identified with the completely physical Fokker-Planck equation (cf., e.g., Mathematical Encyclopedia [in Russian], Vol. 2, p. 958). The asymptotic expansion corresponding to the Fokker−Planck process of effective diffusion can be obtained as here considering the ideas of [7].

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JACOBI EQUATIONS ON MINIMAL HOMOGENEOUS SUBMANIFOLDS IN HOMOGENEOUS RIEMANNIAN SPACES

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INTRODUCTION

There are many articles devoted to the problem of stability of minimal surfaces. The subject attracted interest, in particular, because of the following circumstances. Firstly, the influence of the topology and the Riemannian curvature tensor of a manifold M upon the stability of its minimal submanifolds was discovered (Simons and Lawson [14], Le Khong Van [9], Aminov [2]). Secondly, stable minimal surfaces provide an intermediate link between minimal surfaces, which are numerous, and global minimal surfaces, which are rare and very difficult to describe and classify [4, 8]. The study of stability for minimal surfaces can be reduced, in the end, to the study of the spectrum of the elliptic Jacobi differential operator I corresponding to the second variation formula for the volume functional. The sum of dimensions of all eigensubspaces of I that correspond to negative eigenvalues is called the index of a minimal submanifold. Important results have been obtained in the study of the index for two-dimensional minimal surfaces in R^n [11, 18]. In specific cases where the minimal surface has a major symmetry group, a technique has been developed for studying the stability of the surface [13, 15]. In [9, 10] the method of relative scaling was proposed in order to obtain a lower estimate of the second variation of the volume functional for minimal surfaces.

The contents of this article are the following. In Sec. 1 we write down an explicit formula for the Jacobi equation on a minimal homogeneous submanifold H/L in a homogeneous Riemannian space G/K in terms of the induced representation of the group H from the subgroup L acting on the normal fiber $m^\perp \subset T_e(G/K)$. The idea that the space $C^\infty(H, m^\perp)_L$ (see Sec. 1) can be used to evaluate various invariant operators (in particular, the Laplace and Jacobi operators) goes back to Smith [17]. This article, as well as the recent article [15] by Onita, is concerned only with the case of a totally geodesic imbedding of H/L in a symmetric space G/K equipped with a canonical metric that generates a connection, which is easy to evaluate.

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In Sec. 1 we also prove Theorem 1.2 on invariant subspaces of finite measure for the Jacobi operator.

In Sec. 2, applying the results of Sec. 1, we solve the long-standing problem of classification of stable minimal simple subgroups in classical Lie groups. In Sec. 3 we estimate from below the indices of some homogeneous minimal surfaces in the space SU_{m-1}/T_m equipped with the Killing metric. In Sec. A we prove the theorem on "tiny" irreducible components of the tensor product of irreducible representations of a compact Lie group and some technical lemmas, which are used in Secs. 2 and 3.

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1. LIFTING THE JACOBI OPERATOR ONTO THE SPACE $C^\infty(H, m^\perp)_L$

Let an isotopic variation $f_t(N)$ be given on a minimal submanifold $N \subset M$. Then it is known [16] that the second variation of the volume functional can be expressed as

$$\frac{d^2}{dt^2}\Big|_{t=0} \operatorname{vol}(f_t(N)) = \int_N \langle IW^\perp, W^\perp\rangle\, dx.$$

Here $W^\perp$ is the orthogonal projection of the vector field $W(x) = (d/dt)\,|_{t=0} f_t(x)$, associated with the variation f_t onto the normal bundle $\mathcal{N}(N) \subset T_* M$, and I is called the Jacobi operator of the minimal submanifold N. It is known [16] that

$$I(W) = -\bar\nabla^2(W) + R(W) - A(W). \tag{1.1}$$

Here $-\bar\nabla^2 = \Delta$ is the Laplace operator on the normal bundle (locally), which can be expressed as $\Delta(W) = \sum_i (\bar\nabla_{E_i}\bar\nabla_{E_i} - \bar\nabla_{\nabla_{E_i}E_i}) W$, where $\{E_i\}$ are (locally) orthogonal vector fields on N forming a basis, and $\bar\nabla$ is the covariant derivative in the normal bundle $\mathcal{N}(N)$. Moreover, R is the Ricci transformation in $\mathcal{N}(N)$: $R(W) = \sum_i (R(E_i, W)E_i)^\perp$, where R is the Riemannian curvature tensor on M and $(\ )^\perp$ denotes the orthogonal projection onto $\mathcal{N}(N)$. Finally, A(W) is the second fundamental form of the submanifold $N \subset M$ in the direction of W, namely $A(W)(X, Y) = \langle -\nabla_X W, Y\rangle$, where ∇ is the covariant derivative on M, and $\langle A(W), V\rangle = \langle A(W), A(V)\rangle$ for any V.

In what follows we shall consider only the case where $M = G/K$ is a homogeneous space with a G-invariant Riemannian metric and N denotes its minimal homogeneous submanifold H/L, where $L = H \cap K$. We denote by $m^\perp$ the normal fiber over the point $x_0 = \{eL\}/L$, so that $m^\perp \subset T_{x_0}(G/K) \subset lG$. It is obvious that the normal bundle $\mathcal{N}(H/L)$, on which H acts on the left, is H-equivalent to the bundle $H \times_{\mathrm{Ad}} m^\perp$, which is factorized according to the action of Ad(L). With each section $\Psi \in \Gamma(\mathcal{N}(H/L))$ we associate an $m^\perp$-valued function $\bar\psi \in C^\infty(H, m^\perp)$ on H such that

$$\bar\psi(h) = h_*^{-1}\psi(h/L). \tag{1.2}$$

It is clear that $\bar\psi$ satisfies the following condition:

$$\bar\psi(hl) = \mathrm{Ad}(l^{-1})\bar\psi(h). \tag{1.3}$$

We denote by $C^\infty(H, m^\perp)_L$ the subspace in $C^\infty(H, m^\perp)$ defined by (1.3). It follows from what is said above that the correspondence between the space of normal sections $\Gamma(\mathcal{N}(H/L))$ and $C^\infty(H, m^\perp)_L$ given by (1.2) is a one-to-one correspondence. Therefore, any operator (in particular, the Laplace and Jacobi operators) $I: \Gamma(\mathcal{N}(H/L)) \to \Gamma(\mathcal{N}(H/L))$ can be lifted to an operator $\bar I$:

$$C^\infty(H, m^\perp)_L \to C^\infty(H, m^\perp)_L.$$

Before formulating the basic theorem, we introduce some new notation. We consider the following orthogonal decompositions:

$$lG = lK + m,\quad lH = lL + p,\quad lL = lK \cap lH,\quad m = p_m + m^\perp, \tag{1.4}$$

where p_m is the orthogonal projection of the tangent space $p = T_{x_0}(H/L)$ onto the tangent space $m = T_{x_0}(G/K)$. It is obvious that the projection π_m is a 1–1 function from p to p_m. Thus, the metric induced from $p_m \subset m$ can be defined on p.

We also define a linear operator θ: $m \to \mathrm{End}(m)$ in the following way:

$$\langle \theta (v)X, Y\rangle = \langle [v, X]_m, Y\rangle + \langle [Y, v]_m, X\rangle + \langle [Y, X]_m, v\rangle, \tag{1.5}$$

where $\langle\ ,\ \rangle$ is a K-invariant metric on m, being the restriction of the G-invariant Riemannian metric on G/H to the tangent space $T_{x_0}(G/H) = m$. We set $e_0 = (1/2)\sum_i \theta(e_i)e_i$, where e_i, $i = \overline{1, s}$ is an orthonormal basis in p_m.

Proposition 1.1. a) The second functional form at $x_0 \in H/L$ can be expressed as

$$A(W)(X, Y) = (1/2)\langle -\theta (X_m)W, Y_m\rangle.$$

b) The Ricci transformation of the normal bundle $m^\perp$ can be expressed as follows (x_m and $x_{m^\perp}$ are the orthogonal projections of x onto m and $m^\perp$, respectively):

$$\bar{R}(e_i, W)e_i = (([\theta (e_i), \theta (W)] - \theta [e_i, W])e_i)_{m^\perp}.$$

THEOREM 1.1. The lifting of the Jacobi operator I onto $C^\infty(H, m^\perp)_L$ has the form

$$\bar{I}(\bar\psi) = -C^{\mathrm{diff}}_{H,L}(\bar\psi) - \sum_{i=1}^{s} E_i(\theta(e_i)\bar\psi) + E_0'(\bar\psi) + \left(\frac{1}{2}\theta(e_0) - \frac{1}{4}\sum_{i=1}^{s}\theta^2(e_i) + \bar{R} - \tilde{A}\right)(\bar\psi).$$

Here the left-invariant fields $\{E_i\}$ on H, which are regarded first-order differential operators $\{e_i = E_i(x_0)_m,\ i = \overline{1, s}\}$, form an orthonormal basis in p_m, $C^{\mathrm{diff}}_{H,L} = \sum_{i=1}^{s} E_iE_i$, $\bar R$ и $\tilde A$ the curvature operator, and $\tilde{A}$ is the operator of the second fundamental form.

It is obvious that $\bar{I}$ can be extended to a linear operator on the space $C^\infty(H, m^\perp_{\otimes c})_L$, which, by the Peter–Weyl theorem and by the Frobenius duality principle (see also [6, 15]), is isomorphic to the direct sum $\oplus_{\lambda\in\mathcal{D}(H)} \tau(V_\lambda \otimes \mathrm{Hom}_L(V_\lambda, m^\perp_{\otimes c}))$, where $\tau(v_\lambda \otimes T)(h) = T(\lambda(h^{-1})v_\lambda)$. Here $\mathcal{D}(H)$ is the set of all irreducible complex representations λ of H, and Hom_L is the set of L-invariant linear operators.

THEOREM 1.2. $\tau(V_\lambda \otimes \mathrm{Hom}_L(V_\lambda, m^\perp_{\otimes c}))$ is an invariant subspace of the Jacobi operator I, which acts on this subspace as follows: $\bar{I}\tau(v_\lambda \otimes T) = \tau(v_\lambda \otimes \bar{I}_* T)$, where $\bar{I}_*$ is a linear operator on the space $\mathrm{Hom}_L(V_\lambda, m^\perp_{\otimes c})$.

If we consider the special case where H/L is a totally geodesic submanifold in the space G/K equipped with the Killing metric, then we have $p_m = p$, $\theta(X) = (\mathrm{ad}\,X)_m$, $\tilde{A} = 0 = e_0$. Therefore, Theorems 1.1 and 1.2 can be rewritten as follows.

COROLLARY 1.1. In the case in question the lifting of the Jacobi operator I has the form

$$\bar{I}(\bar\psi) = -C^{\mathrm{diff}}_{H,L}(\bar\psi) - \sum_{i=1}^{s}(E_i[e_i, \bar\psi]_m - [e_i, [e_i, \bar\psi]_{lk}]).$$

The induced operator $\bar{I}_*$ on $\mathrm{Hom}_L(V_\lambda, m^\perp_{\otimes c})$ has the form

$$(\bar{I}_* T)v = -\sum_{i=1}^{s}(T(\lambda^2(e_i)v) + [e_i, T(\lambda(e_i)v)]_m + [e_i, [e_i, Tv]_{lk}]_m).$$

The remaining part of the present section is devoted to the proofs of Proposition 1.1 and Theorems 1.1 and 1.2.

Proof of Proposition 1.1. First, we choose a local system of coordinates near the point $x_0 = \{eK\}/K \in G/K$ with the aid of the exponential mapping $\exp: \mathcal{I}_\varepsilon \to U_\varepsilon(x_0) \subset G/K$, where $\mathcal{I}_\varepsilon$ is a neighborhood of the point {0} in $m' = p + m^\perp$. In what follows, unless it may lead to misunderstandings, we shall identify each point $\{yK\}/K$ with its representative $y \in \exp \mathcal{I}_\varepsilon$ in the contiguity class (for example, $x_0 = e$). Next, we choose local sections $\{E_i\} \subset T_*(G/K)$ over $U_\varepsilon(x_0)$ so that $E_i(x_0) = v_i$, $i = \overline{1, s}$ is an orthonormal basis in p, $E_j(x_0) = v_j$, $j = \overline{s+1, r}$ is an orthonormal basis is $m^\perp$, and $E_i(\exp x/K) = \exp x_* E_i(x_0)$, $i = \overline{1, r}$. Along with the vector fields $\{E_i\}$ on $U_\varepsilon(x_0)$, we also consider the following vector fields on G/K: $\tilde{E}_i(x) = (\partial/\partial t)|_{t=0}\{\exp tv_i x\}/K$. The following lemma is obvious.

LEMMA 1.1. For any point $x \in U_\varepsilon(x_0)$, the following relations hold:

a) $\langle E_i, E_j\rangle = \delta_{ij}$;

b) $\langle \tilde{E}_k, E_l\rangle = \langle \mathrm{Ad}_{x^{-1}} v_k, v_l\rangle = a^l_k(x)$;

c) $\{\tilde{E}_i, \tilde{E}_j\} = -\partial/\partial t|_{t=0}(\exp t[v_i, v_j]x)/K$.

With the aid of Lemma 1.1, we evaluate the second-order derivative following the known prescription [7, p. 155]. To do this, we define the functions $\gamma_{ij}^{p} = \langle\{E_i, E_j\}, E_p\rangle$ on $U_\varepsilon(x_0)$. Then we have [7] $2\langle\nabla_{E_i}E_j, E_k\rangle = \beta_{ij}^{k}$, where

$$\beta_{ij}^{k} = \gamma_{ij}^{k} + \gamma_{ki}^{j} + \gamma_{kj}^{i}. \tag{1.5}$$

We set $d_{kl}^{p}(x) = \langle(\mathrm{Ad}_{x^{-1}}[v_k, v_l])_m, v_p\rangle$, and $f_{kl}^{j}(x) = \langle[\mathrm{Ad}_{x^{-1}}v_k, v_l]_m, v_j\rangle$. Direct computation shows that the following equalities hold:

$$\gamma_{ij}^{p} = \sum_{l,k}(E_i(c_j^l)a_l^p - c_j^l c_i^k d_{kl}^p - E_j(c_k^l)a_l^p), \tag{1.6}$$

$$E_i(a_k^j) = f_{ki}^j, \tag{1.7}$$

$$E_i(c_k^j) = -\sum_{l,s} c_l^j f_{si}^l c_{ks}^s \tag{1.8}$$

where $(c_i^k(y)) = (a_j^i(y))^{-1}$; and $E_j = c_j^l \bar{E}_l$.

Scrupulous calculations involving Lemma 1.1 and formulas (1.5)-(1.8) show that the following lemma holds.

LEMMA 1.2. At the point x_0 we have $\nabla_{E_p}\gamma_{ij}^{k} = 0 = \nabla_{E_l}\beta_{ij}^{k}$ for all i, j, k, and p.

The final part of the proof of Proposition 1.1. Part a) follows from formula (1.5). Part b) follows from Lemma 1.2, formula (1.5), and the following formula:

$$\bar{R}(e_i, W)e_i = [(\nabla_{E_i}\nabla_W - \nabla_W\nabla_{E_i} - \nabla_{\{W, E_i\}})E_i]_{m\perp}.$$

Proof of Theorem 1.1. It is obvious that Theorem 1.1 follows from the propositions stated below.

Proposition 1.2. The operator I defined in Theorem 1.1 is an H-invariant operator. Moreover, I transforms the space $C^\infty(H, m^\perp)_L$ into itself.

Proposition 1.3. For any section $\psi \in \Gamma(\mathcal{N}(H/L))$ the equality $(\overline{I\psi})|_e = I(\bar{\psi})|_e$, holds, where $\bar{\psi}$ and $\overline{I\psi}$ denote the liftings of ψ and $I\psi$ onto $C^\infty(H, m^\perp)_L$, respectively.

Proof of Proposition 1.3. Let ψ be a section of the normal bundle $\mathcal{N}(H/L)$. ψ can be represented as $U_\varepsilon(x_0)\psi$ in a neighborhood $U_\varepsilon(x_0)$, where $\{\alpha_i\}$ are functions on $V_\varepsilon(x_0)$ and $\{E_i\}$ are the sections defined in Lemma 1.1 for i = s + 1, r. Using formulas (1.1), (1.5), and Lemma 1.2, we get

$$\overline{I\psi}|_e = \sum_{i,j}(-\nabla_{E_i}\nabla_{E_i}(\alpha_j)v_j - \nabla_{E_i}(\alpha_j)\theta(v_i)v_j + \nabla_{E_0}(\alpha_j)v_j + [(1/2)\theta(v_0) - (1/4)\sum_i\theta^2(v_i) + \bar{R} - \bar{A}]\alpha_j v_j). \tag{1.9}$$

We define the lifting of a section E_i, where i = s + 1; r, onto the space $C^\infty(H, m^\perp)_L$ by $\bar{E}_i(h) = h^{-1}E_i(h/L) = v_i$. It is obvious that $E_i(\bar{E}_j) = 0$, where each E_i is a left-invariant field on H/L with the value v_i at e. Therefore, we have

$$I(\bar{\psi})|_e = -\sum_{i,j}(E_iE_j(\alpha_j)v_j - E_i(\alpha_j)\theta(v_i)v_j + E_0(\alpha_j)v_j + \alpha_j[(1/2)\theta(v_0) - (1/4)\sum_i\theta^2(v_i) + \bar{R} - \bar{A}]v_j). \tag{1.10}$$

Comparing (1.9) with (1.10), we find that $\overline{I(\psi)}|_e = I(\bar{\psi})|_e$.

Proof of Proposition 1.2. The first assertion is trivial. Next, we remark that the metric on m is L-invariant, and so θ commutes with the action of the group L. Taking into account that $\bar{\psi} \in C^\infty(H, m^\perp)_L$ [i.e. $\bar{\psi}(l\exp tv_i) = \mathrm{Ad}(l^{-1})\bar{\psi}(\exp t(\mathrm{Ad}(l)v_i))$], we can easily get the following lemma.

LEMMA 1.3. $\mathrm{Ad}(l)(I\bar{\psi})(l) = (-c_{H,L}^{\mathrm{diff}}(\bar{\psi}) - \sum_i E_i'(\theta(e_i')\bar{\psi}) + E_0'(\bar{\psi}) + ((1/2)\theta(e_0') + \sum_i(-1/4)\theta^2(e_i') + \bar{R} - \bar{A})\bar{\psi})(e)$,

where $e_i' = \mathrm{Ad}\,l(e_i)$ is a new orthonormal basis in p_m.

By virtue of Proposition 1.3, the right-hand side of the equality in Lemma 1.3 is equal to $\overline{I\psi}(e)$. It follows that the identity

$$\mathrm{Ad}(e)[I\bar{\psi}(hl)] = [I\bar{\psi}](h) \tag{1.11}$$

holds for h = e. As mentioned above, $\bar{I}$ is an H-invariant operator. Thus, the fact that (1.11) holds for h = e and for any ψ implies that (1.11) holds for all h. Hence, we get immediately the second assertion of Proposition 1.2.

Proof of Theorem 1.2. Let $\bar\psi(h) = T(v_\lambda \otimes T)$. Then $E_i\psi(h) = (d/dt)|_{t=0} T(\lambda((\exp(te_i))^{-1}h^{-1}v_\lambda) = -T(\lambda(e_i)\lambda(h^{-1})v_\lambda)$. It follows that

$$\bar I \tau(v_\lambda \otimes T)(h) = (\bar I_* T)(\lambda(h^{-1})v_\lambda),$$

where $\bar I_*$ is a linear endomorphism of the space $\mathrm{Hom}(V_\lambda, m^\perp_{\otimes C})$. Since $\bar\psi \in C^\infty(H, m^\perp_{\otimes C})_L$, we find that $\bar I_*$ belongs to the space End $\mathrm{Hom}_L(V_\lambda, m^\perp_{\otimes C})$. The theorem is proved.

2. CLASSIFICATION OF STABLE MINIMAL SIMPLE SUBGROUPS IN CLASSICAL LEE GROUPS

Every compact Lee group G is a globally symmetric space with respect to the **Riemannian** structure generated by the Killing form on the algebra ℓG. If H is a compact subgroup of G, then H is a totally geodesic submanifold, and so it is a locally minimal **submanifold** in G.

THEOREM 2.1. Let G be a classical Lee group, and let p: H → G be an imbedding of a simple compact group H. The subgroup p(H) is a stable minimal submanifold in G if and only if

a) $G = SU_{m+1}$, $H = SU_{n+1}$, and p is the canonical imbedding, or $H = Sp_n$ and p is the canonical imbedding;

b) $G = SO_m$, $H = SU_n$, or Sp_n, and p is the composition of the canonical **imbeddings** ρ_1 and ρ_2, where $\rho_1: SU_n \to SO_{2n}$ (or $\rho_1: Sp_n \to SU_{2n} \to SO_{4n}$) and $\rho_2: SO_{2n} \to SO_m$ (or $\rho_2: SO_{4n} \to SO_m$ for $H = Sp_n$),

or $H = SO_n$ and p is the canonical imbedding (for n = 7, 8, 16, we have additional **semi**-spinor imbeddings),

or $H = G_2$, F_4, E_8 and p is a representation of the last dimension,

or $H = E_0$, E_7, F_4 and p is the composition of the adjoint representation $\mathbf{Ad_H}$ and the canonical imbedding $SO(lH) \to SO_m$;

c) $G = Sp_m$, $H = SP_n$, and p is the canonical imbedding.

Before proving Theorem 2.1a), we introduce new notation and we rewrite Corollary 1.1 in a form that will be suitable for our purposes. First, we remark that the compact Lie group G acts on itself by transitive left and right translations, which preserve the Killing metric on G. It follows that G, regarded as a homogeneous Riemannian space, can be represented in the form of a factor space $G \times G / G \sim G \times G / \mathrm{Diag}(G \times G)$. The canonical involution T on G, which acts in the following way: $\tau(g_1, g_2) = (g_2, g_1)$, defines the structure of a global symmetric space on M = G x G/Diag (G x G). Therefore, in this case Corollary 1.1 reads as follows (see also Lemma 3.2 of [15]):

Proposition 2.1. The Jacobi operator I on the bundle $C^\infty(H \times H, m^\perp)_{\mathrm{diag}(H\times H)}$ **has the** form

$$I(\bar\psi) = -C^{\mathrm{diff}}_{H\times H}(\bar\psi) + C^{\mathrm{alg}}_{H\times H}(\bar\psi),$$

where H x H acts on the algebra $\ell G + \ell G$ by means of the adjoint representation $\mathrm{Ad}(\rho \bullet p)$, where p is the imbedding H → G, and $C^{\mathrm{diff}}_{H\times H}$ and $C^{\mathrm{alg}}_{H\times H}$ are the differential and the algebraic Casimir operators on H x H, respectively.

We denote by m the orthogonal complement of the subalgebra $\rho(\ell H)$ in the algebra ℓG. m can be decomposed into irreducible components of the representation $\mathrm{Ad}_G\rho$, namely $m = \oplus_i m_i$. It is well known [15, 23] that the Casimir operator of the representation Ad p of H on m_i can be diagonalized, i.e., the equality

$$C^{\mathrm{alg}}_H|_{m_i} - a_i \cdot Id \qquad (2.1)$$

holds. We recall that the imbedding of the tangent space $lM \to l(G \times G)$ is antidiagonal, i.e., $lM = \{(x, -x),\ x \in lG\}$. It is obvious that the normal fiber over the point $e \in p(H \times H)$ coincides with the subspace $m - \{(x, -x),\ x \in m\} \subset lM$.

We set $\bar m_i \simeq \bar m \cap (m_i \oplus m_i)$. The following lemma follows immediately from Proposition 2.1.

LEMMA 2.1. The space $C^\infty(H \times H, \bar m)_H$ can be written as a direct sum of subspaces $\oplus_i C^\infty(H \times H, \bar m_i)_H$, all of which are invariant subspaces for the Jacobi operator I.

Moreover, on $C^\infty(H \times H, m_i)_H$ the Jacobi operator has the form $I(\psi) = -C^{\mathrm{diff}}_{H\times H}(\bar\psi) + a_i\bar\psi$, where a_i is the eigenvalue of the Casimir operator with respect to the irreducible representation $\mathrm{Ad}_G\,p$ of H on m_i [see formula (2.1)].

Thus, we get the following

COROLLARY 2.1. The subgroup $\rho(H)$ is a stable minimal submanifold in G if and only if for any irreducible complex representation $\lambda \in \mathcal{Y}(H \times H)$ of the subgroup $H \times H$ and for any irreducibility component $m_i \subset m$, the following inequality holds:

$$(-c_\lambda + a_i) \dim \mathrm{Hom}_H (V_\lambda, m_{i\times c}) \geqslant 0,$$

where c_λ (a_i, respectively) is the value of the Casimir operator of the group $H \times H$ (of the group H) on the space V (on the space m_i) of the representation λ (of the representation $\mathrm{Ad}_G \rho$).

The eigenvalue c_λ of the Casimir operator of the irreducible representation λ of H can be evaluated from the following formula [12, 20]:

$$c_\lambda = -\langle \mathrm{do}(\lambda) + \delta, \mathrm{do}(\lambda)\rangle, \tag{2.2}$$

where $\mathrm{do}(\lambda)$ is the dominant weight of λ, and δ is the sum of all positive roots of the group H. The following proposition holds.

Proposition 2.2 [12]. The equality

$$0 > c_\lambda + c_\gamma > c_{\lambda+\gamma}$$

holds for all irreducible representations λ and γ. (Here $\lambda + \gamma$ is the Cartan composition of irreducible representations λ and γ [15].)

Proposition 2.3. If among the irreducibility components m_i there is a representation that is distinct from the fundamental representations π_i of H, then the subgroup $\rho(H)$ is not a stable minimal submanifold in G.

Proof. Let the irreducible component $\sigma_i = \sigma|_{m_i}$ be not a fundamental representation of H. Then $\sigma_i = \sigma_i^1 \mp \sigma_i^2$, where σ_i^1 and σ_i^2 are nontrivial irreducible representations of H. We denote by m_i^1 and m_i^2 the spaces of the representations σ_i^1 and σ_i^2, respectively. It is easily seen that $m_i^1 \otimes m_i^2$ is the space of the irreducible representation $\sigma_i^1 \hat{\otimes} \sigma_i^2$ of the group $H \times H$, and the eigenvalue of the Casimir operator of this representation is equal to $c(\sigma_i^1) + c(\sigma_i^2)$. It is obvious that the space $\mathrm{Hom}_H (m_i^1 \otimes m_i^2, m_i)$ is nonempty since it contains the orthogonal projection from $m_i^1 \otimes m_i^2$ onto m. Hence, using Corollary 2.1 and Proposition 2.2, we obtain Proposition 2.3 immediately.

Proof of Theorem 2.1.

a) Classification of stable minimal simple subgroups in SU_m.

Proposition 2.4. a) Let ρ be an irreducible representation of the group SU_{n+1} in SU_m that is distinct from the representations π_1 and $\pi_\pi \simeq \pi_1$. Then the subgroup $\rho(SU_{n+1})$ is a stable minimal submanifold in SU_m.

b) Let ρ be an irreducible representation of SO_n with $n \geqslant 7$ in SU_m. Then the subgroup $\rho(SO_n)$ is not a stable minimal submanifold in SU_m.

c) Let ρ be an irreducible representation of the group Sp_n in SU_m that is distinct from π_1. Then $\rho(Sp_n)$ is not a stable minimal submanifold in SU_m.

d) Let H be a singular Lee group and let ρ be an irreducible representation of H in SU_m. Then $\rho(H)$ is not a stable minimal submanifold in SU_m.

Proof. By Theorem 3.8 of [19] the representation $\mathrm{Ad}\,\rho$ on the complexification $SU_{m\otimes C}$ is equivalent to the representation $\rho \otimes \rho^* - \mathrm{Id}$ [Id is the trivial representation on the subspace $\mathrm{Diag}\,(gl_m(\mathbf{C}))$]. This means that the restriction of the representation $\mathrm{Ad}_{SU_m} \cdot \rho$ to the subspace $m_{\otimes C}$ is equivalent to the representation $\sigma(\rho) = \rho \otimes \rho^* - \mathrm{Id} - \mathrm{Ad}_H$. Let $\rho(H)$ be one of the representations mentioned above. Then $\rho + \rho^*$ differs from Ad_H, and so it does not appear in the decomposition of $\sigma(\rho)$. Since $\rho + \rho^*$ is not a fundamental representation of H, using Proposition 2.3, we get Proposition 2.4 immediately.

Proposition 2.5. Let $\rho: H \to SU_m$ be a sum of k irreducible representations, among which there are at least two components that differ from the trivial representation. Then $\rho(H)$ is not a stable minimal submanifold in SU_m.

Proof. Let $\rho = \oplus_i \rho_i$. It follows from Proposition 2.3 that if at least one subgroup $\rho_i(H)$ is not a stable minimal submanifold in SU_{m+1}, then $\rho(H)$ is not a stable minimal submanifold in SU_{m+1} either. Hence, taking Proposition 2.4 into account, we derive Proposition

2.5 by direct verification in the case where H = SU_{n+1} or Sp_n, and $\rho_1 \oplus \rho_2$ is equal either to $\pi_1 \oplus \pi_1$, or to $\pi_1 \oplus \pi_1^*$, or to $\pi_1^* \oplus \pi_1^*$. The last assertion is easy to obtain with the aid of Proposition 2.3.

It is obvious that Theorem 2.1 follows from Propositions 2.4 and 2.5, and from direct verification of the fact that the imbeddings enumerated in Theorem 2.1a) are stable minimal imbeddings. It is easy to establish this fact with the aid of Corollary 2.1, taking into account that $|c_{\pi_1}(H)|$, for H = SU_n, Sp_n is the least number among the values $|c_\lambda(H)|$, where $\{\lambda\}$ is an irreducible representation of H [20].

b) Classification of Stable Minimal Simple Subgroups in SO_n. The proof for this series of spaces is similar to that in case a) but more delicate. First we restrict our considerations to those irreducible representations ρ of H in SO_n that are irreducible in the real domain. By Theorem 3.8 of [19], the restriction of the representation Ad ρ to $m_{\otimes C}$ is equivalent to the representation $\sigma(\rho) = \Lambda^2(\rho) - Ad_H$. Using [5] and 2.3, we find at once that all representations ρ of simple groups H that are irreducible in the real domain, except for the representations enumerated in Theorem 2.1b) and also the (semi)spinor orthogonal representations of SO_n for n = 8q, 8q + 1, satisfy the criterion for nonstable minimal submanifolds (Proposition 2.3).

We denote by β a (semi)spinor representation of H, where H is equal either to SO_{8q}, where $q \geqslant 3$, or to SO_{8q+1} or SO_{8q+7}, where $q \geqslant 1$. Then $\lambda = \beta \hat{\otimes} \beta$ is an irreducible representation of the group H × H. Let r = rkH − 2 if H = SO_{8q}, and let r = rkH − 1 if H = SO_{8q+1}, SO_{8q+7}. It follows from Table 5 of [3] that $\mathrm{Hom}_H(V_\lambda, V_{\pi_r}) = 1$, and it follows from Table 3 of [20] that $c_\lambda = 2c_\beta > c_{\pi_r}$. This means that the subgroup β(H) is not a stable minimal submanifold in SO_n if H = SO_{8q} for $q \geqslant 3$, or SO_{8q+1}, SO_{8q+7} for $q \geqslant 1$. To complete the proof of Theorem 2.1b), we will need to prove that the subgroups enumerated in Theorem 2.1b) are stable.

We denote by $c_m(H)$ the eigenvalue of the Casimir operator for the fundamental representations of H that has the least absolute value.

LEMMA 2.2. Let λ be an irreducible representation of the group H × H. Let π_i be a fundamental representation of the algebra ℓH and let $|c_{\pi_i}| \leqslant 3c_m(H)$. If $-c_\lambda + c_{\pi_i} < 0$, then λ must be one of the representations $\pi_s \hat{\otimes} \pi_q$ of H × H, where π_s and σ_q are fundamental representations of H such that $|c_{\pi_s} + c_{\pi_q}| < |c_{\pi_i}|$.

Proof. We have $\lambda = \varphi \hat{\otimes} \psi$, where φ, and ψ are irreducible representations of H, and $c_\lambda = c_\varphi + c_\psi$. Let φ be not a fundamental representation of the algebra ℓH. Then, by Proposition 2.2, $|c_\varphi| > |2c_m(H)|$. Hence, we find that $|c_\varphi + c_\psi| > |3c_m(H)| > c_{\pi_i}$. The lemma is proved.

Continuation of the Proof of Theorem 2.b). Direct selection with the aid of Lemma 2.2 and formula (2.2) (see Table 3 of [20]) shows that the representations ρ named above satisfy the inequality in Corollary 2.1. The proof is completed.

c) Classification of Stable Minimal Simple Subgroups in Sp_n. It follows from Theorem 3.8 of [19] that the restriction of the representation $AdS_p\rho$ to $m_{\otimes C}$ is equivalent to the representation $S^2(\rho) - Ad_H$. Since the component ρ + ρ = 2ρ appears in the decomposition of the representation $S^2(\rho)$, the remaining part of the proof is the same as that in case a). Namely, it is easy to verify with the aid of [20] that any symplectic representation of a simple group H satisfies the criterion for nonstable minimal submanifolds (see Proposition 2.3) except for the case where H = Sp_n and $\rho = \pi_1 \oplus [k]\pi_0$. In this case we have $\sigma(\rho) \simeq [k]\pi_1 \oplus [k(k+1)]\pi_0$ and, by virtue of Corollary 2.1, $\rho(Sp_n)$ is a stationary minimal imbedding in the group Sp_m. The theorem is proved.

3. LOWER ESTIMATE FOR THE INDICES OF SOME HOMOGENEOUS MINIMAL SPACES IN SU_{m+1}/T_m

$\mathfrak{C}$-spaces G/C(t), i.e., spaces that can be realized as orbits of the adjoint representations of compact Lie groups, lend themselves well to investigations in geometry [1] and in representation theory [3]. They are equipped with numerous invariant Riemannian structures [1, 8], including the Einstein–Kähler metric [1] and the Killing metric. The latter generates the canonical connection of genus two [7]. Any suborbit in G/C(t) is a complex submanifold, and so a global minimal and a stable minimal submanifold [4] in G/C(t) is equipped with the Killing metric, the picture changes sharply. In this section we consider only the case where G = SU_{m+1} and C(t) is its maximal tore T_m.

We recall that the sum of dimensions of whose eigensubspaces in $\Gamma(\mathcal{N}(N))$, that correspond to negative eigenvalues of the Jacobi operator on N is what we call the index of a minimal submanifold $N \subset M$.

THEOREM 3.1. (i) Any suborbit H/T_H, where T_H is the maximal tore in H and $H \subset SU_{m+1}$, is a totally geodesic submanifold in SU_{m+1}/T_m equipped with the Killing metric.

(ii) Let $H = SU_{n+1}$ and let the imbedding $\rho \cdot H \to SU_{m+1}$ be an irreducible representation that differs from each of the fundamental representations $\pi_i(H)$. Then the index of the submanifold $\rho(H/T_H)$ is not less than $\dim H = (n+1)^2 - 1$.

(iii) Let $H = SO_{2n+1}$, where $n \geqslant 3$. Then the index of the submanifold $\rho(H/T_H)$ is not less than $2n + 1$.

Proof of Theorem 3.1 (i). We consider the suborbit $H/T_H \to SU_{m+1}/T_m$, where $T_H = H \cap T_m$. Let W be the orthogonal complement of the Cartan subalgebra ℓT_H in the algebra ℓH and let V be the orthogonal complement of the Cartan algebra ℓT_m in SU_{m+1}. It is easy to convince oneself that W is orthogonal to ℓH, i.e., W belongs to V. Since the metric on V is the Killing metric, it follows that H/T_H is a totally geodesic submanifold in SU_{m+1}/T_m.

(ii) We choose the canonical basis in $V(\pi_1) \sim \mathbf{C}^{n+1}$ consisting of normalized weighting vectors v_{x_1}. We denote the dual basis in $V(\pi_n)$ by $\{v^*_{-x_i}\}$. We imbed the space $V(2\pi_1 + 2\pi_n)$ in $V(2\pi_1 \otimes 2\pi_n)$. For convenience we denote $v_{x_i} \otimes v_{-x_j}$ by $v_{x_i - x_j}$, $v_{x_i} \cdot v_{x_j} \in V(S^2(\pi_1))$ by $v_{x_i + x_j}$ etc.

LEMMA 3.1. The linear operator defined as follows:

$$\hat{L}(v_{x_i - x_j}) = c_{i_1} c_{j_1} v_{x_i + x_1} \otimes v^*_{-x_1 - x_j} - c_{i_2} c_{j_2} v_{x_i + x_2} \otimes v^*_{-x_2 - x_j},$$

$$L(v_{x_1 - x_1}) = (1/2)\, v_{2x_1} \otimes v^*_{-2x_1} - v_{x_1 + x_2} \otimes v_{-x_2 - x_1},$$

$$\hat{L}(v_{x_2 - x_2}) = v_{x_2 + x_1} \otimes v_{-x_1 - x_2} - (1/2)\, v_{2x_2} \otimes v^*_{-2x_2},$$

where $(2 - \delta_{ij} - \delta_{i1} - \delta_{i2})(2 - \delta_{ij} - \delta_{j1} - \delta_{j2}) \neq 0$, and $c_{k\ell} = 1$ for $k \neq i$, and $c_{k\ell} = 1/2$ otherwise, transforms the space $V(\pi_1 + \pi_n)$ into $V(2\pi_1 + 2\pi_n)$.

Proof. We define the convolution operator $\tau\colon V(2\pi_1 \otimes 2\pi_n) \to V(\pi_1 \otimes \pi_n)$ as in the proof of Theorem 4.1. It is easy to check that $\operatorname{Ker} \tau = V(2\pi_1 + 2\pi_n)$ and $\tau\hat{L}(V(\pi_1 + \pi_n)) = 0$, which yields our assertion.

By Proposition 4.3 (a) stated below, the component $V(2\pi_1 + 2\pi_n)$ appears in the decomposition of the representation $\operatorname{Ad} \rho - \operatorname{Ad}_{SU_{n+1}}$. We denote by $\pi_{m^\perp}$ the orthogonal projection onto $m^\perp_{\otimes \mathbf{C}}$. Then Lemma 3.1 and the fact that $\hat{L}$ and $\pi_{m^\perp}$ transform weighting vectors into weighting ones and preserve the weight imply that the operator $L = \pi_{m^\perp}\hat{L}$ belongs to the space $\operatorname{Hom}_{T_n} \times (V(\pi_1 + \pi_n), m^\perp_{\otimes \mathbf{C}})$. There is a natural metric on the space $\operatorname{Hom}_{T_n}(V, W)$ induced by the metrics on V and W. Direct verification based on Corollary 1.1 shows that

$$\langle I_* L, L \rangle = \sum_{\alpha \in \Delta(SU_{n+1})} \langle I_* L v_\alpha, L v_\alpha \rangle = -(n) \sum_{j=1}^{n+1} (|L v_{x_1 - x_j}|^2 + |L v_{x_j - x_1}|^2 + |L v_{x_2 - x_j}|^2 + |L v_{x_j - x_2}|^2) - (n+1)(|L v_{x_1 - x_2}|^2 + |L v_{x_2 - x_1}|^2) < 0. \tag{3.1}$$

It follows from (3.1) and Corollary 1.1 that $\operatorname{ind}(\rho(SU_{n+1}/T_n)) \geqslant \dim_{\mathbf{C}} T(V(\pi_1 + \pi_n) \otimes L) = (n+1)^2 - 1$.

(iii) We choose Witt's basis in $\mathbf{C}^{2n+1}$ consisting of the base vectors $v_{\pm x_i}$, and v_0, where $|v_{\pm x_i}| = 1 = (1/2)|v_0|$, for the representation π_1 of SO_{2n+1}. We define $\hat{L}\colon V(\pi_1) \to V(S^2(\pi_1))$ in the following way:

$$\hat{L}(v_{\pm x_i}) = v_{\pm x_i} v_0,$$

$$\hat{L}(v_0) = (v_0 \cdot v_0) + (2/n) \sum_{i=1}^{n} v_{x_i} v_{-x_i}.$$

It is easy to convince oneself that $L(V(\pi_1)) \subset V(2\pi_1) \subset V(S^2(\pi_1))$.

By Proposition 4.3 (b) stated below, the component $2\pi_1$ appears in the decomposition of the representation $\operatorname{Ad} \rho - \operatorname{Ad}_{SO_{2n+1}}$. We denote by $\pi_{m^\perp}$ the orthogonal projection onto $m^\perp_{\otimes \mathbf{C}}$. Then, arguing as in case (ii), we find that the operator $L - \pi_{m^\perp}\hat{L}$ belongs to the subspace $\operatorname{Hom}_{T_n} \times (V(\pi_1), m^\perp_{\otimes \mathbf{C}})$. Direct computation, in which Corollary 1.1 is used, shows that $\langle I_* L, L \rangle < 0$. As in (ii), we have ind $(\rho(SO_{2n+1}/T_n)) > \dim V(\pi_1) = 2n + 1$. The theorem is proved.

A. TECHNICAL LEMMAS ON DECOMPOSITION OF THE TENSOR PRODUCT OF REPRESENTATIONS OF COMPACT LEE GROUPS

The following proposition is well known.

Proposition 4.1 [19, Theorem 3.8]. (i) Let ρ be a representation of a group G in SU_{n+1}. Then the representation Ad_ρ in the algebra $gl_n(C) \sim u_{(n+1)}\otimes C$ is equivalent to the representation $\rho \otimes \rho^*$, where ρ^* is the contragradient representation corresponding to ρ.

(ii) Let ρ be a representation of G in the group SO_n. Then the representation Ad_ρ in the algebra $so_n(C)$ is equivalent to the representation $\Lambda^2(\rho)$.

(iii) Let ρ be a representation of G in the group Sp_n. Then the representation Ad_ρ in the algebra $sp_n(C)$ is equivalent to the representation $S^2(\rho)$.

It is well known [3, 5] that the Cartan composition $\varphi \otimes \psi$ appears in the decomposition of the product $\psi + \varphi$ of irreducible representations. Moreover,

$$\varphi \otimes \varphi = S^2(\varphi) \oplus \Lambda^2(\varphi),$$

and the following proposition holds.

Proposition 4.2 [5]. Let ρ be an irreducible representation of a group G with dominant weight $do(\rho)$. Moreover, let α be a simple root of the group G such that $\langle do(\rho), \alpha\rangle > 0$. Then the irreducible component $\xi(\rho, \alpha)$ with dominant weight $2do(\rho) - \alpha$ appears in the decomposition of the representation $\Lambda^2(\rho)$.

Let φ and ψ be irreducible representations of the semisimple algebra ℓG with dominant weights Λ and M, respectively. Following Dynkin, we shall say that ψ is a subordinate representation to φ, if for each simple root $\alpha_i \in A(G)$ we have $\Lambda_{\alpha_i} \geqslant M_{\alpha_i}$.

THEOREM 4.1. Let ψ be a subordinate irreducible representation to an irreducible representation φ.

(i) Then the component $\varphi \otimes \varphi^*$ appears in the decomposition of the tensor representation $\psi \otimes \psi^*$.

(ii) Suppose that ψ and φ are self-adjoint representations. Then the component $\Lambda^2(\varphi)$ $[S^2(\varphi)]$ appears in the decomposition of the representation $\Lambda^2(\psi)$ $[S^2(\psi)]$ if ψ and φ are simultaneously orthogonal or simultaneously symplectic. Otherwise, the following inclusions hold:

$$\Lambda^2(\psi) \subset S^2(\varphi);\ S^2(\psi) \subset \Lambda^2(\varphi).$$

Proof. We denote by $do(\psi)$ and $do(\varphi)$ the dominant weights of ψ and φ, respectively. Then $do(\varphi) - do(\psi)$ is the dominant weight of some irreducible representation γ. We denote by $E(\psi)$, $E(\varphi)$, and $E(\gamma)$ the spaces of the representations ψ, φ, and γ, and we denote by $E^*(\psi)$, $E^*(\varphi)$, and $E^*(\gamma)$ their adjoint spaces, respectively. We define the convolution transformation from

$$E(\psi) \otimes E(\gamma) \otimes E^*(\psi) \otimes E^*(\gamma) \text{ в } E(\psi) \otimes E^*(\psi)$$

as follows

$$\sigma(x \otimes y \otimes x^* \otimes y^*) = y^*(y)\, x \otimes x^*. \tag{4.1}$$

It is clear that σ is an ℓG-invariant transformation. Since $do(\psi) + do(\varphi) - do(\gamma)$, we can imbed $E(\varphi)$ in the tensor product $E(\psi) \otimes E(\gamma)$ as the dominant irreducible component of the latter, and in the same way we can imbed $E^*(\psi)$ in $E^*(\psi) \otimes E^*(\gamma)$). We denote by $U(\ell G)$ the enveloping algebra of the Lee algebra ℓG. It is a well-known fact that $E(\psi) \otimes E^*(\psi)$ is generated as a $U(\ell G)$-module by the vector $v_{do}(\psi) \otimes v_{ml}(\psi^*)$, where $v_{do}(\psi^*)$ and $v_{ml}(\psi^*)$ are the dominant vector and the minor vector in $E(\psi)$ and $E^*(\psi)$, respectively. Taking into account that

$$v_{do}(\psi) \otimes v_{mi}(\psi^*) \in \operatorname{Im} \sigma\, (E(\psi) \otimes E^*(\psi)),$$

we find that the restriction of σ to the subspace $E(\varphi) \otimes E^*(\varphi)$ is a surjective mapping onto $E(\psi) \otimes E^*(\psi)$. Hence, we obtain at once the first assertion of Theorem 4.1. To prove assertion (ii), it suffices to note that if φ and ψ are self-adjoint representations, then γ is also a self-adjoint representation. Moreover, γ is an orthogonal representation if φ and ψ are either simultaneously symplectic. Otherwise, γ is a symplectic representation. Therefore, the transformation given by (4.1) assumes the following form:

$$a(x \otimes y \otimes x' \otimes y') = \sigma_\gamma(y, y')\, x \oslash x', \qquad (4.2)$$

where σ_γ is a nonsingular ℓG-invariant biliner form on $E(\gamma)$, σ_γ is a symmetric form if γ is an orthogonal representation, and it is a skew-symmetric form if γ is symplectic. Regarding $S^2(E(\psi))$ and $\Lambda^2(E(\psi))$ as the subspaces of symmetric and skew-symmetric tensors in $E(\psi)$ * $E(\psi)$, we can easily deduce (ii) from (i) with the aid of (4.2) and of what is said above. The theorem is proved. Using Theorem 4.1 and the data in Table 5 of [5], we obtain the following

Proposition 4.3. a) If an irreducible representation ψ of the algebra su_{n+1} differs from the fundamental representations π_i, then the component $2\pi_i \dotplus 2\pi_n$ appears in the decomposition of the representation

$$\psi \otimes \psi^* .$$

b) The component $2\pi_1$ appears in the decomposition of the representation

$$\psi \otimes \psi^*,$$

where ψ is an arbitrary irreducible representation of the algebra so_{2n+1}.

Finally, we include the following lemma on decomposition of the tensor product of reducible representations.

LEMMA 4.1 [20]. Let φ, and ψ be representations of H. Then the following relations hold:

$$\Lambda^2(\varphi \oplus \psi) = \Lambda^2(\varphi) \oplus \Lambda^2(\psi) \oplus [\varphi \otimes \psi],$$
$$S^2(\varphi \oplus \psi) = S^2(\varphi) \oplus S^2(\psi) \oplus [\varphi \otimes \psi].$$

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A SEMIGROUP OF OPERATORS IN THE BOSON FOCK SPACE

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A boson Fock space with n degrees of freedom is a space of holomorphic functions on n n-dimensional Hilberts space with the scalar product:

$$\langle f, g\rangle = \iint f(z)\overline{g(z)} \exp(-(z, z))\, dz\, d\bar{z}.$$

We are interested in operators of the form

$$Bf(z) = \iint \exp\left\{\frac{1}{2}(z\bar{u})\begin{pmatrix} K & L \\ M & N\end{pmatrix}\begin{pmatrix} z^t \\ \bar{u}^t\end{pmatrix}\right\} f(u) \exp(-(u,u))\, du\, d\bar{u}. \quad (0.1)$$

The main problem considered in this article is the problem of the boundedness of these operators.

Unitary operators of the form (0.1) appeared in [1], in such a form Berezin has written down the automorphisms of the canonical commutation relations. In numerous papers of the years 70-80 (we mention only [2-5]) the fundamental role of the automorphisms of canonical commutation and anticommutation relations in the representation theory for infinite dimensional groups has been clarified (this role is the same as for the operators of variables exchange and multiplication by a function in the representation theory of Lie groups). After it had been discovered that a representation of an infinite dimensional group is, in fact, the visible part of a representation of an essentially bigger and invisible with the unaided eye semigroup (see [6]), and, actually, even not a semigroup, but a category, at first a problem of semigroup with the Weil representation has arisen. Ol'shanskii indicated that this semigroup is semigroup BO of all operators of the form (0.1), and then a problem has arisen concerning the algebraic nature of this semigroup, as well as the problem of the boundedness of the operators. It turns out (Ol'shanskii), that for $n < \infty$ the boundedness of the operators (0.1) is equivalent to the pair of conditions: 1) $\left\|\begin{pmatrix} K & L \\ M & N\end{pmatrix}\right\| \leqslant 1$; 2) $\|K\| < 1$, $\|N\| < 1$ (here, as everywhere in this paper, under the norm of a matrix we understand the Euclidean norm). In the joint paper by Ol'shanskii, Nazarov, and the author [9] it has been clarified that the considered semigroup is isomorphic to some semigroup of linear relations.

In Sec. 1 of this paper we introduce an accurate definition of operators of the form (0.1), in Sec. 2 we discuss a realization of the semigroup BO as a semigroup of linear relations, and a semigroup of generalized fraction-linear Krein transformations of an infinite dimensional matrix ball. In Secs. 3 and 4 we formulate and prove theorems on the boundedness of the operators. In Sec. 5 we consider a somewhat more general class of operators.

For applications of the semigroup BO to the representation theory of the Virasoro algebra, and to the conformal quantum field theory (cf. [7; 10]), see the Fermion analog of this paper (cf., [8]).

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