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MINIMAL Φ-LAGRANGIAN SURFACES IN ALMOST HERMITIAN MANIFOLDS

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ABSTRACT. A general method of calibrations is developed for the study of minimal Φ-Lagrangian surfaces in almost-Hermitian manifolds. A criterion for minimality of Φ-Lagrangian surfaces is given, along with a lower bound for the second variation of the volume functional on minimal Φ-Lagrangian surfaces in Hermitian manifolds. The generalized Maslov index of these surfaces is shown to be trivial.

Bibliography: 11 titles.

§1. Introduction

The study of minimal Lagrangian surfaces in Kähler manifolds M^{2n} was begun by Harvey and Lawson [9] for the case $M^{2n} = \mathbf{R}^{2n} = \mathbf{C}^n$ with the standard Kähler structure. They discovered that any minimal Lagrangian surface $L \subset \mathbf{R}^{2n}$ is globally minimal (and consequently stable). Later Bryant [8] considered an arbitrary Kähler manifold M^{2n} and proved a criterion for minimality of Lagrangian surfaces $L \subset M^{2n}$. He showed that the restriction of the first Chern form to any minimal Lagrangian surface $L \subset M^{2n}$ is equal to zero. A. T. Fomenko conjectured that any minimal Lagrangian surface in the symplectic space $\mathbf{R}^{2n}$ has the following topological property: its characteristic classes, which generalize the well-known Maslov classes, are equal to zero. Then Fomenko and the author [4] discovered a criterion for minimality of Φ-Lagrangian surfaces in an arbitrary Hermitian manifold and proved that the Maslov index is trivial for any minimal Lagrangian surface $L \subset \mathbf{R}^{2n}$. A natural question arises as to whether minimal Φ-Lagrangian surfaces are stable and how to describe their topological classes. In this paper we give an answer to this question. For this we use the general theory of calibrations that we have developed.

We recall that a fundamental 2-form Φ on an almost-Hermitian manifold $(M^{2n}, J, \langle\ ,\ \rangle)$ is defined as follows: $\Phi(X, Y) = \langle X, J, Y\rangle$. A submanifold $L^n \subset M^{2n}$ is called a *minimal Φ-Lagrangian submanifold* if it is minimal with respect to the metric $\langle\ ,\ \rangle$ and the restriction of Φ to L^n is annihilated. The manifold M^{2n} is called an *Hermitian manifold* if the complex structure J is integrable, and it is called an *almost-Kähler manifold* if Φ is closed. Finally, an Hermitian manifold M^{2n} with a closed form Φ is called a *Kähler manifold*. Any almost-Kähler manifold is symplectic, and its Φ-Lagrangian submanifolds are simply Lagrangian.

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The plan of the paper is as follows. In §2 we develop the method of calibrations to study local minimality and stable minimality of submanifolds of Riemannian manifolds. In §3 we construct a canonical almost-Hermitian connection on an almost-Hermitian manifold M^{2n}, after which we give a criterion for the minimality of Φ-Lagrangian surfaces $L^n \subset M^{2n}$ in terms of a certain differential 1-form on L. In §4 we introduce a lower bound for the second variation of the volume functional on minimal Φ-Lagrangian surfaces L in an Hermitian manifold M^{2n}. In particular, we obtain the following theorem.

THEOREM. *A minimal Φ-Lagrangian submanifold of an Hermitian manifold M^{2n} is stable if the first Chern form Ω on M^{2n} is nonpositive* (*that is,* $\Omega(X, JX) \leq 0$ *for any vector* $X \in T(M^{2n})$).

We note that in the case of an Hermitian manifold with positive first Chern form there are examples of unstable minimal Lagrangian manifolds.

In §5 we introduce the construction of the generalized Maslov index, proposed by V. V. Trofimov. We then prove the triviality of the generalized Maslov-Trofimov index of any minimal Φ-Lagrangian surface in an Hermitian manifold M^{2n}.

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§2. Quasiregular calibrations and local minimality of submanifolds of a Riemannian manifold

DEFINITION 2.1. A differential n-form φ on a Riemannian manifold is called a *calibration* if its comass is not greater than 1. A submanifold $N^n \subset M^m$ is called a *φ-submanifold* if for all points $x \in N$ we have

$$\varphi(\overrightarrow{T_x N}) = \|\overrightarrow{T_x N}\|,$$

where $\| \ \|$ denotes the norm of the mass (for the details see [9]).

REMARK 2.1. (i) If we require the closure of the form φ in Definition 2.1, then φ is a calibration in the usual sense. The method of calibrations was developed by Dào Trọng Thi [2] and Harvey and Lawson [9] for the study of globally minimal surfaces in Riemannian manifolds. In [4] Fomenko and the author developed this method for the study of minimal Φ-Lagrangian submanifolds of Hermitian manifolds.

(ii) It is easy to see that any submanifold $N^n \subset M^m$ is locally a φ-submanifold with respect to some local calibration φ on M. In particular, for φ we can choose a simple differential n-form. The existence of a global calibration φ with respect to which N is a φ-submanifold follows from the next proposition.

PROPOSITION 2.1. *Let $\{U_i\}$ be a locally finite covering on a manifold M and $\{h_i\}$ the family of functions of a partition of unity with respect to $\{U_i\}$; that is,* $\operatorname{supp} h_i \subset U_i$, $\sum h_i(x) = 1$ $\forall x \in M$, *and* $h_i \geq 0$ $\forall i$. *Assume also that on each domain U_i there is given a calibration φ_i. Then $\varphi = \sum h_i \varphi_i$ is a calibration on M. Let $N^n \subset M^m$ be a submanifold with the property that $N \cap U_i$ is a φ_i-submanifold for any i. Then N is a φ-submanifold.*

PROOF. Clearly,

$$\|\varphi_x\|^* \leq \sum |h_i(x)| \|\varphi_i\|^* \leq \sum h_i(x) = 1,$$

so φ is a calibration. The second part of the proposition is obvious.

We mention the simplest property of φ-submanifolds.

PROPOSITION 2.2. *Let N^n be a φ-submanifold of M^m and $(v_1, \dots, v_n)$ an (oriented) orthonormal basis in T_xN^n. Let w be any vector in T_xM orthogonal to all the v_i. Then the following relations hold:*

(i) $\varphi(v_1 \wedge \cdots \wedge v_n) = 1$,

(ii) $\varphi(w, v_1, \dots, \hat{v}_i, \dots, v_n) = 0$ *for any* $i = 1, \dots, n$.

PROOF. Equality (i) follows directly from the definition:

$$\varphi(v_1 \wedge \cdots \wedge v_n) = |v_1 \wedge \cdots \wedge v_n| = 1.$$

Equality (ii) follows from the fact that φ attains a maximum on the multivector $v_1 \wedge \cdots \wedge v_n$ among unit simple vectors $v_1' \wedge \cdots \wedge v_n' \in G_{m,n}$ (the Grassmann manifold of n-planes in $\mathbf{R}^m \simeq T_xM^m$).

Henceforth we shall consider only a special class of calibrations.

DEFINITION 2.2. Let φ be a calibration on M^m. We denote by $G_x\varphi$ the set of n-planes $l^n \in T_xM^m$ such that $\varphi(\vec{l}) = \|\vec{l}\|$. We say that a calibration is *regular* if all the sets $G_x\varphi$ are diffeomorphic; that is, the bundle $(G_\varphi(M), M, G_x\varphi, p)$, where p is the natural projection onto the base M, is locally trivial. We shall say that a calibrated φ is *quasiregular* if φ can be represented as a sum $\sum h_i\varphi_i$, as in Proposition 2.1, and all the calibrations φ_i are regular.

PROPOSITION 2.3. *Let φ be a quasiregular calibration on a Riemannian manifold M^m, and let $N^n \subset M^m$ be a φ-submanifold. Then, for any normal vector field X on N,*

$$\left.\frac{d}{dt}\right|_{t=0} \operatorname{vol}(N_t) = \left.\frac{d}{dt}\right|_{t=0} \left(\int_{N_t} \varphi\right),$$

where the family N_t is the image of N under the action of the one-parameter subgroup of diffeomorphisms generated by the vector field X.

PROOF. We denote the mean curvature of N by H. Then

$$\text{(A)} \quad \left.\frac{d}{dt}\right|_{t=0} \operatorname{vol}(N_t) = \int_N \langle -H, X\rangle + \int_{\partial N} (X \lrcorner \operatorname{vol}) = \int_N \langle -H, X\rangle,$$

$$\text{(B)} \quad \left.\frac{d}{dt}\right|_{t=0} \left(\int_{N_t} \varphi\right) = \int_N (X \lrcorner d\varphi) + \int_{\partial N} (X \lrcorner \varphi) = \int_N (X \lrcorner d\varphi).$$

Since $d\varphi = d(h_i\varphi_i) = dh_i \wedge \varphi_i + h_i d\varphi_i$, we have

$$X \lrcorner d\varphi = (X \lrcorner dh_i) \wedge \varphi_i - dh_i \wedge (X \lrcorner \varphi_i) + h_i(X \lrcorner d\varphi_i).$$

Since $\varphi_i|_N = \operatorname{vol}|_N$, we obtain

$$(X \lrcorner d\varphi)|_N = dh_i(X)\operatorname{vol} + h_i(X \lrcorner d\varphi_i)$$

(the term $dh_i \wedge (X \lrcorner \varphi_i)$ is equal to zero by Proposition 2.3(ii)). Finally we have

$$X \lrcorner d\varphi = \sum h_i(X \lrcorner d\varphi_i). \tag{2.1}$$

Taking (A), (B), and (2.1) into account, we derive Proposition 2.3 from the following lemma.

LEMMA 2.1. *For any point* $x \in (N \cap V_i)$,

$$\langle -H, X\rangle = (X \lrcorner d\varphi)(v_1, \dots, v_n),$$

where $\{v_i\}$ *is an oriented orthonormal basis in* $T_x N$.

PROOF OF THE LEMMA. For convenience we redenote φ_i by φ and $(N \cap V_i)$ by N. We also choose n oriented orthonormal vector fields $v_1, \dots, v_n$ on some neighborhood $N_\varepsilon(x)$ of the fixed point x. Since the multivector $v_1(y) \wedge \cdots \wedge v_n(y) \in G_y(\varphi)$ for any point $y \in N_\varepsilon(x)$, these fields v_i determine a section $p \colon N_\varepsilon(x) \to \tilde{G}_\varphi(M)$, where $\tilde{G}_\varphi(M)$ denotes the principal bundle over $G_\varphi(M)$ with structural group SO_n. We consider the following commutative diagram:

$$\begin{array}{ccc} & N_\varepsilon & \xrightarrow[p]{} & \tilde{G}_\varphi(M) \\ & q\downarrow & & \downarrow j. \\ (N_\varepsilon \times I^{m-n}) \simeq & M_\varepsilon(N_\varepsilon) & \xrightarrow[i]{} & M \end{array}$$

On this diagram q is the identity embedding of N_ε into some neighborhood $M_\varepsilon(N_\varepsilon) \simeq N_\varepsilon \times I^{m-n}$ and j is the natural projection from the bundle $\tilde{G}_\varphi(M)$ onto the base M. By means of the covering homotopy theorem we conclude that there is a section $\bar{p} : M_\varepsilon(N_\varepsilon) \to \tilde{G}_\varphi(M)$ such that $p = \bar{p} \cdot q$. Consequently, there is an extension V_i of the vector fields v_i from N_ε to $M_\varepsilon(N_\varepsilon)$ such that

(i) $\langle V_i, V_j\rangle = \delta_{ij}$, and
(ii) $\varphi(V_1, \dots, V_n) = 1$.

We now calculate the value of $(X\_d\varphi)(v_1, \dots, v_n)$ at the point x:

$$\begin{aligned}(X \lrcorner d\varphi)&(v_1, \dots, v_n) \\ &= \sum_{i=1}^{n}(-1)^i v_i(\varphi(X, v_1, \dots, \hat{v}_i, \dots, v_n)) - X(\varphi(V_1, \dots, V_n)) \\ &\quad + \sum_{1\le i<j\le n} (-1)^{i+j}\varphi([v_i, v_j], X, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_n) \\ &\quad + \sum_{i=1}^{n}(-1)^i\varphi([X, v_i], \dots, \hat{v}_i, \dots, v_n).\end{aligned}$$

The first and third terms are equal to zero by Proposition 2.2. The second terms are also equal to zero by (ii). Finally we obtain

$$\begin{aligned}(X \lrcorner d\varphi)(v_1, \dots, v_n) &= \sum_{i=1}^{n}(-1)^i\varphi([X, v_i], \dots, \hat{v}_i, \dots, v_n) \\ &= -\sum_i\langle [X, v_i], v_i\rangle = \sum_i\langle -\nabla_X v_i + \nabla_{v_i} X, v_i\rangle \\ &= \sum_i\langle \nabla_{v_i} X, v_i\rangle = \langle -H, X\rangle,\end{aligned}$$

as required.

COROLLARY 2.1. *Let* φ *be a quasiregular calibration on a Riemannian manifold* M^m, *and let* $N^n \subset M^m$ *be a minimal* φ*-submanifold. Then, for any normal vector* *field* X *on* N^n,

$$\frac{d^2}{dt^2}\bigg|_{t=0} \operatorname{vol}(N_t) \geq \frac{d^2}{dt^2}\bigg|_{t=0} \left(\int_{N_t} \varphi\right).$$

PROOF. We put $V(t) = \operatorname{vol}(N_t)$ and $F(t) = \int_{N_t} \varphi$. Then

(a) $V(0) = F(0)$, and
(b) $V'(0) = F'(0) = 0$.

The first identity is satisfied because N is a φ-submanifold. The second is satisfied by Proposition 2.3. Since $V(t) \geq F(t)$ for all t, we obtain $V''(0) \geq F''(0)$. This completes the proof.

The class of minimal Φ-Lagrangian submanifolds is quite wide. Harvey and Lawson [9] constructed a set of examples of minimal Lagrangian submanifolds in $\mathbf{R}^{2n}$. Fomenko and the author [4] constructed examples of such surfaces in Hermitian manifolds. Bryant [8] proved the local existence of such surfaces in the case of Kähler-Einstein manifolds.

§3. A canonical almost-Hermitian connection and a criterion for minimality of Φ-Lagrangian surfaces

In this section we construct a canonical almost-Hermitian connection on an almost-Hermitian manifold. Then, by using the method of calibrations developed in §2, we prove a criterion for minimality of Φ-Lagrangian surfaces in an arbitrary almost-Hermitian manifold.

We first recall the basic definitions. A *principal bundle* $U(M)$ over an almost-Hermitian manifold M^{2n} consists of all unitary frames $\{e_1, Je_1, \dots, e_n, Je_n\}$ with structural group U_n. We assume that $\theta^i = (e_i)^* + \sqrt{-1}(Je_i)^*$ are canonical complex 1-forms on $U(M)$. Also, let (ω^i_j) be a connection form on $U(M)$ and T its curvature tensor. We write out the Cartan structural equations [3]:

$$\text{(I)}\quad d\theta^i = -\omega^i_j \wedge \theta^j + T^i_{jk}\theta^j \wedge \theta^k + T^i_{\bar{j}k}\bar{\theta}^j \wedge \theta^k + T^{\bar{i}}_{\bar{j}\bar{k}}\bar{\theta}^j \wedge \bar{\theta}^k;$$

$$\text{(II)}\quad d\omega^i_j = -\omega^i_k \wedge \omega^k_j + \Omega^i_j,$$

where Ω is the curvature tensor of the given connection.

THEOREM 3.1. (i) *On the bundle* $U(M)$ *there is a unique connection* ω^i_j *such that its torsion tensor has zero components* $T^i_{\bar{j}k}$, *called the canonical connection.*

(ii) *The complex structure* J *is integrable if and only if all the components* $T^{\bar{i}}_{\bar{j}\bar{k}}$ *of the torsion tensor of the canonical connection are equal to zero.*

(iii) *The fundamental 2-form* Φ *is closed if and only if all the components* T^i_{jk} *of the torsion tensor of the canonical connection are equal to zero and the following relations are satisfied*:

$$T^{\bar{i}}_{\bar{j}\bar{k}} + T^{\bar{j}}_{\bar{k}\bar{i}} + T^{\bar{k}}_{\bar{i}\bar{j}} = 0 \quad \textit{for any } i, j, k.$$

The following well-known assertion [3] about the Kähler property of the manifold M^{2n} follows immediately from Theorem 3.1.

ASSERTION 3.1. *An almost-Hermitian manifold M^{2n} is a Kähler manifold if and only if there is an almost-Hermitian connection on it with zero torsion tensor.*

We now state a criterion for the minimality of Φ-Lagrangian submanifolds L of almost-Hermitian manifolds M^{2n} in terms of the canonical connection form in $U(M)$.

THEOREM 3.2. *Let L be an arbitrary Φ-Lagrangian submanifold of M^{2n}, and let $v_1, \dots, v_n$ be certain (oriented) orthonormal (local) vector fields on L. Let $\overline{p}: L \to U(M)$ be a local section defined as follows:*

$$\overline{p}(x) = (v_1(x), Jv_1(x), \dots, v_n(x), Jv_n(x)).$$

Then L is minimal if and only if the induced form

$$\overline{p}^*(\overline{\psi})\overline{\psi} = -\left(\sqrt{-1}\sum \omega_i^i + 2\,\mathrm{Im}\left(\sum T_{\overline{ik}}^{i}\overline{\theta}^k\right)\right)$$

is identically zero on L.

We can get rid of the local character of Theorem 3.2 by considering the Gaussian map $p: L \to \mathrm{Lag}(M)$ [4]. We recall that $\mathrm{Lag}(M)$ denotes the bundle of Φ-Lagrangian planes on M^{2n} and p assigns to each point $x \in L$ its tangent plane T_xL. Clearly, $U(M)$ is the principal bundle over $\mathrm{Lag}(M)$ with structural group O_n and natural projection j:

$$j(v_1, Jv_1, \dots, v_n, Jv_n) = v_1 \wedge \cdots \wedge v_n.$$

We also have $p = j \cdot \overline{p}$. Direct calculations show that the form $\overline{\psi}$ is annihilated on the fibers $j^{-1}(l)$, $l \in \mathrm{Lag}(M)$, and it is invariant under the action of the group O_n. Thus, we have the following result.

PROPOSITION 3.1. *The form*

$$\overline{\psi} = -\left(\sqrt{-1}\sum \omega_i^i + 2\,\mathrm{Im}\left(\sum T_{ik}^{i}\overline{\theta}^k\right)\right)$$

is induced by some form ψ on $\mathrm{Lag}(M)$: $\overline{\psi} = j^*(\psi)$.

We can thus restate Theorem 3.2 as follows.

THEOREM 3.2′. *A Φ-Lagrangian submanifold L of an almost-Hermitian manifold M^{2n} is minimal if and only if the induced 1-form $p^*(\psi)$ is identically zero on L.*

REMARK 3.1. When M^{2n} is Hermitian we can write out a simple explicit formula for ψ by means of local complex coordinates on M^{2n} [4]. We recall that $\psi = Jdf + d\theta$, where f is a function lifted from M^{2n} to $\mathrm{Lag}(M)$ and θ is a function on $\mathrm{Lag}(M)$ with period 2π.

Let us assume that M^{2n} is an Hermitian manifold; then from part (ii) of Theorem 3.1 it follows that $\overline{\psi} = -\sum\sqrt{-1}\omega_i^i$. Therefore the first Chern form $\Omega = (\sqrt{-1}/2\pi)d(\omega_i^i) = (-1/2\pi)d\overline{\psi}$ is identically zero on minimal Φ-Lagrangian submanifolds $L \subset M^{2n}$. Hence we obtain the following result.

COROLLARY 3.1. *Let L be a manifold Φ-Lagrangian submanifold of an Hermitian manifold M^{2n}. Then the restriction of the first Chern form to L is equal to zero.*

REMARK 3.2. Corollary 3.1 was first established by Bryant [8] for the case of a Kähler manifold M^{2n}.

In the general case the differential $d\overline{\psi}$ is not a horizontal form. Scrupulous calculation, taking account of the second structural equation II, shows that

$$d\overline{\psi} = -2\pi\Omega + 2\,\mathrm{Im}\{(T^{l}_{\overline{ki}} + T^{i}_{\overline{kl}})\omega^{i}_{l} \wedge \overline{\theta}^{k} + (R^{i}_{s\overline{kl}} - T^{i}_{sp}T^{p}_{jk})\theta^{s} \wedge \overline{\theta}^{k} + (T^{i}_{\overline{jp}}\overline{T}^{p}_{ks} + T^{i}_{\overline{sp}}\overline{T}^{p}_{jk} - T^{i}_{\overline{pk}})\overline{\theta}^{s} \wedge \overline{\theta}^{k} + T^{i}_{\overline{ik}}(\overline{T}^{k}_{\overline{ps}}\theta^{p} \wedge \theta^{s} + \overline{T^{k}_{ps}}\overline{\theta}^{p} \wedge \overline{\theta}^{s})\} \quad (3.1)$$

(here $R^{i}_{s\overline{kl}}$ are the components of the curvature form Ω^{i}_{s}).

From (3.1) and parts (ii) and (iii) of Theorem 3.1 we immediately have the following proposition.

PROPOSITION 3.2. *Let M^{2n} be an almost-Kähler manifold. Then the form $d\psi$ is horizontal if and only if the complex structure J is integrable; that is, M^{2n} is a Kähler manifold.*

Almost all the assertions of Theorem 3.1 are well known (see [6], Chapter IV, §112). As for (iii), according to Lichnerowicz it holds only in the integrable case (then one of the two conditions in (iii) is satisfied automatically). Without vouching for novelty, we say by way of an addition to Lichnerowicz a few words about (iii) in the general case.

Let us write out the condition for the form $\overline{\Phi}$ to be closed. Let σ be a local section of $M \to U(M)$. Let θ^{i} denote the induced forms $\sigma(\theta^{i})$. Then we have

$$2\Phi = \sqrt{-1}\sum \theta^{i} \wedge \overline{\theta}^{i}, \qquad 2d\Phi = \sqrt{-1}\sum(d\theta^{i} \wedge \overline{\theta}^{i} - \theta^{i} \wedge d\overline{\theta}^{i}). \qquad (3.2)$$

Substituting in (3.2) the formulas for $d\theta$ and $d\overline{\theta}$ (the first structural equation (1) and its dual version

$$d\overline{\theta}^{i} = -\overline{\omega}^{i}_{j} \wedge \overline{\theta}^{j} + \overline{T}^{i}_{jk}\overline{\theta}^{j} \wedge \overline{\theta}^{k} + \overline{T^{i}_{\overline{jk}}}\theta^{j} \wedge \theta^{k}),$$

we conclude that $d\Phi = 0$ if and only if $T^{i}_{jk} = 0$ and the following condition is simultaneously satisfied for any i, j, and k:

$$T^{i}_{\overline{jk}} + T^{j}_{\overline{ki}} + T^{k}_{\overline{ij}} = 0.$$

This proves the theorem.

We now consider Φ-Lagrangian submanifolds L of an almost-Hermitian manifold M^{2n}. We first recall that a form $\varphi \in \Lambda^{n}\mathbf{R}^{2n}$ is said to be *specially Lagrangian* if φ has the form $\varphi = \mathrm{Re}(e^{i\theta}dz_1 \wedge \cdots \wedge dz_n)$, where $\{z_i\}$ is a unitary basis in $\mathbf{C}^n = \mathbf{R}^{2n}$ [9]. A differential form on an almost-Hermitian manifold M^{2n} is called an *SL-form* if for any point $x \in M$ the restriction of φ to T_xM^{2n} is specially Lagrangian [4]. Below we construct in a neighborhood of any Φ-Lagrangian submanifold a family of quasiregular calibrations of type SL.

Let us proceed with the proof of Theorem 3.2. Let $\mathscr{D}$ be a domain in L on which there is defined a local section $\overline{p}\colon L \to U(M)$. Consider all possible extensions of $\overline{p}$ to some section $\hat{p}\colon M_E(\mathscr{D}) \to U(M)$, where $M_E(\mathscr{D})$ is a normal neighborhood of $\mathscr{D}$ in M^{2n}. We put

$$\mathscr{F}_{\mathscr{D}} = \{\varphi_{\hat{\theta}} = \mathrm{Re}\{\hat{p}^{*}(\theta^{1} \wedge \cdots \wedge \theta^{n})\}\}.$$

The next proposition is obvious.

PROPOSITION 3.3. (i) *The family* $\mathcal{F}_{\mathcal{Z}}$ *consists of regular calibrations, and the submanifold* $\mathcal{D} = (M_\varepsilon(\mathcal{D}) \cap L)$ *is a* $\varphi_{\hat\theta}$*-submanifold for each* $\varphi_{\hat\theta} \in F_{\mathcal{Z}}$.

(ii) *The restriction of the form* $\overline{\varphi}_{\hat\theta} = \operatorname{Im} p^*(\theta^1 \wedge \cdots \wedge \theta^n)$ *to* L *is equal to zero for any* $\varphi_{\hat\theta} \in \mathcal{F}_{\mathcal{Z}}$.

(iii) *Let* $\varphi'_{\hat\theta}$ *be another SL-calibration on* $M_E(\mathcal{D})$ *defined by means of another section* $\overline{p}' : \mathcal{D} \to U(M)$. *Then for any* $x \in \mathcal{D}$ *we have* $\varphi'_{\hat\theta}(x) = \det(\langle v_i, v'_j\rangle)$, *where* $\{v_i\}$ *and* $\{v'_j\}$ *are the orthonormal frames in* T_xL *corresponding to the sections* $\overline{p}$ *and* $\overline{p}'$ *(see Theorem* 3.2).

By means of Proposition 3.3(i) and Lemma 2.1 we deduce Theorem 3.2 from the identity

$$(X \lrcorner\, d\varphi_{\hat\theta})|_L(v_1, \dots, v_n) = \overline{p}^*(\overline{\psi})(JX), \tag{3.3}$$

where $v_1 \wedge \cdots \wedge v_n$ is an oriented unit n-vector in $T_* L$.

For this we calculate the differential $d\varphi_{\hat\theta} = \operatorname{Re} d(\theta^1 \wedge \cdots \wedge \theta^n)$. Here and later we will omit the symbol $\hat p^*$ for convenience. Then

$$d\varphi_{\hat\theta} = \operatorname{Re}\left(\sum_i (-1)^{i+1}\theta^1 \wedge \cdots \wedge d\theta^i \wedge \cdots \wedge \theta^n\right).$$

Taking account of the first structural equation (I), we obtain

$$\begin{aligned} d\varphi_{\hat\theta} = \operatorname{Re}\Bigg(&\left(\sum \omega_i^i\right) \wedge \theta^1 \wedge \cdots \wedge \theta^n \\ &+ (-1)^{i+1}\left(\sum T^i_{\overline{jk}}\overline{\theta}^j \wedge \overline{\theta}^k \wedge \theta^1 \wedge \cdots \underset{i}{\wedge} \cdots \wedge \theta^n\right)\Bigg). \end{aligned} \tag{3.4}$$

Since the form $\sum \omega_i^i$ takes purely imaginary values, we have

$$\operatorname{Re}\left(\left(-\sum \omega_i^i\right) \wedge \theta^1 \wedge \cdots \wedge \theta^n\right) = \overline{\psi} \wedge \operatorname{Im}(\theta^1 \wedge \cdots \wedge \theta^n). \tag{3.5}$$

Direct calculation shows that for any vector X and any $j \neq i$ we have

$$(X \lrcorner \operatorname{Re}(T^i_{\overline{jk}}\overline{\theta}^j \wedge \overline{\theta}^k \wedge \theta^1 \wedge \cdots \underset{i}{\wedge} \cdots \wedge \theta^n))|_L = 0.$$

We finally obtain

$$\begin{aligned} (X \lrcorner\, d\varphi_{\hat\theta})|_L = X\Bigg(\lrcorner\Bigg(&-\sqrt{-1}\sum \omega_i^i \wedge \operatorname{Im}(\theta^1 \wedge \cdots \wedge \theta^n) \\ &- \operatorname{Re}\left(\sum T^i_{\overline{ik}}\overline{\theta}^k \wedge \theta^1 \wedge \cdots \wedge \overline{\theta}^i \wedge \cdots \wedge \theta^n\right)\Bigg)\Bigg)\Bigg|_L. \end{aligned} \tag{3.6}$$

LEMMA 3.1. *Suppose that* φ_θ *is an SL-form, where* $\varphi_\theta = \operatorname{Re}(\theta^1 \wedge \cdots \wedge \theta^n)$. *Let* $\overline{\varphi}_\theta$ *denote the form* $\operatorname{Im}(\theta^1 \wedge \cdots \wedge \theta^n)$. *Then, for any* 1*-form* ψ,

$$\psi \wedge \overline{\varphi}_\theta = -J\psi \wedge \varphi_\theta;$$

here the operator J *acts on the cotangent bundle* T^*M^{2n} *by means of an almost-Hermitian metric as follows:*

$$J\psi(v) = \psi(J^*v) = \psi(-Jv).$$

PROOF OF THE LEMMA. The form $\psi + \sqrt{-1}J\psi$ has degree $(1, 0)$, so $(\psi + \sqrt{-1}J\psi)\wedge\theta^1 \wedge \cdots \wedge \theta^n = 0$. Consequently, the form $\operatorname{Im}((\psi + \sqrt{-1}J\psi)\wedge\theta^1 \wedge \cdots \wedge \theta^n)$ is equal to zero, and so Lemma 3.1 follows.

Using Lemma 3.1, we can obtain

$$\sum_i \left(-\operatorname{Re} \sum_k T^i_{\overline{ik}} \overline{\theta}^k \wedge \theta^1 \wedge \cdots \wedge \overline{\theta}^i \wedge \cdots \wedge \theta^n \right) (X_1, v_1, \ldots, v_n)$$

$$= \sum_i \left(-2 \operatorname{Re} \sum_k T^i_{\overline{ik}} \overline{\theta}^k \right)(X) = 2 \operatorname{Im} \left(\sum_{i,k} T^i_{\overline{ik}} \overline{\theta}^k \right)(JX), \tag{3.7}$$

where $\{v_1, \ldots, v_n\}$ is an orthonormal basis in $T_* L$.

From (3.6) and (3.7) by means of Lemma 3.1 we immediately obtain (3.3). Thus (3.3) is true. This proves Theorem 3.2.

§4. A lower bound for the second variation of the volume functional on minimal Φ-Lagrangian submanifolds of Hermitian manifolds

The aim of this section is to prove the following theorem.

THEOREM 4.1. *Let $L \subset M^{2n}$ be an orientable minimal Φ-Lagrangian submanifold of an Hermitian manifold M^{2n}, and X a normal vector field with compact support on L. Then*

$$XX\left(\int_L \mathrm{vol} \right) \geq -2\pi \int_L \Omega(X, JX),$$

where Ω is the first Chern form on M^{2n}.

To this end we consider the family of quasiregular calibrations $\mathcal{F}(L)$ corresponding to the functions of a partition of unity with respect to the covering $\{\mathcal{D}_i\} \subset L$:

$$\mathcal{F}(L) = \left\{ \varphi = \sum h_i \varphi_i ,\ \varphi_i \in \mathcal{F}_{\mathcal{L}_i} \right\}.$$

By Proposition 3.3, L is a φ-submanifold for each $\varphi \in \mathcal{F}(L)$. Let $X(\varphi)$ denote the Lie derivative of the tensor φ in the direction X. Let us calculate the second variation

$$\left.\frac{d^2}{dt^2}\right|_{t=0} \left(\sum_i \int_{L_i} h_i \varphi_i \right) = \sum_i \int_L X \lrcorner\, d(X(h_i \varphi_i)) = \sum_i \int_L X \lrcorner\, X(d(h_i \varphi_i)).$$

Since

$$X(d(h_i \varphi_i)) = X(dh_i) \wedge \varphi_i + dh_i \wedge X(\varphi_i) + h_i X(d\varphi_i) + X(h_i)\, d\varphi_i,$$

we have

$$\begin{aligned} X \lrcorner\, X(d(h_i \varphi_i)) = {} & (X \lrcorner\, X(dh_i))\varphi_i - X(dh_i) \wedge (X \lrcorner\, \varphi_i) + X(\lrcorner\, dh_i) \wedge X(\varphi_i) \\ & - dh_i \wedge (X \lrcorner\, X(\varphi_i)) + h_i (X \lrcorner\, X(d\varphi_i)) \\ & + X(h_i)(X \lrcorner\, d\varphi_i). \end{aligned} \tag{4.1}$$

Substituting $\sum_i dh_i = 0 = (X \lrcorner\, \varphi_i)|_L = (X \lrcorner\, d\varphi_i)|_L$ in the right-hand side of (4.1), we obtain

$$X \lrcorner\, X(d(h_i \varphi_i)) = (X \lrcorner\, dh_i) \wedge X(\varphi_i) - dh_i \wedge (X \lrcorner\, X(\varphi_i)) + h_i (X \lrcorner\, X(d\varphi_i)).$$

Making the substitution $X(\varphi_i) = (X \lrcorner\, d\varphi_i) + d(X \lrcorner\, \varphi_i)$ and integrating in parts, we obtain

$$\int_L (X \lrcorner\, dh_i) \wedge X(\varphi_i) = -\int_L d(X \lrcorner\, dh_i) \wedge X(\varphi_i) = 0.$$

We also have

$$\frac{d^2}{dt^2}\bigg|_{t=0}\left(\int_{L_t}\sum_i h_i\varphi_i\right)=\sum_i\int_L dh_i\wedge(X\lrcorner X(\varphi_i))+h_i(X\lrcorner X(d\varphi_i)). \qquad (4.2)$$

We split the rest of the proof into three steps.

Step 1. Let us calculate the integral

$$\int_L h_i(X\lrcorner X(d\varphi_i)).$$

From (3.4), (3.5), and Theorem 3.1(ii) we have $d\varphi_i=\psi_i\wedge\overline{\varphi}_i$. Therefore

$$X\lrcorner X(d\varphi_i)|_L=((X\lrcorner X(\psi_i))\wedge\overline{\varphi}_i-X(\psi_i)\wedge(X\lrcorner\overline{\varphi}_i)+(X\lrcorner\psi_i)\wedge X(\overline{\varphi}_i)-\psi_i\wedge(X\lrcorner X(\overline{\varphi}_i)))|_L.$$

Since $\overline{\varphi}_i|_L=0=\psi_i|_L$, by Lemma 3.1 we obtain

$$\begin{aligned}X\lrcorner X(d\varphi_i)|_L&=(-X(\psi_i)\wedge(X\lrcorner\overline{\varphi}_i)+\psi_i(X)X(\overline{\varphi})\\&=((X\lrcorner d\psi_i)+d(\psi_i(X)),JX)\varphi_i+\psi_i(X)X(\overline{\varphi}_i))|_L.\end{aligned} \qquad (4.3)$$

LEMMA 4.1. *For each domain $\mathscr{D}_i$ there is a calibration $\varphi_i\in\mathscr{F}_{\mathscr{D}_i}$ such that $\psi_i(X)$ is identically zero on the minimal Φ-Lagrangian submanifold L.*

PROOF. We first choose coordinates on some neighborhood $M_\varepsilon(\mathscr{D}_i)\simeq\mathbf{R}\times\mathbf{R}^{n-1}\times\mathbf{R}^n$ so that (a) $(0,0,y)\in L$, and (b) $d/dt|_{t=0}(t,0,y)|X|=X$.

Consider an arbitrary section $\tilde{p}:M_\varepsilon(\mathscr{D}_i)\to U(M)$ that extends the Gaussian map $p:\mathscr{D}_i\to\mathrm{Lag}(M)$: $p|_{\mathscr{D}_i}=j\cdot\tilde{p}$. Suppose that $\hat{p}$ is the required section (that is, the calibration $\varphi_{\hat{\theta}}$ corresponding to it satisfies $\psi(X)\equiv 0$). We form the equation for $\hat{p}$:

$$\begin{cases}\text{(A)} & \hat{p}(t,z,y)=\tilde{p}(t,z,y)g(t,z,y),\\ \text{(B)} & \psi(\hat{p}_t(0,0,y)\partial t)=0,\end{cases}$$

where g belongs to U_n, the structural group, which acts on the fibers $f^{-1}(x)\in\mathrm{Lag}(M)$ by right shifts: $(v_1\wedge\cdots\wedge v_n)\cdot g=(v_1\cdot g\wedge\cdots\wedge v_n\cdot g)$. Clearly, $g(0,0,y)=e$.

From (A) and (B) we see that $\psi(X)=0$ is equivalent to the condition

$$0=\psi(\tilde{p}_t(0,0,y)\partial t)-\sqrt{-1}\,\mathrm{tr}(g_t(0,0,y)). \qquad (4.4)$$

We put $g(t,z,y)=\exp(-t\psi(\tilde{p}_t(0,0,y)\partial t)h_0)$, where h_0 is the diagonal element $(\sqrt{-1},0,\dots,0)\in u_n$. Then we can verify immediately that (4.4) is satisfied. Hence the calibration $\varphi_{\hat{\theta}}$ corresponding to the section $\hat{p}:M_\varepsilon(\mathscr{D}_i)\to U(M)$ that we have constructed is the required one. This proves the lemma.

Conclusion of Step 1. Suppose that the calibrations φ_i are chosen as in Lemma 1. Then from (4.3) it follows immediately that

$$X\lrcorner X(d\varphi_i)(v_1,\dots,v_n)=(X\lrcorner d\psi_i)JX=-2\pi\Omega(X,JX). \qquad (4.5)$$

Here $v_1\wedge\cdots\wedge v_n$ is an oriented unit n-vector in T_*L.

Step 2. In this step we assert that the calibrations φ_i, chosen as in Lemma 4.1, satisfy the following identity:

$$\sum_i\int_L dh_i\wedge(X\lrcorner X(\varphi_i))=0. \qquad (4.6)$$

Let us fix a calibration φ_0 and a point $x_0\in\mathscr{D}_0$. Let $\{\varphi_0,\dots,\varphi_k\}$ be the set of all calibrations from the chosen family such that $\varphi_i(x_0)\neq 0$, that is, $x_0\in\mathscr{D}_i$, $i=1,\dots,k$. Then (4.6) follows from the next lemma.

LEMMA 4.2. *For any fixed point* $x_0 \in L$,

$$\sum_{i=0}^{k}(dh_i \wedge (X \lrcorner X(\varphi_i))) = 0. \tag{4.7}$$

PROOF OF LEMMA 4.2. Since the φ_j are SL-forms, we can write $\varphi_j = \mathrm{Re}(e^{i\alpha_j}\omega_0)$, where ω_0 is a complex form of degree $(n, 0)$ and α_j is a real function in some neighborhood $\mathscr{D}_\varepsilon(x)$. By Proposition 3.3(iii) and the fact that L is orientable we may assume that $\alpha_j(y) = 0$ for any point $y \in \mathscr{D}_\varepsilon(x) \cap L$ and any $j = 0, \dots, k$. Then at y we have

$$X(\varphi_j) = \mathrm{Re}(X(e^{i\alpha_j}\omega_0)) = \mathrm{Re}\, X(\omega_0) - X(\alpha_j)\,\mathrm{Im}\,\omega_0. \tag{4.8}$$

At any point $x \in \mathscr{D}_\varepsilon(x_0)$, on the one hand, we have

$$d\varphi_j = \mathrm{Re}(d\omega_0 + \sqrt{-1}\, d\alpha_j \wedge \omega_0) = \psi_0 \wedge \overline{\varphi}_0 - d\alpha_j \wedge \overline{\varphi}_0 = (\psi_i - d\alpha_j) \wedge \overline{\varphi}_j,$$

and on the other hand we have $d\varphi_j = \psi_j \wedge \overline{\varphi}_j$. Therefore $(\psi_j - \psi_0 + d\alpha_j) \wedge \overline{\varphi}_j = 0$. It is easy to verify that the latter equality holds if and only if $\psi_j = \psi_0 - d\alpha_j$. Consequently, at the point $y \in \mathscr{D}_\varepsilon(x) \cap L$ we have

$$X(\alpha_j) = (d\alpha_j, X) = \psi_j(X) - \psi_0(X) = 0. \tag{4.9}$$

From (4.8) and (4.9) it follows immediately that

$$\sum_{i=0}^{k}(dh_i \wedge (X \lrcorner X(\varphi_i))) = \sum_{i=0}^{k}(dh_i \wedge (X \lrcorner \mathrm{Re}\, X(\omega_0))) = 0. \tag{4.10}$$

Lemma 4.2 follows immediately from (4.10), and hence (4.6) follows.

Step 3. From (4.2), (4.5), (4.6), and (4.10) it follows immediately that

$$\left.\frac{d^2}{dt^2}\right|_{t=0}\left(\int_{L_t}\sum h_i\varphi_i\right) = \int_L -2\pi\Omega(X, JX). \tag{4.11}$$

Then Theorem 4.1 follows immediately from (4.11) and Corollary 2.1.

The form Ω is said to be negative (positive) if the associated symmetric bilinear form Ω_J, $\Omega_J(X, Y) = \Omega(X, JY)$, is negative (positive). In the case of Kähler manifolds M^{2n} the fact that the first Chern form is positive (negative) is equivalent to the fact that the Ricci tensor is positive (negative) on M^{2n}.

COROLLARY 4.1. *A minimal Φ-Lagrangian submanifold L of an Hermitian manifold M^{2n} is stable if the first Chern form on M^{2n} is nonpositive.*

The class of Kähler manifolds with nonpositive first Chern form is quite broad. In particular, it contains $\mathbf{C}^n$, bounded homogeneous domains $\mathscr{D} \subset \mathbf{C}^n$, and all complex submanifolds of a Kähler manifold M^{2n} with zero curvature form [3]. On the other hand, if the first Chern form is positive, then M^{2n} may have unstable minimal Φ-Lagrangian submanifolds. For example, $M^{2n} = \mathbf{C}P^n$ and $L = \mathbf{R}P^n$.

COROLLARY 4.2. *Let λ_0 denote the lower bound of the eigenvalues of the Hessian of the volume functional on L. We put $c_\theta = \inf_{x\in M}\{c_0(x)$, where $c_0(x)$ is the least eigenvalue of the symmetric bilinear form $-\Omega_J(X)\}$. Then $\lambda_0 \geq 2\pi c_0$.*

REMARK 4.1. The inequality in Corollary 4.2 is best possible. For example, consider a (real) torus T^n and its complexification T^n_c. Then $L = T^n$ is a minimal

Lagrangian submanifold of T_c^n, and in this case $\lambda_0 = c_0 = 0$. Another example is $L = \mathbf{R}P^n$, $M^{2n} = \mathbf{C}P^n$, and in this case $\lambda_0 = 2\pi c_0 = -\frac{1}{2}$. In these examples we use the classical formula for the second variation to calculate the value of λ_0:

$$XX\left(\int_L \mathrm{vol}\right) = \int_L \langle -\nabla^2 X + \overline{R}(X) - \overline{A}(X), X\rangle$$

(for the details see [10] and [11]). In particular, for the pairs (L, M) under consideration there is a vector field X on L such that the inequality in Theorem 3.1 becomes an equality.

§5. The generalized Maslov-Trofimov class of a minimal Φ-Lagrangian surface

We first give the construction of the generalized Maslov class proposed by Trofimov. Let H denote the holonomy group of an almost-Hermitian connection on M^{2n}. Let $H_0 \subset H$ be the subgroup generated by parallel displacements along loops on a Φ-Lagrangian surface $L \subset M^{2n}$. Then we can map L into Λ^+/H_0, where $\Lambda^+ \simeq U_n/\mathrm{SO}_n$ is the Lagrangian Grassmannian, by means of actions of the subgroup H_0.

THEOREM 5.1. *Let L^n be a minimal Φ-Lagrangian submanifold of an Hermitian manifold M^{2n}.*

(i) *The subgroup H_0 is contained in the group SU_n.*

(ii) *The composition $\det j\colon L \to \Lambda^+/H_0 \to S^1$ takes L into a point on S^1.*

COROLLARY 5.1. *Let $[\alpha]$ be the generating element of the cohomology group $H^1(S^1, \mathbf{Z})$. Then the induced cohomology class $j^* \det^*[\alpha]$ is annihilated on L.*

REMARK 5.1. In the case $M^{2n} \simeq \mathbf{C}^n$ the class $j^* \det^*[2\alpha]$ is the Maslov index of the Lagrangian surface L. Thus, we again obtain the theorem on the triviality of the Maslov index in this case [4].

PROOF OF THEOREM 5.1. We observe that the restriction of the first Chern form to L is equal to zero. We now give the infinitesimal version of part (i) of Theorem 5.1.

PROPOSITION 5.1$'$. *Let L be a submanifold of an Hermitian manifold M^{2n}, and H_0 the subgroup of the holonomy group $H(M^{2n})$ generated by parallel displacements along loops in L. Then the algebra lH_0 belongs to the algebra su_n if and only if the restriction of the first Chern form to L is annihilated.*

This proposition generalizes Theorem 4.6 of [3], Chapter IX, for the special case when $L = M^{2n}$ and M^{2n} is a Kähler manifold. It is proved in the same way as in [3].

Continuation of the proof of Theorem 5.1. Clearly, Theorem 5.1 has the following equivalent formulation.

THEOREM 5.1$''$. *For any pair (y, y_0) and any path $u(y, y_0) \subset L$ joining the points y and y_0, there is an element $g \in \mathrm{SU}_n \subset \mathrm{U}_n$ that acts on the tangent space $T_{y_0}M^{2n}$ such that*

$$T_y L = (T_{y_0} L) \cdot \overline{u}(y_0, y) \cdot g,$$

where $\overline{u}(y_0, y)$ denotes the parallel displacement along the part $u(y, y_0)$.

PROOF OF THEOREM 5.1″. Consider an arbitrary path $u_t \subset L : u_0 = y_0$, $u_1 = y$. Let $\overline{p} : L \to U(M)$ be a (local) section. Then by Theorem 3.2 we have $\overline{\psi}(\overline{p}(u_t)) = 0$. Suppose that $\overline{u}_t$ is the horizontal lift of the path u_t in the bundle $U(M)$ with initial condition $\overline{u}_0 = \overline{p}(u_0)$. Then we can write

$$\overline{u}_t = \overline{p}(u_t) \cdot g_t, \qquad g \in U_n, \; g_0 = e.$$

Obviously, the g_t must satisfy the equation

$$\omega(\dot{\overline{u}}_t) = \mathrm{ad}(g_t^{-1})\omega(\overline{p}(\dot{u}_t)) + g_t^{-1}\dot{g}_t,$$

where ω is the Hermitian (canonical) connection form on the principal bundle $U(M)$. Therefore the path $\overline{u}_t$ is horizontal if and only if

$$\omega(\dot{\overline{u}}_t) = 0 = \mathrm{ad}(g_t^{-1})\omega(\overline{p}(\dot{u}_t)) + g_t^{-1}\dot{g}_t,$$

which is equivalent to

$$\dot{g}_t g_t^{-1} = \omega(\overline{p}(\dot{u}_t)). \tag{5.1}$$

Since $\overline{\psi}(\overline{p}(\dot{u}_t)) = 0$, we have $\omega(\overline{p}(\dot{u}_t)) \in \mathrm{su}_n$. Since the group SU_n is simply-connected, it follows from (5.1) that $g_t \in \mathrm{SU}_n$ for any t. This means that

$$T_y L \cdot g_1 = j(\overline{p}(u_1) \cdot g_1) = j(\overline{u}_1) = \overrightarrow{T_{y_0} L} \cdot \overline{u}(y, y_0),$$

or

$$T_1 L = T_{y_0} L \cdot \overline{u}(y, y_0) \cdot g_1^{-1}, \quad \text{where } g_1 \in \mathrm{SU}_n \tag{5.2}$$

(here j is the natural projection $U(M) \to \mathrm{Lag}(M)$; see Proposition 3.1).

Theorem 5.1 follows immediately from (5.2).

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