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Curvature estimate for the volume growth of globally minimal submanifolds

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Introduction

It is well known that in each homology class of a Riemannian manifold there exists a cycle of the least volume (or simply speaking, a globally minimal submanifold). These globally minimal cycles yield many information of geometry and topology of their ambient manifold, however, to detect them the existence (and almost regularity) theorems can not help us so much. Intuitively, one knows that globally minimal submanifolds would occupy a position of "maximal curvature" in their ambient manifold. In Fomenko's and author's announcement [LF] we gave a mathematical formulation of this conjecture. The aim of this note is to complete the proof of our announcement [LF]. In particular, we obtain an estimate in terms of upper curvature bounds for the volume growth of globally minimal submanifolds in Riemannian manifolds, new isoperimetric inequalities for these submanifolds, an explicit formula of the least volumes of closed submanifolds in symmetric spaces. As a result, we prove that every Helgason's sphere in a compact irreducible simply connected symmetric space M is a globally minimal submanifold. Note that some Helgason's spheres realize a torsion element in the corresponding homology group of M, therefore, one cannot use the calibration method for proving their global minimality [HL].

The main idea is to compare the volume growth rate of a family of exhaustion subsets A_t in a globally minimal submanifold $X \subset M$ with the volume growth rate of certain cones with base ∂A_t (see [Fo 1] and the proof of Theorem 2.3 below). On the other hand, our results and technique are also related to the field of "comparison theorems in Riemannian geometry". In fact, the essential part of our technique relies on the Rauch-Bishop comparison-monotonicity estimate for Jacobi fields. We refer to survey articles [K, S] for more information in this field. Sakai's survey contains a very extensive bibliography, and Karcher's survey gives a complete exposition on the subject with the help of distance functions and the Riccati equation instead of Jacobi fields. We also refer to Remark 2.4 below on other related results.

1 Geodesic defect of Riemannian manifolds and the volume of globally minimal submanifolds

(a) Let $B_r(x)$ be the ball of radius r centered at x in a tangent space T_xM. Recall that the injectivity radius $R(x)$ of a Riemannian manifold M at a point x is defined as follows:

$$R(x) = \sup\{r \mid Exp: B_r(x) \to M \text{ is a diffeomorphism}\} .$$

The injectivity radius $R(M)$ of M is defined as: $R(M) = \inf_{x \in M} R(x)$. Now we fix a point $x_0 \in M$. We define *k-dimensional deformation coefficient* $\chi_k(x > x_0)$ as follows (cf. [Fo 2]). Suppose that Π_x^{k-1} is a $(k-1)$-plane through x in the tangent space T_xM. Denote D_ε^{k-1} the disk of radius ε in Π_x^{k-1}, and by S_ε the disk $Exp(D_\varepsilon^{k-1})$. We consider the cone CS_ε formed by geodesics joining the vertex x_0 and the base S_ε. We put

$$\chi(x > x_0, \Pi_x^{k-1}) = \lim_{\varepsilon \to 0} \frac{\operatorname{vol}_k CS_\varepsilon}{\operatorname{vol}_{k-1} S_\varepsilon} ,$$

$$\chi(x > x_0) = \max_{\Pi^{k-1} \subset T_xM} \chi_k(x, \Pi^{k-1}) .$$

(b) Let $f(x)$ be the function which measures the distance between a point $x \in M$ and the fixed point x_0. We set

$$q(x_0, r) = \exp\left(\int_0^r \left(\max_{x \in \{f = t\}} \chi_k(x > x_0) \right)^{-1} dt \right) . \tag{1.1}$$

We put

$$\Omega_k(x_0, r) = \lambda_k q(x_0, r) ,$$

$$\Omega_k(M) = \inf_{x_0 \in M} \Omega_k(x_0, R(x_0)) ,$$

where λ_k is volume of the unit ball in $\mathbf{R}^k$.

The defined value is called *the k^{th} geodesic defect* of a Riemannian manifold M. The following theorem was obtained by Fomenko in 1972 [Fo 2].

Theorem 1.1. *Let $X^k \subset M^n$ be a globally minimal submanifold. Then the following inequality holds*

$$\operatorname{vol}_k(X^k) \geqq \Omega_k(M) \geqq 0 .$$

Remark 1.2. Theorem 1.1 has a clear geometric interpretation. It is a consequence of the fact that logarithmic volume growth rate (with respect to the "exterior" distance function t) of a globally minimal submanifold X in M is greater than the integrand in right hand side of (1.1). This derivative $d/dt(\ln \operatorname{vol} X_t)$ equals the "isoperimetric" relation $\operatorname{vol} \partial X_t / \operatorname{vol} X_t$ (see also the proof of Theorem 2.3). The injectivity radius of M is involved because X is a globally minimal submanifold in M.

2 Lower bound for geodesic defects of Riemannian manifolds. New isoperimetric inequalities

Suppose that $a^2 (a \in \mathbf{R}$ or $a \in \sqrt{-1} \otimes \mathbf{R})$ is an upper curvature bound of a Riemannian manifold M.

Theorem 2.1 [LF]. *Lower bound of geodesic defects.*
(a) *If $a^2 \geqq 0$ and $R(M)a \leqq \pi$, then we have:*

$$\Omega_k(M) \geqq k\lambda_k a^{1-k} \int_0^{R(M)} (\sin at)^{k-1}\, dt\,.$$

(b) *If $a^2 > 0$ and $R(M)a > \pi$, then we have:*

$$\Omega_k(M) > \mathrm{vol}(S^k(r = 1/a))\,.$$

(c) *If $a = 0$, then we have $\Omega_k(M) \geqq \lambda_k R(M)^k$.*
(d) *If $a^2 \leqq 0$, then we have:*

$$\Omega_k(M) \geqq k\lambda_k |a|^{1-k} \int_0^{R(M)} (\sinh|a|t)^{k-1}\, dt\,.$$

Theorem 2.2 [LF]. *Upper bound of the deformation coefficient. Let r be the distance between x and x_0.*
(a) *If $a^2 \geqq 0$ and $r \leqq \pi/a$, then we have:*

$$\chi_k(x > x_0) \leqq \frac{\int_0^r (\sin at)^{k-1}\, dt}{(\sin ar)^{k-1}}\,.$$

(b) *If $a = 0$, then we have:*

$$\chi_k(x > x_0) \leqq \frac{r}{k}\,.$$

(c) *If $a^2 \leqq 0$, then we have*

$$\chi_k(x > x_0) \leqq \frac{\int_0^r (\sinh|a|)^{k-1}\, dt}{(\sinh|a|r)^{k-1}}\,.$$

Theorem 2.3 [LF]. *Isoperimetric inequality. Assume that X^k is a globally minimal submanifolds through a point $x \in M$. Let $B_x(r)$ be the geodesic ball of radius r centered at x. Denote A_r^{k-1} the boundary of the intersection $X^k \cap B_x(r) = X_r^k$.*
(a) *If $a^2 > 0$ and $r \leqq \min(R(M), \pi/a)$, then we have:*

$$\frac{\mathrm{vol}(A_r^{k-1})}{\mathrm{vol}(X_r^k)} \geqq \frac{(\sin ar)^{k-1}}{\int_0^r (\sin at)^{k-1}\, dt}\,.$$

Consequently, the following inequality holds

$$\mathrm{vol}(A_r^{k-1}) \geqq k\lambda_k a^{1-k} \sin^{k-1}(ar)\,.$$

(b) *If $a = 0$ and $r \leqq R(M)$, then we have:*

$$\mathrm{vol}(A_r^{k-1}) \geqq k\lambda_k r^{k-1} = \text{the volume of the standard } k\text{-dimensional sphere } S^k \text{ of radius } r.$$

Hence we obtain the following inequalities:

$$\mathrm{vol}(A_r^{k-1}) \geqq (kr)^{-1}\,\mathrm{vol}(X_r^k)\,,$$

$$\mathrm{vol}(X_r^k) \leqq (k)^{\frac{k}{1-k}}(\lambda_k)^{\frac{1}{1-k}}(\mathrm{vol}_{k-1}A_r)^{\frac{k}{k-1}}\,.$$

(c) *If* $a^2 < 0$ *and* $r \leqq R(M)$, *then we have*:

$$\frac{\mathrm{vol}(A_r^{k-1})}{\mathrm{vol}(X_r^k)} \geqq \frac{(\sinh|a|r)^{k-1}}{\int_0^r (\sinh|a|t)^{k-1}\,dt}\,.$$

Hence we get

$$\mathrm{vol}(A_r^{k-1}) \geqq k\lambda_k|a|^{1-k}\sinh^{k-1}(|a|t)\,.$$

Remark 2.4. (a) The estimates in Theorems 2.1 and 2.2 are sharp. First, we note that the equalities in these estimates hold if M and N are spaces of constant curvature. As Theorem 3.6 and Proposition 3.12 show, the equalities hold even in some case when M is not of constant curvature, (but N must be of constant curvature). Roughly speaking, these theorems tell us that globally minimal submanifolds tend to a position of "maximal curvature" in their ambient manifold.

(b) As a particular case of Theorem 2.1 we obtain the Bishop lower estimate for the volume of a Riemannian manifold in term of its upper curvature bounds [BC]. In fact, the Bishop argument almost coincides with ours in the case $N = M$. The slight difference is that we get the estimate for the volume growth rate in term of the deformation coefficient, while Bishop obtains it in term of the Jacobian of the exponential map. This is only an analytic modification, but it is necessary in our case since when $\dim N < \dim M$ we need to make a surgery for N (cone-construction) in our proof and therefore, we have to use boundary value Jacobi estimates instead of initial value Jacobi estimates (see the proofs of Theorem 2.2 and Theorem 2.3).

(c) We also like to mention the lower estimate for the volume of a closed submanifold $N \subset M$ obtained by Heintze and Karcher [HK]. Their estimates are expressed in term of lower curvature bounds, $\mathrm{vol}\,M$, and the length of the mean curvature of N. When N is a point, their theorems imply the Bishop upper estimate for the volume of M in term of lower curvature bounds. In this sense, their estimates and ours are symmetric. Note that their estimate is obtained with the help of the initial value Jacobi estimates for the Jacobian of the exponential map on the normal bundle of a submanifold $N \subset M$.

(d) There are many kind of isoperimetric inequality for minimal submanifolds in space forms (see [Ch], [ChGu] and references in those papers). The argument of Choe and Gulliver is rather close to ours by the cone-construction comparison. (By the way, they observe that the value of such a comparison was first noticed by Blanschke). In particular, they have proved that if M is $\mathbf{R}^n$ or $\mathbf{H}^n$ then the volume of a compact minimal submanifold $X^k \subset M$ with boundary ∂X^k is less than the volume of a cone with base ∂X^k and vertex at a point $x_0 \in X^k$. Combining this result and ours yields that our theorems are still valid for a (locally) minimal submanifolds in $\mathbf{R}^n$ or $\mathbf{H}^n$. It would be interesting to extend this result in the case of non-positively curved manifold M.

Let us show an immediate consequence of Theorem 2.1.

Corollary 2.5. *If M is a compact simply-connected symmetric space with an upper curvature bound a, then the volume of any k-dimensional non-trivial cycle in M is greater than or equal to the volume of k-dimensional sphere of curvature a.*

In the course of the proof of Theorems 2.1 and 2.2 we obtain an estimate for the volume growth of a globally minimal submanifold $X \subset M$ with respect to the distance function on M (see (2.5)). Comparing this "exterior" distance function with the "interior" distance function on X yields the following consequence on *the volume growth of a globally minimal submanifold* $X \subset M$.

Corollary 2.6. *Let X^k be a globally minimal submanifold in a complete non-compact Riemannian manifold M of non-positive curvature. Then the function $V(r) = \mathrm{vol}_k B_X(r)$ grows at least as a polynomial of r of degree k, where $B_X(r)$ is a geodesic ball of radius r in X^k. If the curvature of M has an upper bound strictly less than zero then the function $V(r)$ grows at least as the exponent of r.*

Remark 2.7. (a) It is well known that there is a close relationship between the curvature of a Riemannian manifold M and the growth of its volume [BC]. As a consequence, we obtain the estimate for the growth of its fundamental group (see [M]), and other topological and geometrical invariants of M such as the Betti numbers, the eigenvalues of the Laplace operator and the Gromov invariants [Br 1, Br 2, Gr 1].

(b) We note that our volume growth estimates for globally minimal submanifolds are also valid for locally minimal submanifolds provided that the radius variable r in these estimates is sufficiently small. For example, taking into account Corollary 2.6 (more precisely, the formula (2.5) below) and the Bishops volume estimate, we obtain that if X is a minimal submanifold in a Riemannian manifold M of non positive curvature then the Ricci curvature of X at every point $x \in X$ cannot be positive.

We also obtain from Theorem 2.1 the following corollary which is often used in the Gromov theory of moduli space of holomorphic curves in a compact symplectic (almost Kählerian) manifold.

Corollary 2.8 [Gr 2]. *There exists a positive number $\hbar > 0$ such that the area of every non-constant holomorphic curve in a compact almost Kählerian manifold M is greater than or equal to $\hbar$.*

Proof of Theorem 2.2. Let us write down an explicit formula for the coefficient $\chi_k(x > x_0, \Pi_x^{k-1})$. Suppose $\lambda(t)$ is a geodesic curve, parametrized by its length, joining the points $x_0 = \lambda(0)$ and $x = \lambda(r)$. First, we note that it suffices to consider only the case $\Pi_x^{k-1} \perp \dot{\lambda}(r)$, otherwise we should take into account the angle between $\dot{\lambda}(r)$ and Π_x^{k-1}. (Obviously, if the maximal value of the deformation coefficient is reached on the subspace Π_x^{k-1} then the latter must be perpendicular to $\dot{\lambda}(r)$. Actually, in our proof (see the proof of Theorem 2.3) we need to consider only those subspaces Π_x^{k-1} which are perpendicular to the corresponding $\dot{\lambda}(r_x)$.) We can write $\lambda(t) = Exp_{x_0}(tv)$, where $v = \dot{\lambda}(0) \in T_{x_0}M$. By our assumption, x is not conjugate with x_0, therefore, Exp_{x_0} is a local diffeomorphism at the point $rv \in T_{x_0}M$. In particular, we can choose $(k-1)$ vectors $\{w_1, \ldots, w_{k-1}\}$ such that the differential $dExp_{x_0}$ at rv sends them to an orthonormal basis $\{Y_1(r), \ldots, Y_{k-1}(r)\}$ of $\Pi_x^{k-1} \subset T_x M$:

$$\langle Y_i(r), Y_j(r) \rangle = \delta_{ij};\ Y_i(r) \in \Pi_x^{k-1}. \qquad (*)$$

We extend $Y_i(r)$ as a Jacobi vector field along $\lambda(t)$ such that $Y_i(0) = 0$. It is easy to see that

$$\chi_k(x > x_0, \Pi^{k-1}) = \frac{\int_0^r |Y_1(t) \wedge \ldots \wedge Y_{k-1}(t)| dt}{|Y_1(r) \wedge \ldots \wedge Y_{k-1}(r)|} . \tag{2.1}$$

Put $F(t) = |Y_1(t)| \ldots |Y_{k-1}(t)|$. Clearly, $|Y_1(t) \wedge \ldots \wedge Y_{k-1}(t)| \leqq F(t)$, and in view of $(*)$, the equality holds at $t = r$. Therefore, the formula (2.1) yields

$$\chi_k(x > x_0, \Pi^{k-1}) \leqq \frac{\int_0^r F(t)\, dt}{F(r)} . \tag{2.2}$$

We need the following lemmas.

Lemma 2.9. *Suppose $F(t)$ be as in (2.2). If for all t and Y_j the section curvature $S(\lambda(t), Y_j(t)) \leqq a^2$, where $a > 0$, then the function $F(t)/G(t)$ is monotone increasing on the interval $[0, r]$. Here $G(t) = (\sin at)^{k-1}/(\sin ar)^{k-1}$.*

Lemma 2.10. *Suppose the function $F(t)$ and $G(t)$ be as in Lemma 2.9. Then the following inequality holds*

$$\frac{\int_0^r F(t)\, dt}{F(r)} \leqq \frac{\int_0^r G(t)\, dt}{G(r)} .$$

Proof of Lemma 2.9. By our assumption, $Y_j(t)$, $j = \overline{1, k-1}$, are orthogonal Jacobi vector fields along $\lambda(t)$. The Rauch-Bishop comparison theorem [BC] states that the function $f_j(t) = |Y_j(t)/\sin at$ is monoton increasing on the interval $[0, r]$. Hence, the function $F(t)/G(t) = \prod f_j$ is such a function.

Proof of Lemma 2.10. Since the function $F(t)/G(t)$ increases on the interval $[0, r]$, we get $F(x_i)G(r) \leqq G(x_i)F(r)$ for every $0 \leqq x_i \leqq r$. Hence we obtain

$$\sum_{k=0}^{n} F(kr/n)G(r) \leqq \sum_{k=0}^{n} G(kr/n)F(r) .$$

Letting $n \to \infty$ we infer easily Lemma 2.10 from the above inequality.

Let us continue the proof of Theorem 2.2.

Taking into account (2.2) and Lemmas 2.9, 2.10 we get

$$\chi_k(x, \Pi^{k-1}) \leqq \frac{\int_0^r F(t)\, dt}{F(r)} \leqq \frac{\int_0^r (\sin at)^{k-1}\, dt}{(\sin ar)^{k-1}} .$$

This completes the proof of the first part in Theorem 2.2. In the same way we prove the remaining parts (b) and (c).

Proof of Theorem 2.1. Put

$$\Phi_k^a(r) = k\lambda_k a^{1-k} \int_0^r (\sin at)^{k-1}\, dt .$$

Clearly, $\Phi_k^a(r)$ denotes the volume of the geodesic ball of radius r in the k-dimensional sphere of curvature a [BC]. This function can be rewritten in another way

$$\Phi_k^a(r) = \lambda_k \exp \int_0^r \frac{(\sin at)^{k-1}\, dt}{\int_0^t (\sin a\tau)^{k-1}\, d\tau} . \tag{2.3}$$

To be correct (and in the same time to prove (2.3)) we note that the right hand side of (2.3) can be defined with the help of the following tautology

$$f(r) = \left\{ \frac{f(\varepsilon)}{g(\varepsilon)} \cdot \exp \int_\varepsilon^r (\ln f(t))' - (\ln g(t))' \, dt \right\} \cdot g(r)$$

which can be roughly read as follows. A positive function which vanishes at zero is defined by ist logarithmic derivative and its behavior near zero. (By the way, the behavior of Φ_k^a near zero does not depend on a. This fact can be obtained by taking $\varepsilon - \delta$ limit, or simply by noticing that the volume of any geodesic ball of radius r tends to the volume of the Euclidean ball (of the same dimension) when $r \to 0$. In the same way we define the function $\Omega_k(x, r)$ by putting its "density" at zero equal to the "density" of Φ_k^a at the same point. The volume of any k-dimensional geodesic ball of radius r centered at x in M is an upper bound for $\Omega_k(x, r)$.)

Let us recall the definition

$$\Omega_k(x_0, r) = \lambda_k \exp \int_0^r \left(\max_{x \in \{f = t\}} \chi_k(x > x_0) \right)^{-1} dt .$$

Theorem 2.2(a) yields (taking into account the above remark)

$$\Omega_k(x_0, r) \geqq \lambda_k \exp \int_0^r \frac{(\sin at)^{k-1}\, dt}{\int_0^t (\sin a\tau)^{k-1}\, d\tau} .$$

Combining the above inequality with (2.3) we obtain immediately Theorem 2.1(a).

The remaining parts (c), (d) are proved in the same way. The part (b) follows from the fact that if $R(M) > \pi/a$ then we have $\Omega_k(M) > \Omega_k(x_0, \pi/a) \geqq \text{vol}(S^k, 1/a)$. This completes the proof of Theorem 2.1.

Proof of Theorem 2.3. Let r be as in Theorem 2.3. We denote CA_r^{k-1} the geodesic cone with the base A_r^{k-1} and with its vertex at the point x. Since X_r^k is a globally minimal submanifold, and the cone CA_r^{k-1} is homological to X_r^k, we have $\text{vol}(X_r^k) \leqq \text{vol}(CA_r^{k-1})$. Hence we conclude

$$\frac{\text{vol}(A_r^{k-1})}{\text{vol}(X_r^k)} \geqq \frac{\text{vol}(A_r^{k-1})}{\text{vol}(CA_r^{k-1})}$$

$$\geqq \left(\max_{y \in A_r} \chi_k(y > x) \right)^{-1} \geqq \frac{(\sin ar)^{k-1}}{\int_0^r (\sin at)^{k-1}\, dt} . \tag{2.4}$$

(The second inequality in (2.4) is inferred from the following formula

$$\mathrm{vol}(CA_r^{k-1}) = \int_{A_r^{k-1}} \chi_k(y > x, \Pi_y^{k-1})\,dy\,,$$

where Π_y^{k-1} denotes the tangent space to A_y^{k-1} at y. The third inequality in (2.4) is a consequence of Theorem 2.2(a).)

We observe that the "density" of the function $\mathrm{vol}(X_r^k)$ at $r = 0$ is greater than or equal to that of $\Phi_k^a(r)$ at $r = 0$, and the equality holds if x_0 is a regular point of X^k[Fo 2]. Now, integrating the inequality (2.4) (see also Remark 1.2) and taking into account (2.3) we obtain the following *estimate for the volume growth of* X^k

$$\mathrm{vol}(X_r^k) \geqq \lambda_k \exp \int_0^r \frac{(\sin at)^{k-1}\,dt}{\int_0^t (\sin a\tau)^{k-1}\,d\tau} = \Phi_k^a(r)\,. \tag{2.5}$$

We infer from (2.4) and (2.5) the following inequality

$$\mathrm{vol}(CA_r^{k-1}) \geqq \mathrm{vol}(X_r^k)\frac{(\sin ar)^{k-1}}{\int_0^r (\sin at)^{k-1}\,dt} \geqq \Phi_k^a(r)\frac{(\sin ar)^{k-1}}{\int_0^r (\sin at)^{k-1}\,dt}\,. \tag{2.6}$$

Combining (2.6) and Theorem 2.1.a yields

$$\mathrm{vol}\,A_r^{k-1} \geqq k\lambda_k a^{1-k}(\sin ar)^{k-1}\,.$$

This completes the proof of Theorem 2.3(a). The remaining part of Theorem 2.3 is proved in the same way.

Proof of Corollary 2.5. It is well known that a compact simply-connected symmetric space satisfies the relation: $R(M)a = \pi$. So we get Corollary 2.4 from Theorem 2.1.

Proof of Corollary 2.8. It is well-known that these holomorphic curves are globally minimal cycles in M since they are ω-cycles, where ω is the symplectic form in M [HL, Gr 2]. Now, our statement follows immediately from Theorem 2.1. This statement can be also obtained from the Heintze-Karcher lower estimate for the volume of minimal submanifolds [HK].

3 Explicit formula for geodesic defects of symmetric spaces. Global minimality of Helgason's spheres

Suppose M is a compact symmetric space. Let us compute the deformation coefficient associated with the fixed point $e \in M$. Without loss of generality we compute this coefficient at a point $Exp\,tx \in M$, where x is a vector in a Cartan space H_{lM} of the tangent space lM to M at e. We shall redenote $\chi_k(Exp\,rx) := \chi_k(Exp\,rx > e)$.

Theorem 3.1. *Let* $\{\alpha_i\}$ *be the roots systems of the symmetric space M with respect to* H_{lM}. *Suppose x is a vector of unit length in* H_{lM}. *Without loss of generality we assume that* $\alpha_1(x) \geqq \ldots \geqq \alpha_p(x) = \alpha_{p+1}(x) = \ldots = 0$.

(a) *If $K < p$, then the following equality holds*

$$\chi_k(Exp\, rx) = \frac{\int_0^r \sin(\alpha_1(x)t) \cdot \ldots \cdot \sin(\alpha_k(x)t)\, dt}{\sin \alpha_1(x) r \cdot \ldots \cdot \sin \alpha_k(x) r} .$$

(b) *If $k \geqq p$, then the following equality holds*

$$\chi_k(Exp\, rx) = \frac{\int_0^r \sin(\alpha_1(x)t) \cdot \ldots \cdot \sin(\alpha_{p-1}(x)t) \cdot t^{k-p}\, dt}{\sin(\alpha_1(x)r) \cdot \ldots \cdot \sin(\alpha_{p-1}(x)r) \cdot r^{k-p}} .$$

Lemma 3.2. *Let $\{v_1, \ldots, v_k\} \in M$ be an orthonormal frame which consists of the eigenvectors associated with the eigenvalues $\{\alpha_1^2(x), \ldots, \alpha_p^2(x), 0, \ldots, 0\}$ of the operator ad_x^2. Denote $V_i(t)$ the parallel vector field along the geodesic $Exp\, tx$ such that $V_i(0) = v_i$, and denote $W_i(t)$ the Jacobi vector field along $Exp\, tx$ such that $W_i(0) = v_i$. Then we have the following equalities.*
– If $i < p$ then $W_i(t) = \alpha_i(x)^{-1} \sin(\alpha_i(x)t) V(t)$.
– If $i \geqq p$ then $W_i(t) = t V_i(t)$.

Proof of Lemma 3.2. In the tangent space lM the vector field tv_i is a Jacobi field along the ray tx. Observe that the vector field $dExp_{|tx}(tv_i)$ is also a Jacobi vector field along the geodesic $Exp\, tx \subset M$. Let us write down an explicit formula for the differential of the exponential mapping at the point tx. We will identify M with the quotient G/U, where G is the isometry group of M and U is the isotropy subgroup of the point e. We identify the tangent space lM with the orthogonal complement to the algebra lU in the algebra lG. We denote exp the exponential mapping from the algebra lG to the group G. Then $\exp tx$ is an element in G acting on M and we denote $d\tau(\exp tx)$ the differential of this action. We have [He]

$$\begin{aligned} dExp_{|tx}(tv) &= d\tau(\exp tx) \sum_{n=0}^{\infty} \frac{t^2 ad_x^2(tv_i)}{(2n+1)!} \\ &= d\tau(\exp tx) \sum_{n=0}^{\infty} \frac{(t^2 \alpha_i^2(x))^n (-1)^n}{(2n+1)!} (tv_i) \\ &= d\tau(\exp tx) \frac{\sin \alpha_i(x) t}{t\, \alpha_i(x)} tv_i = \frac{\sin \alpha_i(x) t}{\alpha_i(x)} (d\tau(\exp tx) v_i) . \end{aligned} \tag{3.1}$$

Now we observe that the parallel vector field V_i is obtained from the vector v_i by the shift $d\tau(\exp)$ along the geodesic $Exp\, tx$, that is, $V_i(t) = d\tau(\exp tx) v_i$. Hence we get Lemma 3.2 from (3.1).

Proof of Theorem 3.1. Now we compute the coefficient $\chi_k(Exp\, rx, \Pi^{k-1})$. We observe that the tangent space $\Pi^{k-1}(t)$ to the normal section of the cone CD_ε^{k-1} at the point $Exp\, tx$ can be represented as the sum $\sum a_i \Pi_i^{k-1}(t)$, where a_i are constant, and $\{\Pi_i^{k-1}(t)\}$ is the basis in the space $\Lambda_{k-1}(T_{Exp\, tx} M)$ such that $\Pi_i^{k-1}(t)$ is generated by the orthonormal frame of the vectors $\{W_i(t) \in T_{Exp\, tx} M\}$. Using the formula (2.1) we get

$$\chi_k(Exp\, rx, \Pi^{k-1}) = \frac{\int_0^r |\Pi^{k-1}(t)|\, dt}{|\Pi^{k-1}(r)|} = \frac{\sum_i \int_0^r a_i |\Pi_{k-1}^i(t)|\, dt}{\sum_i a_i |\Pi_{k-1}^i(r)|} .$$

Hence we obtain

$$\chi_k(Exp\, rx) = \max_i \frac{\int_0^r |\Pi^i_{k-1}(t)|\, dt}{\Pi^i_{k-1}(r)}.$$

Combining Lemma 3.2, Lemma 2.9 and Lemma 2.10 we get

$$\max_i \frac{\int_0^r |\Pi_i^{k-1}(t)|\, dt}{|\Pi^i_{k-1}(r)|} = \frac{\int_0^r \sin(\alpha_1(x)t). \dots . \sin(\alpha_k(x)t)\, dt}{\sin(\alpha_1(x)r). \dots . \sin(\alpha_k(x)r)}$$

if $k < p$. In the same way we prove the theorem in the case $k \geqq p$. This completes the proof of Theorem 3.1.

Corollary 3.3. *If M is a symmetric space of rank $= 1$, that is, $\dim H_{lM} = 1$, then the deformation coefficient $\chi_k(Exp\, rx)$ depends only on r.*

(a) *For $M = S^n$ (or $\mathbf{R}P^n$) we have $\chi_k(r) = \int_0^r (\sin t)^{k-1}\, dt/(\sin r)^{k-1}$.*

(b) *For $M = \mathbf{C}P^n$ we have*

$$\chi_k(r) = \frac{\int_0^r (\sin\sqrt{2}t)(\sin t)^{2k-2}\, dt}{\sin\sqrt{2}r(\sin r)^{2k-2}}.$$

(c) *For $M = \mathbf{H}P^n$ we have*

$$\chi_k(r) = \frac{\int_0^r (\sin\sqrt{2}t)^3(\sin t)^{4k-4}\, dt}{(\sin\sqrt{2}r)^3(\sin r)^{4k-4}}.$$

From Corollary 3.3 and by definition (see (1.1)) it follows that the k-dimensional geodesic defect $\Omega_k(\mathbf{R}P^n)$ of a real (resp., complex, quaternionic) projective space depends only on k, and therefore, equals to $\Omega_k(\mathbf{R}P^k)$ (resp. for complex and quaternionic cases). It is easy to see (cf. the proof of Theorem 2.3) that $\Omega_k(\mathbf{R}P^k)$ (resp. for complex and quaternionic cases) is equal to the volume of $\mathbf{R}P^k$. Now, from Theorem 2.1 we obtain immediately the following consequence.

Corollary 3.4 [Fo 1]. *For any $k \leqq n$ the canonically embedded space $\mathbf{R}P^k \subset \mathbf{R}P^n$ (and $\mathbf{C}P^k \subset \mathbf{C}P^n$, $\mathbf{H}P^k \subset \mathbf{H}P^n$ resp.) is a globally minimal submanifold.*

Remark. The operator ad_x^2 coincides with the Ricci transformation $R_x: y \to R_{xy}x$ in the tangent space lM. Therefore, the deformation coefficient $\chi_k(Exp\, rx)$ get the maximal value, if and only if the plane Π^{k-1} is an eigenspace with the maximal eigenvalue of the induced Ricci transformation in the space $\Lambda_{k-1} T_{Exp\, rx} M$. Roughly speaking, the curvature at point $Exp\, rx$ in the direction (rx, Π^{k-1}) get the maximal value.

It is well known that in a simply connected irreducible compact symmetric space M there are totally geodesic spheres of curvature a^2, where a^2 is the curvature upper bound of M. Further, any such sphere lies in some totally geodesic Helgason's sphere of maximal dimension $i(M)$. All Helgason's spheres are equivalent under the action of the isometry group $\mathrm{Iso}(M)$. Moreover, they are of the same curvature a^2[He]. Now we obtain immediately from Corollary 2.5 the following proposition.

Proposition 3.5. *If a Helgason sphere $S(M)$ realizes a non-trivial cycle in the homology group of a simply connected symmetric space M, then it is a globally minimal submanifold in M.*

First, we write the list of Helgason's spheres realizing a non-trivial cycles in real homologies of compact irreducible simply-connected symmetric spaces.

1) If M is a simple compact group, then $i(M) = 3$, and $S(M)$ is a subgroup associated to a highest root of the group M.

2) $M = SU_{l+m}/S(U_l \times U_m)$, $i(M) = 2$, $S(M) = SU_2/S(U_1 \times U_1)$.

3) $M = SO_{l+2}/SO_l \times SO_2$, $i(M) = 2$, $S(M) = SO_3/SO_2$.

4) $M = SU_{2n}/Sp_n$, $i(M) = 5$, $S(M) = SU_4/Sp_2$.

5) $M = Sp_{m+n}/Sp_m \times Sp_n$, $i(M) = 4$, $S(M) = \mathbf{H}P^1$.

6) $M = SO_{2n}/U_n$, $i(M) = 2$, $S(M) = SO_4/U_2$.

7) $M = Sp_n/U_n$, $i(M) = 2$, $S(M) = Sp_1/U_1$.

8) $M = F_4/Spin_9$, $i(M) = 8$, $S(M) = Spin_9/Spin_8$.

9) $M = \mathrm{Ad}\, E_6/T^1\, Spin_{10}$, $i(M) = 2$, $S(M) = SU_2/T^1$.

10) $M = \mathrm{Ad}\, E_7/T^1\, E_6$, $i(M) = 2$, $S(M) = SU_2/T^1$.

11) $M = E_6/F_4$, $i(M) = 9$, $S(M) = Spin_{10}/Spin_9$.

Remark. In all listed cases, if the dimension of Helgason's spheres $i(M) = 2$, the corresponding symmetric space are Kälerian manifolds, and their Helgason's spheres are diffeomorphic to $\mathbf{C}P^1$. The global minimality of the Helgason sphere in 1) was first proved by Fomenko [Fo 1], and then by Dao Chong Thi [Da 1], Tasaki [Ts], the author [Le 1] by the calibration method. The global minimality of the Helgason sphere in 8) was proved by Fomenko [Fo 1] by the method of geodesic defect and by Berger [Be] by the calibration method. It would be interesting to find calibrations which calibrate the Helgason spheres in 4) and 11). It is well known that all characteristic classes on spaces M in 4) and 11) are trivial [Ta 2]. We think a suitable calibration may be chosen among induced invariant differential forms from the isometry group $I(M)$ to M (see also the proof below). We also conjecture that all Helgason's spheres are M^*-minimal submanifolds (see [Le 2]).

Proof of our classification. By looking at the table of real homologies of irreducible globally symmetric spaces [Ta 1, Ta 2], and the table of Helgason's spheres in these spaces [O], comparing dimensions, we conclude that all other Helgason's spheres not in the above list are trivial cycles in real homologies of their ambient spaces. By the above remark, to complete the classification, it suffices to show that the Helgason spheres in 4) and 11) are non-trivial cycles of real homologies. There are many methods to verify if a given submanifold realizes a non-trivial cycle of real homologies of its ambient manifold. In particular, in [Fo 2] Fomenko and Dao Chong Thi gives a complete classification of totally geodesic spheres which realize non-trivial cycles of real homologies of their ambient symmetric spaces. Fomenko's method depends on the explicit description of Bott's periodicity, and it is rather complicated to apply these results to our concrete case. Dao's method is simpler but he computes only for exceptional cases where Bott's periodicity does not hold.

Our verification below makes use of Dao's method. First, we consider the case 4) $S(M) = SU_4/Sp_2 \to SU_{2n}/Sp_n$. We have the following commutative diagram

$$\begin{array}{ccc} SU_4/Sp_2 & \longrightarrow & SU_{2n}/Sp_n \\ \downarrow & & \downarrow \\ SU_4 & \longrightarrow & SU_{2n} . \end{array}$$

Here the embedding $SU_{2k}/Sp_k \to SU_{2k}$, $k = 2$ or n, is the Cartan embedding of the symmetric space under consideration. We note that $S^5 = SU_4/Sp_2$ realizes a non-trivial cycle in SU_4, since so does the corresponding subgroup Sp_2. Therefore, the sphere S^5 also realizes a non-trivial cycle in SU_{2n} since the subgroup SU_4 is totally non-homologous to zero in SU_{2n}. Hence we conclude that the Helgason sphere S^5 realizes a non-trivial cycle of real homologies of SU_{2n}/Sp_n.

The fact, that the Helgason's sphere S^9 realizes a non-trivial cycle of real homologies of E_6/F_4 was proved in Dao's paper [D2]. (To see it we consider the following sequence of mappings

$$S^9 \to E_6/F_4 \to E_6 \to SU_{27} .$$

It is easy to see that the resulting map $\rho: S^9 \to SU_{27}$ is a composition of two maps ρ_1 and ρ_2, where $\rho_1(S^9) \subset Spin_{10}$ is a primitive cycle, and ρ_2 is a spinor representation of $Spin_{10}$ which sends the primitive cycle $\rho_1(S^9)$ to a non-trivial cycle in SU_{27} [Dy, Da 2]. Therefore, we conclude that the Helgason sphere S^9 realizes a non-trivial cycle of real homologies of E_6/F_4.)

Theorem 3.6. *Every Helgason's sphere in a compact irreducible simply connected symmetric space is a globally minimal submanifold in its* $\mathbf{Z}$ *(resp.* $\mathbf{Z}_2$*) homology class.*

Remark 3.7. As a simple corollary of our theorem we obtain that all Helgason spheres in irreducible simply connected symmetric spaces are stable minimal. This corollary was obtained by Ohnita [O] with the help of analyzing the spectrum of the Jacobi operator on these spheres.

Proof. In view of our classification it suffices to show that the Helgason spheres not in the above list realize non-trivial cycles of $\mathbf{Z}_2$ (resp. $\mathbf{Z}$) homologies in their ambient symmetric spaces. All of them are of dimension 2 [He, O]. Since their ambient spaces M are simply connected, and besides, in the considered cases we have $\pi_2(M) = \mathbf{Z}_2$ [Ta 1], it suffices to show that these spheres realize non-trivial elements of the second homotopy group $\pi_2(M)$. Let $M = G/U$, where G is a simply connected group. Our proof is based on the exact sequence [Ta 1]

$$0 = \pi_2(G) \to \pi_2(G/U) \to \pi_1(U) \to \pi_1(G) = 0 .$$

Thus, the map $j: \pi_2(G/U) \to \pi_1(U)$ is an isomorphism. Therefore, the Helgason sphere realizes a non-trivial element in $\pi_2(G/U)$ if and only if its image via j is a non-trivial circle $S^1 \subset U$ in the fundamental group $\pi_1(U)$. Let us recall a geometrical realization of the map j. Assume S^2 is a sphere in G/U. Fix a point $x \in S^2$. Let us realize the sphere S^2 as a suspension over S^1 such that one of its vertices is the fixed point x, and the other is some point $y \in S^2$. This means that we are given a homotopy $F: [0, 1] \times S^1 \to S^2$ such that: $F(0 \times S^1) = x$, and $F(1 \times S^1) = y \in S^2$. Let $\tilde{y}$ be a point in G whose projection $p(\tilde{y}) = y$. According to the covering homotopy theorem there exists a homotopy $\tilde{F}: [0, 1] \times S^1 \to G$ such that

$F(1 \times S^1) = \tilde{y}$, and $p \cdot \tilde{F} = F$. Clearly, $\tilde{F}$ realizes a relative sphere whose boundary S^1 lies in the fiber $p^{-1}(x)$. Hence, this circle is the image of sphere S^2 via the map j. With the above geometric realization j_F of the map j we will show that the image $j_F(S^2)$ of the Helgason sphere $S^2 \in G/U$ may be chosen as a geodesic circle $S^1 \subset U$. To do this we consider the following orthogonal decomposition of the Lie algebra $lG = lU \oplus V$, where V is identified with the tangent space of the symmetric space G/U at e. We note that the totally geodesic subspace $\exp V$ coincides with the Cartan embedding $C(G/U)$ of the symmetric space G/U into G. Consider a highest root α of the algebra lG. It is known that its restricted root $\bar{\alpha}$ is a highest root of the symmetric space G/U. Fix a Cartan algebra $H_V \subset V$. Let $h_{\bar{\alpha}} \in H_V$ be the dual vector to $\bar{\alpha}$, and $v_{\bar{\alpha}} \in V$ the corresponding eigenvector. This implies that

$$h_{\bar{\alpha}} = \sqrt{-1}(1/2)(H_\alpha - H_{\alpha^\theta}), \mathbf{R}v_{\bar{\alpha}} = V \cap \mathbf{C}(X_\alpha - \theta X_\alpha)\,, \tag{3.2}$$

where H_α denotes the vector in the Cartan algebra H_{ClG} corresponding to the root α, $X_\alpha \in ClG$ is the corresponding eigenvector, and θ is the involutive authomorphism defining the symmetric space G/U [He]. Recall that in our case Helgason's sphere is of dimension 2. Therefore, the multiplicity of $\bar{\alpha}$ equals 1 and $v_{\bar{\alpha}}$ is defined uniquely, moreover, the plane $\operatorname{span}(h_{\bar{\alpha}}, v_{\bar{\alpha}})$ is a Lie triple. Indeed, this plane is the tangent plane to the Helgason sphere $S^2 \subset G/U$; it is also the tangent space to the Cartan embedding $C(S^2)$ of this sphere into G. Now we put $w_\alpha = [h_{\bar{\alpha}}, v_{\bar{\alpha}}]$. Since the multiplicity of $\bar{\alpha}$ equals 1 we have $w_\alpha \in lU \cap \mathbf{C}(X_\alpha + \theta X_\alpha)$ (see [He, p. 336]). Taking into account (3.2) we see that the vectors $h_{\bar{\alpha}}, v_{\bar{\alpha}}, w_\alpha$ form a basis of the Lie subalgebra in lG corresponding to the root α. Denote $SU_2(\alpha)$ the corresponding subgroup in G. We note that the subgroup $SU_2(\alpha)$ contains the sphere $C(S^2)$. Further, we observe that the intersection between the group $SU_2(\alpha)$ and U is an one-dimensional compact subgroup $S^1(\alpha)$ generated by the vector w_α.

Lemma 3.8. *There exists a geometrical realization F_j such that F_j sends the Helgason sphere S^2 to the geodesic circle $S^1(\alpha)$.*

Proof. Let $\bar{e}$ denote the antipodal point of e in the sphere $C(S^2)$. Let $S^1(\bar{e})$ be the equator on $C(S^2)$ consisting of those points $g \in S^2_C \subset G$ such that $g^2 = \bar{e}$. We claim that the natural projection $q: G \to G/U$ sends this equator to a point. In fact, this claim is a consequence of the following assertion.

Proposition 3.9 [Fo 2, p. 124]. *Let gU be an arbitrary coset relative to U in G, and besides, $g \in C(G/U)$. Then $gU \cap C(G/U) = \{\sqrt{g^2}\} \cap C(G/U)$.*

(This assertion can be obtained from the following explicit expression for the Cartan embedding $C: G/U \to G$; $gU \to g\sigma(g^{-1})$, where σ denotes the corresponding involutive automorphism of the group G.)

From Proposition 3.9 and the above claim we deduce immediately that the semisphere $S^{2+} \subset C(S^2)$ with the boundary $S^1(\bar{e})$ and containing the point e is a relative sphere of the fibration $U \to G \to G/U$; moreover, its projection into G/U coincides with the Helgason sphere $S^2 \subset G/U$. Now, it is easy to see that there exists a geometric realization F_j which sends the Helgason sphere S^2 to the equator $S^1(\bar{e})$. Suppose z is a point of $S^1(\bar{e})$. Then the shift L_z^{-1} sends the equator $S^1(\bar{e})$ to a geodesic circle $T^1(\alpha)$. By definition $T^1(\alpha)$ is also a geometric realization of the image $j(S^2)$. To complete the proof of Lemma 3.8 it suffices to show that $T^1(\alpha) = S^1(\alpha)$. In fact, the shift L_z^{-1} sends the fiber containing $S^1(\bar{e})$ to the subgroup U; and on the other hand, the subgroup $SU_2(\alpha)$ is invariant under the action L_z^{-1}. Hence, $T^1(\alpha)$ belongs to the intersection between $SU_2(\alpha)$ and U. This implies that $T^1(\alpha) = S^1(\alpha)$.

Corollary 3.10. *$S^1(\alpha)$ is a shortest closed geodesic on the group G, and therefore, on the group U.*

Proof. By construction $SU_2(\alpha)$ is the subgroup corresponding to the highest root α of G. Since G is simply connected the circle $S^1(\alpha)$ is of minimal length [He].

Let $U = SO_n$. It is known that a shortest closed geodesic on SO_n is conjugate under the action of the group $\mathrm{Iso}(SO_n)$ with the standardly embedded sub-group SO_2 which generates a non-trivial element in the fundamental group $\pi_1(SO_n)$. Hence, from Corollary 3.10 we get immediately the following consequence.

Corollary 3.11. *Helgason's spheres in symmetric spaces SU_n/SO_n; E_8/SO_{16}, G_2/SO_4 realize non-trivial elements in $\mathbf{Z}_2$-homologies of their ambient spaces.*

In other cases we have to look more carefully. Our aim is to show that the geodesic circle $S^1(\alpha)$ realizes a non-trivial element in the fundamental group $\pi_1(U)$. Let w_α belong to a Cartan algebra H_{lU} which is contained in a Cartan algebra H_{lG}. Let $h_\alpha \in \mathbf{R}w_\alpha$ be the vector corresponding to the root α. It is known that the vector $h(\alpha) = 4\pi h_\alpha/|\alpha|^2$ belongs to the unit lattice $\Gamma(G, H_{lG})$ of the group G. Let $\tilde{U}$ denote the universal covering of the group U. The condition that the geodesic circle $S^1(\alpha)$ realizes a non-trivial element in $\pi_1(U)$ is equivalent to the following that $h(\alpha)$ does not belong to the unit lattice $\Gamma(\tilde{U}, H_{lU})$ of the group $\tilde{U}$. It is known that the unit lattice Γ of the simply connected group $\tilde{U}$ is $\mathrm{span}_{\mathbf{Z}}\{h(\beta_j)\}$, where $\{\beta_j\}$ is a fundamental system of roots of lU, and $h(\beta_j) = 4\pi h_{\beta_j}/|\beta_j|^2$ (see [He, Ta 1]).

Let us now consider a symmetric space $M = G/U$, where lU is a direct sum of two simple Lie algebras lU_1 and lU_2. In our case M is one of the following spaces: $SO_{m+n}/(SO_n \times SO_m)$, $E_6/(SU_2 \cdot SU_6)$, $E_7/(SU_2 \cdot Spin_{12})$, $E_8/(SU_2 \cdot E_7)$, $F_4/(SU_2 \cdot Sp_3)$. (Except the case of real Grassmannians, other products listed above, $U = U_1 \cdot U_2$, are not direct. Namely, the intersection of U_1 and U_2 consists of 2 points [Ta1]). We note that the vector $h(\alpha)$ does not lie in any algebra lU_i, $i = 1$ or 2, otherwise, the subgroup $SU_2(\alpha)$ lies in the group $U_i \subset U$ entirely. This contradicts to our observation that $SU_2(\alpha)$ meets U at only a circle $S^1(\alpha)$. Hence, in the case $lU = so_n \oplus so_m$, the root α can be written as $x_i \pm x_j$, where $x_i \in H^*_{so_n}$ and $x_j \in H^*_{so_m}$. Thus, $h(\alpha)$ does not belong to the unit lattice of $Spin_n \times Spin_m$. In the same way we verify that for all listed above M the Helgason sphere S^2 realizes a non-trivial element in $\pi_2(M) = H_2(M, \mathbf{Z}) = H_2(M, \mathbf{Z}_2) = \mathbf{Z}_2$.

In order to complete the proof of Theorem 3.6 we need to consider the cases $M = E_6/PSp_4$ and $M = E_7/SU_8^*$. Straightforward calculation shows that if a closed geodesic of minimal length in group $\tilde{U}/\{\pm 1\}$, $\tilde{U} = Sp_4$, SU_8, then it is conjugate under $I_0(U)$ with either the circle $S^1(\beta)$ generated by a highest root β or the closed geodesic S^1_* whose pull back into the covering group U is the shortest geodesic joining two elements $(+1) = (e)$ and (-1). Since the group $SU_2(\alpha)$ does not lie in U, we get that α is not a highest root of $lU_1 \oplus lU_2$. Hence, we deduce easily that the circle $S^1(\alpha)$ is conjugate with S^1_*. Thus, $S^1(\alpha)$ realizes a non-trivial element in $\pi_1(U)$. This completes the proof.

In conclusion we show a consequence of Theorem 2.1 for non-compact symmetric spaces. It is well known that the upper bound of section curvature of these spaces is zero [He].

Proposition 3.12. *Let X be a flat totally geodesic submanifold in a non-compact symmetric space M. Then X is a globally minimal submanifold.*

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