf{R}^{l-k+1}(x')$ the linear subspace associated with the fiber S^{l-k} over the point x'. Using integration along fibres we deduce from (3.2)

$$\begin{aligned} cd(e, V) &= \lambda_{k,l} \int_{SO_l/SO_{l-k+1}} \int_{S^{l-k}(x')} |\langle \bar{V}, j(x', y) \wedge \cdots \wedge I^{m-1}(j(x', y))\rangle| \, dy \, dx' \\ &= \lambda_{k,l} \int_{SO_l/SO_{l-k+1}} \Big\{ |\langle V, j(x') \wedge \cdots \wedge I^{m-1}(j(x'))\rangle| \\ &\quad \cdot \int_{S^{l-k}(x)} |\langle V^{\perp}(x'), y \wedge \cdots \wedge I^{m-1}(y)\rangle| \, dy \Big\} \, dx'. \end{aligned} \tag{3.3}$$

where $|\langle V, z\rangle|$ denotes the volume of the orthogonal projection of a simple polyvector z on the plane V; and $V^{\perp}(x')$ is the intersection of V with the space $\mathbf{R}^{l-k+1}(x') \oplus \cdots \oplus I^{m-1}(\mathbf{R}^{l-k+1}(x'))$.

Proposition 3.4. *Let $p \leq q$. For each mp-plane $V \subset \mathbf{R}^q \oplus \cdots \oplus I^{m-1}(\mathbf{R}^q)$, where $\mathbf{R}^q \subset \mathbf{R}^l_1$, we put*

$$M(V) = \int_{S^{q-1}} |\langle V, x \wedge \cdots \wedge I^{m-1}(x)\rangle| \, dx.$$

Then $M(V)$ reaches its maximal value if and only if $V = V^p \wedge \cdots \wedge I^{m-1}(V^p)$, where $V^p \subset \mathbf{R}^q$.

Repeating the reduction process (3.3) and applying Proposition 3.4 we obtain the following proposition immediately.

Proposition 3.5. *The deformation coefficient $cd(e, V)$ attains its maximum at V_0 if and only if there exists $\tilde{x} \in SO_l$ such that $V_0 = Ad_{\tilde{x}} W$.*

Proof of Proposition 3.4. Obviously, we have

$$M(V) \leq \int_{S^{q-1}} |\langle V, x\rangle| \cdot \ldots \cdot |\langle V, I^{m-1}(x)\rangle| \, dx. \tag{3.4}$$

Applying the theorem about geometric and arithmetic means we infer from (3.4)

$$M(V) \leq \left(\frac{1}{m}\right)^{m/2} \int_{S^{q-1}} \left(\sum_r |\langle V, I^r(x)\rangle|^2\right)^{m/2} dx. \tag{3.5}$$

Now we study the projection $I^r_V(x)$ of $I^r(x)$ on V and its length $|\langle V, I^r(x)\rangle|$. Let B_r denote the symmetric bilinear form on $\mathbf{R}^q$ defined by $B_r(x,x) = \langle I^r_V(x), I^r_V(x)\rangle$. Let θ^r_j be the eigenvalues of B_r, $j = 1, \ldots, q$. Evidently, $0 \leq \theta^r_j \leq 1$.

Lemma 3.6. *The following identity holds*

$$\sum_{r,j} \theta^r_j = \sum_r \operatorname{tr}(B_r) = \dim V = mp.$$

Proof. Let Π_r be the bilinear form on V defined by: $\Pi_r(x,x) = \langle \pi_r(x), \pi_r(x)\rangle$, where π_r denotes the orthogonal projection on $I^r(\mathbf{R}^q)$. We will show that $\operatorname{tr}(B_r) = \operatorname{tr}(\Pi_r)$. Without loss of generality we can assume that $\dim V \geq \dim I^r(\mathbf{R}^q)$. Now we consider the eigenvectors $\{f^r_i\} \in I^r(\mathbf{R}^q)$ of B_r corresponding to θ^r_i. Then $\{f^r_i\}$ can be chosen as an orthonormal basis in $I^r(\mathbf{R}^q)$. Clearly, we have

$$\langle f^r_i, I^r_V(f^r_j)\rangle = \langle I^r_V(f^r_i), I^r_V(f^r_j)\rangle = \delta_{ij}\theta^r_i. \tag{3.6}$$

We want to find the orthogonal projection $\widehat{I^r_V(f^r_j)}$ of the vector $I^r_V(f^r_j) \in V$ on $I^r(\mathbf{R}^q)$. We note that this projection is defined uniquely, up to multiplication by a constant, by the hyperplane orthogonal to it in the subspace $I^r(\mathbf{R}^q)$. Obviously, this hyperplane H^r_j is defined by the following equation

$$H^r_j = \operatorname{span}\{z | \langle I^r_V(f^r_j), z\rangle = 0\}. \tag{3.7}$$

Now, comparing (3.7) with (3.6), it is easy to see that $\widehat{I^r_V(f^r_j)} \in \operatorname{span}\{f^r_j\}$. Therefore, the orthogonal projection of the vector $I^r_V(f^r_j)/|I^r_V(f^r_j)| \in V$ on the subspace $I^r(\mathbf{R}^q)$ is $\theta^r_j f^r_j$. Note that for any vector w in the orthogonal complement to $\operatorname{span}\{I^r_V(f^r_j)\}$ in V we have $\langle w, f^r_i\rangle = 0$. Hence, in view of (3.7), we have that θ^r_j, $j = 1, \ldots, q$, and 0 with multiplicity $mp - q$ are eigenvalues of Π_r, and then we have $\operatorname{tr}(B_r) = \operatorname{tr}(\Pi_r)$. Further we note that $\sum \Pi_r(x,x) = \langle x, x\rangle$. Therefore $\sum \operatorname{tr}(B_r) = \sum \operatorname{tr}(\Pi_r) = \dim V$. This completes the proof of Lemma 3.6.

Let us continue the proof of Proposition 3.4. From the proof of Lemma 3.6 we know that

$$\sum_{r=0}^{m-1} |\langle V, I^r(x)\rangle|^2 = \sum_{r=0}^{m-1} B_r(x,x).$$

We set $B(x,x) = \sum B_r(x,x)$. Since $B_r(x,x)$ are symmetric bilinear forms whose eigenvalues belong to the segment $[0,1]$, the symmetric bilinear form $B(x,x)$ is also positive, moreover, its eigenvalues belong to the segment $[0,m]$. Denote these eigenvalues by η_i, $i = 1, \ldots, q$. From Lemma 3.6 we know that $\sum \eta_i = \mathrm{Tr}(B) = \sum \mathrm{Tr}(B_r) = \dim V = pm$. Let w_i be the eigenvectors corresponding to η_i. Obviously, we can choose w_i as an orthonormal basis in $\mathbf{R}^q$. So, we write (3.5) as follows

$$M(V) \le \left(\frac{1}{m}\right)^{m/2} \int_{S^{q-1}} \left(\sum_j \eta_j (x_j)^2\right)^{m/2} dx, \tag{3.8}$$

where x_j is the j-th coordinate of $x \in S^{q-1}$ with respect to the basis of vectors $\{w_i\}$. Let $F(\eta_1, \ldots, \eta_q)$ be the function in the right-hand side of (3.8) whose variables satisfy the following condition:

$$\eta_i \in [0,m]; \qquad \sum \eta_i = mp. \tag{C}$$

We want to find the maximum of F. To see this we choose any two variables η_1 and η_2 among η_j and fix the others. So, we have $\eta_2 = c - \eta_1$, where c is some constant. Straightforward calculation yields:

$$\frac{d^2}{d\eta_1^2}(F) = \left(\frac{1}{m}\right)^{m/2} \int_{S^{q-1}} \left(\frac{m}{2} - 1\right) \cdot \frac{m}{2} \cdot \left\{\sum_j \eta_j (x_j)^2\right\}^{(m-4)/2} \cdot (x_1^2 - x_2^2)^2 \, dx.$$

If $m \ge 3$ the above formula shows that F is a convex function with respect to η_1. Therefore, F attains its maximal value at only "boundary" variables. This means that under the condition C we have

$$F(\eta_1, \ldots, \eta_q) \le F(m, \ldots, m, 0, \ldots, 0).$$

This formula shows that $M(V)$ attains its maximal value if and only if the eigenvalues of $B(x,x) = \sum B_r(x,x)$ are $(m, \ldots, m, 0, \ldots, 0)$. Since $\theta_j^r \in [0,1]$ we immediately obtain that for every r the eigenvalues of B_r are $(1, \ldots, 1, 0, \ldots, 0)$, moreover $B_i = B_j$ for all i, j. Consequently, we have $V = V_1 \wedge I(V_1) \wedge \cdots \wedge I^{m-1}(V_1)$. If $m = 2$ then F is a linear function with respect to η_j. In this case it suffices to consider two inequalities (3.4) and (3.5) to obtain our assertion. This completes the proof of Proposition 3.4. Now we study the intersection between Grassmannian submanifolds in $G_l(\mathbf{R}^{l+m})$.

Proposition 3.7. *For almost all (in dimension sense)* $y \in (M)^* = SO_{l+m}/S(O_k \times O_{l-k+m})$ *the space* $N_y = \tilde{y} \cdot G_{l-k}(\mathbf{R}^{l-k+m})$ *meets* $G_k(\mathbf{R}^{k+m})$ *at only one point.*

Proof. Geometrically, the embedding $G_k(\mathbf{R}^{k+m}) \to G_l(\mathbf{R}^{l+m})$ can be described as follows:

$$G_k(\mathbf{R}^{k+m}) \ni x \mapsto x \wedge v_{l-k} \in G_l(\mathbf{R}^{l+m}),$$

where v^{l-k} denotes the subspace orthogonal to $\mathbf{R}^k$ in $\mathbf{R}^l$. So, the intersection $T(y)$ of the considered Grassmannians consists of those l-dimensional subspaces W^l such that:

$$W^l \in (G_k(\mathbf{R}^{k+m}) \wedge v^{l-k}) \cap (G_{l-k}(\tilde{y} \cdot \mathbf{R}^{l-k+m}) \wedge \tilde{y} \cdot v^k). \tag{3.9}$$

Clearly, the following lemmas yield Proposition 3.7.

Lemma 3.8. *The set of all elements $y \in (M)^*$ such that the dimension of $\tilde{y} \cdot \mathbf{R}^k \cap \mathbf{R}^{l-k}$ is greater than or equal to* 1 *has codimension* 1.

Lemma 3.9. *If $\tilde{y} \cdot \mathbf{R}^k \cap \mathbf{R}^{l-k}$ contains only the origin in $\mathbf{R}^{l+m}$ then $T(y)$ contains only one element.*

Proof of Lemma 3.8. If suffices to prove that the set of $\tilde{y} \in SO_{l+m}$ with the above property has codimension greater than or equal to 1 in SO_{l+m}. Let $\tilde{y}$ belong to this set. Then its entries (we consider $\tilde{y}$ as a matrix) satisfy the equation:

$$\mathrm{vol}(\tilde{y} \cdot v^k \wedge v^{l-k}) = 0. \tag{3.10}$$

The solution to (3.10) is an algebraic hypersurface in SO_{l+m}. This completes the proof.

Proof of Lemma 3.9. Let $W^l \in T(y)$. According to (3.9) W^l contains both $\mathbf{R}^{l-k}$ and $\tilde{y} \cdot \mathbf{R}^k$. By our assumption W^l must be their span. This yields the assertion.

Let us complete the proof of Theorem 3.1. Suppose V is a submanifold of $G_l(\mathbf{R}^{l+m})$ representing the same homology class as $G_k(\mathbf{R}^{k+m})$. Then V meets every submanifold $N_y = \tilde{y} \cdot G_{l-k}(\mathbf{R}^{l-k+m})$ at least one time. Hence, our theorem immediately follows from Proposition 3.5, Proposition 3.7 and Corollary 2.3.

Proof of Theorem 3.1′. Let N be a volume-minimizing cycle in the homology class $[G_k(\mathbf{R}^{m+k})]$. First, we observe that N is almost everywhere smooth (see [7]) and then we can apply Corollary 2.2 to N. On the other hand, since $G_k(\mathbf{R}^{m+k})$ satisfies the condition in Corollary 2.3, we conclude that the cycle N also satisfies this condition. In particular, we obtain that for almost all $x \in N$ (in dimension sense) the tangent space T_xN to N satisfies the condition of maximal deformation coefficient: $cd(x, T_xN) = \sigma(M)^*_{km}$. In view of Proposition 3.6 we obtain that the tangent space T_xN is also tangent to some sub-Grassmannian $g \cdot G_k(\mathbf{R}^{k+m})$. Then we can apply Proposition 3.2 in [11], which states that such a submanifold must be one of the sub-Grassmannians $g \cdot G_k(\mathbf{R}^{k+m})$. Indeed, Proposition 3.2 in [11] is stated for the case of Grassmannian of oriented planes $G_k^+(\mathbf{R}^{k+m})$, but their Grassmannian and ours locally isometric, so their Proposition is still valid in our case. This completes the proof of Theorem 3.1′.

Proof of Theorem 3.2. The proof of this theorem is similar to that of Theorem 3.1. First we will prove the Integral Wirtinger Inequality for arbitrary k (cf. Proposition 2.10.a).

Proposition 3.10. *Let N^{2k} be a manifold in $\mathbf{C}P^n$. Then its volume can be estimated from below by*

$$\operatorname{vol}(N^{2k}) \geq \zeta_k^C \cdot \int_{U_{n+1}/(U_{n-k+1}\times U_k)} \#(N \cap \mathbf{C}P^{n-k}(x))\mu_x,$$

where ζ_k^C is the constant in Proposition 2.10.a. *Moreover, the inequality becomes an equality if and only if N^{2k} is a complex submanifold.*

Proof. As in the proof of Proposition 2.11, it suffices to show that the deformation coefficient $\chi_{2k}(e, V^{2k})$, related to the family of complex projective subspaces of dimension $(n-k)$ in $\mathbf{C}P^n$, reaches its maximal value iff V^{2k} is a complex space. According to (2.8) we obtain (see also Proposition 2.10.a):

$$cd(e, V^{2k}) = \int_{G_k(T_e\mathbf{C}P^n)} |\langle \overline{V^{2k}}, Ad_{\tilde{x}}(\overline{W})\rangle|\, dx,$$

where W is the tangent space to the (fixed) complex projective space $\mathbf{C}P^k$. Now we consider the complex Grassmannian $G_{k-1}(T_e\mathbf{C}P^n)$. We associate to each point $x \in G_{k-1}(T_e\mathbf{C}P^n)$ the fibre $q(x)$ of complex lines in the complex $(n-k+1)$-dimensional orthogonal complement to the space span(x) in $T_e\mathbf{C}P^n$. As a result we get a fibre bundle over $G_{k-1}(T_e\mathbf{C}P^n)$ whose fibres are diffeomorphic to $\mathbf{C}P^{n-k}$. Let us denote this fibre bundle by $T^1_{k-1,n}$. Obviously, $T^1_{k-1,n}$ is also a fibre bundle over the complex Grassmannian $G_k(T_e\mathbf{C}P^n)$ with the natural projection $p: (v, x) \mapsto v \wedge x$. So we have the following fibrations

$$\mathbf{C}P^{k-1} \to T^1_{k-1,n} \to G_k(T_e\mathbf{C}P^n),$$

$$\mathbf{C}P^{n-k} \to T^1_{k-1,n} \to G_{k-1}(T_e\mathbf{C}P^n).$$

We observe that the invariant metric on $T^1_{k-1,n} \simeq U_n/(U_{k-1} \times U_{n-k} \times U_1)$, obtained from the bi-invariant metric on U_n factorized by the action of its subgroup $U_{k-1} \times U_{n-k} \times U_1$, coincides with those which are obtained by lifting the invariant metric on $G_{k-1}(T_e\mathbf{C}P^n)$ via q, and the one on $G_k(T_e\mathbf{C}P^n)$ via p. Therefore we get

$$cd(e, V^{2k}) = A_{k,n} \int_{G_{k-1}(T_e(\mathbf{C}P^n))} \int_{\mathbf{C}P^{n-k}(y)} |\langle \overline{V^{2k}}, \overline{y \wedge x}\rangle|\, dx\, dy,$$

where $A_{k,n}$ is a constant which depends only on n and k.

For any point $y \in G_{k-1}(T_e\mathbf{C}P^n)$ are denoted by $\Pi_V y$ the orthogonal projection of y on the subspace V^{2k}. Let $\Pi_V y^\perp$ denote the orthogonal complement to the projection

$\Pi_V y$ in V^{2k}. Then we get

$$\int_{\mathbf{C}P^{n-k}(y)} |\langle \overline{V^{2k}}, \overline{y \wedge x}\rangle| \, dx = |\langle V, \bar{y}\rangle| \cdot \int_{\mathbf{C}P^{n-k}(y)} |\langle \overline{\Pi_V y^{\perp}}, \bar{x}\rangle| \, dx. \tag{3.11}$$

From the proof of Proposition 2.11 we conclude that the right-hand side of (3.11) is less than or equal to $|\langle V, \bar{y}\rangle|$. Moreover, the equality holds if and only if $\Pi_V y^{\perp}$ is a complex line. Repeating the reduction procedure as above we obtain Proposition 3.10 from the following lemma.

Lemma 3.11. *Let V^{2k} be a subspace of real dimension $2k$ in $\mathbf{C}^{n+1}$. For every $x \in \mathbf{C}P^n$ let us denote $|\langle V^{2k}, x\rangle|$ the volume of the projection of the unit complex line $x \in \mathbf{C}P^n$ on the space V^{2k}. Then the function*

$$M_C(V^{2k}) = \int_{\mathbf{C}P^n} |\langle V^{2k}, x\rangle| \, dx$$

reaches its maximal value if and only if V^{2k} is a complex subspace.

Proof. We consider the Hopf fibration $S^{2n+1} \to \mathbf{C}P^n$. As in the proof of Proposition 2.11 we conclude that

$$M_C(V^{2k}) = C_n \int_{S^{2n+1}} |\langle V^{2k}, x' \wedge Jx'\rangle| \, dx' = C_n \int_{S^{2n+1}} \operatorname{vol}(\Pi_V x' \wedge \Pi_V Jx') \, dx',$$

where $C_n = \operatorname{vol}(U_1)^{-1}$, and $\Pi_V x'$ denotes the orthogonal projection of the unit vector $x' \in S^{2n+1}$ on the subspace V^{2k}. Therefore we obtain

$$M_C(V^{2k}) \leq C_n \cdot \int_{S^{2n+1}} |\Pi_V x'| \cdot |\Pi_V Jx'| \, dx', \tag{3.12}$$

and besides, the equality holds iff $\Pi_V x'$ is perpendicular to $\Pi_V Jx'$ for every $x' \in S^{2n+1}$. That condition is equivalent to the complexity of V^{2k}. Note that the group SO_{2n+2} acts on the Grassmannian of real $2k$-dimensional planes in $\mathbf{R}^{2n+1} = \mathbf{C}^{n+1}$ transitively. Applying the Schwarz inequality for integrals to the right-hand side of (3.12) we get

$$M_C(V^{2k}) \leq C_n \left(\int_{S^{n+1}} |\Pi_V x'|^2 \, dx' \right)^{1/2} \left(\int_{S^{2n+1}} |\Pi_V Jx'|^2 \, dx' \right)^{1/2} = C_n \int_{S^{2n+1}} |\Pi_V x'|^2 \, dx'.$$

Moreover, the inequality becomes an equality if and only if V is a complex plane (and in this case we also have $|\Pi_V x'| = |\Pi_V Jx'|$). This completes the proof of Lemma 3.11 and then the proof of Proposition 3.10.

Continuation of Proof of Theorem 3.2. The remaining part of this proof can be carried out in the same way as in the proof of Theorem 3.1. It is easy to see that the following key lemma is an analog of Proposition 3.4.

Lemma 3.12. *For each real plane $V^{2pm} \subset \bigoplus_{r=0}^{m-1} I^r(\mathbf{C}^q)$ we put*

$$M(V) = \int_{\mathbf{C}P^{q-1}} |\langle \overline{V}, x \wedge \cdots \wedge I^{m-1}(x) \rangle| \, dx.$$

Then $M(V)$ reaches its maximal value if and only if $V = V_1 \wedge \cdots \wedge I^{m-1}(V_1)$, where V_1 is some complex subspace in $\mathbf{C}^q$.

Proof. Applying the Schwarz inequality and the technique in the proof of Proposition 3.4 we get

$$M(V) \leq C_{q,m}\left(\int_{S^{2q-1}} B(x,x)^{m/2}\, dx\right)\left(\int_{S^{2q-1}} B(Jx,Jx)^{m/2}\, dx\right),$$

where $C_{q,m}$ is some constant and $B(x,x)$ is a symmetric bilinear form as in the proof of Proposition 3.4. Now, the condition that $M(V^{2k})$ reaches its maximal value is the combination of the following two: V^{2k} is product of $I^r(\mathbf{R}^{2p})$ and V^{2k} is a complex subspace. This completes the proof of Lemma 3.12.

Proof of Theorem 3.3. We follow the proof of Theorem 3.2. To do this we consider the Hopf fibration $S^{4q-1} \to \mathbf{H}P^{q-1}$ and apply the Hölder inequality for integrals (instead of the Schwarz inequality).

4. Properties of $(M)^*$-minimal Cycles

Let N be a k-cycle in Riemannian manifold M^m provided with a family $(M)^*$ of submanifolds N_λ^* in M realizing a cycle $[N^*]$ as in Corollary 2.2. If the inequality in this corollary for the volume of N becomes an equality, we wil call N a $(M)^*$-minimal cycle. Corollary 2.3 states that a $(M)^*$-minimal cycle is homologically volume-minimizing. The homological class $[N] \in H_*(M)$ of such a cycle will be called a $(M)^*$-class. First we show that there is an analog of *Equidistribution Theorem* for homologically volume-minimizing cycles in a $(M)^*$-homology class.

Theorem 4.1. *Equidistribution Theorem. Let N' be a homological volume-minimizing cycle in a $(M)^*$-homology class. The the set of $N_\lambda^* \in (M)^*$ such that $\#(N_\lambda^* \cap N') \neq \chi$ is of measure zero in $(M)^*$. Here χ equals the intersection number of cycles $[N]$ and $[N^*]$.*

Proof. By our assumption and taking into account Corollary 2.2 we conclude that N' also satisfies the condition in Corollary 2.3. Namely we have

$$\mathrm{vol}(N') = \chi \cdot (\sigma(M)_k^*)^{-1} \cdot \mathrm{vol}(M)^*.$$

Theorem 2.1 implies that the above equality holds if and only if N' satisfies the following two conditions

1) For almost all $x \in N'$ we have $cd(x, T_x N) = \sigma(M)_k^*$.
2) For almost all $y \in (M)^*$ the actual intersection number $\#(N_y \cap N')$ equals the algebraic intersection number χ.

Now Theorem 4.1 follows from the second condition.

Applying Theorem 4.1 to complex submanifolds in the complex projective manifolds $\mathbf{C}P^n$ we obtain the following corollary. Recall that the homology group $H_{2k}(\mathbf{C}P^n, \mathbf{Z}) = \mathbf{Z}$ is generated by the element $[\mathbf{C}P^k]$.

Corollary 4.2. *Let r be a positive integer, and let N^{2k} be a complex submanifold realizing the element $r[\mathbf{C}P^k] \in H_{2k}(\mathbf{C}P^n, \mathbf{Z})$. Then the set of $(2n - 2K)$-dimensional projective spaces $\mathbf{C}P_\lambda^{n-k} \subset \mathbf{C}P^n$ such that $\#(\mathbf{C}P_\lambda^{n-k} \cap N^{2k}) \neq r$ is of measure zero in the set of all $\mathbf{C}P_\lambda^{n-k}$ which is identified with $SU_n/S(U_{n-k} \times U_k)$ provided with the invariant measure.*

Proof. Applying Proposition 2.10.a to the cycle $r\mathbf{C}P^k$ we get that all homology classes in $H_*(\mathbf{C}P^n, Z)$ are $(M)^*$-homology classes. It is well known that the complex submanifold N^{2k} is volume-minimizing in its homology class. Hence we infer Corollary 4.2 from Theorem 4.1.

Volume-minimizing cycles in an $(M)^*$-homology class possess some properties similar to those of ϕ-currents, where ϕ is a calibration on M. First, we note that the cycles under consideration are also $(M)^*$-minimal. Further, the tangent space to a $(M)^*$-minimal cycle belongs to a certain distribution of k-planes in TM. Namely at every point $x \in M$ we put

$$I(x) = \{V \in G_k(T_x M) | cd(x, V) = \sigma(M)_k^*\}.$$

Then $(M)^*$-minimal cycles are integral submanifolds of the distribution $I(x)$. Recall that ϕ-submanifolds are integral submanifolds of the distribution $G_\phi(M) = \{V \in TM | \phi(\overline{V}) = 1\}$. When $M = G/H$ is a compact homogeneous Riemannian space, we find a striking relation between these distributions. Let ϕ be an invariant calibration on M. Then its restriction to the tangent space of M at the point $\{eH\}$ is a H-invariant form. Therefore, the value of ϕ at a k-vector $\overline{V} \subset T_{\{eH\}} G/H$ can be expressed as follows

$$\phi(\overline{V}) = \int_H \langle \overline{V}, Ad_{\tilde{x}} \overline{W} \rangle \, d\tilde{x},$$

where $\overline{W}$ is some k-vector in the space $T_{\{eH\}}M$. Obviously, the value $\phi(\overline{V})$ depends only on the orbits of the H-action on $\bigwedge_k T_{\{eH\}}M$ (cf. Proposition 2.6). Moreover, let us denote by L the isotropy group of the H-action at the k-vector $\overline{W}$. Then we have

$$\phi(\overline{V}) = \int_{H/L} \langle \overline{V}, Ad_{\tilde{x}} \overline{W} \rangle \, dx. \tag{4.1}$$

This formula is similar to the one we used for computing deformation coefficient $cd(\{eH\}, V)$, (see (2.8)). Further, the distribution G_ϕ is the set of all k-dimensional tangent subspaces whose associated unit simple k-vectors maximize $\phi(\overline{V})$; the distribution I is the set of all k-dimensional tangent subspaces whose associated unit simple k-vectors maximize value $cd(x, \overline{V})$. In many cases, for example, for a Kähler form and its powers ϕ, we can choose a corresponding $\overline{W}$ as a simple polyvector.

The similarity between $(M)^*$-cycle and ϕ-currents also appears in the following theorem.

Theorem 4.3. *Let N be a $(M)^*$-minimal cycle realizing a torsion free element in the homology group $H_k(M, \mathbf{Z})$. If M is a compact manifold, then N is a ϕ-current for some calibration ϕ on M and the homology class $[N]$ is stable.*

Remark. In many cases, for example, for $M = \mathbf{C}P^n$, there is a unique (up to multiplication by a constant) invariant calibration of a given dimension on the manifold M (see also [21]). In such cases, in view of Theorem 4.3, we can obtain a calibration on M with the help of integral geometry. As it was discussed above, the two kinds of involved integral inequalities are similar but not equivalent. For instance, we consider the deformation coefficient as in Proposition 3.5. It is easy to see that if m is even, then the integrand $|\langle \overline{V^{km}}, Ad_{\bar{x}}\overline{W}\rangle|$ equals $\langle \overline{V^{km}}, Ad_{\bar{x}}\overline{W}\rangle$ for all V^{km} which belongs to the distribution of maximal deformation coefficient. Therefore, such a plane V^{km} also belongs to the distribution of the calibration associated with W as it was discussed above (see (4.1)).

Proof of Theorem 4.3. Let us recall the Federer Stability Theorem.

Theorem. [8] *For every $\alpha \in H_k(M, G)$ we put*

$$\text{mass}(\alpha) = \min\{\text{vol}\, X^k \subset M | [X^k] = \alpha\}.$$

Then the following equality holds for $\alpha \in H_k(M, \mathbf{Z})$.

$$\lim_{n\to\infty} \frac{\text{mass}(n\alpha)}{n} = \text{mass}(\alpha_{\mathbf{R}}),$$

where $\alpha_{\mathbf{R}}$ denotes the image of α under the map $H_k(M, \mathbf{Z}) \to H_k(M, \mathbf{R})$.

If for some $n \in \mathbf{Z}^+$ we have $\text{mass}(n\alpha)/n = \text{mass}(\alpha_{\mathbf{R}})$ we say that the homology class α is stable.

Now assume N is as in Theorem 4.3. We observe that the cycle pN is also a $(M)^*$-cycle for all $p \in \mathbf{Z}^+$. So we get

$$\text{mass}(p[N])/p = \text{mass}([N]).$$

Therefore, according to the Federer Stability Theorem, the homology class $[N]$ must be stable, and N is a volume-minimizing cycle in the class $[N]_{\mathbf{R}} \in H(M, \mathbf{R})$. It is well-known that there is a calibration ϕ on M which calibrates N (cf. [4], [21]). Applying Theorem 4.3 to Theorem 3.1 we obtain the following corollary.

Corollary 4.4. [11] *If the Grassmannian of oriented planes $G_k^+(\mathbf{R}^{k+m})$ realizes a non-trivial element in the homology group $H_{km}(G_l^+(\mathbf{R}^{l+m}), \mathbf{R})$ with real coefficients, then $G_k^+(\mathbf{R}^{k+m})$ is a volume-minimizing cycle in its homology class with real coefficients.*

Proof. Obviously, $G_k(\mathbf{R}^{k+m})$ and its 2-sheeted covering $G_k^+(\mathbf{R}^{k+m})$ have the same homology groups with real coefficients. By Theorem 4.3, $G_k(\mathbf{R}^{k+m})$ is a volume-minimizing real current. Its is well known that in this case there exists an invariants calibration ϕ on $G_l(\mathbf{R}^{l+m})$ such that ϕ calibrates $G_k(\mathbf{R}^{k+m})$. It is easy to see that the lifted calibration ϕ^* on $G_l^+(\mathbf{R}^{l+m})$ must calibrate $G_k^+(\mathbf{R}^{k+m})$ too. This means that $G_k^+(\mathbf{R}^{k+m})$ is a globally minimal submanifold.

Finally we conjecture that every homology class in $H_8(F_4/\mathrm{Spin}_9, \mathbf{Z})$ is a $(M)^*$-class. A. T. Fomenko and M. Berger proved that the Helgason sphere S^8 realizing the generating element of this group is a globally minimal submanifold [9], [1]. We also conjecture that every canonically embedded sub-Grassmannian $G_k(\mathbf{F}^l) \subset G_{k+m}(\mathbf{F}^{l+n})$ is volume minimizing in its $\mathbf{Z}_2$ homology, where $\mathbf{F} = \mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$ (see also [11] for the case of oriented $G_{k+m}^+(\mathbf{R}^{l+n})$).

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