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Cup-length estimates for symplectic fixed points

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CONTENTS

1 Introduction

In 1965 Arnold conjectured that the number $\#(Fix(\phi))$ of fixed points of an exact symplectomorphism ϕ on a compact symplectic manifold M^{2n} is at least as many as the number of critical points of a smooth function on M^{2n}. In homological terms this implies that $\#(Fix(\phi))$ is greater than or equal to the cup-length $cl(M^{2n}, \mathbf{F})$ of the cohomology ring $H^*(M^{2n}, \mathbf{F})$. Recall that $cl(M^{2n}, \mathbf{F})$ is the maximal integer $l+1$ such that there exist classes $\alpha_1, \dots, \alpha_l \in H^*(M^{2n}, \mathbf{F})$ of positive dimension with $\alpha_1 \smile \dots \smile \alpha_l \neq 0$. If all the fixed points are non-degenerate we should have a better estimate in which the cup-length is replaced by the sum of the Betti numbers. The Arnold conjecture for non-degenerate fixed points has been verified in several cases [E], [Sik], [C-Z],[F1] - [F3], [H-S], [O]. This conjecture for degenerate fixed points was proved in the case of M^2 by Nikishin, Simon, Eliashberg, Sikorav, Floer, of the torus T^{2n} by Conley and Zehnder, then in the case of symplectic manifolds with vanishing second homotopy group by Hofer, Floer [H], [F4], and of $\mathbf{C}P^n$ by Fortune, Floer [Fo], [F3].

Floer initiated his homology theory for an indefinite functional, which is now called Floer homology theory, and proved the Arnold conjecture for non-degenerate symplectic fixed points in monotone symplectic manifolds. His method has been developed in [S-Z], [H-S] and [O]. He also proved the conjecture for degenerate fixed points in some cases by using the cap action of $H^*(M, \mathbf{Z}_2)$ on the Floer homology group, which is defined only in the non-degenerate case. To get an estimate in the degenerate case, he approximated the given symplectomorphism by non-degenerate ones.

In this note we define the cap action for weakly-monotone symplectic manifolds and prove the associativity of the action under a certain condition. As a result we obtain the following theorem.

Main Theorem *Let (M^{2n}, ω) be a closed symplectic manifold of dimension $2n$ satisfying the following property:*

$$c_1(M)|_{\pi_2(M)} = \lambda \cdot \omega|_{\pi_2(M)},$$

with some negative constant λ and the minimal Chern number is greater than or equal to n. Suppose that ϕ is an exact symplectomorphism on M. Then the number of fixed points of ϕ is at least the cup-length $cl(M, \mathbf{Z}_2)$.

We would like to emphasize the following fact.

The associativity of the action breaks down because of the presence of non-trivial holomorphic spheres. But in the Floer proof for $\mathbf{C}P^n$ the presence of holomorphic spheres is necessary. (In the finite dimensional situation Floer's approach could also serve as a new proof for the cup-length estimate of critical points of a smooth function on a compact manifolds.) Note that associativity of quantum cup-product has been established by Ruan and Tian. Our associativity fails, in general, because we ignore "quantum contribution".

After finishing this manuscript, we received a preprint by Floer, Hofer and Salamon "Transversality in elliptic Morse theory for the symplectic action", which overlaps Appendix 1 in this paper.

2 Preliminaries

First recall that for a given exact symplectomorphism ϕ in a compact symplectic manifold M^{2n} there exists a periodic Hamiltonian $H \in C^\infty(S^1 \times M^{2n})$ such that the fixed points of ϕ are in one-to-one correspondence with the 1-periodic solutions of the following equation

$$\dot{x}(t) = X_H(t, x(t)), \tag{2.1}$$

where X_H is the Hamiltonian vector field of H (i.e. $\omega(\xi, X_H) = dH(\xi) \forall \xi$). A 1-periodic solution of (2.1) is called **non-degenerate** if $\det(I - d\phi(x(0))) \neq 0$. Now we collect some known facts on the Floer homology of non-degenerate 1-periodic Hamiltonian systems. Details are found in [F3],[H-S],[S-Z].

Let $\mathcal{P}(H)$ denote the set of all contractible loops satisfying (2.1). If $\langle \omega, \pi_2(M) \rangle = 0$, the equation (2.1) is the Euler-Lagrange equation of the **action functional** $\mathcal{A}_H$ on the space $\mathcal{L}(M^{2n})$ of contractible loops in M:

$$\mathcal{A}_H(x) = -\int_{D^2} u^*\omega + \int_0^1 H(t, x(t))dt, \tag{2.2}$$

where u is the bounding disk of x, i.e. $u|_{\partial D^2} = x$. If $\langle \omega, \pi_2(M)\rangle \neq 0$, the first term of the right-hand side of (2.2) is single-valued after taking the covering space $\tilde{\mathcal{L}}(M)$ of $\mathcal{L}(M)$ corresponding to the homomorphisms $\phi_\omega, \phi_{c_1} : \pi_2(M) \to \mathbf{R} : \phi_\omega(A) = \int_A \omega$, $\phi_{c_1}(A) = \int_A c_1$. More precisely,

$$\tilde{\mathcal{L}}(M) = \{(x,u) |\, x \in \mathcal{L}(M), u : D^2 \to M \text{ such that } x = u|_{\partial D^2}\}/\sim$$

$$(x,u) \sim (y,v) \Leftrightarrow \begin{cases} x = y, \\ \int_{D^2} u^*\omega = \int_{D^2} v^*\omega, \\ \int_{D^2} u^* c_1 = \int_{D^2} v^* c_1. \end{cases}$$

The covering transformation group of $\tilde{\mathcal{L}}(M) \to \mathcal{L}(M)$ is

$$\Gamma = \frac{\pi_2(M)}{\ker \phi_{c_1} \cap \ker \phi_\omega}. \tag{2.3}$$

Geometrically, $\pi_2(M)$ acts on $\tilde{\mathcal{L}}(M)$ by connected sum of 2-spheres with the bounding disk.

Let $\tilde{\mathcal{P}}(H)$ denote the inverse image of $\mathcal{P}(H)$ by the projection $\tilde{\mathcal{L}}(M^{2n}) \to \mathcal{L}(M^{2n})$, then $\tilde{\mathcal{P}}(H)$ is the critical set of the functional $\mathcal{A}_H$. Fix an almost complex structure J calibrated by ω, that is, $g_J(v,w) = \omega(v, Jw)$ defines a Riemannian metric on M^{2n} (in particular, we have: $X_H = J\nabla H$). Then one can define the "minus gradient flow" of $\mathcal{A}_H$ by the solution $u : \mathbf{R}\times S^1 \to M^{2n}$ of the following equation

$$\bar{\partial}_{J,H}(u) = \frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} + \nabla H(t,u) = 0. \tag{2.4}$$

The linearization D_u of $\bar{\partial}_{J,H}(u)$ is a Fredholm operator, and we call its index $\mu(u)$ the **relative index** of u. We have $\mu(u) = \mu([x^-,u^-]) - \mu([x^+,u^+])$, where $\mu([x^-,u^-])$ is the **Conley-Zehnder index** of $[x^-,u^-]$. On the set $\mathcal{P}(H)$ the Conley-Zehnder index $\mu(x)$ is well-defined modulo $2N$, where N is the minimal Chern number of M^{2n}. We denote by $\mathcal{M}([x^-,u^-],[x^+,u^+],H,J)$ the space of **connecting orbits** u which satisfy (2.4) and the limit condition:

$$\lim_{s\to-\infty} u(s,t) = x^-(t), \lim_{s\to+\infty} u(s,t) = x^+(t), \tag{2.4.1}$$

with

$$(x^+, u^- \sharp u) \sim (x^+, u^+). \tag{2.4.2}$$

Using weak-compactness argument one shows that $\lim_{s\to\pm\infty} u(s,t)$ exists if and only if the **energy**

$$E(u) = \int_{-\infty}^{\infty}\int_0^1 \|\frac{\partial u}{\partial s}\|^2\, dt\, ds$$

is finite. Applying the Sard-Smale theorem one shows that there exists a generic set of pair (J,H) such that $\mathcal{M}([x^-,u^-],[x^+,u^+],H,J)$ is a smooth manifold of dimension $\mu(u)$ (see also Proposition 3.2.i). Moreover, this manifold is invariant under the $\mathbf{R}$-action by translation in s-variable. Now one can define the Floer chain complex $CF_*(H,J)$ with $\mathbf{Z}_2$ coefficients on a **weakly-monotone symplectic manifold** as follows. Recall that a $2n$-dimensional symplectic manifold (M,ω) is called weakly monotone if it satisfies $\omega(A) \leq 0$ for any $A \in \pi_2(M)$ with $3 - n \leq c_1(A) < 0$. This condition yields non-existence of J-holomorphic spheres of negative Chern number for a **regular almost complex structure** J, which is generic in the sense of Baire. This fact combined with a "transversality property" of H makes sure that the moduli space $\mathcal{M}([x^-,u^-],[x^+,u^+],H,J)/\mathbf{R}$ is compact if its dimension equals zero, and compact up to splitting into two connecting orbits if its dimension equals one.

Denote by CF_k the $\mathbf{Z}_2$-vector space consisting of $\sum_{\mu(\tilde{x})=k} \xi(\tilde{x})\cdot\tilde{x}$, where $\tilde{x} \in \tilde{\mathcal{P}}(H)$ and the coefficients $\xi(\tilde{x}) \in \mathbf{Z}_2$ satisfy the following finiteness condition:

$$\{\tilde{x} \mid \xi(\tilde{x}) \neq 0, \text{ and } \mathcal{A}_H(\tilde{x}) > c\} \text{ is a finite set for all } c \in \mathbf{R}.$$

The boundary operator is defined as follows:

$$\partial\tilde{x} := \sum_{\mu(\tilde{y})=\mu(\tilde{x})-1} n_2(\tilde{x},\tilde{y})\cdot\tilde{y};$$

here $n_2(\tilde{x},\tilde{y})$ is the modulo 2-reduction of the cardinality of $\mathcal{M}(\tilde{x},\tilde{y},H,J)/\mathbf{R}$. The complex $CF_*(H,J) := (CF_*,\partial)$ is called the **Floer chain complex** associated to (H,J). Its homology group $HF_*(H,J)$ is called **Floer homology group**. This group is a finitely generated module, in each degree, over the Novikov ring Λ^0_ω which is the completion of the group ring

$$\Gamma_0 = \frac{\ker\phi_{c_1}}{\ker\phi_{c_1}\cap\ker\phi_\omega} \subset \Gamma_1$$

over the field $\mathbf{Z}_2$ with respect to the weight homomorphism $\phi_\omega : \pi_2(M) \to \mathbf{R}$. Furthermore, Floer homology group does not depend on the choice of a generic pair (H,J). Hofer and Salamon showed that if the minimal Chern number N of M^{2n} is at least n, then there is an isomorphism

$$HF_{k+n} \cong \bigoplus_{j=k(\mathrm{mod}2N)} H_j(M,\mathbf{Z}_2)\otimes\Lambda^0_\omega.$$

3 Cap action of cohomology group $H^*(M,\mathbf{Z}_2)$ on Floer homology $HF_*(H,J)$

Let $\alpha \in H^k(M^{2n},\mathbf{Z}_2)$ and $\alpha^\sharp : \cup\Delta^{2n-k} \to M$ be a singular chain representing the Poincaré dual class of α. Then the action of α via $\alpha^\sharp$ on a Floer chain complex $CF_*(H,J)$ can be described as follows:

$$\alpha^\sharp \cap \tilde{x} := \sum_{\mu(\tilde{y})=\mu(\tilde{x})-k} m^{\alpha^\sharp}(\tilde{x},\tilde{y})\,\tilde{y}, \qquad (3.1)$$

where $\tilde{x},\tilde{y} \in \tilde{\mathcal{P}}(H)$ are generators of $CF_*(H,J)$, and $m^{\alpha^\sharp}(\tilde{x},\tilde{y})$ denotes the modulo 2-reduction of the cardinality of the set

$$\mathcal{M}^{\alpha^\sharp}(\tilde{x},\tilde{y},H,J) := \{u \in \mathcal{M}(\tilde{x},\tilde{y},H,J)|\, u(0,0) \in \mathrm{Im}(\alpha^\sharp)\}.$$

In the following Proposition 3.2 we will give a precise definition of a regular triple $(H,J,\alpha^\sharp)$, which ensures the finiteness of the number $m^{\alpha^\sharp}(\tilde{x},\tilde{y})$ in (3.1).

First, recall that given a regular complex structure J the set of **J-regular Hamiltonians** (i.e. whose 1-periodic solutions are non-degenerate and have no intersection with any J-holomorphic sphere of Chern number less than or equal to 1) on M is generic (in the sense of Baire) in the Banach affine space of all smooth functions $H+h : S^1 \times M \to \mathbf{R}$ with the norm

$$\|h\|_\varepsilon = \sum_{k=0}^{\infty} \varepsilon_k \|h\|_{C^k(S^1\times M)} < \infty,$$

where $\varepsilon = \{\varepsilon_k\}$ is a sufficiently rapidly decreasing sequence ([F5]). In fact, combining their proof with an argument from general topology gives us the following

Lemma 3.1 *Let H be a non-degenerate Hamiltonian and $\Upsilon_\delta(H)$ the set of all Hamiltonians H' with $\|H'-H\|_\varepsilon < \delta$. Then there exists a generic (in the sense of Baire) set of Hamiltonians $H'' \in \Upsilon_\delta(H)$ such that the set of regular almost complex structures J satisfying the following condition (*) is generic in the sense of Baire*
(*) all J-holomorphic spheres of Chern number at most 1 have no intersection with 1-periodic solutions $x \in \mathcal{P}(H'')$.

Proof First we recall that for the proof of genericity (in the sense of Baire) of regular pairs (H,J) Hofer and Salamon showed that the evaluation map

$$\mathcal{M}_s(A;J)\times_G S^2 \times S^1 \times \mathcal{P} \to M \times M : ([v,z],t,x,H) \mapsto (v(z), x(t))$$

is transversal to the diagonal in $M \times M$. Here J is a fixed regular complex structure, $\mathcal{P}$ is the Banach manifold of all pairs (x,H) of a non-degenerate 1-periodic solution $x \in \mathcal{P}(H)$ and $H \in \Upsilon_\delta(H)$, and $\mathcal{M}_s(A,J)$ is the set of simple J-holomorphic spheres, realizing class $A \in H_2(M,\mathbf{Z})$. It follows that, if we replace $\mathcal{M}_s(A;J)$ by the infinite dimensional Banach space $\mathcal{M}_s(A)$ of all pairs (v,J) of an almost complex structure J and a simple J-holomorphic sphere v in the class A, then the corresponding map is also transversal. Thus,

by the Sard-Smale theorem, there is a generic set $T(A)$ of regular values of the projection from the Banach manifold

$$\mathcal{N}^\infty(A) = \{([J,v,z],t,x,H) \mid v(z) = x(t), (x,H) \in \mathcal{P}\}$$

onto the product space $\Upsilon_\delta(H) \times \mathcal{J}$: $([J,v,z],t,x,H) \mapsto (H,J)$. Now with the help of a fact from general topology (see Appendix 1, Claim A.1.11) we get that, there is a generic (in the sense of Baire) set $\mathcal{H}(A)$ of Hamiltonians H'' in $\Upsilon_\delta(H)$ such that the set $\{J \in \mathcal{J}_{reg} \mid (H'',J) \in T(A)\}$ is generic (in the sense of Baire) in the space of calibrated almost complex structures. Taking the countable intersection of all $\mathcal{H}(A)$, where the Chern number of A is less than or equal to 1, we get the required set. □

Let H_0 be one of H'' in Lemma 3.1. We also call such a Hamiltonian **H-S-regular**. We choose disjoint compact neighborhoods $U_1,\ldots,U_m \subset S^1 \times M$ of the graphs of the finitely many contractible 1-periodic solutions of (2.1). We denote by $\mathcal{V}_\delta(H_0)$ the set of all Hamiltonians with $\|H-H_0\|_\varepsilon < \delta$ and $H = H_0$ on U_j for $j = 1,\ldots,m$. If $\delta > 0$ is sufficiently small then there are no contractible 1-periodic solutions of (2.1) outside the set $\{U_j\}$ for $H \in \mathcal{V}_\delta(H_0)$.

Further we call a pair (H,J) **H-S-regular** if the following conditions hold:
(i) H is a J-regular Hamiltonian.
(ii) The space $\mathcal{M}(x,y,H,J)$ of connecting orbits is a finite dimensional manifold for all $x,y \in \mathcal{P}(H)$. More precisely, the cross section $\bar{\partial}_{J,H}$ is transversal to the zero section.
(iii) If u is a connecting orbit with $\mu(u) \le 2$ then the image $u(\mathbf{R}\times S^1)$ has no intersection with J-holomorphic spheres of Chern number zero.

Proposition 3.2 *Given any triple $(H_0,J_0,\alpha^\sharp)$ with a H-S-regular Hamiltonian H_0 and a map $\alpha^\sharp : \cup\Delta^{2n-k} \to M$ there are a neighborhood $\mathcal{U}_\delta(J_0)$ of J_0 and a generic set $S(\alpha^\sharp) \subset \mathcal{V}_\delta(H_0)\times\mathcal{U}_\delta(J_0)$ such that the following holds for $(H,J) \in S(\alpha^\sharp)$.*
(i) The pair (H,J) is H-S-regular.
(ii) The map $\alpha^\sharp$ meets the evaluation map $e : \mathcal{M}(x,y,H,J) \to M$, $e(u) = u(0,0)$, transversally.
(iii) There is no connecting orbit $u \in \mathcal{M}(x,y,H,J)$ of relative index less than or equal to $k+1$, where $k = \dim\alpha$, and besides, satisfying one of the following conditions (a) and (b):
(a) The image $u(\mathbf{R}\times S^1)$ intersects with one of holomorphic spheres of Chern number zero, and moreover, $u(0,0) \in \mathrm{Im}\alpha^\sharp$.
(b) There are $m \ge 1$ holomorphic spheres $v_1,\ldots,v_m$ and $2m$ points $z_1^\pm,\ldots,z_m^\pm \in S^2$ such that $u(0,0) = v_1(z_1^+), v_1(z_1^-) = v_2(z_2^+),\ldots,v_m(z_m^-) \in \mathrm{Im}\alpha^\sharp$, and besides, the sum of the Chern numbers of the spheres v_i is less than or equal to $\frac{1}{2}\cdot(k+1-\mu(u))$.

From Proposition 3.2, combined with the Gromov compactness argument, we easily get the following corollary.

Corollary 3.3 *For a pair* $(H, J) \in S(\alpha^\sharp)$ *as in Proposition 3.2 the intersection number* $m^{\alpha^\sharp}(\tilde{x}, \tilde{y})$ *is finite.*

In fact, for the proof of Corollary 3.3 we need only the conditions $\mu(u) \le \dim \alpha = k$ and $\sum_i (c_1(v_i)) \le \frac{1}{2} \cdot (k - \mu(u))$ in Proposition 3.2.(iii). Transversality with $\mu(u) = k+1$ is used for the proof of "invariance properties" of the cap action (see Proposition 3.7, Proposition 3.8). In general, the transversality with $\mu(u) > k+1$ breaks down for the same reason that prevents the associativity of the action.

Proof of Proposition 3.2 Let us prove the first part (i). Note that the proof of this statement has been sketched in [H-S]. But their proof is based on a similar result in [S-Z], the detailed proof of which is not written down. For the sake of completeness we shall carry out a detailed proof here (and in Appendix 1). Write $S(0) = \{(H', J') \in \mathcal{V}_\delta(H_0) \times \mathcal{U}_\delta(J_0) | H' \text{ is } J'\text{-regular.}\}$. We fix a pair $x^\pm \in \mathcal{P}(H_0)$. Denote by $\mathcal{P}(x^-, x^+)$ the Banach manifold of $W^{1,p}_{loc}$ maps $u : \mathbf{R} \times S^1 \to M$ which satisfy the limit condition (2.4.1) in $W^{1,p}$ sense with $p > 2$. Let $\mathcal{B} = \mathcal{P}(x^-, x^+) \times \mathcal{V}_\delta(H_0) \times \mathcal{U}_\delta(J_0)$ and $\mathcal{E}$ be the bundle over $\mathcal{B}$ whose fiber $\mathcal{E}_{(u,H,J)} = L^p(u^*TM)$. Recall that the space of connecting orbits is the zero set of the cross section $\mathcal{F} : \mathcal{B} \to \mathcal{E}$ defined by

$$\mathcal{F}(u, H, J) = \bar{\partial}_{J,H} u.$$

The differential of this section at zero (u, H, J) is the linear operator given by

$$D\mathcal{F}(u, H, J)(\xi, h, Y) = D_u \xi + \nabla h(t, u) + Y(\frac{\partial u}{\partial t} + X_H). \tag{3.2}$$

It was shown that D_u is a Fredholm operator of index $\mu(u)$ ([F3, S-Z], see also Appendix 1, Fact A.1.10). Moreover we have the following (see the proof in Appendix 1, Prop. A.1.1).

Lemma 3.4 (cf. [S-Z, Theorem 8.4]) *The section* $\mathcal{F}$ *is transversal to the zero section.*

This Lemma implies that the set

$$\mathcal{M}(x^-, x^+) := \{(u, H, J) \in \mathcal{B} | \, \bar{\partial}_{J,H} u = 0\}$$

is a separable infinite dimensional Banach manifold. Denote by $S(1)$ the set of regular values of the projection from $\mathcal{M}(x^-, x^+)$ to the second and third factors $\mathcal{V}_\delta(H_0) \times \mathcal{U}_\delta(J_0)$. Then the inverse image $\mathcal{M}(x^-, x^+, H, J)$ of $(H, J) \in S(1)$ by this projection is a smooth manifold of connecting orbits between 1-periodic solutions $x^-, x^+ \in \mathcal{P}(H)$.

Now we shall show that there is a dense set $S(2) \subset S(1)$ such that for $(H, J) \in S(2)$ the pair (H, J) is H-S-regular. Denote by $\mathcal{M}_s(A, x^-, x^+)$ the set of all quadruples $[v, u, H, J]$ such that v is a simple J-holomorphic sphere

in the homology class $A \in H_2(M, \mathbf{Z})$ and u is an element in $\mathcal{M}(x^-, x^+, H, J)$. Clearly, $\mathcal{M}_s(A, x^-, x^+)$ is the zero set of the section $\mathcal{K}$ from the Banach space $W^{1,p}(S^2, M) \times \mathcal{B}$ to the bundle $\mathcal{G}$ over it whose fibre at $[v, u, H, J]$ is the direct sum of $\Lambda_v(J)$ and $\mathcal{E}_{(u,H,J)}$. Here $\Lambda_v(J)$ consists of all L^p sections of the vector bundle over S^2 whose fibre at $z \in S^2$ is the space of J-anti-linear maps $T_z S^2 \to T_{v(z)} M$ and

$$\mathcal{K}([v, u, H, J]) = \bar{\partial}_J(v) \oplus \mathcal{F}(u, H, J).$$

Using McDuff's result which states that the differential $D\bar{\partial}(v, J)$ is surjective [MD], and Floer's, Salamon-Zehnder's result which states that the differential $D\mathcal{F}(u, H)$ is surjective ([F3, S-Z], see also Appendix 1, Proposition A.1.1), we easily show that the differential $D\mathcal{K}$ is surjective. Now we consider the evaluation map

$$E_1 : \mathcal{M}_1(A) = \mathcal{M}_s(A, x^-, x^+) \times_G S^2 \times S^1 \to M \times M,$$

given by

$$([v, u, H, J], z, t) \mapsto (v(z), u(0, t)).$$

We shall show that the subspace

$$\mathcal{N}_1(A) = \{([v, u, H, J], z, t) \mid v(z) = u(0, t)\}$$

is an infinite dimensional Banach submanifold in $\mathcal{M}_1(A)$. To do this, it is sufficient to prove that the evaluation map E_1 is transversal to the diagonal $\Delta_M \subset M \times M$.

Claim 3.5 [H-S] *The evaluation map*

$$e_t : \mathcal{M}_s(A, x^-, x^+) \to M, \; e_t([v, u, H, J]) = u(0, t)$$

is a submersion for every $t \in S^1$.

In the proof of this claim (see Appendix 1, Proposition A.1.4) we use only perturbations of Hamiltonians H. It follows that the evaluation map E_1 is transversal to the diagonal $\Delta_M \subset M \times M$.

Now we choose $S(A)$ as the set of regular values of the projection $\mathcal{N}_1(A) \to \mathcal{V}_\delta(H_0) \times \mathcal{U}_\delta(J_0)$ onto the factors (H, J). The Fredholm index of this projection is $2c_1(A) + \mu(u) - 3$, which is negative if $c_1(A) = 0$ and $\mu(u) \le 2$. Choose $S(2, x^-, x^+)$ as the intersection of $S(1)$ and $S(A)$ when A runs over all spheres of Chern number zero. Then we take $S(2)$ as the intersection of all the sets $S(2, x^-, x^+)$ where $x^-, x^+ \in \mathcal{P}(H_0)$. Clearly, the set $S(3) := S(2) \cap S(0)$ is the required set for the part (i).

In order to prove the remaining parts (ii) and (iii) of Proposition 3.2 we consider two evaluation maps

$$E_a : \mathcal{M}_2 = \mathcal{M}_s(A, x^-, x^+) \times_G S^2 \times \mathbf{R} \times S^1 \times \cup \Delta^{2n-k} \to M \times M \times M \times M,$$

$$([v,u,H,J],z,s,t,q) \mapsto (v(z), u(s,t), u(0,0), \alpha^\sharp(q)),$$

and $E_b : \mathcal{M}_3 \to M \times \ldots_{(2m+2\text{times})} \times M$, where

$$\mathcal{M}_3 = \mathcal{M}_s(A_1,\ldots,A_m,x^-,x^+) \times_G (S^2 \times S^2) \ldots_{(m\text{times})} \times_G (S^2 \times S^2) \times \cup\Delta^{2n-k},$$

$$([v_1,\ldots,v_m,u,H,J],z_1^\pm,\ldots,z_m^\pm,q) \mapsto u(0,0),\ldots v_i(z_i^-), v_i(z_i^+),\ldots,\alpha^\sharp(q).$$

Here the space $\mathcal{M}_s(A_1,A_2,\ldots,A_m,x^-,x^+)$ is defined as is $\mathcal{M}_s(A,x^-,x^+)$; namely $v_1,\ldots,v_m$ are simple J-holomorphic spheres and u is a connecting orbit with respect to (H,J). Now we show that the maps E_a and E_b are transversal to the product of diagonals in the target spaces respectively.

Claim 3.6 (a) *For each (s,t) the evaluation map*

$$e_{s,t} : \mathcal{M}_s(A,x^-,x^+) \to M \times M,\ [v,u,H,J] \mapsto (u(0,0), u(s,t))$$

is a submersion provided $u(0,0) \neq u(s,t)$ or $t \neq 0$.
(b) *For each $(z_1^\pm,\ldots,z_m^\pm)$ the evaluation map*

$$e_z : \mathcal{M}_s(A_1,A_2,\ldots,A_m,x^-,x^+) \to M \times \ldots_{(2m\text{times})} \times M,$$

$$(v_1,\ldots,v_m,u,H,J) \mapsto (v_1(z_1^-), v_1(z_1^+),\ldots,v_m(v_m^-), v_m(z_m^+)),$$

is a submersion provided any two of $\{v_j(z_j^-), v_j(z_j^+)\}$ do not coincide.

For the proof of Claim 3.6(a) we use only perturbations of Hamiltonians H (similar to that one of Claim 3.5). The argument is standard and therefore omitted here. It implies that the map E_a is transversal to the product of two diagonals in the target space at points where $u(0,0) \neq u(s,t)$ or $t \neq 0$.

So the remaining case is that $v(z) = u(0,0) = u(s,0) = \alpha^\sharp(q)$. In this case, we also use perturbations of almost complex structures (see the proof of Proposition A.1.5). That is we use perturbations of almost complex structures outside of a neighborhood of the image of u to show the transversality to the first factor diagonal and we use perturbation of Hamiltonians to show the transversality to the second factor diagonal. Note that perturbation of almost complex structures outside of a neighborhood of the image of u does not effect the connecting orbit u. Thus, it follows that the space

$$\mathcal{N}_2 = \{([v,u,H,J],z,s,t,q) |\ v(z) = u(s,t), u(0,0) = \alpha^\sharp(q)\}$$

is an infinite dimensional Banach submanifold of $\mathcal{M}_2$. The projection from $\mathcal{N}_2$ to the factors (H,J) is a Fredholm map of index $2c_1(A) - 2 + \mu(u) - k$. This number is negative by the condition in Proposition 3.2.(iii). Denote by $S(4A)$ the set of regular values of the projection $\mathcal{N}_1 \to \mathcal{V}_\delta(H_0) \times \mathcal{U}_\delta(J_0)$. Let $S(4)$ be the countable intersection of all the set $S(4A)$ with $S(3)$ when A runs over all spheres of Chern number zero. Then we get that for any pair $(J,H) \in S(4)$ the conditions (i), (ii) and (iiia) in Proposition 3.2 hold.

To prove Claim 3.6(b) we use only perturbations of almost complex structures J (see Appendix 1, Proposition A.1.5). Thus, it follows that the map E_b is transversal to the diagonal in the target space at points where all $v_j(z_j^\pm)$ are distinct. If some of $v_j(z_j^\pm)$ coincide, we have $j_1,\ldots,j_{m-1} \in \{1,\ldots,m\}$ such that $u(0,0) = v_{j_1}(z_{j_1}^-), v_{j_1}(z_{j_1}^+) = v_{j_2}(z_{j_2}^-),\ldots,v_{j_{m-1}}(z_{j_{m-1}}^-) = \alpha^\sharp(q)$. Hence the problem reduces to the one for $m-1$. (Note that $c_1(v_j) \geq 0$.) Consequently, the space $\mathcal{N}_3$ given by

$$\begin{aligned}\mathcal{N}_3 = \{([v_1,\ldots v_m,u,H,J],z_1^\pm,\ldots z_m^\pm,q) |\ & u(0,0) = v_1(z_1^-), v_1(z_1^+)\\ & = v_2(z_2^-),\ldots,v_m(z_m^-) = \alpha^\sharp(q)\}\end{aligned}$$

is a infinite dimensional Banach submanifold of $\mathcal{M}_3$. The projection of $\mathcal{N}_3 \to \mathcal{V}_\delta(H_0) \times \mathcal{U}_\delta(J_0)$ on the factors (H,J) is a Fredholm map of index $2\sum_i c_i + \mu(u) - k - 2m$, where c_i is the Chern number of A_i. Our condition in Proposition 3.2.(iii b) implies that this number is negative. We take the set $S(5)$ of generic values of each above projection corresponding to each m-ple $(A_1,\ldots,A_m)$. Now, $S(\alpha^\sharp) = S(5) \cap S(4)$ is the set we are looking for. □

A pair (H,J) satisfying conditions (i)–(iii) in Proposition 3.2 is called α-regular. It is easy to see that the set of intersection of all α-regular pair (H,J), when α runs over the homology group $H^*(M,\mathbf{Z}_2)$, is generic in $\mathcal{V}_\delta(H_0) \times \mathcal{U}_\delta(J_0)$. From now on we consider only such a pair (H,J) which we call a **F-generic pair**. For the sake of simplicity, in the remaining part of this note we denote the space $\mathcal{M}(x,y,H,J)$ by $\mathcal{M}(x,y)$, since no confusion may arise.

Next we prove that the action of α on a Floer chain complex descends on its Floer homology. Moreover, this action does not depend on a cycle $\alpha^\sharp(\cup\Delta^{2n-k})$ representing the Poincaré dual class of α. This statement follows

Proposition 3.7 *Let c be an element in Floer chain complex (H,J). Then we have*

(i) $$\alpha^\sharp \cap \partial c = \partial(\alpha^\sharp \cap c).$$

(ii) Suppose $\alpha^\sharp(\cup\Delta^{2n-k})$ is a boundary: $\alpha^\sharp = \partial\beta^\sharp$, where $(H,J,\beta^\sharp)$ is also a regular triple. Then we have

$$\alpha^\sharp \cap c = \partial(\beta^\sharp \cap c) + \beta^\sharp \cap \partial c.$$

Proof (i) Fix a pair $\tilde{x},\tilde{y}$ of 1-periodic solutions of Conley-Zehnder index difference $k+1$. Proposition 3.2 and Floer's gluing argument (see Fact A.1.6) imply that the space $\mathcal{M}^{\alpha^\sharp}(\tilde{x},\tilde{y})$ is a 1-dimensional manifold whose ends are the union of the set $\mathcal{M}(\tilde{x},\tilde{z}) \times \mathcal{M}^{\alpha^\sharp}(\tilde{z},\tilde{y})$ satisfying $\mu(\tilde{z}) = \mu(\tilde{x}) - 1$ and $\mathcal{M}^{\alpha^\sharp}(\tilde{x},\tilde{z}) \times \mathcal{M}(\tilde{z},\tilde{y})$ satisfying $\mu(\tilde{z}) = \mu(\tilde{y}) + 1$. Since the number of these ends are even we get

$$m^{\alpha^\sharp}(\partial\tilde{x},\tilde{y}) = n_2(\alpha^\sharp \cap \tilde{x},\tilde{y})$$

which proves the part (i) of Proposition 3.7 immediately.

(ii) Fix a pair $\tilde{x}, \tilde{y}$ of 1-periodic solutions of Conley-Zehnder index difference $(k+1)$. Proposition 3.2 and Floer's gluing argument (see Fact A.1.6) imply that the space $\mathcal{M}^{\beta^\sharp}(\tilde{x}, \tilde{y})$ is an 1-dimensional submanifold whose ends are the union

$$\mathcal{M}^{\alpha^\sharp}(\tilde{x}, \tilde{y}) \cup \{\mathcal{M}^{\beta^\sharp}(\tilde{z}, \tilde{y}) \times \mathcal{M}(\tilde{x}, \tilde{z}) | \mu(\tilde{z}) = \mu(\tilde{x}) - 1\}$$
$$\cup \{\mathcal{M}^{\beta^\sharp}(\tilde{x}, \tilde{z}) \times \mathcal{M}(\tilde{z}, \tilde{y}) | \mu(\tilde{z}) = \mu(\tilde{y}) + 1\}.$$

The rest of the proof continues in the same way. □

By Proposition 3.7 we can denote the cap action of $\alpha \in H^*(M, \mathbf{Z}_2)$ simply by $\alpha\cap$. We now prove the naturality of this action in the category of Floer homology. Recall that given two H-S-regular pairs (H, J) and (H', J') there is a natural chain homomorphism Θ between the corresponding Floer chain complexes $CF_*(H, J)$ and $CF_*(H', J')$. This chain homomorphism Θ can be defined by counting the number of solutions of the "chain homomorphism equation"

$$\frac{\partial u}{\partial s} + J(s, u)\frac{\partial u}{\partial t} + \nabla H(s, t, u) = 0$$

which is an s-dependent analog of the connecting orbit equation (2.4). Therefore, transversality, compactness and gluing arguments for connecting orbits can be applied for the chain homomorphism Θ ([F3], [S-Z]).

Proposition 3.8 *Suppose that $(H, J, \alpha^\sharp)$ and $(H', J', \alpha^\sharp)$ are regular triples and Θ is a natural chain homomorphism between the corresponding Floer chain complexes $CF_*(H, J)$ and $CF_*(H', J')$.*
(i) For $c \in CF_*(H, J)$ we have $\alpha^\sharp \cap (\Theta c) = \Theta(\alpha^\sharp \cap c)$.
(ii) Consequently, for $c \in HF_*(H, J)$ we have $\alpha \cap (\Theta c) = \Theta(\alpha \cap c)$

Proof The proof of this lemma is similar to the previous one.

Fix two critical points $\tilde{y} \in \tilde{\mathcal{P}}(H)$ and $\tilde{y}' \in \tilde{\mathcal{P}}(H')$ of Conley-Zehnder index difference k, where $k = \dim \alpha$. Consider the space

$$\mathcal{M}_\Theta^{\alpha^\sharp}(\tilde{y}, \tilde{y}') = \{(u, a) \in \mathcal{M}_\Theta(\tilde{y}, \tilde{y}') \times \mathbf{R} | u(a, 0) \in \mathrm{Im}\alpha^\sharp\}.$$

This space is a 1-dimensional manifold, whose ends are the union of the set $\mathcal{M}^{\alpha^\sharp}(\Theta(\tilde{y}), \tilde{y}')$ and the sets $\mathcal{M}^{\alpha^\sharp}(\tilde{y}, \tilde{z}) \times \mathcal{M}_\Theta(\tilde{z}, \tilde{y}')$ satisfying $\mu(\tilde{z}) = \mu(\tilde{y})$ (see also Fact A.1.7 on gluing maps). The rest of the proof continues in the same way. □

4 Associativity of the cap action

Associativity of the cap action means that $(\alpha \smile \beta) \cap c = \alpha \cap (\beta \cap c)$ holds for any $\alpha, \beta \in H^*(M, \mathbf{Z}_2)$ and $c \in HF_*(H, J)$. In fact, associativity does not hold

in general. For instance, associativity fails for complex projective spaces (see [F3]). The purpose of this section is to prove the associativity under certain assumptions. Write $d_J(M) = \min\{c_1(S) | S$ is a J-holomorphic sphere$\}$. The following proposition is predicted by Floer under a slightly stronger condition, namely $\omega_{|\pi_2} = 0$ [F3].

Proposition 4.1 *Let (M, ω) be a weakly monotone symplectic manifold and (H, J) a F-generic pair of a time-dependent Hamiltonian and a calibrated almost complex structure. Moreover, suppose that for $\alpha \in H^{k_1}(M; \mathbf{Z}_2), \beta \in H^{k_2}(M; \mathbf{Z}_2)$ the triple $(H, J, \alpha^\sharp \cap \beta^\sharp)$ is also regular. If $k_1 + k_2 < d_J(M)$, then we have*

$$(\alpha \smile \beta) \cap c = \alpha \cap (\beta \cap c)$$

for all $c \in HF_*(H, J)$.

In particular, we get

Corollary 4.2 *If there are no J-holomorphic spheres, then the associativity holds.*

Proof of Proposition 4.1 Choose cycles $\alpha^\sharp : \cup\Delta^{2n-k_1} \to M$ and $\beta^\sharp : \cup\Delta^{2n-k_2} \to M$ which represent the Poincaré dual of α and β respectively. Recall that for $\tilde{x}, \tilde{y}, \tilde{z} \in \tilde{\mathcal{P}}(H)$, the evaluation maps $\mathcal{M}(\tilde{x}, \tilde{y}) \to M$ and $\mathcal{M}(\tilde{y}, \tilde{z}) \to M$, given by $u \mapsto u(0, 0)$, are transversal to $\beta^\sharp$ and $\alpha^\sharp$ respectively (Proposition 3.2.(ii)). In particular, the images of these maps are disjoint from lower dimensional strata of $\alpha^\sharp$ and $\beta^\sharp$ if $\mu(\tilde{x}) - \mu(\tilde{y}) = k_2$, $\mu(\tilde{y}) - \mu(\tilde{z}) = k_1$. Since we are only interested in connecting orbits whose values at $(0, 0)$ belong to the images of $\alpha^\sharp$ or $\beta^\sharp$, we may regard $\alpha^\sharp$ and $\beta^\sharp$ as if they are smooth maps from smooth manifolds.

For $\tilde{x}, \tilde{z} \in \tilde{\mathcal{P}}(H)$ with $\mu(\tilde{x}) - \mu(\tilde{z}) = k_1 + k_2$, we write

$$\overset{\circ}{\mathcal{M}}{}^{\alpha^\sharp, \beta^\sharp}(\tilde{x}, \tilde{z}) =$$
$$\{(u, a) \in \mathcal{M}(\tilde{x}, \tilde{z}) \times \mathbf{R} | u(a, 0) \in \mathrm{Im}\alpha^\sharp,\ u(-a, 0) \in \mathrm{Im}\beta^\sharp \text{ and} a > 0\}.$$

By an argument similar to the proof of Proposition 3.2 we can assume that the map $\Phi : \mathcal{M}(\tilde{x}, \tilde{z}) \times \mathbf{R} \to M \times M$, given by $\Phi(u, a) = (u(a, 0), u(-a, 0))$, is transversal to $\alpha^\sharp \times \beta^\sharp$. Thus, $\overset{\circ}{\mathcal{M}}{}^{\alpha^\sharp, \beta^\sharp}(\tilde{x}, \tilde{z})$ is a manifold of dimension 1. To prove the associativity, we have to investigate the end of $\overset{\circ}{\mathcal{M}}{}^{\alpha^\sharp, \beta^\sharp}(\tilde{x}, \tilde{z})$.

Let $\{(u_i, a_i)\}$ be a sequence in $\overset{\circ}{\mathcal{M}}{}^{\alpha^\sharp, \beta^\sharp}(\tilde{x}, \tilde{z})$. The weak-compactness argument shows that after taking a subsequence, the image of u_i converges to the image of a connecting orbit uniformly or splits into images of connecting orbits. Note that $\mu(\tilde{x}) - \mu(\tilde{z}) = k_1 + k_2 < d_J(M)$. Thus no J-holomorphic bubbles appear for a F-generic pair (H, J).

If $\{a_i\}$ is bounded from above, we can assume that the limit exists, and we denote this limit by a_∞. In this case, $\{u_i\}$ converges (without bubbling) to a connecting orbit u_∞ such that $u_\infty(a_\infty, 0) \in \mathrm{Im}\ \alpha^\sharp$ and $u_\infty(-a_\infty, 0) \in \mathrm{Im}\ \beta^\sharp$. The argument goes as follows. Let u_∞ be the limit of a u_i in C^∞_{loc}-topology. Since the convergence is uniform on compact subsets, $u_\infty(a_\infty, 0) \in \mathrm{Im}\ \alpha^\sharp$ and $u_\infty(-a_\infty, 0) \in \mathrm{Im}\ \beta^\sharp$. If the image of u_i splits into images of at least two connecting orbits, the relative index of u_∞ is less than $k_1 + k_2$. Thus the dimension counting argument shows that we can avoid such a situation. Hence we have $u_\infty \in \mathcal{M}(\tilde{x}, \tilde{z})$. If $a_\infty > 0$, we get $u_\infty \in \overset{\circ}{\mathcal{M}}{}^{\alpha^\sharp,\beta^\sharp}(\tilde{x}, \tilde{z})$. If $a_\infty = 0$, then $u_\infty(0,0) \in \mathrm{Im}\ \alpha^\sharp \cap \mathrm{Im}\ \beta^\sharp$ and $(u_\infty, 0)$ is an end of $\overset{\circ}{\mathcal{M}}{}^{\alpha^\sharp,\beta^\sharp}(\tilde{x}, \tilde{z})$.

If $\{a_i\}$ is not bounded, $\{u_i\}$ split into l connecting orbits $v_1, \dots, v_l$ with $l \geq 2$. Using F-genericity of (H, J) again (Proposition 3.2.(ii)) we obtain that $l = 2$.

Therefore there are two types of ends of $\overset{\circ}{\mathcal{M}}{}^{\alpha^\sharp,\beta^\sharp}(\tilde{x}, \tilde{z})$.

Case 1. $\{u_i\}$ converges to $u_\infty \in \mathcal{M}(\tilde{x}, \tilde{z})$ such that $u_\infty(0,0) \in \mathrm{Im}\alpha^\sharp \cap \mathrm{Im}\beta^\sharp$.

Case 2. $\{u_i\}$ splits into $v_1 \in \mathcal{M}^{\beta^\sharp}(\tilde{x}, \tilde{y})$ and $v_2 \in \mathcal{M}^{\alpha^\sharp}(\tilde{y}, \tilde{z})$ for some $\tilde{y}$ with $\mu(\tilde{x}) - \mu(\tilde{y}) = k_2$ and $\mu(\tilde{y}) - \mu(\tilde{z}) = k_1$.

In fact, the set of these limits is the boundary of $\overset{\circ}{\mathcal{M}}{}^{\alpha^\sharp,\beta^\sharp}(\tilde{x}, \tilde{z})$. The gluing argument gives the collar neighborhood of limit points in Case 2 (see Fact A.1.8). For the existence of the collar neighborhoods of limit points in Case 1, it suffices to show that 0 is a regular value of the projection of $\Phi^{-1}(\mathrm{Im}\alpha^\sharp \times \mathrm{Im}\beta^\sharp) \subset \mathcal{M}(\tilde{x}, \tilde{z}) \times \mathbf{R}$ to the second factor $\mathbf{R}$. Suppose that 0 is not a regular value, we can choose a path (u_τ, a_τ) in $\Phi^{-1}(\mathrm{Im}\alpha^\sharp \times \mathrm{Im}\beta^\sharp)$ such that $d/d\tau|_{\tau=0} a_\tau = 0$ and $d/d\tau|_{\tau=0} u_\tau(0,0) \neq 0$. Then $d/d\tau|_{\tau=0} u_\tau(0,0)$ is tangent to both of the images of $\alpha^\sharp$ and $\beta^\sharp$, hence the image of the intersection cycle $\gamma^\sharp$ of $\alpha^\sharp$ and $\beta^\sharp$. Since $\mathcal{M}(\tilde{x}, \tilde{z})$ and $\gamma^\sharp$ are of complementary dimension, $d/d\tau|_{\tau=0} u_\tau(0,0)$ cannot be tangent to $\gamma^\sharp$. Otherwise it contradicts to the transversality of the evaluation map to $\gamma^\sharp$ (Proposition 3.2.(ii)). Hence 0 is a regular value.

Since the end of $\overset{\circ}{\mathcal{M}}{}^{\alpha^\sharp,\beta^\sharp}(\tilde{x}, \tilde{z})$ is either limits in Case 1 or Case 2, we get

$$m^{\alpha \smile \beta}(\tilde{x}, \tilde{z}) = \sum_y m^\beta(\tilde{x}, \tilde{y}) \cdot m^\alpha(\tilde{y}, \tilde{z}).$$

Hence we get the associativity. □

Remark 4.3 *The proof of Proposition 3.2 also implies that the action of α and β are commutative in the sense of graded algebra, if $k_1 + k_2 < d_J(M)$.*

5 Proof of the main theorem

We identify the symplectic fixed points with the 1-periodic solutions of a periodic Hamiltonian H. Let $\mathcal{P}(H)$ be the subset of all contractible 1-periodic solutions. We want to estimate the number of such solutions. Since regular Hamiltonians are dense, we take a sequence of regular Hamiltonians $\{H_i\}$ converging to H:

$$\lim_{i \to \infty} \| H_i - H \|_{C^2} = 0.$$

It is easy to see that the limits of 1-periodic solutions of H_i are also 1-periodic solutions of H (Lemma 5.1). To distinguish these limits we use the cap action of cohomology ring $H^*(M^{2n}, \mathbf{Z}_2)$ on the Floer homology group $HF_*(H_i, J)$ (Lemma 5.2). Finally, the computation of the Floer homology group associated to a C^2-small time-independent function on M^{2n} gives us the non-triviality of the cap action of the fundamental cocycle $[M] \in H^{2n}(M^{2n}, \mathbf{Z}_2)$ on the Floer homology group (Lemma 5.3).

First let us recall that given any (time-dependent) Hamiltonian H, whose periodic solutions are isolated, there exists a positive constant $\hbar_0$ such that the energy of each non-trivial connecting orbit u satisfying (2.4) is greater than or equal to $\hbar_0$, and besides, $\hbar_0$ is a lower bound for energy of any J-holomorphic sphere in M [H-S].

Lemma 5.1 (i) *For any $\varepsilon > 0$, there exists i_0 such that if $i \geq i_0$ and $z : S^1 \to M$ be a 1-periodic solution of $\dot{z} = X_{H_i}(z)$, then z satisfies $\| z - z_0 \|_{C^1} < \varepsilon$ for some $z_0 \in \mathcal{P}(H)$.*
(ii) *Suppose that 1-periodic solutions of H are isolated and $\varepsilon_0 > 0$ is the minimal C^0-distance between distinct elements in $\mathcal{P}(H)$. For any $\varepsilon \in (0, \varepsilon_0)$ there exists an integer i_1 satisfying the following:*
If $i \geq i_1$ and u is a connecting orbits between 1-periodic solutions z and z' $\in \mathcal{P}(H_i)$ with

$$E(u) \leq \hbar_0/2 \mathrm{and} \|z - z_0\|_{C^1} \leq \varepsilon/2 \mathrm{forsome} z_0 \in \mathcal{P}(H),$$

then the image $u(\mathbf{R} \times S^1)$ is contained in 2ε-neighborhood of z_0.

Proof (i) By the Ascoli-Arzela theorem, there exists $\delta > 0$ satisfying the following condition.

If a loop $x : S^1 \to M$ satisfies $\| \dot{x} - X_H(x) \|_{C^0} < \delta$, then $\| x - y \|_{C^1} < \varepsilon$ holds for some $y \in \mathcal{P}(H)$.

By the choice of $\{H_i\}$, there exists a positive integer i_0 such that $\| X_{H_i} - X_H \|_{C^0} < \delta$. Hence we have

$$\| \dot{z} - X_H(z) \|_{C^0} \leq \| \dot{z} - X_{H_i}(z) \|_{C^0} + \| X_{H_i}(z) - X_H(z) \|_{C^0} \leq \delta.$$

Therefore there exists $z_0 \in \mathcal{P}(H)$ such that $\| z - z_0 \|_{C^1} < \varepsilon$.

(ii) By the definition of ε_0, each limit of any subsequence of the sequence $\{z_i \in \mathcal{P}(H_i) | \|z_i - z_0\|_{C^1} \leq \varepsilon/2\}$, when H_i converges to H, is z_0. Suppose that the statement is false. Then we have a sequence of connecting orbits $\{u_{i_j}\}$ such that one of the end is ε-close to z_0, the energy $E(u_{i_j}) < \hbar_0/2$ and the image is not contained in ε-neighborhood of z_0. Then after translation in $\mathbf{R}$-direction, we may assume that u_{i_j} converges to a connecting orbit u_∞ such that $u_\infty(0,t)$, for some t, is outside of the ε-neighborhood of z_0. Hence u_∞ is non-trivial connecting orbit for (H, J). Thus $E(u_\infty) > \hbar_0$, which contradicts to the fact that $E(u_\infty) < \liminf\ E(u_{i_j}) < \hbar_0/2$. □

Note that under the condition of Main Theorem, the Floer homology can be defined on M^{2n}. Moreover the action of $H^*(M^{2n}, \mathbf{Z}_2)$ on the Floer homology group is associative, since there is no J-holomorphic sphere for a regular J and we can apply Corollary 4.2. Therefore, we obtain our Main Theorem immediately from the following two lemmas.

Lemma 5.2 *Let (M^{2n}, ω) be a symplectic manifold as in the Main Theorem. Suppose that $\alpha_i \in H^{\kappa_i}(M^{2n}, \mathbf{Z}_2)$, $i = 1, \ldots, l$, are elements of positive degree. If the composition $(\alpha_1 \cap) \circ \ldots (\alpha_l \cap)$ of the actions on the Floer homology group of M^{2n} is non-trivial, then the number of distinct elements in $\mathcal{P}(H)$ is at least $l + 1$.*

Lemma 5.3 *If M^{2n} satisfies the condition of Main Theorem, then the action of the fundamental class $[M] \in H^{2n}(M^{2n}, \mathbf{Z}_2)$ on the Floer homology group is non-trivial.*

Proof of Lemma 5.2 We assume that all the periodic solutions are isolated (otherwise, Lemma 5.2 is trivial). Choose H_i as above. Lemma 5.1.(i) tells us that any 1-periodic solution for H_i is close to one of 1-periodic solutions for H in C^1-topology. If two loops y and z are sufficiently close in C^0-topology, we can make a bounding disk of z from a bounding disk of y, unique up to homotopy.

Namely, we choose a homotopy $S^1 \times [0, 1] \to M$ between two loops y and z in an ε neighborhood of y and glue this cylinder with the bounding disk of y. The resulting map is a bounding disk of z. If ε is small enough (for instance, smaller than the injectivity radius of M), we can show the uniqueness up to homotopy. Hence we can compare Conley–Zehnder indices of these 1-periodic solutions with the bounding disks as above.

Claim 5.4 *Suppose that $z, z' \in \mathcal{P}(H_i)$, where i is sufficiently large, are sufficiently close in C^1-topology. Then the Conley-Zehnder index difference $|\mu([z, v]) - \mu([z', v'])|$ is bounded by $2n$ if v and v' are bounding disks obtained by the procedure above.*

Proof Recall that the Conley-Zehnder index $\mu([z, v])$ equals, up to an additive constant, the analytical index of the Fredholm operator P_v on the bounding disk v

$$P_v\xi = \nabla_{\frac{\partial}{\partial r}}\xi + J(v)\nabla_{\frac{\partial}{\partial t}}\xi + \rho(\nabla_\xi J(v)\frac{\partial v}{\partial t} + \nabla_\xi \nabla H(t, v)),$$

where (r, t) are the polar coodinates of the disk v and ρ is a cut-off function supported nearby the boundary ∂D^2 (see Fact A.1.10). The Atiyah-Patodi-Singer index theorem implies that the difference index ind P_v − ind $P_{v'}$ equals the spectral flow of the elliptic operator

$$A_z\xi = J(z)(\nabla_{\frac{d}{dt}}\xi - \nabla_\xi X_{H_s}) + \nabla_\xi J(z)\frac{dz}{dt}$$

from the periodic solution z to the periodic solution z'. When z and z' are C^1-close to $z_0 \in \mathcal{P}(H)$ enough, the spectral flow comes only from the zero eigenvalue (with counting multiplicity) of the linearization of the operator $\dot{x} - X_{H_0}(x(t))$. That is the dimension of the solutions of a linear ordinary differential operator acting on vector fields along z_0, therefore, it is at most $2n$. Hence we get the Claim 5.4.

Under the assumption of Lemma 5.2 there is a sequence of elements $\{\tilde{z}_i^j\} \in \tilde{\mathcal{P}}(H_i), j = 0, \ldots, l$, such that $\mu(\tilde{z}_i^j) - \mu(\tilde{z}_i^{j+1}) = \kappa_j$, and $m^{\alpha_j}(\tilde{z}_i^j, \tilde{z}_i^{j+1}) \neq 0$. Let z_0^j be a limit of a subsequence of $\{z_i^j\}$, which is also denoted by $\{z_i^j\}$ for the sake of convenience. We want to show that if $j < k$ then $z_0^j \neq z_0^k$. Let us assume the contrary, that is, $z_0^j = z_0^k$. There are two cases:

(a) $\lim_{i\to\infty}(\mathcal{A}_{H_i}(\tilde{z}_i^j) - \mathcal{A}_{H_i}(\tilde{z}_i^k)) \leq \hbar_0/2$,

(b) $\lim_{i\to\infty}(\mathcal{A}_{H_i}(\tilde{z}_i^j) - \mathcal{A}_{H_i}(\tilde{z}_i^k)) > \hbar_0/2$.

Consider the case (a). In this case we also have

$$\lim_{i\to\infty}(\mathcal{A}_{H_i}(\tilde{z}_i^j) - \mathcal{A}_{H_i}(\tilde{z}_i^{j+1})) \leq \hbar_0/2.$$

From Lemma 5.1 (ii) we know that for i large enough the image of all connecting orbits u_i^j between $\tilde{z}_i^j$ and $\tilde{z}_i^{j+1}$ lies in a small neighborhood U_ε of z_0^j. Since z_0^j is contractible we can choose another cycle $\alpha_j^\sharp$ which has no intersection with U_ε. But this implies that $m^{\alpha_j}(\tilde{z}_i^j, \tilde{z}_i^{j+1}) = 0$, which is a contradiction.

Now we consider the case (b). Write $\tilde{z}_i^j = [z_i^j, v_i^j]$. Then the bounding disk $v_0^j = \lim_{i\to\infty} v_i^j$ equals a connected sum $v_0^k = \lim_{i\to\infty} v_i^k$ with a non-trivial element g_{jk} of $\pi_2(M^{2n})$. Note that we have

$$\lambda^{-1}c_1(g_{jk}) = \omega(g_{jk}) = \mathcal{A}_{H_0}(\tilde{z}_0^j) - \mathcal{A}_{H_0}(\tilde{z}_0^k)) > 0,$$

by the assumption. Therefore, $c_1(g_{jk}) < 0$. Next, using Claim 4.4, we get

$$\mu(\tilde{z}_i^j) - \mu(\tilde{z}_i^k) \leq 2n + 2c_1(g_{jk}),$$

which is less than or equal to zero, since $c_1(g_{jk}) \leq -n$. But it contradicts to the other assumption that $\mu(\tilde{z}_i^j) - \mu(\tilde{z}_i^k) = \kappa_j + \ldots + \kappa_{k-1} > 0$. □

Proof of Lemma 5.3 Since the action is compatible with the natural isomorphism between Floer homology groups for generic pairs, it is sufficient to deal with the time-independent C^2-small Hamiltonian case. Let h be a C^2-small Morse function such that

(1) all 1-periodic solutions are constant loops at critical points of h,

(2) the gradient flow is of Morse-Smale type and the linearization of the connecting orbit equation at these solutions are surjective,

(3) all connecting orbits of energy less than $|(\lambda)^{-1}/2|$ should be gradient trajectories,

(4) $|h| < |\lambda^{-1}/4|$.

The existence of a function h with properties (1), (2),(4) is known (see [F3], [S-Z], [H-S]). To see that such a function h, after multiplication with a small positive number, satisfies condition (3), we use the Salamon–Zehnder theorem.

Namely, the integration of the symplectic form on such a connecting orbit is zero and we can apply their theorem. Note that properties (1), (2), (4) are preserved under multiplication by a small positive number. So we can define Floer chain complex $CF_*(h, J)$ [F3], [H-S].

Now let us calculate the action of the fundamental class $[M]$ on $CF_*(h, J)$. The Poincaré dual of $[M]$ is represented by one point. We choose a point p in generic position, there is one and only one gradient trajectory passing through p and connecting a local maximum q_+ and a local minimum q_- of the function h. Let $\tilde{q}_+, \tilde{q}_-$ be q_+, q_- with trivial bounding disks. Note that the energy of this gradient trajectory (as a connecting orbit) is less than $|\lambda^{-1}/2|$. Hence all the connecting orbits between $\tilde{q}_+$ and $\tilde{q}_-$ are gradient trajectories, which implies that $m^{[M]}(\tilde{q}_+, \tilde{q}_-) = 1$.

Note that the formal sum of all local maxima (with trivial bounding disk) gives the fundamental class in Morse homology of M and any two of local minima are homologous. Thus, the equality $m^{[M]}(\tilde{q}_+, \tilde{q}_-) = 1$ proves the non-triviality of the action of the fundamental cocycle on the Floer homology group $HF_*(h, J)$. □

Appendix 1. Auxiliary technical lemmas

On the transversality argument

Let $H : S^1 \times M \to \mathbf{R}$ be a Hamiltonian such that all the 1-periodic solutions are non-degenerate. Recall that for 1-periodic solutions $x, y \in \mathcal{P}(H)$ and $p > 2$ we denote by $\mathcal{P}(x, y)$ the Banach manifold consisting of $u \in W^{1,p}_{loc}(\mathbf{R}\times S^1, M)$ such that

$$u(s,t) = \exp_{x(t)}\xi^-(s,t) \text{ for } s < -R$$

$$u(s,t) = \exp_{y(t)}\xi^+(s,t) \text{ for } s > R$$

for a sufficiently large real number R,

$$\xi^- \in W^{1,p}((-\infty, -R] \times S^1, (x \circ pr_2)^*TM)$$

and

$$\xi^+ \in W^{1,p}([R, \infty) \times S^1, (y \circ pr_2)^*TM),$$

where $pr_2 : \mathbf{R} \times S^1 \to S^1$ is the second factor projection (see [F3,F5] for details.)

For each regular almost complex structure J and a J-regular Hamiltonian H_0 we denote by $\mathcal{B}_J$ the subspace $\mathcal{P}(x, y) \times \mathcal{V}_\delta(H_0) \times \{J\} \subset \mathcal{B}$ (see section 3). We denote the restriction of the bundle $\mathcal{E}$ on $\mathcal{B}_J$ by the same symbol $\mathcal{E}$ and the restriction of the section $\mathcal{F}$ also by the same symbol $\mathcal{F}$. Thus, for a solution u of $\mathcal{F}(u, H') = 0$, the linearization of $\mathcal{F}$ at (u, H') is given by $D\mathcal{F}(\xi, h) = D_u\xi + \nabla h$ (compare with (3.2)), where D_u is an elliptic partial differential operator given by

$$D_u\xi = \nabla_s\xi + J(u)\nabla_t\xi + \nabla_\xi J(u)\frac{\partial u}{\partial t} + \nabla_\xi\nabla H'(t, u).$$

Clearly Lemma 3.4 is a direct consequence of the following

Proposition A.1.1 ([S-Z]) *The linearization $D\mathcal{F} : W^{1,p}(u^*TM) \times \mathcal{V}_\delta(H_0) \to L^p(u^*TM)$ is surjective.*

For the proof of this proposition, we need the following

Lemma A.1.2. *Let $u : \mathbf{R} \times S^1 \to M$ be a connecting orbit for (J, H') joining distinct 1-periodic solutions x and y.*
(1) For each $t_0 \in S^1$, the set $\{s \in \mathbf{R} | \ \partial u/\partial s(s, t_0) = 0\}$ is nowhere dense.
(2) The set $C_u = \{(s,t) \in \mathbf{R} \times S^1 | $ there exists $s' \in \mathbf{R}$ such that $u(s,t) = u(s',t)\}$ is nowhere dense.

Proof (1) Note that the section $\xi = \partial u/\partial s$ is a solution of $D_u\xi = 0$, i.e.

$$\nabla_s\xi + J(u)\nabla_t\xi + (\nabla_\xi J)(u)\frac{\partial u}{\partial t} + \nabla_\xi\nabla H'(t, u) = 0.$$

If ξ vanishes on $(a, b) \times \{t_0\}$, then the higher derivatives $\partial^k u/\partial s^k(s_0, t_0) = 0$ for $s_0 \in (a, b)$ and positive integer k. Hence the equation $D_u\xi = 0$ implies that $\nabla_t\xi(s_0, t_0) = 0$, which deduces that the 1-jet of ξ vanishes at (s_0, t_0). Differentiating the both sides of $D_u\xi = 0$, we can show that the k-jet of ξ vanishes at (s_0, t_0) inductively. Then the unique continuation theorem [J, Theorem 2.6.1] implies that $\xi = 0$ everywhere, which contradicts to the fact that u is a connecting orbit joining distinct periodic solutions.

(2) If not, there is an open subset U contained in C_u. Shrinking the open set U, if necessary, we can assume that $u(U) = \mathrm{Im}\ u \cap W$ for some open set $W \subset M$. We decompose $u^{-1}(W)$ into connected components V_i. We put $V_i' = \{(s',t) \in V_i | u(s',t) = u(s,t) \text{ for some } (s,t) \in U\}$ and $U_i' = \{(s,t) \in U | u(s,t) = u(s',t) \text{ for some } (s',t) \in V_i'\}$. Applying the Baire category theorem to $U = \cup U_i'$, there exists i such that U_i' contains a non-empty open subset. Summing up, we can assume that there exist open subsets U and V such that $u(U) = u(V)$. The first part of the lemma implies that $\bar{u} : \mathbf{R} \times S^1 \to M \times S^1, \bar{u}(s,t) = (u(s,t),t)$ is an immersion on a dense subset. By the implicit function theorem, after shrinking U and V if necessary, there is a diffeomorphism $\phi : U \to V$ such that $u|_U = u|_V \circ \phi$. Moreover, we can assume that $\partial u/\partial s \neq 0$ on U and V. Note that ϕ preserves the t-coordinate. Therefore we get

$$\frac{\partial u}{\partial s}(\phi(s,t)) = f(s,t)\frac{\partial u}{\partial s}(s,t) \text{ for some function } f \text{ on } U,$$

and

$$\frac{\partial u}{\partial t}(\phi(s,t)) = g(s,t)\frac{\partial u}{\partial s}(s,t) + \frac{\partial u}{\partial t}(s,t) \text{ for some function } g \text{ on } U.$$

Substituting these identities to

$$\frac{\partial u}{\partial s} \circ \phi + J(u \circ \phi)\frac{\partial u}{\partial t} \circ \phi + \nabla H'(t, u \circ \phi) = 0,$$

we get

$$f \cdot \frac{\partial u}{\partial s} + J(u \circ \phi)(g \cdot \frac{\partial u}{\partial s} + \frac{\partial u}{\partial t}) + \nabla H'(t,u) = 0.$$

Comparing this equation with the equation for connecting orbits, we get

$$\{(f-1) + gJ(u)\}\frac{\partial u}{\partial s}(s,t) = 0.$$

Since $\partial u/\partial s \neq 0$ on U, we have $f(s,t) = 1$ and $g(s,t) = 0$ on U, which implies that ϕ is a translation in the s-variable, i.e. $\phi(s,t) = (s+\alpha, t)$ for some non-zero real number α. The unique continuation theorem [J, Theorem 2.6.1] implies that $u \circ \phi = u$ on $\mathbf{R} \times S^1$, which implies that u cannot converges to 1-periodic solutions uniformly as s tends to $\pm\infty$. This is a contradiction. Consequently, C_u does not contain interior points. □

From Lemma A.1.2 we easily get the following

Lemma A.1.3 *The mapping $\bar{u} : \mathbf{R} \times S^1 \to M \times S^1$ is somewhere injective, i.e. there exists an open subset $U \subset \mathbf{R} \times S^1$ such that*
(1) $\bar{u}|_U$ is an embedding,
(2) $U = \bar{u}^{-1}(\bar{u}(U))$.

Moreover, for any non-empty open subset $V \subset \mathbf{R} \times S^1$, we can choose U, as above, contained in V.

We sketch the proof of Proposition A.1.1 due to Salamon and Zehnder. If $D\mathcal{F}$ is not surjective, there is a non-zero element η in the dual space $L^q(u^*TM)$ which annihilates the image of $D\mathcal{F}$. By the unique continuation theorem [J, Theorem 2.6.1], the set of zeros of η is nowhere dense.

Next one can show that η is proportional to $\partial u/\partial s$, i.e. $\eta(s,t) = \lambda(s,t) \cdot \partial u/\partial s$. The proof of this fact relies on Lemma A.1.3. Namely, if η and $\partial u/\partial s$ are linearly independent, one can find a perturbation h of the Hamiltonian such that the pairing of η and the Hamiltonian vector field of h is not zero, which contradicts to the fact that η annihilates the image of $D\mathcal{F}$.

By a similar argument, one can show that the function $\lambda(s,t)$ is independent of s-variable. Then one can show that $\int_0^1 \langle \eta, \partial u/\partial s \rangle dt$ has constant sign.

On the other hand, the s-derivative of $\int_0^1 \langle \eta, \partial u/\partial s \rangle dt$ is zero, because of the equations $D_u \partial u/\partial s = 0$ and $D_u^* \eta = 0$, where D_u^* is the formal adjoint of D_u. However the pairing of η and $\partial u/\partial s$ is finite, which is a contradiction. □

For $x, y \in \mathcal{P}(H_0)$ and a generic almost complex structure J, we write $\mathcal{M}_J(x,y) = \{(u,H) \in \mathcal{P} \times \mathcal{V}_\delta(H_0) |\ \bar{\partial}_{J,H} u = 0\}$. Clearly Claim 3.6 is an immediate consequence of the following

Proposition A.1.4 [H-S] *The evaluation map at (s_0,t_0)*

$$ev_{(s_0,t_0)} : \mathcal{M}_J(x,y) \to M$$

is a submersion.

Proof We shall prove that for any $\xi_0 \in T_{u(s_0,y_0)}M$, there exists a section $\xi \in W^{1,p}(u^*TM)$ and $h \in C_\epsilon^\infty$ such that

(1) $\xi(s_0,t_0) = \xi_0$,

(2) $D\mathcal{F}\xi = \nabla h$.

Write $X = \{\zeta \in W^{1,p}(u^*TM) | \zeta(s_0,t_0) = 0\}$ and restrict the linear operator $D\mathcal{F}$ to $X \times \mathcal{V}_\delta(H_0)$. First we shall show that $D\mathcal{F} : X \times \mathcal{V}_\delta(H_0) \to L^p(u^*TM)$ is surjective.

As in the proof of Proposition A.1.1, we can show that $\eta \in L^q(u^*TM)$ is a weak solution of $(D\bar{\partial}_{J,H})^*\eta = 0$ on $\mathbf{R} \times S^1 - \{(s_0,t_0)\}$, if η annihilates the image of $D\mathcal{F}$. Lemma A.1.3 and the unique continuation theorem imply that η vanishes except (s_0,t_0). However, since η is L^q, and is even represented by a measurable section, this implies that $\eta = 0$ as an element in $L^q(u^*TM)$.

Let ξ be an element in $W^{1,p}(u^*TM)$ such that $\xi(s_0,t_0) = \xi_0$. By the statement above, $D\mathcal{F}\xi = D\mathcal{F}\zeta + \nabla h$ holds for some $\zeta \in X$ and $h \in C_\epsilon^\infty$. Then $\xi - \zeta$ is the desired section. □

The proof of Claim 3.6.b is carried out in the same way as that of the following Proposition.

Proposition A.1.5. *Let $f : S^2 \to M$ and $g : S^2 \to M$ be distinct J-holomorphic spheres, p a point in S^2. For any tangent vector $v \in T_{f(p)}M$, there exists a 1-parameter family of pairs $\{(f_t, J_t)\}$ such that $(f_0, J_0) = (f, J)$, f_t is J_t-holomorphic, g is J_t-holomorphic for all t and $\partial/\partial t|_{t=0} f_t(p) = v$.*

Proof. We may assume that f is not multiply covered, hence somewhere injective. We shall show that there is an open subset U of S^2 which is disjoint from the inverse image of the set of intersection points and f is an immersion on U. If there are only finitely many intersection points of f and g, then the claim follows immediately. If there are infinitely many intersection points, then they accumulate to a common singular points of f and g [MD]. Since such points are isolated and S^2 is compact, the number of such points is finite. After removing small neighborhoods of such points, there are only finitely many intersection points in the complement. Then the claim follows as in the previous case. Then we can find an open subset W of M such that $W \cap f(S^2) = f(U)$ and $W \cap g(S^2) = \emptyset$. We only consider perturbations of calibrated almost complex structures which coincide J outside of W and denote by $\mathcal{J}_W$ the space of such almost complex structures. Write $Y = \{\eta \in W^{1,p}(f^*TM) | \eta(p) = 0\}$. As in the proof of Proposition A.1.4, the linearization operator of $\bar{\partial}$ restricted to $Y \times \mathcal{J}_W$ is surjective to $L^p(T^*S^2 \otimes f^*TM)$. Hence by the implicit function theorem, we get the Proposition. □

On the gluing argument

In general, the moduli space $\widehat{\mathcal{M}}(x,y) = \mathcal{M}(x,y)/\mathbf{R}$ is not compact. Its ends can be described by means of gluing maps [F3], e.g.

$$\widehat{\Psi} : \widehat{\mathcal{M}}(x,y) \times \widehat{\mathcal{M}}(y,z) \times [R, +\infty) \to \widehat{\mathcal{M}}(x,z)$$

for some $R > 0$. As applications, Floer proved $\partial^2 = 0$ and the invariance of Floer homology under exact deformations.

We shall recall some of Floer's argument. Let $\mathcal{U}$ and $\mathcal{V}$ be compact neighborhoods of $u_1 \in \mathcal{M}(\tilde{x}, \tilde{z})$ and $u_2 \in \mathcal{M}(\tilde{z}, \tilde{y})$ respectively. Then there is a real number R such that

$$u(s,t) = \exp_{y(t)} \xi(s,t) \text{ for } s > R - 1, u \in \mathcal{U}$$

and

$$v(s,t) = \exp_{y(t)} \zeta(s,t) \text{ for } s < -R + 1, v \in \mathcal{V}.$$

We define the "almost gluing map" as follows:

$$\begin{aligned}\Phi(u,v,\rho)(s,t) &= u(s+\rho,t) \text{ for } s \le -1\\ &\quad \exp_{y(t)}\{\beta(-s)\xi(s+\rho) + \beta(s)\zeta(s-\rho)\} \text{ for } s \in [-1,1]\\ &\quad v(s-\rho,t) \text{ for } s \ge 1,\end{aligned}$$

where β is a cut off function which vanishes on $(-\infty, 0]$ and equals 1 on $[1,\infty]$. Then the gluing map Ψ is obtained by applying the Picard iteration procedure (see [F3]). In this note we also use a similar gluing maps for describing ends of certain spaces. Namely in the proof of Proposition 3.7 we need the following fact.

Fact A.1.6. (i) *For any pair of connecting orbits $u_1 \in \mathcal{M}^{\alpha^\sharp}(\tilde{x}, \tilde{y})$ and $u_2 \in \mathcal{M}(\tilde{y}, \tilde{z})$ there is a 1-parameter family of elements $u_\rho \in \mathcal{M}^{\alpha^\sharp}(\tilde{x}, \tilde{z})$ such that $\lim_{\rho\to\infty} u_\rho = u_1$ and there are reparametrizations of u_ρ such that their limit is u_2.*
(ii) *We have the same gluing map for $u_1 \in \mathcal{M}^{\alpha^\sharp}(\tilde{y}, \tilde{z})$ and $u_2 \in \mathcal{M}(\tilde{x}, \tilde{y})$.*

Analogously, for the proof of Proposition 3.8 and Proposition 4.1 we need the following facts, respectively.

Fact A.1.7. *Suppose that $\tilde{y} \in \widetilde{\mathcal{P}}(H)$ and $\tilde{y}' \in \widetilde{\mathcal{P}}(H')$.*
(i) *For any pair of a connecting orbit $u_1 \in \mathcal{M}^{\alpha^\sharp}(\tilde{y}, \tilde{z})$, where $\tilde{z} \in \widetilde{\mathcal{P}}(H)$, and a solution of the "chain homomorphism equation" $u_2 \in \mathcal{M}_\Theta(\tilde{z}, \tilde{y}')$, there is a 1-parameter family of solutions of "chain homomorphism equation" $u_\rho \in \mathcal{M}_\Theta^{\alpha^\sharp}(\tilde{y}, \tilde{y}')$ such that $\lim_{\rho\to\infty} u_\rho = u_1$ and there are reparametrizations of u_ρ such that their limit is u_2.*
(ii) *We have the same gluing maps for solutions of "chain homomorphism equation" $u_1 \in \widehat{\mathcal{M}}_\Theta(\tilde{y}, \tilde{z}')$, where $\tilde{z}' \in \widetilde{\mathcal{P}}(H')$, and a connecting orbit $u_2 \in \mathcal{M}^{\alpha^\sharp}(\tilde{z}', \tilde{y}')$.*

Fact A.1.8. *For any pair of a connecting orbits $u_1 \in \mathcal{M}^{\alpha^\sharp}(\tilde{x}, \tilde{y})$ and $u_2 \in \mathcal{M}^{\beta^\sharp}(\tilde{y}, \tilde{z})$ there is 1-parameter family of elements $u_\rho \in \overset{\circ}{\mathcal{M}}{}^{\alpha^\sharp,\beta^\sharp}(\tilde{x}, \tilde{z})$ such that $\lim_{\rho\to\infty} u_\rho = u_1$ and the limit of reparametrized u_ρ is u_2.*

Proof of Fact A.1.6 It is enough to prove the part (i). We need to show there exists a number R_1 such that for $\rho > R_1$ the almost gluing map Φ at parameter ρ meets the inverse image of $\alpha^\sharp$ by the evaluation map at $(0,0)$ transversally and in an approximate solution $u_1 \sharp_\rho u_2$. Let $\mathcal{U}$ be a compact neighborhood of u_1 in $\mathcal{M}(\tilde{x}, \tilde{y})$ and a closed interval $[-\epsilon, \epsilon]$ such that the evaluation map

$$ev : \mathcal{U} \times [-\epsilon, \epsilon] \to M$$

given by $ev(v,s) = v(s,0)$ meets the image of $\alpha^\sharp$ transversally at one point. In particular, there is a neighborhood N of Im $\alpha^\sharp$ such that the image of the boundary of $\mathcal{U} \times [-\epsilon, \epsilon]$ by the evaluation map and N are disjoint. Since the evaluation map for the gluing map and the evaluation map for the almost gluing map are C^0-close, we can choose a sufficiently large R_1 such that for $\rho > R_1$ we have $\Psi(u, u_2, \rho)(s - \rho, 0)$ is outside of N for (u, s) in the boundary of $\mathcal{U} \times [-\epsilon, \epsilon]$. Then, for instance, the degree theory yields that for each $\rho > R_1$, the algebraic intersection number of the evaluation map and $\alpha^\sharp$ is 1. Hence we have, for each $\rho > R_1$, at least one connecting orbit $v(s,t) = \Psi(u, u_2, \rho)(s-\rho, t)$ from $\tilde{x}$ to $\tilde{z}$ such that $v(0,0) \in \alpha^\sharp$ and $u \in \mathcal{U}$. The uniqueness follows from C^∞_{loc}-convergence of reparametrized glued connecting orbits to u_1. □

Facts A.1.7 and A.1.8 can be proved in the same way.

On the Conley–Zehnder index

The Conley–Zehnder index $\mu : \widetilde{\mathcal{P}}(H) \to \mathbf{Z}$ is characterized up to additive constant by the following condition

$$\mathrm{ind} D\mathcal{F}(u) = \mu(\tilde{x}) - \mu(\tilde{y}),$$

where $u : \mathbf{R} \times S^1 \to M^{2n}$ is a smooth map satisfying the limit condition (2.4.1), (2.4.2).

A topological definition (which is also called the Maslov index) of the Conley–Zehnder index was given in [S-Z]. Here we shall give an analytical definition.

Let v be a bounding disk of $x \in \mathcal{P}(H)$. We consider an elliptic differential operator P_v on v defined as follows

$$P_v\xi = \nabla_{\frac{\partial}{\partial r}}\xi + J(v)\nabla_{\frac{\partial}{\partial t}}\xi + \rho(\nabla_\xi J(v)\frac{\partial v}{\partial t} + \nabla_\xi \nabla H(t, v)),$$

where (r, t) be the polar coordinates of the disk v and ρ is a cut-off function supported nearby the boundary ∂D^2. .

This is a Fredholm operator with the Atyiah-Patodi-Singer boundary condition. We define $\mu'(x, v) := -\mathrm{ind} P_v$. Denote by $-v$ the bounding disk v with opposite orientation. With respect to its polar coordinates compatible with the orientation of $-v$, we define a differential operator P_{-v} by

$$P_{-v}\xi = \nabla_{\frac{\partial}{\partial r}}\xi + J(-v)\nabla_{\frac{\partial}{\partial t}}\xi + \rho(\nabla_\xi J(-v)\frac{\partial(-v)}{\partial t} - \nabla_\xi \nabla H(t, -v)).$$

Fact A.1.9 ind $P_{(-v)} = 2n -$ ind P_v.

Proof We consider the 2-sphere $(-v)\sharp v$. Regard P_v and P_{-v} as differential operators from $(0,0)$-forms to $(0,1)$-forms with values in v^*TM and $(-v)^*TM$ respectively. Then we can glue P_{-v} and P_v along the boundary. This new differential operator $P_{(-v)\sharp v}$ is homotopic (through elliptic differential operators) to the Dolbeault operator tensored with a trivial complex vector bundle of rank n. Hence

$$\mathrm{ind} P_{(-v)\sharp v} = 2n.$$

On the other hand, the Atiyah-Patodi-Singer index theorem implies

$$\mathrm{ind} P_{(-v)} + \mathrm{ind} P_v = \mathrm{ind} P_{(-v)\sharp v}.$$

This proves Fact A.1.9. □

Fact A.1.10 *Let $u : \mathbf{R} \times S^1 \to M$ be a smooth map satisfying the limit condition (2.4.1) and (2.4.2). Then*

$$\mathrm{ind} D\mathcal{F}(u) = \mu'(x^-, u^-) - \mu'(x^+, u^+).$$

Proof We get from the Atiyah–Patodi–Singer index theorem and Fact A.1.9,

$$2n = \mathrm{ind} P_{u^-\sharp u\sharp(-u^+)} = \mathrm{ind} P_{u^-} + \mathrm{ind} D\mathcal{F}(u) + 2n + \mathrm{ind} P_{(-u^+)}.$$

This proves Fact A.1.10. □

A fact from general topology

Here we will give a proof of the statement which is needed for the proof of Lemma 3.1.

Claim A.1.11 *Let X and Y be metric spaces which satisfies the second axiom of countability. Suppose that S is a countable intersection of open dense subsets in the product space $X \times Y$. Consider the space*

$$X_S = \{x \in X | S \cap \{x\} \times Y \text{isacountableintersectionofopendense subsetsin} \{x\} \times Y\}.$$

Then X_S is a countable intersection of open dense subsets in X.

Proof First, we note that it suffices to prove this Claim for an open dense set S. Let U be an open set in Y, then we write

$$X_U(S) = \{x \in X | S \cap \{x\} \times Y \text{isdisjointfrom} \{x\} \times U\}.$$

It is easy to see that $X_U(S)$ is a closed subset, because S is open. Moreover we claim that $X_U(S)$ is nowhere dense. In fact, if not, there is an open subset V in X such that V is contained in $X_U(S)$. This implies that the open set $V \times U$ is disjoint from S. This contradicts to the fact that S is dense.

With our assumption, Y satisfies the second axiom of countability, so we have a countable basis of Y consisting from open subsets $\{U_i\}$. Now we will show the following equality

$$X - X_S = \cup X_{U_i}(S).$$

By definition, for any $x \in X - X_S$ the intersection $S \cap (\{x\} \times Y)$ is not dense in $\{x\} \times Y$. Thus, by the choice of $\{U_i\}$, the set $S \cap (\{x\} \times Y)$ is disjoint from $\{x\} \times U_i$ for some U_i, that is, $x \in X_{U_i}(S)$.

Summing up: $X - X_S$ is a countable union of closed nowhere dense subsets. Therefore, their complements X_S is a countable intersection of open dense subsets. □

Appendix 2. Remarks on other related results

In [F3] Floer gave a proof of the following theorem, which extends a result by Fortune [Fo].

Theorem A.2.1 *For any n let $\mathbf{C}P^n$ be provided with the standard symplectic form ω such that $\omega[g_0] = n+1$ for the generator g_0 of $H_2(\mathbf{C}P^n, \mathbf{Z})$. Then the number of fixed points of a symplectomorphism on a product*

$$P = \times_{i=1}^k \mathbf{C}P^{n_i}$$

is greater than or equal to the greatest common divisor of $\{n_i + 1\}$.

Note that under the normalization $\omega[g_0] = n + 1$ for $\mathbf{C}P^n$ the product of $\mathbf{C}P^{n_i}$ is a monotone symplectic manifold.

Since Floer gave only a brief sketch of the proof, we try here to give a comprehensive explanation of his argument.

First we consider the case of $\mathbf{C}P^n$. We denote by α the generator of $H^2(\mathbf{C}P^n)$.

Claim A.2.2 [F3] *The cap action of α is an isomorphism $HF_{2k}(H,J) \to HF_{2k-2}(H,J)$ for all $k \in \mathbf{Z}$.*

Let us recall that, by "invariant properties", to prove Claim A.2.2 it suffices to verify the claim for the Floer homology groups associated with a (small) time-independent Hamiltonian H. Floer chose H being the function $H(x) = \sum_{i=1}^{n+1} i \cdot |x_i|^2 / \sum_{i=1}^{n+1} |x_i|^2$. To compute the cap action of α on the Floer complex of this Hamiltonian H one has to count the connecting orbits coming from the gradient trajectories of H, as well as the connecting orbit approximating

J-holomorphic spheres. That is why Floer emphasized that the presence of J-holomorphic spheres is necessary for this Claim. Note that all the connecting orbits in this case can be obtained explicitly (see [H-S] for the explicit solution in the case $\mathbf{C}P^1$).

To distinguish degenerate symplectic fixed points in $\mathbf{C}P^n$ we can use a lemma, similar to Lemma 5.2. The key observation is that the cap action $(\alpha\cap)^{n+1}$ on the Floer chain complex is the same as the multiplication by the generator g_0 of the group $\pi_2(\mathbf{C}P^n) = \mathbf{Z}$. So, in an approximating set $\mathcal{P}(H_s)$ we can choose a sequence of periodic solutions $\tilde{x}_s^0, \dots, \tilde{x}_s^{n+1}$ such that $\tilde{x}_s^{n+1} = g_0(\tilde{x}_s^0)$, and $m^\alpha(\tilde{x}_s^i, \tilde{x}_s^{i+1}) = 1$. Since we have the identity

$$\mathcal{A}(\tilde{x}_s^0) - \mathcal{A}(\tilde{x}_s^{n+1}) = \omega(g_0) = n+1$$

the case (b) in the proof of Lemma 5.2 cannot happen. This proves Theorem A.2.1 for the case $k = 1$.

Note that the minimal Chern number of the product of complex projective spaces equals the greatest common divisor of $\{n_i + 1\}$. Thus we get the following

Lemma A.2.3 *Let N be the greatest common divisor of $\{n_i + 1\}$. Then we have*

$$HF_k(\times \mathbf{C}P^{n_i}) \cong \bigoplus_{j=k(\mathrm{mod}2N)} H_j(\times \mathbf{C}P^{n_i}, \mathbf{Z}_2).$$

Now we define the cap action of the cohomology groups $H^*(\times \mathbf{C}P^{n_i}, \mathbf{Z}_2)$ on the Floer homology as in section 3. Denote by $\alpha_i := 1 \oplus \cdots \oplus \underset{i}{\alpha} \cdots \oplus 1 \in H^*(\times \mathbf{C}P^{n_i}, \mathbf{Z}_2)$ the image of the generator $\alpha \in H^*(\mathbf{C}P^{n_i})$. To compute the action of α_i we use the function $\hat{H}(x) = \sum H_i(x_i)$, where H_i is the quadratic function on each factor $\mathbf{C}P^{n_i}$. The following lemma is an analog of Lemma A.2.3 and can be proved in the same way.

Lemma A.2.4 *For all $k \in \mathbf{Z}$ the cap action of α_i is an isomorphism*

$$HF_{2k}(\times \mathbf{C}P^{n_i}) \to HF_{2k-2}(\times \mathbf{C}P^{n_i}).$$

Moreover, the (n_i+1)-th iteration of the cap action of α_i is same as the action of the generator of $\pi_2(\mathbf{C}P^{n_i})$.

We shall prove the following Proposition A.2.5, which is better estimate than Theorem A.2.1 and was proved by Givental [G] for toric manifolds, after following Floer's theoretical approach.

Let P be a product of $(\mathbf{C}P^{n_i}, k_i\omega_i)$ where k_i are positive integers and ω_i is the standard symplectic form with $\omega_i[\mathbf{C}P^1] = 1$.

Proposition A.2.5 *For any symplectomorphism ϕ on P, the number of fixed points of ϕ is at least* max $\{(n_i + 1)/k_i\}$.

Proof Without loss of generality, we may assume that $(n_1+1)/k_1 = \max\{(n_i+1)/k_i\}$. To distinguish degenerate symplectic fixed points in the product we repeat our argument for the case of $\mathbf{C}P^n$. From Lemma A.2.4, the (n_1+1)-th iteration of the cap action of α_1 is same as the action of the generator g_1 of $\pi_2(\mathbf{C}P^{n_1+1})$. Thus we can choose in an approximating set $\mathcal{P}(H_s)$ a sequence of critical points $\tilde{x}_s^0, \ldots, \tilde{x}_s^{n_1+1}$ such that $\tilde{x}_s^{n_1+1} = g_1(\tilde{x}_s^0)$, and $m^\alpha(\tilde{x}_s^i, \tilde{x}_s^{i+1}) = 1$. We also have the identity

$$\mathcal{A}(\tilde{x}_s^0) - \mathcal{A}(\tilde{x}_s^{n_1+1}) = \omega(g_1) = k_1.$$

We divide n_1+1 critical points into k_1 classes, namely,

$$\{\tilde{x}_s^0, \ldots, \tilde{x}_s^{l_1}\},\ \{\tilde{x}_s^{l_1+1}, \ldots, \tilde{x}_s^{l_2}\},\ \ldots, \{\tilde{x}_s^{l_{k_1-1}+1}, \ldots, \tilde{x}_s^{n_1}\}$$

such that the difference of values of the action functional of any two of each class is less than 1. Since the symplectic form has integral periods, any two of each class cannot correspond to the same periodic solution. On the other hand, at least one of k_1 classes has at least $(n_1+1)/k_1$ elements, so we get the conclusion. □

Remark A.2.6 In [V] Viterbo defines the action of cohomology ring $H^*(M, \mathbf{R})$ on the Morse homology of M by means of differential forms.

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