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RELATIVE CALIBRATIONS AND THE PROBLEM OF STABILITY OF MINIMAL SURFACES

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Introduction.

The aim of this paper is to develop a method of relative calibrations for studying stability of minimal surfaces in Riemannian manifolds. The idea of calibrations was originated by Federerer [9]. Then it was exploited by Dao Trong Thi [13], Harvey and Lawson [10] and others [5, 11] for the proof of global minimality of surfaces in Riemannian manifolds. The well-known Stock's theorem gives us a close connection between the traditional calibrations - the closed differential forms of comass 1 and the globally (homologically) minimal surfaces. As a matter of fact, the method of calibrations has a deeper justification: the linear functional S_φ, which is the integral $\int_N \varphi^k$ where φ^k is a calibration, minimizes the volume functional vol_k. Choosing φ such that $\int \varphi^k$ approximates up to a second term the functional vol_k on a given minimal surface N^k we can estimate from below the second variation of the functional vol_k on surface N^k in M. This approach does require a closeness of the form φ and the condition of its comass 1.

The plan of the paper is as follows. In §1 we develop the method of relative calibrations. Then in §2 we apply the developed method to obtain a lower estimate of the second variation of volume functional on minimal Φ-Lagrangian surfaces in Hermitian manifolds. In particular we prove the following result.

Theorem. A minimal Φ-Lagrangian submanifold in Hermitian manifold M^{2n} is stable if the first Chern form Ω on M^{2n} is non-positive (i.e. $\Omega(X, JX) \leqslant 0$ for any tangent vector $X \in T(M^{2n})$).

Note that there exist unstable minimal Φ-Lagrangian submanifolds in Hermitian manifolds M^{2n} with the positive first Chern form.

In §3 we investigate the stability of closed subgroups in semisimple Lie groups. In particular we have the following theorems.

Theorem A. The representation $\Phi : SU_2 \to G$, where G is a semisimple compact Lie group of the classical type, is stable if and only if it is two-dimensional into $G = SU_n$ or Sp_n, it is the sum of a two-dimensional representation $\Phi_2 + \Phi_2$ or a three-dimensional representation Φ_3 for $G = SO_n$.

Theorem B. The canonically embedded subgroup $SU_n \to SU_m$ (correspondingly submanifold $SL_n(C)/SU_n \to SL_m(C)/SU_m$) is stable with respect to the Riemannian structure corresponding to the Killing form. The canonically imbedded primitive Pontryagin cycles $P_1, \dots P_n \to SU_m$ are stable minimal surfaces.

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1. Quasi regular calibrations and relative calibrations.

We denote by $G_n(M^m)$ the Grassmannian fibre bundle whose fibre consists of oriented n-planes $\xi \subseteq TM$. Using the Riemannian metric on M one can identify $G_n(M^m)$ with the bundle whose fiber consists of unit simple tangent n-vectors. Thus every differential n-form φ on M defines a function, denoted also by φ on $G_n(M^m)$.

Definition 1.1. A pair (φ^n, $C\varphi(M)$) for which φ^n is a differential n-form on M and $C\varphi(M)$ is a locally trivial fibre subbundle of $G_n(M^m)$ will be called a relative calibration if the following conditions hold.

(A) φ^n takes the value 1 on $C\varphi(M)$

(B) the set $C\varphi(M)$ is a critical set of the function φ^n and besides for every $x \in C\varphi(M)$ the value $\varphi(x)$ is local maximum on $C\varphi(x)$ (i.e. the quadratic form $\mathrm{Hess}\,\varphi\,|_x$ is negative semidefinite on $T_x G_n(M)$).

Not always do we succeed in constructing globally a calibration on the entire manifold M. We often have to sew together local calibrations using the partition of unity. It is easy to prove the following proposition.

Proposition 1.1. Let $\{U_i\}$ be a locally finite covering of a manifold M and $\{h_i\}$ be the partition of unity: $\mathrm{supp}\, h_i \subset U_i$, $h_i \geqslant 0$, $h_i(x) = 1$ $\forall x \in M$. Suppose that on every domain U_i there exists a relative calibration $(\varphi_i^n, C\varphi_i(U_i))$. We extend φ_i and C_i from U_i to M as follows: for all $j \neq i$ we set $\varphi_i \mid U_j = 0$, $C_i(U_j) = G_n(U_j)$. Then the pair ($\varphi = \sum h_i \varphi_i$, $C\varphi = \bigcap_i C\varphi_i$) satisfies the conditions (A) and (B).

Definition 1.2. A pair (φ , $C\varphi$) constructed as in Proposition 1.1. is said to be a quasi regular relative calibration. A submanifold $N^n \hookrightarrow M^m$ will be called φ - submanifold if for all $x \in N^n$ we have inclusion $\overrightarrow{T_x N^n} \in C\varphi$.

Example of φ - submanifolds. From Definition 1.2 it immediately follows that N^n is φ - submanifold if and only if for every i the submanifold $(N^n \cap U_i)$ is a φ_i - submanifold. Clearly one can always choose (locally) φ_i such that for every $x \in N^n \cap U_i$ the set $C\varphi(x)$ consists of the unique element associated with the tangent space $T_x N^n$.

We note simplest properties of φ -submanifolds.

Proposition 1.2. Let N^n be a φ -submanifold in M and $(v_1, \ldots v_n)$ is an oriented orthonormal basis in $T_x N^n$. Assume w be a vector in $T_x M$ orthogonal to all v_i. Then the following equalities hold

(i) $\varphi(v_1 \wedge \ldots \wedge v_n) = 1$

(ii) $\varphi(w \wedge v_1, \ldots \wedge \hat{v}_i \wedge \ldots \wedge v_n) = 0$ for every $i = \overline{1,n}$.

Proof. The equality (i) follows from Definition 1.2. The equality (ii) is implied from the fact, that φ gets a critical value at the polyvector $v_1 \ldots v_n$.

Proposition 1.3. Let φ be a quasi regular relative calibration on Riemannian manifold M^m and $N^n \subset M^m$ be a φ -submanifold. Then for any normal vector field X on N we have the following equality

$$\frac{d}{dt}\Big|_{t=0} \mathrm{vol}\,(N_t) = \frac{d}{dt}\Big|_{t=0} \left(\int_{N_t} \varphi \right)$$

where N_t is the image of N under the action of the one-parameter group of diffeomorphisms generated by the vector field X.

Proof. By H we denote the mean curvature of manifold N. We have then the following identities.

$$\frac{d}{dt}\Big|_{t=0} \mathrm{vol}(N_t) = \int_N \langle -H,X \rangle + \int_{\partial N} (X \lrcorner \mathrm{vol}) = \int_N \langle -H,X \rangle , \quad (I)$$

$$\frac{d}{dt}\Big|_{t=0} \left(\int_{N_t} \varphi \right) = \int_N (X \lrcorner d\varphi) + \int_{\partial N} (X \lrcorner \varphi) = \int_N (X \lrcorner d\varphi) \quad (II)$$

Since $d\varphi = d(h_i \varphi_i) = dh_i \wedge \varphi_i + h_i d\varphi_i$ we get

$$X \lrcorner d\varphi = (X \lrcorner dh_i) \wedge \varphi_i - dh_i \wedge (X \lrcorner \varphi_i) + h_i(X \lrcorner d\varphi_i)$$

And because of $\varphi_{i\,|\,N} = \mathrm{vol}_N$ we obtain

$$X \lrcorner\, d\varphi\,|_N = dh_i(X)\ \ \mathrm{vol} + h_i\ (X \lrcorner\, d\varphi_i)$$

(the term $dh_i \wedge (X \lrcorner\, \varphi_i)$ equals zero by Proposition 1.2.(ii))
Finally we have

$$X \lrcorner\, d\varphi \;=\; \sum h_i (X \lrcorner\, d\varphi_i)$$

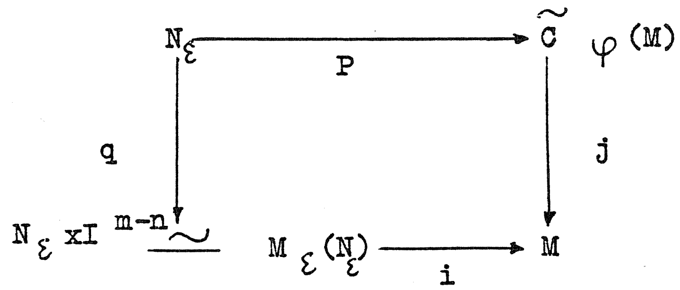
Taking (I), (II),(1.1) into account we will infer Proposition 1.3 from the following lemma.

<u>Lemma 1.1</u>. The following equality is satisfied at all points $x \in (N \cap V_i)$

$$\langle -H,\ X\rangle = (X \lrcorner\, d\varphi_i)\ (v_1, \ldots, v_n),$$

where $\{v_i\}$ is an oriented orthonormal basis in T_xN.

<u>Proof</u>. For the sake of convenience we redenote φ_i by φ and $(N \cap V_i)$ by N. Further we choose n orthonormal vector fields on some neighbourhood $N(x)$ of the fixed point x. Because the polyvector $v_1(y) \wedge \ldots \wedge v_n(y)$ is in $C\,\varphi(y)$ for all $y \in N_\varepsilon(x)$ these vector fields v_i define the section $P : N_\varepsilon \to \widetilde{C}\,\varphi(M)$, where $\widetilde{C}\,\varphi(M)$ is the principal fibre bundle on $C\,\varphi(M)$ with structural group SO_n. We consider the commutative diagram



In this diagram q is the identical embedding of N_ε into some neighbourhood $M_\varepsilon(N_\varepsilon) \simeq N_\varepsilon \times I^{m-n}$ and j is the natural projection of the fibre bundle $C\,\varphi(M)$ into the base M. By virtue of the covering homotopy theorem we conclude that there exists a section $p : M_\varepsilon(N_\varepsilon) \to \widetilde{C}\,\varphi(M)$ satisfying $P = p \cdot q$. Consequently, there is an extension V_i of vector fields v_i from N_ε into $M_\varepsilon(N_\varepsilon)$ such that

$$(*)\ \ \langle V_i,\ V_j\rangle \;=\; \delta_{ij}\,,$$

$$(**)\qquad \varphi(V_1, \ldots, V_n) = 1 \quad .$$

We now compute the value $(X \lrcorner d\varphi)(v_1,\ldots, v_n)$ at x.

$$(X \lrcorner d\varphi)(v_1,\ldots,v_n) = \sum_{i=1}^{n} (-1)^i v_i(\varphi(X,v_1,\ldots,\hat{v}_i,\ldots,v_n))$$

$$-X(\varphi(V_1,\ldots,V_n)) + \sum_{1\leq i<j\leq n} (-1)^{i+j}\varphi([v_i,v_j],X,\ldots,v_i,\ldots$$

$$\ldots \hat{v}_j,\ldots,v_n) + \sum_{i=1}^{n} (-1)^i \varphi([X,v_i],\ldots, v_i,\ldots, v_n) \quad .$$

The first and third terms are zeros by Proposition 1.2. The second terms are also zeros by means of the identity $(**)$.
Finally we get

$$(X \lrcorner d\varphi)(v_1,\ldots,v_n) = \sum_{i=1}^{n} (-1)^i \varphi([X,v_i],\ldots,\hat{v}_i,\ldots,v_n)$$

$$= \sum_i \langle [X,v_i], v_i\rangle = \sum_i \langle -\nabla_X v_i + \nabla_{v_i} X, v_i\rangle$$

$$= \sum_i \langle \nabla_{v_i} X, v_i\rangle = \langle -H,X\rangle$$

Q.E.D.

Theorem 1.1. Let φ be a quasi regular calibration on Riemannian manifold M^m and $N^n \subset M^m$ be a minimal φ-submanifold. Then for any normal vector field X with a compact support on N^n the following inequality holds

$$\frac{d^2}{dt^2}\Big|_{t=0} \mathrm{vol}\,(L_t) \geqslant \frac{d^2}{dt^2}\Big|_{t=0} \left(\int_{L_t} \varphi\right) \quad .$$

Proof. Since the vector field X has the compact support we can assume that for every $t \in (-\alpha, \alpha)$ the submanifold N_t lies in a finite union of domains $U_1,\ldots,U_k$. On every intersection $N_t \cap U_i = N_{i,t}$ we define two sections $p_{o,t} : N_{i,t} \longrightarrow G_n(U_i)$ and $P_{i,t} : N_{i,t} \longrightarrow C\,\varphi_i(U_i)$. The section $p_{o,t}$ is the restriction on $N_{i,t}$ of the section $p_o : N_t \longrightarrow G_n(M)$, $x_t \longmapsto T_{x_t}N_t$, the section $P_{i,t}$ is some extension of the section $P_{i,o} = p_{o,t}$ (such an extension exists by the covering homotopy theorem, see the proof of Lemma 1.1.). Since $P_{i,t}(x)$ is a critical point with the negative semidefinite Hessian we have

$$\varphi_i(p_{o,t}(x)) = \varphi_i(P_{i,t}(x)) + T_i(\rho(t))\,\rho^2(t) \text{ and } T_i(0) \leqslant 0 ,$$

where $\rho(t)$ is the distance between the points $p_{o,t}(x)$ and $P_{i,t}(x)$.

Since $\varphi_i(P_{i,t}(x)) = 1$, we get

$$F(t) = \int_{N_t} \sum h_i \varphi_i(p_{o,t}(x)) = \int_{N_t} 1 + \sum h_i T_i(\rho(t))\,\rho^2(t) =$$

$$= V(t) + \int_{N_t} \sum h_i T_i \cdot \rho^2(t) .$$

Clearly, $F(t) = \int_{N_t} \varphi$ and $V(t) = \mathrm{vol}(N_t)$. Finally, we obtain

$$\frac{d^2}{dt^2}\Big|_{t=0} (F(t) - V(t)) = \int_N \frac{d^2}{dt^2}\Big|_{t=0} \left(\sum h_i T_i(\rho(t))\,\rho^2(t)\right)$$

$$= \int_N 2T_i(0)\,(\rho'(0)^2) \leqslant 0.$$

The proof is complete .

Corollary 1.1. If a form φ is closed and N^n is a minimal φ-submanifold, then N is a stable submanifold.

Proof. The required inequality follows immediately from Theorem 1.1.

$$\frac{d^2}{dt^2}\Big|_{t=0} \mathrm{vol}(N_t) \geqslant \frac{d^2}{dt^2}\Big|_{t=0} \left(\int_{N_t} \varphi\right) = 0 .$$

2. Lower estimate for the second variation of the volume functional on Φ-minimal Lagrangian submanifolds.

Recall that the fundamental 2-form Φ on a Hermitian manifold $(M^{2n}, J, \langle , \rangle)$ is defined as: $\Phi(X,Y) = \langle X, JY \rangle$. A submanifold $L^n \subset M^{2n}$ is said to be minimal Φ-Lagrangian if it is minimal with respect to the metrics $\langle , \rangle$ and the restriction of the form Φ on L vanishes. This section is devoted to the proof of the following result.

Theorem 2.1. Let $L \subset M^{2n}$ be a minimal Φ-Lagrangian submanifold in Hermitian manifold M^{2n} and X be a smooth normal vector field with the compact support on L then the following inequality holds

$$XX\left(\int_L vol\right) \geq -2\pi \int_L \Omega(X, JX)$$

where Ω is the first Chern form on M^{2n}.

We choose a covering $\{D_i\}$ on some neighbourhood of the submanifold $L \subset M$ such that every domain $\{D_i\}$ is diffeomorphic to a ball $B^n \subset R^n$. Further we define on every intersection $L_i = L \cap D_i$ a section P_i from L_i into the fibre bundle $U(D_i)$ whose fibre consists of unitary bases, as follows: $P_i(x) = (v_1(x), Jv_1(x), \ldots, v_n(x), Jv_n(x))$ where $v_1, \ldots v_n$ are certain orthonormal fields on L_i and J is a complex structure operator. We consider all possible extensions $\widehat{P}_i$ of the section P_i from L_i to $M(L_i)$ where $M(L_i)$ is some normal neighbourhood of L_i in D_i. We assume

$$F_{L_i} = \left\{ \varphi_{\hat{\theta}} = \operatorname{Re} \widehat{P}_i^{*} (\theta^1 \wedge \theta^2 \wedge \ldots \wedge \theta^n)\right.$$

Here θ^i are the canonical complex 1-forms on the bundle $U(D_i)$.

Proposition 2.1. (i) The family F_{L_i} consists of relative calibrations and furthermore L_i is a $\varphi_{\hat{\theta}}$-submanifold for every $\varphi_{\hat{\theta}} \in F_{L_i}$.

(ii). The restriction of the form $\overline{\varphi}_{\hat{\theta}} = \operatorname{Im} P_i(\theta^1 \wedge \ldots \wedge \theta^n)^1$ on L vanishes on L.

(iii). For every $x \in L_i$ and every $\varphi, \psi \in F_{L_i}$ the forms $\varphi(x)$ and $\psi(x)$ are coincident.

Proof. We set $C\varphi_{\hat{\theta}} = \left\{ v \in G_n(D_i) : \varphi_{\hat{\theta}}(v) = 1 \right\}$. Then the pair $(\varphi_{\hat{\theta}}, C\varphi_{\hat{\theta}})$ is a calibration [10] and $C\varphi_{\hat{\theta}} \simeq SU_n/SO_n$. Following [10] we call $\varphi_{\hat{\theta}}$ a special Lagrangian (SL) form. Because of the work of Harvey and Lawson [10] we can easily deduce all the statements in Proposition 2.1.

Consider the following family of quasi regular calibrations

$$F_L = \left\{ \varphi = \sum h_i \varphi_i, \quad \varphi_i \in F_{L_i} \right\}.$$

Clearly L is a φ-submanifold for every $\varphi \in F_L$. We now compute the second variation along the vector field X

$$\frac{d^2}{dt^2}\Big|_{t=0} \left(\sum_i \int_{L_t} h_i \varphi_i \right) = \sum_i \int_L X \lrcorner\, d(X(h_i \varphi)) =$$

$$= \sum_i \int_L X \rfloor X(d(h_i \varphi_i)) \ .$$

We have $X \rfloor X(d(h_i \varphi_i)) =$

$$= ((X \rfloor X(dh_i)) \varphi_i - X(dh_i) \wedge (X \rfloor \varphi_i) + (X \rfloor dh_i) \wedge X(\varphi_i)$$

$$- dh_i \wedge (X \rfloor X(\varphi_i)) + h_i(X \rfloor X(d\varphi_i)) + X(h_i)(X \rfloor d\varphi_i) \ . \quad (2.1)$$

Substituting $\sum_i dh_i = 0 = (X \rfloor \varphi_i)|_L = (X \rfloor d\varphi_i)|_L$ into the right part of (2.1) we get

$$X \rfloor X(d(h_i \varphi_i)) = (X \rfloor dh_i) \wedge X(\varphi_i) - dh_i \wedge (X \rfloor X(\varphi_i)) +$$

$$+ h_i(X \rfloor X(d\varphi_i)).$$

Applying the identity $X(\varphi_i) = X \rfloor d\varphi_i + d(X \rfloor \varphi_i)$ and using the rule of integration by parts we obtain

$$\int_L (X \rfloor dh_i) \wedge X(\varphi_i) = -\int_L d(X \rfloor dh_i) \wedge (X \rfloor \varphi_i) = 0 \ .$$

Hence it follows that

$$\frac{d^2}{dt^2}\Big|_{t=0} \left(\int_{L_t} \sum h_i \varphi_i\right) = \sum_i \int_{L_i} dh_i \wedge (X \rfloor X(\varphi_i)) +$$

$$+ h_i(X \rfloor X(d\varphi_i)) \ . \qquad (2.2)$$

We divide the estimate of the right part of (2.2) into three steps.

<u>Step 1.</u> We compute the integral

$$\int_L h_i(X \rfloor X(d\varphi_i)) \ .$$

We need the following lemmas whose proof can be carried out by direct calculation.

<u>Lemma 2.1</u>. For every SL-form $\varphi_{\hat{\theta}} \in F_{L_i}$ the following identity holds at all points $x \in M_\varepsilon(L_i)$

$$d\varphi = \hat{P}^*\left(-\sum \omega_i^i\right) \wedge \mathrm{Im}\, \hat{P}^*(\theta^1 \wedge \ldots \wedge \theta^n) \ ,$$

where (ω^i_j) is the 1-form of the Hermitian connection on the fibre bundle $U(D_i)$.

Lemma 2.2. Let φ_θ be a SL-form, $\varphi_\theta = Re(\theta^1 \wedge \ldots \wedge \theta^n)$. By $\bar{\varphi}_\theta$ we denote the form $Im(\theta^1 \wedge \ldots \wedge \theta^n)$. Then for any 1-form ψ we have the following equality

$$\psi \wedge \bar{\varphi}_\theta = - J\psi \wedge \varphi_\theta$$

where the operator J acts on the cotangent bundle T^*M^{2n} as follows: $J\psi(v) = \psi(-Jv)$.

Put $\hat{P}^*_i(-\sum \omega^k_k) = \psi_i$. Then $d\varphi_i = \psi_i \wedge \bar{\varphi}_i$ by Lemma 2.1. Consequently, we obtain

$$X \lrcorner X(d\varphi_i)_{|L} = (X \lrcorner X(\psi_i)) \wedge \bar{\varphi}_i - X(\psi_i) \wedge (X \lrcorner \bar{\varphi}_i)$$
$$+ (X \lrcorner \psi_i) \wedge X(\bar{\varphi}_i) - \psi_i \wedge (X \lrcorner X(\bar{\varphi}_i)).$$

Since $\bar{\varphi}_{i|L} = 0 = \psi_{i|L}$ (the first equality follows from Proposition 2.1. and the second follows from Proposition 1.3, Lemma 2.1 and from the assumption of minimality of L) we get the following equality by applying Lemma 2.2

$$X \lrcorner X(d\varphi_i)_{|L} = ((X \lrcorner d\psi_i) + d(\psi_i(X)), JX)\ \varphi_i + \psi_i(X)X(\bar{\varphi}_i) \quad (2.3)$$

Lemma 2.3. For every L_i there exists a calibration $\varphi_i \in F_{L_i}$ such that the function $\psi_i(X)$ is identically equal to zero on minimal Φ-Lagrangian submanifold L.

Proof. First, we choose a coordinate system on some neighbourhood $M_\varepsilon(L_i) \simeq R \times R^{n-1} \times R^n$ such that the following conditions are satisfied

a) $(0,0,y) \in L$

b) $(d/dt)_{|t=0}\ (t,o,y)\ |X| = X$.

Then the equation of the section $\hat{P}$ for which the corresponding SL-form $\varphi_{\hat{\theta}}$ satisfies the condition $\psi(X) = 0$, gets the following form

$$\text{(A)} \quad \hat{P}(t,z,y) = P(t,z,y) \cdot g(t,z,y),$$

$$\text{(B)} \quad \psi(\hat{P}_t(o,o,y) \cdot \partial t) = 0$$

where P is some fixed section $M_\varepsilon(L_i) \xrightarrow{P} U(D_i)$ constructed as in the beginning of this section and g is in the structural group U_n. Clearly, $g(o, o, y) = e$. From (A), (B) we conclude that the condition $\psi(X) = 0$ is equivalent to the following equation

$$0 = \psi(P_t(o, o, y)\ t) + \sqrt{-1}\,(-1)\ \mathrm{tr}\,(g_t(o, o, y)) \quad . \qquad (2.4)$$

Let $g(t,z,y) = \exp(-t\ \psi(P_t(o,o,y)\ \partial t)\ h_o)$, where h_o is the diagonal element $(\sqrt{-1}, o, \ldots, o) \in u_n$. One can verify directly that $g(t,z,y)$ is a solution of the equation (2.4). Hence it follows that the calibration ψ_θ corresponding to the constructed section $\hat{P}$ is the required one. The proof is complete.

Continuation of Step 1. Suppose that the calibrations φ_i are chosen as in Lemma 2.3. for every i. From (2.2) we obtain the following equality

$$X \lrcorner X(d\ \varphi_i)\,|_L = (X \lrcorner d\ \psi_i)\ JX = -\,2\pi\ \Omega(X,\ JX) \quad . \qquad (2.5)$$

Indeed, $d\ \psi_i = -\,d(\sum \omega_i^i) = -\,2\pi\Omega$, Ω is the first Chern form.

Step 2. We claim that the calibrations chosen as in Lemma 2.3 satisfy the following equation

$$\sum_i \int_L dh_i \wedge (X \lrcorner X(\varphi_i)) = 0 \quad . \qquad (2.\bar{6})$$

We fix the calibration φ_0 and the point $x_o \in L_o$. Let $\{\varphi_o, \varphi_1, \ldots, \varphi_s\}$ be the set of such calibrations from the chosen family that $\varphi_i(x_o) = 0$, i.e. $x_o \in L_i$ for all $i = \overline{1, s}$. Then the identity is implied from the following assertion.

Lemma 2.3. For every point $x_o \in L$ one has

$$\sum_{i=o}^{s} (dh_i \wedge (X \lrcorner X(\varphi_i))) = 0$$

Proof. Since φ_j are SL-forms we can write $\varphi_j = \mathrm{Re}(e^{i\alpha_j}\ \omega_o)$ where ω_o is some C-valuable n-form of degree (n,o), and α_j is a real function on some neighbourhood $D_\varepsilon(x_o) = U_o \cap U_i$. By Proposition 2.1 (iii) we can assume that $\alpha_j(y) = 0$ for every $y \in D_\varepsilon(x_o) \cap L$ and all $j = \overline{o, s}$. Then we have

$$X(\varphi_i) = \mathrm{Re}(X(e^{i\alpha_j}\ \omega_o)) = \mathrm{Re}\ X(\omega_o) - X(\alpha_j)\ \mathrm{Im}\,\omega_o \quad . \qquad (2.7)$$

At any point $y \in D_\varepsilon(x_0)$, on the one hand, the following equality is satisfied.

$$d\varphi_j = \mathrm{Re}(d\omega_0 + \sqrt{-1}\, d\alpha_j \wedge \omega_0)$$

$$= \psi_0 \wedge \bar{\varphi}_0 - d\alpha_j \wedge \bar{\varphi}_0 = (\psi_0 - d\alpha_j) \wedge \bar{\varphi}_j.$$

On the other hand, we have $d\varphi_j = \psi_j \wedge \bar{\varphi}_j$, therefore $(\psi_j - \psi_0 + d\alpha_j) \wedge \bar{\varphi}_j = 0$. It is easy to verify that the last equation is equivalent to the equation $\psi_j = \psi_0 - d\alpha_j$. Consequently at any point $y \in D_\varepsilon(x) \cap L$ we get

$$X(\alpha_j) = d\alpha_j(X) = \psi_j(X) - \psi_0(X) = 0 \quad . \qquad (2.8)$$

Combining (2.8) and (2.7) we obtain

$$\sum_{i=o}^{s} (dh_i \wedge (X \lrcorner X(\varphi_i)) = \sum_{i=o}^{s} dh_i \wedge (X \quad \mathrm{Re}X(\omega_0)) = 0 \, . \qquad (2.9)$$

Q.E.D.

Step 3. Applying (2.2), (2.5), (2.6) and Lemma 2.4 we get

$$\frac{d^2}{dt^2}\Big|_{t=o} \left(\int_{L_t} \sum h_i \varphi_i \right) = -2\pi \Omega(X, JX) \quad . \qquad (2.10)$$

Then we infer Theorem 2.1 from (2.10) and Theorem 1.1.

A form Ω is said to be negative (resp. positive) if the associated symmetric bilinear form Ω_J, $\Omega_J(X,Y) = \Omega(X,JY)$, is negative (resp. positive). In the case of Kähler manifold M^{2n}, positivity (resp. negativity) of the first Chern form implies positivity (resp. negativity) of the Ricci tensor and vice versa.

Corollary 2.1. A minimal Φ-Lagrangian submanifold $L \subset M^{2n}$ is stable if the first Chern form on M_{2n} is non-positive.

The class of Kähler manifolds with the non-positive first Chern form is large. In particular, it contains C^n, the bounded homogeneous domain $D^n \subset C^n$ and all the complex submanifolds of arbitrary Kähler manifold M^{2n} of zero curvature [4]. The Bryant's theorem [8] states that all Kähler-Einstein manifolds (among them there are manifolds of non-positive first Chern form and manifolds of an opposite type) contain local minimal Φ - Lagrangian submanifolds. Minimal

Φ-Lagrangian submanifolds in Hermitian manifolds M^{2n} possess a trivial characteristic Maslov' class (this was proved by A.T.Fomenko and the author for the case $M^{2n} = R^{2n}$ [6] and by the author for the general case [13].

Remark. The inequality in Theorem 2.1 is sharp. For example, consider the imbedding of a real torus T^n into its complexification T^n_c. Then T^n is a minimal Lagrangian submanifold in T^n_c and there exists a normal vector field X on L such that the inequality in Theorem 2.1 turns into the equality. In the same way one considers the canonical imbedding $R P^n \subset C P^n$.

3. Stable subgroups in semisimple Lie groups.

Let $\Phi : H \to G$ be a representation of a semi-simple group H into a semisimple group G. We shall use our method for finding out the representations Φ such that subgroups $\Phi(H)$ are stable in group G. (Naturally we consider the Riemannian metrics on G corresponding to the Killing form on $1G = g$. It is well-known that any semisimple subgroup $H \subset G$ is totally geodesic, therefore it is a minimal submanifold). We are interested in the case of compact groups H and G. One can apply the Poincare duality theorem of the case of non-compact spaces H^*, G^*. Most proofs of the mentioned below theorems are only sketched or not given. The complete proofs and further applications of the method will appear in next papers.

In fact, we can think that pair (φ^n, $C\varphi$) is determined by the form φ^n. Every locally maximal level $C\varphi$ of the function φ on $G_n(M)$, provided φ takes constant value $C > 0$ on $C\varphi$, gives us the gauge ($\bar{\varphi}^n$, $C\bar{\varphi}$), where $\bar{\varphi} = c^{-1}\varphi^n$ (we assume the requirement of regularity of the pair ($\bar{\varphi}^n$, $C\bar{\varphi}$) to be satisfied). With the help of the following proposition we can recognize the φ-submanifolds.

Proposition 3.1. Let $v_1,\ldots,v_n, v_{n+1},\ldots,v_m$ be an orthonormal basis in the Euclidean space R^m and $v_1,\ldots,v_m$ the dual basis in $\Lambda^1 R^m$. We shall identify the tangent space $T v_1 \ldots v_n G(R^m)$ with the linear span of the vectors v_{ij} ($i=\overline{1, n}$, $j=\overline{n+1,m}$) acting on form φ as follows: $v_{ij}(\varphi) = v_j^* \wedge (v_i \lrcorner \varphi)(v_1 \wedge \ldots \wedge v_n)$.

(i) The point $v_1 \wedge \ldots \wedge v_n$ is a critical point of the function $\varphi = a_{12\ldots n} v_1 \wedge \ldots \wedge v_n + a_{i_1\ldots i_n} v_{i_1} \wedge \ldots \wedge v_{i_n}$ if and only if the intersection of any index set $(i_1,\ldots,i_n)$ and the

set $(1,\dots,n)$ contains no more than $(n-2)$ elements.
(ii) Assume $v_1 \wedge \dots \wedge v_n$ to be a critical point of some form φ and $(v_1 \wedge \dots \wedge v_n) = 1$. Then the symmetric bilinear form $(\text{Hess}\ \varphi)\, v_1 \ \dots \ v_n$ is defined as follows

$$\text{Hess}\ \varphi\ (v_{ij}, v_{ij}) = -1 \ ,$$

$$\text{Hess}\ \varphi\ (v_{ij}, v_{kl}) = v_{kl}(v_{ij}(\varphi)) \ .$$

We now return to the semisimple compact Lie groups. Consider the simplest case, when $\deg \varphi = 3$, φ is a bi-invariant form on group G. Clearly, φ is defined by its restriction on the algebra $g = 1G$.

Theorem 3.1. The form $(X,Y,Z) = \langle X, [Y,Z] \rangle$. is a Ad_G-invariant 3-form on g. A point $v_1 \wedge v_2 \wedge v_3$ is a critical point of φ on $G_3(g)$ if and only if the span (v_1, v_2, v_3) forms a Lie subalgebra. Hence, φ-submanifolds in G are either totally geodesic 3-spheres or totally geodesic real three-dimensional projective spaces.

Proof. The first assertion is trivial. By Proposition 3.1 the point $v_1 \ \ v_2 \ \ v_3$ is a critical point if and only if the following relations are satisfied for every $i = \overline{4,N}$, where $N = \dim G$.

$$v_{1i}(\varphi) = \langle v_i, [v_2, v_3] \rangle = 0 \ ,$$

$$v_{2i}(\varphi) = \langle v_1, [v_i, v_3] \rangle = 0 \ ,$$

$$v_{3i}(\varphi) = \langle v_1, [v_2, v_i] \rangle = 0 \ .$$

Clearly these relations hold if and only if $[v_1, v_2]$, $[v_2, v_3]$ $[v_3, v_1]$ are in span $(v_1, v_2, v_3) \subset g$. The second assertion is proved. The third follows from the second.

Any 3-dimensional totally geodesic non-flat submanifold in semisimple Lie group G is determined by a representation $\Phi : SU_2 \longrightarrow G$.

Proposition 3.2. $v_1 \wedge v_2 \wedge v_3 = \Phi(su_2)$ is a critical point realizing locally maximal value of the form $\varphi(X,Y,Z) = \langle X, [Y,Z] \rangle$ defined on the algebra g if and only if Φ is a two-dimensional representation for $g = \mathbf{su}_n$ or $\mathbf{sp}_n$ and the sum of two-dimensional representation $\Phi_2 + \Phi_2$ for $g = \mathbf{so}_n$.

Thus one can see that the critical points realizing a locally maximal value of the function φ on $G_3(g)$, where g is a simple Lie algebra, realize absolutely maximal value of the considered function φ. In [7] Brother showed that the canonical embedding SO_3

$\to SO_n$ is stable; his statement about the stability of $SO_3 \to U_n$ is not true. Hence we have the following

<u>Proposition 3.3</u>. The irreducible representation Φ_2 of the group SU_2 is stable in SU_n, the representations $\Phi_2 + \Phi_2$ and Φ_3 of SU_2 are stable in SO_n, the representation $\Phi_2(SU_2)$ is stable in Sp_n.

The above list of representations of the group SU_2 really exhausts all its stable representations. To prove the instability of other representations we need the following lemma

<u>Lemma 3.1</u>. Let Φ be a representation of SU_2 in a group G, and Xo, X_+, X_- be a standard basis of the subalgebra $\Phi(su_2) \subset g$: $[X_0, X_+] = X_-$, $[X_+, X_-] = X_0$, $[X_-, X_0] = X_+$, $|X_0| = |X_+| = |X_-|$. By $2\pi l$ we denote the length of the circle $S(X_0) = \exp t X_0$. Further, we suppose that there exist orthonormal vectors V, W in the orthogonal supplement of $\Phi(su_2)$ in g such that the following relations hold

$$\langle [V, W], X_+ \rangle = \langle [V, W], X_- \rangle = 0 ,$$

$$C = \frac{|\langle [V,W], X_0 \rangle|}{|X_0|} \geqslant \frac{5}{41} ,$$

then the subgroup $\Phi(SU_2)$ is unstable in G.

<u>Proof</u>. We shall construct an explicit variation, decreasing the volume of $\Phi_2(SU_2)$ in G. The variation field has the following type $X(x) = a(x)\bar{V} + b(x)\bar{W}$, where $\bar{V}, \bar{W}$ - the left continuation of the vectors $V, W \in g$ on the sub-group $\Phi(SU_2)$. Before constructing the function a(x) and b(x) on $\Phi(SU_2)$ we consider the Hopf's fibrations:

$$p : S^3 \xrightarrow{S^1 (X_0)} S^2 \qquad \text{or} \qquad p : RP^3 \xrightarrow{S^I (X_0)} S^2,$$

where the base S^2 is provided with the standard Riemannian metrics, for which the projection p is a Riemannian immersion. Fix some point $y_0 \in S^2$. Clearly, there exists a section $q : S^2 \setminus y_0 \to S^3$. Then the functions a(x) and b(x) can be chosen as follows. For every $x \in p^{-I}(y_0)$ we put

$$a(x) = f(p(x)) \cos(x, q(p(x))),$$

$$b(x) = f(p(x)) \sin(x, q(p(x))) .$$

Here the bracket (x, q (p (x)) denotes the oriented (by action of the

$$2k^{-1/2}(k+1)^{1/2}\,k\,\theta^{2k-1}(X_1,\dots,X_{2k-1}) = \begin{cases} (-1)^{k/2}\,\mathrm{Re}\sum\limits_{\sigma\in S_{2k-1}}\varepsilon_\sigma \mathrm{Tr}(X_{\sigma(1)}\cdots X_{\sigma(2k-1)}) & \text{if } k \text{ is even} \\ (-1)^{k+1/2}\,\mathrm{Im}\sum\limits_{\sigma\in S_{2k-1}}\varepsilon_\sigma \mathrm{Tr}(X_{\sigma(1)}\cdots X_{\sigma(2k-1)}) & \text{if } k \text{ is odd.} \end{cases}$$

We now introduce some notation concerning the elements of $gl_n(C)$. By j we denote the element $\sqrt{-1}\,E_{jj}$, by $\overrightarrow{ij}^{\,+}$ the element E_{ij}, by ij^- the element $\sqrt{-1}\,E_{ij}$, by ij^+ the element $(\overrightarrow{ij}^{\,+} - \overrightarrow{ji}^{\,+})/\sqrt{2}$, by ij^- the element $(\overrightarrow{ij}^{\,-} - \overrightarrow{ji}^{\,-})/\sqrt{2}$, by h_n the element $\sqrt{-1}\,\mathrm{diag}(-1,\dots,1,-n) \in \mathbf{su}_{n+1}$.

Proposition 3.4. The $(2n-1)$-vector $P_{2n-1} = \sqrt{n(n+1)}\,(h_n \wedge 1n^+ \wedge 1n^- \dots \wedge (n-1)n^+ \wedge (n-1)n^-)$ is a critical point realizing locally maximal positive value of the function θ^{2n-1} on the Grassmannian $G_{2n-1}(\mathbf{su}_n)$.

Proof. We choose the basis in the tangent space $TP_{2n-1}(G_{2n-1}(SU_n))$ as in Proposition 3.1., i.e. the vectors $ij^{\pm} \lrcorner\, kl^{\pm}$ ($j=n$; $i,k,l < n$) act on the linear function θ^{2n-1} as follows: $ij^{\pm}\ kl^{\pm}(0) = kl^{\pm}(ij^{\pm} \lrcorner\, \theta)(P_{2n-1})$. It is easy to verify that $\theta^{2n-1}(P_{2n-1}) = 1$ and the point P_{2n-1} is critical. To compute the matrix Hess $\theta^{2n-1}(P_{2n-1})$ it suffices to calculate the values $kl^{\pm} \wedge k'l'^{\pm} \wedge (i'j'^{\pm} \lrcorner\ ij^{\pm} \lrcorner\, \theta)(P_{2n-1})$. We use the following rules:

a) Closeness principle: $\mathrm{Tr}(\overrightarrow{i_1j_1} \times \overrightarrow{i_2j_2} \times \dots \times \overrightarrow{i_nj_n}) \neq 0$ if and only if $j_k = i_{k+1}$, $i_1 = j_n$. Consequently if $\mathrm{Tr}(i_1j_1 \times \dots \times i_nj_n) \neq 0$ then each index i_k occurs an even number of times and the next vectors have at least one common index.

b) Symmetry principle: the form θ^{2n-1} is invariant with respect to permutation $\sigma(i,j)$ of the indices i and j.

c) Antisymmetry rule: $\mathrm{Hess}(ij \lrcorner kl,\ i'j' \lrcorner\, k'l') = -\mathrm{Hess}(i'j' \lrcorner\, kl,\ ij \lrcorner\, k'l')$ if $(ij \lrcorner\, kl) \neq (i'j' \lrcorner\, k'l')$.

d) Sign concordance condition: if $\theta^{2n-1}(i_1j_1 \wedge \dots \wedge i_{2n-1}j_{2n-1}) \neq 0$, then the number of indices with the sign (-) equals n (mod 2).

We exhibit a part of the matrix Hess$\theta^{2n-1}(P_{2n-1})$ in tables 1,2 where $\alpha = -(2(n-1))^{-1}$. The remaining terms of the matrix Hess can be easily calculated by using rules a), b), c), d) and tables 1,2.

A straightforward computation shows that the matrix Hess θ^{2n-1} is negative semi-definite and its null-index equals the dimension of the orbit $SU_n(P_{2n-1})$ on the Grassmannian $G_{2n-1}(\mathbf{su}_n)$.

group S^1 $(X_0))$ arc, joining $q(p(x))$ and x in $p^{-1}(x)$, $f(y)$ is a function on S^2, depending on the distance $\rho(y,y_0)$ and besides $f(y_0) = 0$. Clearly, the functions $a(x)$ and $b(x)$ continue on the entire S^3 (RP^3). Applying the second variation formula for minimal orbits [7] (in the considered situation the subgroup $\Phi(SU_2)$ acts on G by the left multiplication)

$$\delta^2(X) = \int_{\Phi(SU_2)} \| da \|^2 + \| db \|^2 - \langle [\bar{V}, \bar{W}] , adb - bda \rangle .$$

We easily get (using Brother's technique [7])

$$\delta^2(X) = \frac{\pi 1}{2} \int_{S^2(1)} (1 - cl) f^2(y) + \| df(y) \|^2 . \quad (3.1)$$

Analysing the Euler-Lagrangian equation for (3.1) one can show that if $cl \geqslant 5/4$ there exists a function f on S^2, $f(y_0) = 0$, such that the integral (3.1) is less than zero. The proof is completed.

It is easy to verify that all irreducible representations of dimension $k \geqslant 4$ of the group SU_2, the representation $\Phi_3(SU_2) \simeq SO_3 \longrightarrow SU_n$, as well as its reducible representations different from the one mentioned in Proposition 3.3 satisfy the assumption in Lemma 3.1. Consequently, they are unstable.

Theorem 3.2. A representation $\Phi : SU_2 \longrightarrow G$ is stable if and only if $\Phi = \Phi_2$ for $G = SU_n$, $\Phi = \Phi_2 + \Phi_2$ or Φ_3 for $G = SO_n$ and finally $\Phi = \Phi_2$ for $G = Sp_n$.

Any representation $\Phi : H \longrightarrow G$, where G is a semisimple Lie group, is determined by the family $\Phi_i : H \longrightarrow G_i$, where G_i are the simple components of the group G. One can easily prove the following theorem.

Theorem 3.3. (i) Let the subgroup $\Phi(SU\ \ )$ be diffeomorphic to sphere S^3. Then $\Phi(SU\ \ )$ is stable in G if and only if there exists index i that $\Phi_i(SU_2)$ is stable in G_i and for all $j \neq i$ we have $\Phi_j(SU_2) \simeq e$.
(ii) Let the subgroup $\Phi(SU_2)$ be diffeomorphic to projective space RP^3. Then $\Phi(SU_2)$ is stable in G iff for all i the subgroup $\Phi_i(SU_2)$ is stable in G_i.

Thus, Theorems 3.2 and 3.3. give us a classification of the stable simple compact three-dimensional groups of semi-simple compact Lie groups of a classical type.

We now deal with the calibrations $(\varphi, C\varphi)$ where φ is bi-invariant form of a high degree on the group SU_n. It is well-known that the exterior algebra of a bi-invariant form on SU_n possesses the generators θ^3, θ^5, ..., θ^{2n-1}. Here the restriction of the form θ^{2k-1} on the algebra su_n has the following type

Theorem 3.4. The canonically imbedded subgroups $SU_n \subset SU_m$ and the Pontrjagin primitive cycles $P_{2n-1} \subset SU_m$ are stable in the group SU_m.

Proof. Let $\psi = \theta^1 \wedge \ldots \wedge \theta^{2n-1}$. It is easy to verify that the polyvector $\overrightarrow{su_n}$ is a critical point of ψ and the matrix Hess at su_n is decomposed into a direct sum of the matrix of type $-I_{(m-n)(m-n-1)(n^2-1)}$ and $(m-n)$ matrices H_i, one of them is equivalent to the matrix Hess φ at $\overrightarrow{su_n}$ on the Grassmanian $G_{n^2-1}(su_{n-1})$. Using the Poincare's duality one remarks that the last one is equivalent to Hess θ^{2n+1} at the point P_{2n+1}, therefore it is negative semidefinite. Then we define a calibration (ψ, C_ψ) as follows : $C_\psi(e) = \{Ad_G(su_n)\}$, $C_\psi(g) = g\, C_\psi(e)$. Then (ψ, C_ψ) is a regular relative calibration on SU_m and SU_n is ψ-submanifold. Hence it is (strongly) stable in SU_m. The same is true for the Pontrjagin cycles. Note that one can repeat these constructions in the case of non-compact symmetric spaces. In particular we obtain the following result.

Proposition 3.5. Canonically imbedded symmetric subspace $Sl_n(C)/SU_n$ are stable minimal submanifolds in the space $Sl_m(C)/SU_m$.

Table I.

	$1n^+ \rfloor 12^-$	$1n^- \rfloor 12^+$	$2n^+ \rfloor 1\text{-}2$	$2n^+ \rfloor 1\text{-}3$	$3n^+ \rfloor 32^-$	$3n^- \rfloor 32^+$
$1n^+ \rfloor 12^-$	-1	2α	2α	α	$-\alpha$	α
$1n^- \rfloor 12^+$	2α	-1	-2α	$-\alpha$	α	$-\alpha$
$2n^+ \rfloor 1\text{-}2$	2α	-2α	-1	0	α	$-\alpha$
$2n^+ \rfloor 1\text{-}3$	α	$-\alpha$	0	-1	$-\alpha$	α
$3n^+ \rfloor 32^-$	$-\alpha$	α	α	$-\alpha$	-1	2α
$3n^- \rfloor 32^+$	α	$-\alpha$	$-\alpha$	α	2α	-1

Table 2.

	$1n^+ \rfloor 12^+$	$1n^- \rfloor 12^-$	$2n^- \rfloor 1\text{-}2$	$2n^- \rfloor 1\text{-}3$	$3n^+ \rfloor 32^+$	$3n^- \rfloor 32^-$
$1n^+ \rfloor 12^+$	-1	-2α	2α	α	$-\alpha$	$-\alpha$
$1n^- \rfloor 12^-$	-2α	-1	2α	α	$-\alpha$	$-\alpha$
$2n^- \rfloor 1\text{-}2$	2α	2α	-1	0	α	α
$2n^- \rfloor 1\text{-}3$	α	α	0	-1	$-\alpha$	$-\alpha$
$3n^+ \rfloor 32^+$	$-\alpha$	$-\alpha$	α	$-\alpha$	-1	-2α
$3n^- \rfloor 32^-$	$-\alpha$	$-\alpha$	α	$-\alpha$	-2α	-1

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