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Almost Complex Structures Which Are Compatible with Kähler or Symplectic Structures

FRANK CONNOLLY[1], LÊ HÔNG VÂN[2]⋆ and KAORU ONO[3]⋆⋆
[1]*Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556, U.S.A.;*
[2]*Max-Planck-Institut für Mathematik, Gottfried-Claren-Straße 26, 53225 Bonn, Germany;*
[3]*Department of Mathematics, Ochanomizu University, Otsuka, Tokyo 112, Japan*

Abstract. In this note we prove that half of all homotopy classes of almost complex structures on M is not compatible with any symplectic structure for a certain class of oriented compact 4-manifolds M. In particular, half of all homotopy classes of almost complex structures on an oriented 4-manifold is not compatible to any Kähler structure.

Key words: almost complex structure, Kähler structure, Seiberg–Witten invariant, symplectic structure

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1. Introduction

On a symplectic manifold (M^{2n}, ω) there is an almost complex structure J_ω compatible to ω (i.e. $\omega(J_\omega x, J_\omega y) = \omega(x, y)$ and $\omega(x, J_\omega x) > 0$). It is well known that the homotopy class $[J_\omega]$ is a symplectic invariant of (M^{2n}, ω). The questions concerned in this note are:

S: Given a homotopy class $[J]$ of an almost complex structure on a compact 4-manifold M^4 is there a symplectic structure ω which is compatible with $[J]$?

K: An analogous question for the existence of a compatible Kähler structure.

Remark. We would like to mention some results related to the questions S and K.

(1) A recent result of Taubes [12] states that a necessary condition for the existence of such a compatible $[J]$ is that the Seiberg–Witten invariant (or one of its values in the case $b_2^+(M^4) = 1$) of the canonical spin^c-structure associated to J must be ± 1 (see the next section for more details). Hence we get many examples of oriented manifolds M^4 which admit almost complex structures but no symplectic structures.

(2) Using the Yang–Mills Instanton theory, Donaldson showed that there is a homotopy class of almost complex structures on $K3$ surfaces which does not contain any complex structure [2].

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(3) Hirzebruch conjectured that (integrable) complex structures on $S^2 \times S^2$ and $\mathbf{C}P^2 \# \overline{\mathbf{C}P^2}$ are unique up to diffeomorphisms and deformation equivalence. This conjecture was recently proved by Friedmann and Qin [5]. Thus the existence of an almost complex structure which is not compatible with a Kähler structure on Hirzebruch's surfaces, follows straightforward from their result combined with an argument in [2]. A similar classification theorem of symplectic structures on minimal rational and ruled surfaces was very recently proved by Taubes (for $\mathbf{C}P^2$) [13] and Lalonde and McDuff.

In [3] Donaldson showed that there is a free involution p on the set of homotopy classes of almost complex structures on a compact oriented closed manifold M^4. Using this we shall prove the following theorems

THEOREM 1.1. *Let M^4 be a closed oriented manifold such that $b_+^2(M) \geq 2$, or $b_+^2(M) = 1$ and $b_1(M) = 0$, or M is diffeomorphic to a ruled surface or a hyperelliptic surface. Suppose that a homotopy class $[J]$ on M^4 is compatible with a symplectic structure. Then the homotopy class $p[J]$ is not compatible with any symplectic structure.*

The following theorem follows from Theorem 1.1 except the case of properly elliptic surface with $p_g = 0$ and $q = 1$. It extends a result of Donaldson we mentioned above to all oriented 4-manifolds.

THEOREM 1.2. *Let M^4 be a closed oriented manifold. Suppose that a homotopy class $[J]$ on M^4 is compatible with a Kähler structure. Then the homotopy class $p[J]$ is not compatible with any Kähler structure.*

Let M be a symplectic (respectively Kähler) 4-manifold considered in Theorem 1.1 (respectively in Theorem 1.2). It follows from our theorems that the action of the orientation preserving diffeomorphism group of M^4 on the set of homotopy classes of almost complex structures on M^4 is not transitive.

A proof of our theorems will be given in Section 3. In Section 2 some facts on almost complex structures on 4-manifolds and the Seiberg–Witten equation (which is the main tool of our proof) will be collected.

2. Preliminaries

2.1. Homotopy Classes of Almost Complex Structures on an Oriented Closed 4-Manifold M^4

(a) It is a classical result due to Ehresmann and Wu [14] that two cohomology classes $c_1 \in H^2(M^4, \mathbf{Z})$ and $c_2 \in H^4(M^4, \mathbf{Z})$ are the first and second Chern classes of an almost complex structure J compatible with a given orientation on M^4, if and only if c_1 and c_2 satisfy the following conditions

$$c_2 = e(M^4),$$

$$c_1 = w_2(M) \bmod 2,$$

$$c_1^2 = 3\tau(M) + 2e(M),$$

where e denotes the Euler class, w_2 the second Whitney class, and τ the signature of M^4.

(b) An almost complex structure J on M^4 can be considered as a section of the associated (SO_4/U_2)-bundle over M^4. Given an almost complex structure J on M we denote by $p(J)$ an almost complex structure on M which coincides with J outside a small ball in M^4, and, moreover, the homotopy class $[p(J)]$ differs from the homotopy class $[J]$ by the non-zero element of $H^4(M^4, \pi_4(SO_4/U_2)) = \pi_4(S^2) = \mathbf{Z}_2$. Using the obstruction theory we see easily that p is defined uniquely up to homotopy. It is easy to see that $c_1(J) = c_1(p(J))$. In [2], Donaldson detected the difference of these two homotopy classes of almost complex structures in terms of a cohomological orientation. Namely, he considered the elliptic operator

$$\delta := d^* \oplus d^+ : \Omega^1 \to (\Omega^0 \oplus \Omega^2_+).$$

Using Hodge theory one can show that the kernel (respectively cokernel) of δ equals $H^1(M^4, \mathbf{R})$ (respectively $H^0(M^4, \mathbf{R}) \oplus H^2_+(M^4, \mathbf{R})$). An orientation of $\det H^1(M^4, \mathbf{R}) \otimes \det(H^0(M^4, \mathbf{R}) \oplus H^2_+(M, \mathbf{R}))$ of an oriented 4-manifold M^4 is called a cohomological orientation. Given an almost complex structure J on M^4 we can deform the operator δ to a complex linear operator $\delta_J^{1/2} = (1/2)(\delta - J\delta J)$. Thus $\delta_J^{1/2}$ gives a canonical way to define a cohomological orientation $o_{[J]}$ of M^4 preferred by $[J]$.

If M^4 is a Kähler manifold with a Kähler form ω, then we can write

$$H^1(M^4, \mathbf{R}) = H^{1,0}; \quad H^0 \oplus H^2_+(M^4, \mathbf{R}) = \mathbf{R} \oplus \mathbf{R}\omega \oplus H^{2,0}.$$

Donaldson defines a complex structure on $H^0 \oplus H^2_+$ by setting $I \cdot 1 = -\omega/\sqrt{2}$. Clearly this complex structure together with the cohomological orientation of M^4 preferred by J induces an orientation on $H^1(M^4, \mathbf{R})$ which is also called the cohomological orientation preferred by $[J]$.

Claim 2.1.c [2, lemma 6.4], [3, p. 418]. The cohomological orientations $o_{[J]}$ and $o_{p([J])}$ are opposite. Thus p defines a free involution on the set of homotopy classes of almost complex structures on M^4. If M^4 is Kähler, the canonical orientation of $H^1(M^4, \mathbf{R})$ defined by the complex structure J coincides with the cohomological orientation preferred by $[J]$.

If ω is a closed 2-form on M^4, then it induces a 2-form Q_ω on the linear space $H^1(M^4, \mathbf{R})$ as follows

$$Q_\omega(\alpha, \beta) = -\int_M \alpha \wedge \beta \wedge \omega. \tag{2.1}$$

If ω is a Kähler form, then Q_ω is the Hodge–Riemann bilinear form (see, e.g. [14]). Thus a Kähler form ω defines a symplectic form on $H^1(M^4, \mathbf{R})$ and, therefore, induces a natural orientation on it. We get easily the following observation

Remark 2.1.d. Let (M^4, J, ω) be a Kähler manifold. Then the orientations on $H^1(M, \mathbf{R})$ defined by J and Q_ω concide.

2.2. SEIBERG–WITTEN EQUATION FOR SYMPLECTIC 4-MANIFOLDS

For more details, see [6, 7, 11–13]. Let us recall that the Seiberg–Witten equation for a spin^c-structure on a Riemannian 4-manifold M^4 is the pair of the following equations for A and a positive half spinor ϕ.

$$D_A(\phi) = 0 \tag{SW1}$$

$$F_A^+ = q(\phi) \ \left(:= \phi \otimes \phi^* - \frac{|\phi|^2}{2}\, Id \right), \tag{SW2}$$

where A is a connection on the associated line bundle of the spin^c-structure and $q(\phi)$ is a traceless symmetric endomorphism of the positive spinor bundle which can be identified with an imaginary valued self-dual 2-form via the Clifford multiplication. We can also perturb the Seiberg–Witten equation by adding a term $\mu \in i\Omega^2_+(M^4)$ to $q(\phi)$ in the second equation (SW2). If $b_2^+(M^4) \geq 2$, the "number" (or the cobordism type of the moduli space) of the solutions to (SW1–2) (actually to any its perturbed equation) does not depend on metric g and, therefore, defines, roughly speaking, the Seiberg–Witten invariant of the spin^c-structure on M. If $b^2_+(M^4) = 1$, for each spin^c-structure there are exactly two chambers in the space of pairs (g, μ) consisting of a metric g and a perturbation term μ such that the "number" of the solutions of the SW-equation with respect to the metric g and the perturbation term μ depends only on the chamber to which the pair (g, μ) belongs. The wall dividing these two chambers is defined by the equation

$$\int \left(c_1(L) - \frac{i\mu}{2\pi} \right) \omega_g = 0, \tag{2.2}$$

where ω_g is the unique (up to scalar) self dual harmonic form on M^4 and L is the associated line bundle of the spin^c-structure. If $b_1(M) = 0$, then one has a (relatively simple) wall-crossing formula which relates the difference of the Seiberg–Witten invariant in two chambers [7]. In short it says that the difference is ± 1. A general formula in the case $b_1 \neq 0$ may be well known to specialists and can be found, for instance, in [8, 10].

For a symplectic manifold (M^4, ω) (or more generally, for an almost complex manifold M^{2n}) we always have a choice of a canonical spin^c-structure S_{can} [6]. The cohomological orientation preferred by J also defines a canonical orientation of the moduli space of the solutions of the Seiberg–Witten equation. We call the invariant defined by the perturbed Seiberg–Witten equation with $\mu = -ir\omega$, r large enough, the Seiberg–Witten–Taubes invariant. We denote the Seiberg–Witten–Taubes invariant for the canonical spin^c-structure by $\mathrm{SWT}(S_{\mathrm{can}}(\omega))$. Taubes [12] proved that $\mathrm{SWT}(S_{\mathrm{can}}(\omega))$ is ± 1. In fact, Salamon [11] showed that the invariant is 1 (see Appendix).

3. Proof of Theorems

LEMMA 3.1. *Let J be an almost complex structure on M^4. Then the canonical* spin^c*-structures defined by $[J]$ and $p[J]$ are equivalent.*

Proof. Without lost of generality we can assume that two almost complex structures J and $p(J)$ coincide outside a ball B_1 of a point and inside B_1 the complex structure J is standard. Then we have a natural identification of the two spin^c-structures outside of the ball. For two spin^c-structures on a given manifold, the difference of them is detected by a $U(1)$-bundle. Let L be a $U(1)$-bundle detecting the difference of these two spin^c-structures. Take a bit bigger open ball B. Since the two spin^c-structures coincide outside of B_1, the $U(1)$-bundle L is trivial outside of B_1, especially on $B - B_1$. The trivialization on $B - B_1$ automatically extends to the 4-ball B. Hence L is trivial on M. It follows that the two canonical spin^c-structures associated to J and $p(J)$ are equivalent. □

LEMMA 3.2. *Let M^4 be a symplectic manifold with $b_+^2 = 1$. Let ω and ω' be symplectic forms on M^4 such that $[J_\omega] = [p(J_{\omega'})]$. Suppose that $b_1(M^4) = 0$. Then ω and ω' are in the same connected component of the positive cone in $H^2(M; \mathbf{R})$.*

Proof. According to Taubes' theorem we have

$$\begin{cases} 1 = \mathrm{SWT}(S_{\mathrm{can}}(\omega)), \\ 1 = \mathrm{SWT}(S_{\mathrm{can}}(\omega')). \end{cases} \tag{3.1}$$

Recall that $\mathrm{SWT}(S_{\mathrm{can}}(\omega)) = \mathrm{SW}(S_{\mathrm{can}}(\omega), o_{[J]}, -ir\omega)$, where the cohomological orientation $o_{[J]}$ also defines the orientation of the moduli space, and $-ir\omega$ with very large r is the perturbation term in the Seiberg–Witten equation (SW2). By Lemma 3.1 we denote $S_{\mathrm{can}}(\omega') = S_{\mathrm{can}}(\omega) = S_{\mathrm{can}}$. From (3.1) and Claim 2.1.c we get

$$1 = \mathrm{SW}(S_{\mathrm{can}}, o_{[J]}, -ir\omega) = \mathrm{SW}(S_{\mathrm{can}}, -o_{[J]}, -ir\omega'). \tag{3.2}$$

Now we suppose that ω and ω' are not in the same component of $H_+^2(M, \mathbf{R})$. From the last term in (3.2) we get

$$1 = -\mathrm{SW}(S_{\mathrm{can}}, o_{[J]}, ir\omega).$$

Hence we get

$$\mathrm{SW}(S_{\mathrm{can}}, o_{[J]}, -ir\omega) - \mathrm{SW}(S_{\mathrm{can}}, o_{[J]}, ir\omega) = 2. \tag{3.3}$$

But (3.3) contradicts to the wall-crossing formula [7] (see also Section 2). □

Proof of Theorem 1.1. Suppose the opposite, i.e. there are two symplectic structures ω and ω' which are compatible to $[J]$ and $p[J]$. According to Lemma 3.1 we can denote by S_{can} the canonical spin^c-structure for both ω and ω'. Firstly, we consider the case $b^2_+(M^4) \geq 2$. Taubes' theorem tells us that the Seiberg–Witten invariant of the canonical spin^c-structure S_{can} is 1. On the other hand, a result by Donaldson (Claim 2.1.c) tells us that the cohomological orientation preferred by J and $p(J)$ are opposite. Since the preferred cohomological orientation defines the canonical orientation of the moduli spaces of the SW-solutions [11] we obtain a contradiction.

Secondly we consider the case $b_2^+(M) = 1$ and $b_1(M) = 0$. By Lemma 3.2 both ω and ω' are in the same connected component of the positive cone $H^2_+(M^4, \mathbf{R})$. Hence for r large enough we have $\text{SW}(S_{\text{can}}, o_{[J]}, -ir\omega) = \text{SW}(S_{\text{can}}, o_{[J]}, -ir\omega')$. But this equality contradicts to Taubes' theorem and the fact that $o_{[J]} = -o_{[p(J)]}$ (Claim 2.1.c).

Finally, we consider the case when M^4 is diffeomorphic to a ruled surface or a hyperelliptic surface. Since a rational ruled surface is simply connected, that is included in the case that $b_1(M) = 0$, we only deal with irrational ruled surfaces and hyperelliptic surfaces.

Subcase A. M^4 is diffeomorphic to an irrational ruled surfaces.

Subcase B. M^4 is diffeomorphic to a hyperelliptic surface.

Let us consider *Subcase A*. If we imitate the argument in the above case with $b_1(M^4) = 0$ and $b_2^+(M^4) = 1$, then there is a problem arising in computing the wall-crossing formula. We note that a ruled surface admits a positive scalar curvature metric g_0. Therefore the two chambers for the canonical spin^c-structure on M^4 have the following two representatives: one is the pair $(g_0, \mu = 0)$, and for the other chamber a pair of a metric compatible to ω and Taubes' perturbation $\mu = -ir\omega$. Thus ω and ω' should be in the same connected component of the positive cone in $H^2(M, \mathbf{R})$. Now we can proceed as in the case when $b_2^+(M^4) = 1$ and $b_1(M^4) = 0$.

Subcase B. M is diffeomorphic to a hyperelliptic surface. First we want to show that the image of the first Chern class $c_1(M, \omega)$ in $H^2(M, \mathbf{Q})$ is zero. To do this we consider the covering space $\tilde{M}$ of M which is diffeomorphic to a product of two elliptic curves. Let $\tilde{\omega}$ be a symplectic form on $\tilde{M}$ which is the pull-back of the symplectic form ω on M. According to Taubes' theorem the canonical bundle of $(\tilde{M}, \tilde{\omega})$ is a Seiberg–Witten class [12]. (Note that $b_2^+(\tilde{M}) = 3 \geq 2$.) Because the only SW class of a 4-torus is the trivial class, $c_1(\tilde{M}, \tilde{\omega})$ is 0 in $H^2(\tilde{M}, \mathbf{Q})$. Hence the image of $c_1(M, \omega)$ in rational cohomology is also zero.

Since $c_1(M, \omega)$ is zero in $H^2(M, \mathbf{Q})$ the wall-crossing number for the canonical spin^c-structure is also 0 for (M, ω). Now suppose that ω' is a symplectic form which is compatible with $[p(J_\omega)]$. Notice that till the (3.2) the argument of Lemma 3.2 does not depend on the condition $b_1 = 0$. Now suppose that ω and ω' are in the same connected componentent of the positive cone in $H^2(M, \mathbf{R})$. Then (3.2) yields a contradiction. Hence these symplectic forms must be in different components of the positive cone. Considering the above obtained wall crossing formula and (3.3) leads to a contradiction. □

Proof of Theorem 1.2. We consider several cases. Suppose that $b_2^+(M^4) \geq 2$, or $b_2^+(M^4) = 1$ and $b_1(M^4) = 0$, or M^4 is a ruled surface. In these cases Theorem 1.2 follows from Theorem 1.1. By Noether's theorem if M^4 is a minimal surface of general type with $p_g = 0$, then $q(M) = 0$. Hence if M^4 is Kähler with $b_+^2(M^4) = 1$, $b_1 \neq 0$, then M^4 must be an irrational ruled surface or an elliptic surface. Thus by the Enriques-Kodaira classification of complex surfaces it suffices to prove Theorem 1.2 in the case that M^4 is an elliptic surface with $b_1 = 2$ and $b_2^+ = 1$. (Note that if M is a hyperelliptic surface the conclusion follows from Theorem 1.1. But the following argument works also for hyperelliptic surfaces.) □

LEMMA 3.3. *Suppose that $b_2^+(M) = 1$ and ω and ω' are two Kähler forms in the same connected component of the positive cone in $H^2(M, \mathbf{R})$. Then the orientations defined by Q_ω and $Q_{\omega'}$ on $H^1(M^4, \mathbf{R})$ are the same.*

Proof. Our argument is similar to that in [9]. Note that for α, $\beta \in H^1(M; \mathbf{R})$, $\alpha \wedge \beta$ lies in the null-cone of $H^2(M, \mathbf{R})$. Consider a path $\{\omega_t\}$ in the e cone from ω to ω'. Then we have a one-parameter family of bilinear forms Q_{ω_t}. If these bilinear forms are all non-degenerate, then the orientations determined by Q_{ω_t} are constant. Thus Lemma 3.3 is a consequence of the following fact.

Suppose that A and B are in the closure of a connected component of the positive cone in $H^2(M, \mathbf{R})$. Then $A \cdot B \geq 0$. Moreover, if $A^2 > 0$, then the equality $A.B = 0$ holds if and only if $B = 0$. This fact can be easily proved by considering an orthogonal decomposition of A and B as follows: $A = a_0 x_0 + \sum_{i \geq 1} a_i x_i$, $B = b_0 x_0 + \sum_{i \geq 1} b_i x_i$. Here x_0 is a unit vector in $H_+^2(M, \mathbf{R})$ and $\{x_i, | i \geq 1\}$ is an orthonormal basis in $H_-^2(M, \mathbf{R})$. The desired fact follows by applying the Cauchy inequality to the RHS of the following inequality:

$$a_0 b_0 \geq \sqrt{\sum_{i>0} a_i^2} \sqrt{\sum_{i>0} b_i^2}.$$

□

Now we consider two subcases.

(1) Suppose that ω and ω' are in the same connected component of the positive cone in $H^2(M, \mathbf{R})$. The same argument as before tells us that the cohomological orientations defined by J and $p[J]$ are the same, which contradicts to Donaldson's theorem.

(2) Suppose that ω and ω' are in different connected components of the positive cone. Since $b_1(M^4) = 2$, Lemma 3.3 tells us that the orientations on $H^1(M^4, \mathbf{R})$ induced by ω and ω' are opposite. Thus the cohomological orientations defined by

J and $p[J]$ are the same, which is a contradiction. This completes the proof. □

Remark 3.4. The following statement was pointed out to us by the referee. Namely Theorem 1.1 is also valid for a manifold M of the same diffeomorphism type as a proper elliptic surface X with $p_g = 0$ and $q = 1$ over a curve of genus 1. To see this we assume the opposite, i.e. there are almost complex structures J and $p(J)$ such that $[J]$ (respectively $[p(J)]$) is compatible with a symplectic structure ω (respectively ω'). Donaldson's theorem and Taubes' theorem tells us that ω and ω' must be in different components of the positive cone in $H^2(M, \mathbf{R})$. On the other hand, the Seiberg–Witten–Taubes invariant of the canonical spin^c-structure associated to J is non-trivial. It is known (see, e.g., [4]) that in this case the first Chern class $c_1(J)$ in rational cohomology must be a multiple of the image of the canonical class K_X, here X is the corresponding proper elliptic surface. The wall-crossing formula tell us that in this case the crossing number is zero which contradicts to (3.3).

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Appendix. The Sign of the Seiberg–Witten–Taubes Invariant of the Canonical spin^c-Structure of a Symplectic Manifold (M^4, ω)

The following theorem is due to Taubes

THEOREM [11, 12]. *Let (M^4, ω) be a symplectic 4-manifold with its orientation given by $\omega \wedge \omega$ and its cohomological orientation given by a compatible to ω almost complex structure J. Then* $\mathrm{SWT}(M^4, S_{\mathrm{can}}) = 1$.

Actually in [12] Taubes states this theorem only up to sign (without considering cohomological orientation) and Salamon makes it more precise by considering the preferred cohomological orientation [11]. In his proof Salamon gives a detailed argument for a Kähler manifold M^4 and outlines a proof for the symplectic case. For the case of completeness we make his argument more detailed in our Appendix.

First we recall that to define the sign of the Seiberg–Witten–Taubes invariant we have to consider the following family of Fredholm operators $\mathcal{D}_t$ [11]

$$\mathcal{D}_t : \begin{array}{c}\Omega^{0,0}(M^4)\\ \oplus \\ \Omega^{0,1}(M^4) \\ \oplus \\ \Omega^{0,2}(M^4)\end{array} \longrightarrow \begin{array}{c}\Omega^{0,0}(M^4)\\ \oplus \\ \Omega^{0,1}(M^4) \\ \oplus \\ \Omega^{0,2}(M^4)\end{array},$$

where

$$\mathcal{D}_t = \mathcal{D}_0 + t\dot{\mathcal{D}}_0 \tag{A.1}$$

with $\mathcal{D}_0$ being a complex linear operator:

$$\mathcal{D}_0(\tau_0, \tau_1, \tau_2) = (\bar{\partial}^*\tau_1, \bar{\partial}\tau_0 + \bar{\partial}^*\tau_2, \bar{\partial}\tau_1) \tag{A.2}$$

and $\dot{\mathcal{D}}_0$ depending on λ:

$$\dot{\mathcal{D}}_0^{\lambda}(\tau_0, \tau_1, \tau_2) = (-\sqrt{\pi}\lambda\tau_0, \sqrt{4\pi}\tau_1, \bar{\tau}_1 \circ N_J/4 - \sqrt{\pi}\lambda\tau_2). \tag{A.3}$$

Here λ denotes the "size" of the Taubes' perturbation term in the SW-equation and in the last formula N_J denotes the Nijenhuis tensor. Recall that the orientation of the determinant of the real vector space $\wedge^{\max}(\ker \mathcal{D}_0) \otimes \wedge^{\max}(\ker \mathcal{D}_0^*)$ which is defined by the almost complex structure J agrees with the preferred cohomological orientation [2, 3, 11] and to define the sign of the Seiberg–Witten–Taubes invariant we must define the orientation of $\wedge^{\max}(\ker \mathcal{D}_1) \otimes \wedge^{\max}(\ker \mathcal{D}_1^*)$ which is obtained by a trivialization of the determinant line bundle $\wedge^{\max}(\ker \mathcal{D}_t) \otimes \wedge^{\max}(\ker, \mathcal{D}_t^*)$ over $[0, 1]$. In [11] Salamon proved that for all $t > 0$ the operator $\mathcal{D}_t$ is invertible. Thus we need to examine what happens at $t = 0$. If $\mathcal{D}_0$ is also invertible then we are done. Hence we assume that there is a crossing at $t = 0$. It can be shown that the crossing is regular at $t = 0$ in the sense of [11] and we can conclude that the sign of the invariant is $+$ by the argument in the Appendix of [11].

Here is another way to show the invariant is $+1$. We deform the operator $\mathcal{D}_0$ by a $(0, 1)$-form a. Namely, we deform the operator $\bar{\partial}$ by $\bar{\partial} + a$ and $\bar{\partial}^*$ by the adjoint of $\bar{\partial} + a$. By Sard–Smale theorem, the deformed operator is surjective for almost all a. We denote the deformed operator by $\mathcal{D}^a$. For a sufficiently large $\lambda > 0$, we consider a family of linear elliptic differential operators $t\mathcal{D}^a + (1 - t)\mathcal{D}_1(\lambda)$, $0 \leq t \leq 1$. Recall that $\mathcal{D}_1(\lambda)$ is the linearization of the Seiberg–Witten equation perturbed by $\mu = i\lambda\,\omega$ [see Section 2.2 and (A.1), (A.3)]. Then the argument in [11] also implies that these operators are surjective for all $t \in [0, 1]$. Note that $\mathcal{D}^a$ is complex linear. Hence the sign of the invariant is $+$.

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