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SYMPLECTIC FIXED POINTS, THE CALABI INVARIANT AND NOVIKOV HOMOLOGY

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1. INTRODUCTION

A SYMPLECTIC structure ω on a manifold M provides a one-to-one correspondence between closed 1-forms and infinitesimal automorphisms of (M, ω) , i.e. vector fields X satisfying $\mathcal{L}_X \omega = 0$. An infinitesimal automorphism X is called a Hamiltonian vector field, if it corresponds to an exact 1-form. A symplectomorphism φ on M is called exact, if it is the time-one map of a time-dependent Hamiltonian vector field. In fact, one can find a periodic Hamiltonian function such that φ is the time-one map of the Hamiltonian system. For each symplectomorphism φ isotopic to the identity through symplectomorphisms, one can assign a “cohomology class” $Cal(\varphi)$, which is called the Calabi invariant of φ . Banyaga [3] showed that φ is exact if and only if $Cal(\varphi) = 0$. The Arnold conjecture states that the number of fixed points of an exact symplectomorphism on a compact symplectic manifold can be estimated below by the sum of the Betti numbers of M provided that all the fixed points are non-degenerate. Arnol’d came to this conjecture by analysing the case that φ is close to the identity (see [1]). If φ is the time-one map of a time-independent Hamiltonian vector field corresponding to a Morse function f which is C^2 -small, the fixed points coincides with the critical points of f and the validity of the conjecture follows directly from Morse theory.

There are many partial results to the Arnol’d conjecture. Great progress was made by Floer, who combined the variational approach (see [4]) and Gromov’s theory of pseudo-holomorphic curves to prove the Arnol’d conjecture for monotone symplectic manifolds [5]. He developed an analogue of Morse theory for the action functional on the loop space and this led to the notion of Floer homology. The Arnol’d conjecture is derived from the fact that the Floer homology group is isomorphic to the ordinary homology group of M . Recently, Hofer and Salamon [8] defined Floer homology groups for a wider class of symplectic manifolds (which are called weakly monotone symplectic manifolds). An almost complex structure J on M is calibrated by ω , if

$$\langle \xi, \eta \rangle = \omega(\xi, J\eta) \tag{1.1}$$

defines a Riemannian metric on M . Such a J is unique up to homotopy and we denote by $c_1 = c_1(M)$ the first Chern class of the almost complex manifold (M, J) . A symplectic manifold (M, ω) is called monotone, if c_1 is a positive multiple of ω on $\pi_2(M)$. The condition of weak monotonicity [8] implies the non-existence of J -holomorphic spheres with negative

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Chern number for a generic J . Hofer and Salamon computed the Floer homology and verified the Arnol'd conjecture in the case that (M, ω) is either monotone, or $c_1(\pi_2(M)) = 0$, or the minimal Chern number is at least half of the dimension of M . Later, the second author defined the “modified Floer homology groups” and verified the conjecture for weakly monotone symplectic manifolds [10]. However we do not know whether the Floer homology groups defined by Hofer and Salamon and the modified Floer homology groups coincide or not.

In this note, we consider an analogue of the Arnol'd conjecture for non-exact symplectomorphisms isotopic to the identity through symplectomorphisms. In the case of non-exact symplectomorphisms, the fixed point set may be empty. For example, an irrational rotation on an even-dimensional torus with the standard symplectic structure preserves the symplectic form, and has empty fixed point set. For this reason, we have to consider Novikov homology instead of ordinary homology. The aim of this note is to show the following theorem.

MAIN THEOREM. *Let (M, ω) be a closed symplectic manifold of dimension $2n$ which satisfies the following condition*

$$c_1|_{\pi_2(M)} = \lambda\omega|_{\pi_2(M)}, \quad \lambda \neq 0$$

and if $\lambda < 0$, the minimal Chern number N satisfies $N > n - 3$. Suppose φ is a symplectomorphism on (M, ω) which is isotopic to the identity through symplectomorphisms. If all the fixed points of φ are non-degenerate, then the number of fixed points of φ is at least the sum of the Betti numbers of the Novikov homology over \mathbf{Z}_2 associated to the Calabi invariant of φ .

It is well-known that Novikov homology groups are isomorphic for almost all cohomology class in $H^1(M; \mathbf{R})$ and the rank of these groups is minimal in the class of Novikov homology groups associated to all the cohomology class in $H^1(M; \mathbf{R})$ (see Appendix C). Hence we get an estimate of the number of fixed points in terms of Novikov homology for generic 1-forms. To prove the Main Theorem, we reduce the problem to one concerning the 1-periodic solutions of a periodic Hamiltonian system and define Floer homology in this setting. The argument in [5, 8] shows that the Floer homology groups are isomorphic under the deformations preserving the Calabi invariant. However, in order to compute the Floer homology, we also have to consider deformations which change the Calabi invariant. To apply the weak compactness argument, it is necessary to estimate of the energy functional for solutions of the “chain homomorphism” equation. This is done for specific deformations with the help of a variant of the Palais–Smale condition (Section 5).

This note also contains three Appendices. The first one concerns the classification of loops in a symplectic manifold under symplectomorphisms. We prove that two embedded contractible loops are congruent under a time-one map of a time-dependent Hamiltonian flow if and only if their Poincaré integral invariants coincide. The second contains a proof of a fact which is needed in the computation of the Floer homology groups (see also [14]). The third is a note on Novikov homology theory. Since it seems difficult to find a reference containing proofs, we give proofs for the sake of completeness.

2. THE CALABI INVARIANT AND A VARIATIONAL APPROACH

Given a symplectic form ω on M , there is an isomorphism Φ from the space of vector fields to the space of differential forms on M :

$$\Phi(V)(W) = -\omega(V, W). \tag{2.1}$$

Let g_1 be an element of the identity component $Diff_\omega^0 M$ of the symplectomorphism group and $\{g_t\}$ a path connecting the identity element and g_1 . We define a vector field $D_t(g_t)$ by

$$D_t(g_t)(x) = \left(\frac{\partial}{\partial t} g_t \right) (g_t^{-1}x).$$

Clearly, the $\Phi(D_t(g_t))$ are closed differential forms. Recall that g_1 is said to be an *exact symplectomorphism* if all the $\Phi(D_t(g_t))$ are exact 1-forms. The *flux homomorphism* Φ from the universal cover $\widehat{Diff}_\omega^0 M$ to $H^1(M, \mathbf{R})$ is defined as follows (see [3]):

$$\Phi(\tilde{g}_1) = [\int \Phi(D_t(g_t)) dt].$$

This homomorphism was first considered by Calabi and we call the image $\Phi(\tilde{g})$ the *Calabi invariant* $Cal(\tilde{g})$ of an element $\tilde{g} = \tilde{g}_1 \in \widehat{Diff}_\omega^0 M$. Passing to the group $\widehat{Diff}_\omega^0 M$ the Calabi invariant of a symplectomorphism on M is an element of the quotient $H^1(M, \mathbf{R})/\Gamma$, where Γ is the image of the subgroup $\pi_1(Diff_\omega^0 M)$ which is identified with the kernel of the projection from $\widehat{Diff}_\omega^0 M$ to $Diff_\omega^0 M$ under the homomorphism Φ . It is known ([3], see also Lemma 2.1) that Γ is a discrete subgroup in $H^1(M, \mathbf{R})$ if and only if the subgroup of exact symplectomorphisms is closed in $Diff_\omega^0 M$. Kählerian manifolds, or more generally, any symplectic manifold M such that the multiplication by ω^{n-1} induces an isomorphism from $H^1(M, \mathbf{R})$ to $H^{2n-1}(M, \mathbf{R})$, are such examples [3].

DEFORMATION LEMMA 2.1. *Let $[\theta] \in H^1(M, \mathbf{R})$ be the Calabi invariant of an element \tilde{g}_1 . Then there exists a smooth path $\{g_t\}$ in $Diff_\omega^0 M$, joining the identity element Id and g_1 , and a periodic Hamiltonian H_t on M such that $\Phi(D_t(g_t)) = \theta + dH_t$ for all t .*

Proof. First, we show that we can choose path g_t connecting the identity element and g_1 such that each $\theta_t = \Phi(D_t(g_t))$ has cohomology class $[\theta]$ and $\theta_0 = \theta_1$. Put

$$V(x) = \int_0^1 D_t(g_t)(x) dt.$$

Let $G_t(x)$ be the one-parameter subgroup of symplectomorphisms generated by the vector field $V(x)$: $G_t = \exp tV$. Then the Calabi invariant for the path $G_{-t} \circ g_t$ is zero. It is known that there exists a smooth path p_t in the subgroup of exact symplectomorphisms, $Diff_{ex} M$, such that $p_0 = Id$ and $p_1 = G_{-1} \cdot g_1$ (see [3]). By reparametrizing the parameter t , we may assume that p_t is constant around 0 and 1. Now let us consider the path $g'_t = G_t \cdot p_t$, so we have $g'_0 = Id$ and $g'_1 = g_1$. We obtain

$$\Phi(D_t(G_t \circ p_t)) = \Phi((G_t)_*(D_t(p_t)) + \Phi(V). \tag{2.2}$$

From (2.2) we obtain that the form $\Phi(D_t(g'_t)) = \theta_t$ lies in the same cohomology class $[\theta]$ for all t . Let p_t be as above. Then $\Phi(D_t(p_t))$ is the differential dH_t of a smooth function H_t on M . Now, the periodicity condition $\theta_0 = \theta_1$ is equivalent to the following:

$$dH_0 + \Phi(V) = G_1^*(dH_1) + \Phi(V). \tag{2.3}$$

Since p_t is constant around 0 and 1, $dH_0 = dH_1 = 0$, i.e. H_0 and H_1 are constant functions. Hence (2.3) holds. Consequently, we have that $H_1 = H_0 + a$, where a is some constant. Now put $H'_t = H_t + \psi(t)$, where $\psi(0) = 0$, $\psi(1) = -a$ and $\psi'(0) = \psi'(1) = 0$. Then the path g_t associated to $\theta_t = \theta + dH'_t$ satisfies the condition of Deformation Lemma. \square

With the help of our Deformation Lemma we now reduce the problem of finding fixed points of a symplectomorphism g to a variational problem on the loop space $\mathcal{L}M$. Suppose

that g is given by the following equation:

$$g_0 = Id, \quad g_1 = g, \quad \Phi(D_t(g_t)) = \theta_t$$

where θ_t satisfy the condition in Deformation Lemma. Clearly, the fixed points of g are in one-to-one correspondence with the 1-periodic solutions of the following differential equation

$$\dot{x}(t) = X_\theta(t, x(t)) \tag{2.4}$$

where $X_\theta = \Phi^{-1}(\theta)$. (It is easy to see that $\Phi^{-1}(\theta) = J\nabla P_\theta$ where P_θ is a local primitive of the closed 1-form θ , i.e. $\theta = dP_\theta$ holds locally.) The set $\mathcal{P}(\theta_t)$ of 1-periodic solutions of (2.4) coincide with the critical set (i.e. the zero set) of the following closed 1-form $d\mathcal{A}_\theta$ on $\mathcal{L}M$:

$$d\mathcal{A}_\theta(z, \xi) = \int \omega(\dot{z}, \xi) + \theta_t(z(t))(\xi). \tag{2.4'}$$

The “(minus) gradient flow” for the multi-valued functional \mathcal{A}_θ is defined by the following equation.

$$\bar{\partial}_{J, \theta}(u) = \frac{\partial u}{\partial s} + J(u) \left(\frac{\partial u}{\partial t} - X_\theta(u) \right) = 0 \tag{2.5}$$

where $u = u(s, t)$ is a mapping $\mathbf{R} \times S^1 \rightarrow M$. For a solution u of (2.5) we define its energy as follows.

$$E(u) = \frac{1}{2} \int_{-\infty}^{\infty} \int_0^1 \left(\left| \frac{\partial u}{\partial s} \right|^2 + \left| \frac{\partial u}{\partial t} - X_\theta(t, u) \right|^2 \right) dt ds. \tag{2.6}$$

As in the case of exact symplectomorphisms, the space of solutions of (2.5) with bounded energy is the space of connecting orbits, that is, $u: \mathbf{R} \times S^1 \rightarrow M$ such that $\lim_{s \rightarrow \pm \infty} u(s, t) = x^\pm(t)$ where $x^\pm(t)$ are periodic solutions of (2.4).

We will restrict ourselves to the component of contractible loops on M . For the sake of simplicity, henceforth, we also denote this space by $\mathcal{L}M$. We construct an associated covering space $\tilde{\mathcal{L}}M$ such that the action functional $\tilde{\mathcal{A}}_\theta$ on this cover is single-valued. Consider the following commutative diagram:

$$\begin{array}{ccccc} \tilde{\mathcal{L}}M & \xrightarrow{\tilde{j}} & \mathcal{L}\tilde{M} & \xrightarrow{\tilde{e}} & \tilde{M} \\ \downarrow \tilde{\pi} & & \downarrow \pi & & \downarrow \pi \\ \tilde{\mathcal{L}}M & \xrightarrow{j} & \mathcal{L}M & \xrightarrow{e} & M \end{array}$$

Here \tilde{M} denotes the covering space of M associated to the period homomorphism of θ , $I_\theta: \pi_1(M) \rightarrow \mathbf{R}$. This means that the covering transformation group is isomorphic to the quotient group

$$\Gamma_1 = \pi_1(M) / \ker I_\theta.$$

Furthermore, e denotes the evaluation map $x(t) \mapsto x(0)$ and j denotes the projection from the covering space $\tilde{\mathcal{L}}M$ of $\mathcal{L}M$ associated to the action of homomorphisms $\phi_{c_1}, \phi_\omega: \pi_2(M) \rightarrow \mathbf{R}$ defined by evaluation of c_1 and ω respectively. This means that the covering transformation group is isomorphic to the quotient group

$$\Gamma_2 = \frac{\pi_2(M)}{\ker \phi_{c_1} \cap \ker \phi_\omega}.$$

An element of $\tilde{\mathcal{L}}\tilde{M}$ is represented by an equivalence class of pairs (\tilde{x}, \tilde{u}) , where \tilde{x} is a loop in \tilde{M} and \tilde{u} is a disk in \tilde{M} bounding \tilde{x} . A pair (\tilde{x}, \tilde{u}) is equivalent to (\tilde{y}, \tilde{v}) if and only if $\tilde{x} = \tilde{y}$ and the values of ϕ_{e_1} and ϕ_{ω} are zero on $u \# (-v)$, where $u = \pi(\tilde{u})$, $v = \pi(\tilde{v})$ (see [8]). Hence, the covering transformation group of $\tilde{\mathcal{L}}\tilde{M} \rightarrow \mathcal{L}M$ is the direct sum $\Gamma = \Gamma_1 \oplus \Gamma_2$:

$$(\gamma_1 \oplus \gamma_2)[\tilde{x}, \tilde{u}] = [\gamma_1 \cdot \tilde{x}, A_2 \# \gamma_1 \cdot \tilde{u}], \tag{2.7}$$

where A_2 is any representative of γ_2 in $\pi_2(M)$. (Geometrically, u is a disk bounded by $x \in \mathcal{L}M$. By the homotopy lifting property, there exists a unique disk $\tilde{u} \in \tilde{M}$ bounded by \tilde{x} . The second homotopy groups of M and \tilde{M} are same, so we consider g_2 as a sphere in \tilde{M} and $\#$ denotes the connected sum of 2-spheres with the bounding disk (see [8])). To summarise; we have the following Lemma.

LEMMA 2.2. *The group Γ is commutative. In particular,*

$$(\gamma \circ \gamma')[\tilde{x}, \tilde{u}] = (\gamma' \circ \gamma)[\tilde{x}, \tilde{u}]$$

for $\gamma, \gamma' \in \Gamma$.

Since the 1-forms $\theta_t, t \in S^1$, satisfy the condition in Deformation Lemma there exists a periodic Hamiltonian \tilde{H}_t on \tilde{M} such that $d\tilde{H}_t = \pi^*\theta_t$. Clearly, the time-dependent Hamiltonian flow on \tilde{M} generated by \tilde{H}_t is the pull-back of the original symplectic flow on M . In particular, the set of contractible periodic solutions $\mathcal{P}(\tilde{H}_t)$ is the set $\pi^{-1}(\mathcal{P}(\theta_t))$. Furthermore, $\tilde{\mathcal{P}}(\tilde{H}) = \tilde{j}^{-1}(\tilde{H}_t)$ is the critical set of the following functional

$$\mathcal{A}_{\tilde{H}}([\tilde{x}, \tilde{u}]) = - \int_D \tilde{u}^* \omega + \int_0^1 \tilde{H}(t, \tilde{x}(t)) dt. \tag{2.8}$$

We now consider the space of connecting orbits $\tilde{u}: \mathbf{R} \times S^1 \rightarrow \tilde{M}$ on $\tilde{\mathcal{L}}\tilde{M}$ satisfying the lifted equation:

$$\bar{\partial}_{J, \tilde{H}}(\tilde{u}) = \frac{\partial \tilde{u}}{\partial s} + J(u) \left(\frac{\partial \tilde{u}}{\partial t} - X_{\tilde{H}}(\tilde{u}) \right) = 0 \tag{2.9.1}$$

with boundary conditions

$$\lim_{s \rightarrow \pm\infty} \tilde{u}(s, t) = [\tilde{x}^\pm(t), \tilde{u}^\pm] \tag{2.9.2}$$

and

$$[\tilde{x}^+, \tilde{u}^- \# \tilde{u}] = [\tilde{x}^+, \tilde{u}^+]. \tag{2.9.3}$$

The paths in $\mathcal{L}\tilde{M}$ are in one-to-one correspondence with the paths on the covering space $\tilde{\mathcal{L}}\tilde{M}$ modulo the action of Γ_2 (see condition (2.9.3)). Consequently, for these connecting orbits we have the following energy identity (cf. [8, 14]):

$$E(\tilde{u}(s, t)) = \int_{-\infty}^{\infty} \int_0^1 \left| \frac{\partial \tilde{u}}{\partial s} \right|^2 dt ds = \mathcal{A}_{\tilde{H}}([\tilde{x}^-, \tilde{u}^-]) - \mathcal{A}_{\tilde{H}}([\tilde{x}^+, \tilde{u}^+]). \tag{2.10}$$

3. TRANSVERSALITY AND COMPACTNESS

From now on, we will deal with a weekly monotone symplectic manifold M , i.e. M satisfies $\omega(A) \leq 0$ for any $A \in \pi_2(M)$ with $3 - n \leq c_1(A) < 0$ [8]. We also use a generic almost complex structure J calibrated by ω . The weak monotonicity condition yields the

non-existence of J -holomorphic spheres of negative Chern number. Moreover, we denote by $\mathbf{M}_k(c; J)$ the set of points $x \in M$ for which there exists a non-constant J -holomorphic sphere $v: S^2 \rightarrow M$ with $c_1(v) \leq k$, $\omega(v) \leq c$ and $x \in v(S^2)$. The set $\mathbf{M}_0(\infty; J)$ is then a subset of M of codimension 4 and the set $\mathbf{M}_1(\infty; J)$ has codimension 2 [8].

The transversality and compactness theorems in this section can be obtained by the same arguments in [8].

Given any smooth periodic 1-form $\theta_t = \theta + dH_t$ we denote by $\mathcal{U}_\delta(\theta_t)$ the set of all periodic 1-forms $\theta + dH'_t$ with $\|H'_t - H_t\|_\varepsilon < \delta$, where the norm $\|h\|_\varepsilon$ is defined as follows:

$$\|h\|_\varepsilon = \sum_{k=0}^{\infty} \varepsilon_k \|h\|_{C^k(S^1 \times M)}.$$

Here $\varepsilon_k > 0$ is a sufficiently rapidly decreasing sequence [6].

THEOREM 3.1. *There is a dense subset $\Theta_0 \subset \mathcal{U}_\delta(\theta_t)$ such that the following holds for $\theta_t \in \Theta_0$.*

- (i) every periodic solution $x \in \mathcal{P}(\theta_t)$ is non-degenerate;
- (ii) $x(t) \notin \mathbf{M}_1(\infty; J)$ for every $x \in \mathcal{P}(\theta_t)$ and every $t \in \mathbf{R}$.

This is obtained by applying the Sard–Smale theorem to certain Banach manifolds. More precisely, for the proof of (i) we consider the Hilbert manifold \mathcal{B} of contractible $W^{1,2}$ loops $x: S^1 \rightarrow M$ and the bundle $\mathcal{E} \rightarrow \mathcal{B}$ whose fibre at $x \in \mathcal{B}$ is the Hilbert space of L^2 -vector fields along x . Define a section $\mathcal{F}: \mathcal{B} \times \mathcal{U}_\delta(\theta_t) \rightarrow \mathcal{E}$ by

$$\mathcal{F}(x, \theta + dH'_t) = \dot{x} - X_{\theta + dH'_t}(t, x).$$

The differential $d\mathcal{F}(x, \theta + dH'_t)$ is surjective [8]. Hence \mathcal{F} intersects the zero section of \mathcal{E} transversally. Thus, the set

$$\mathcal{P} = \{(x, \theta') \in \mathcal{B} \times \mathcal{U}_\delta(\theta) \mid \mathcal{F}(x, \theta') = 0\}$$

is a separable infinite-dimensional Banach manifold. A periodic form $\theta'_t = \theta + dH'_t \in \mathcal{U}_\delta(\theta_t)$ is a regular value of the projection $\mathcal{P} \rightarrow \mathcal{U}_\delta(\theta_t)$ onto the second factor if and only if every periodic solution $x \in \mathcal{P}(\theta'_t)$ is non-degenerate. By the Sard–Smale theorem the set $\Theta' \in \mathcal{U}_\delta(\theta_t)$ is generic in the sense of Baire.

To prove (ii) we consider the evaluation map

$$e: (\mathcal{M}_s(A; J) \times_G S^2) \times S^1 \times \mathcal{P} \rightarrow M \times M: ([v, z], t, x, \theta_t) \mapsto (v(z), x(t))$$

where $\mathcal{M}_s(A; J)$ denotes the moduli space of simple J -holomorphic spheres realizing a homology class $A \in H_2(M, \mathbf{Z})$ and G is the automorphism group $PGL_2(\mathbf{C})$. It is shown that the evaluation map $e_t: \mathcal{P} \rightarrow M: (x, \theta_t) \mapsto x(t)$, is a submersion for every $t \in S^1$ [8]. Therefore, the map e is transversal to the diagonal Δ_M in $M \times M$. Hence the space

$$\mathcal{N} = \{([v, z], t, (x, \theta_t)) \mid v(z) = x(t), (x, \theta_t) \in \mathcal{P}\}$$

is an infinite-dimensional Banach submanifold of $(\mathcal{M}_s(A; J) \times_G S^2) \times S^1 \times \mathcal{P}$ of codimension $2n$. The projection

$$\mathcal{N} \rightarrow \mathcal{U}_\delta(\theta): ([v, z], x, \theta_t) \mapsto \theta_t$$

is a Fredholm map of Fredholm index $2c_1(A) - 3$. Applying the Sard–Smale theorem we get that the set $\Theta(A)$ of regular values of the above projection is of the second category in the sense of Baire. Denote by Θ_0 the intersection of Θ' with $\bigcap_{A \in \Gamma} \Theta(A)$, where Γ is the countable set of integral (and spherical) 2-homology classes A in M for which $c_1(A) \leq 1$. Then Θ_0 is the desired set for Theorem 3.1. \square

By the previous theorem there exists a periodic 1-form $\theta_t = \theta + dH_t$ in a prescribed cohomology class $[\theta]$ such that all contractible 1-periodic solution of (2.4) are non-degenerate and do not intersect the set $\mathbf{M}_1(\infty, J)$. Choose disjoint compact neighborhoods $U_1, \dots, U_m \subset S^1 \times M$ of the graphs of the finitely many contractible 1-periodic solutions of (2.4). We denote by $\mathcal{V}_\delta(\theta_t)$ the set of all periodic 1-forms $\theta_t = \theta + dH_t$ with $\|H_t - H_t\|_\varepsilon < \delta$ and $H_t = H_t$ on U_j for $j = 1, \dots, m$. If $\delta > 0$ is sufficiently small then there are no contractible 1-periodic solutions of (2.4) outside the set U_j for $\theta_t \in \mathcal{V}_\delta(\theta_t)$. For an element $[\tilde{x}, \tilde{u}] \in \tilde{\mathcal{P}}(\tilde{H})$, we assign an integer $\mu([\tilde{x}, \tilde{u}])$, which is called the Conley–Zehnder index [8, 14].

THEOREM 3.2. *There is a generic set $\Theta_1 \subset \mathcal{V}_\delta(\theta_t)$ containing θ_t such that the following holds for $\theta_t \in \Theta_1$:*

- (i) *the space $\mathcal{M}([\tilde{x}^-, \tilde{u}^-], [\tilde{x}^+, \tilde{u}^+]; \theta_t, J)$ of solutions of (2.9.1), (2.9.2) and (2.9.3) with $[\tilde{x}^\pm, \tilde{u}^\pm] \in \mathcal{P}(\theta_t)$ is a finite dimensional manifold of dimension $\mu([\tilde{x}^-, \tilde{u}^-]) - \mu([\tilde{x}^+, \tilde{u}^+])$;*
- (ii) *$u(s, t) \notin \mathcal{M}_0(\infty; J)$ for every $u \in \mathcal{M}([\tilde{x}^-, \tilde{u}^-], [\tilde{x}^+, \tilde{u}^+]; \theta_t, J)$ with $\mu([\tilde{x}^-, \tilde{u}^-]) - \mu([\tilde{x}^+, \tilde{u}^+]) \leq 2$ and every $(s, t) \in \mathbf{R} \times S^1$.*

The proof of Theorem 3.2 is also obtained by applying the Sard–Smale theorem to certain Banach manifolds [8]. For the proof of (i) we consider for $p > 2$ the Banach manifold \mathcal{B} of $W^{1,p}$ maps $u: \mathbf{R} \times S^1 \rightarrow M$, whose limits are periodic solutions x^\pm of (2.4). Let $\mathcal{E} \rightarrow \mathcal{B}$ be the bundle whose fibre at $u \in \mathcal{B}$ is the Banach space of L^p -vector fields along u . Define a section $\mathcal{F}: \mathcal{B} \times \mathcal{V}_\delta(\theta_t) \rightarrow \mathcal{E}$ as follows

$$\mathcal{F}(u, \theta_t) = \frac{\partial u}{\partial s} + J(u) \left(\frac{\partial u}{\partial t} - X_{\theta_t}(t, u) \right).$$

The linearization of $\mathcal{F}(u, \theta_t)$ coincides with that of the similar operator in [8]. Further arguments in [8] can be repeated word-for-word here.

In the following we consider the lifted Hamiltonian system on \tilde{M} . Note that the eq. (2.9.1) is invariant under translations in the s -variable. We denote the quotient space by $\mathcal{M}([\tilde{x}^-, \tilde{u}^-], [\tilde{x}^+, \tilde{u}^+]; \tilde{H}, J)/\mathbf{R}$.

THEOREM 3.3. *Suppose that J and θ_t are regular (in the sense of Theorem 3.1 and Theorem 3.2). Then $\mathcal{M}([\tilde{x}^-, \tilde{u}^-], [\tilde{x}^+, \tilde{u}^+]; \tilde{H}, J)/\mathbf{R}$ is compact, if $\mu([\tilde{x}^-, \tilde{u}^-]) - \mu([\tilde{x}^+, \tilde{u}^+]) = 1$. $\mathcal{M}([\tilde{x}^-, \tilde{u}^-], [\tilde{x}^+, \tilde{u}^+]; \tilde{H}, J)/\mathbf{R}$ is compact up to splitting into two elements in $\mathcal{M}([\tilde{x}^-, \tilde{u}^-], [\tilde{z}, \tilde{v}]; \tilde{H}, J)/\mathbf{R}$ and $\mathcal{M}([\tilde{z}, \tilde{v}], [\tilde{x}^+, \tilde{u}^+]; \tilde{H}, J)/\mathbf{R}$, if $\mu([\tilde{x}^-, \tilde{u}^-]) - \mu([\tilde{x}^+, \tilde{u}^+]) = 2$.*

Remark 3.4. For the proof of Theorem 3.3 (in the case of an exact Hamiltonian) Hofer and Salamon use the uniform lower bound of the energy for holomorphic spheres and connecting orbits. They prove the existence of such bounds by bubbling analysis (Gromov’s compactness theorem). Alternatively, we can use the (explicit) lower estimate for the volume of (globally) minimal cycles in a compact Riemannian manifold [9] to estimate the energy of holomorphic spheres. Using Cauchy’s integral inequality one can obtain lower bounds for the energy of connecting orbits in terms of distances between periodic solutions. More precisely, we define a distance $\rho(x(t), y(t))$ between loops $x(t)$ and $y(t)$ by

$$\inf \left\{ \int_{-\infty}^{\infty} \int_0^1 \left| \frac{\partial u}{\partial s} \right| dt ds \mid u: \mathbf{R} \times S^1 \rightarrow M, \quad \lim_{s \rightarrow -\infty} u(s, t) = x(t), \quad \lim_{s \rightarrow \infty} u(s, t) = y(t) \right\}.$$

Clearly $\rho(x(t), y(t)) \geq \int_0^1 \text{dist}(x(t), y(t)) dt \geq \min_t \text{dist}(x(t), y(t))$, where “dist” denotes the Riemannian distance on M . If the C^0 -distance $\max_t \text{dist}(x(t), y(t))$ is less than or equal to the injectivity radius R_M of M , then $\rho(x(t), y(t)) \leq \max_t \text{dist}(x(t), y(t))$. Given a finite number of periodic solutions $x_i(t)$ of (2.4) let \hbar denote the minimum of the distance between $x_i(t)$ and $x_j(t)$, $i \neq j$.

LEMMA 3.5. *Given a positive $\varepsilon < \min\{R_M, \hbar/2\}$ there exists a number $c(\varepsilon) > 0$ depending on θ_i such that*

$$E(u) \geq c(\varepsilon)(\hbar - 2\varepsilon)$$

for any connecting orbit u on M satisfying (2.5), whose limits as s tends to $\pm\infty$ are different periodic solutions of (2.4).

Proof. Let $B_i(\varepsilon)$ be the neighborhood of $x_i(t)$ which consists of loops whose distance ρ from $x_i(t)$ is less than or equal to ε . Let $B(\varepsilon) = \cup B_i(\varepsilon)$. By the Palais–Smale condition (see Lemma 5.1) there exists $c(\varepsilon) > 0$ such that the following inequality holds outside of $B(\varepsilon)$

$$\|\dot{x}(t) - X_{\theta_i}(x(t))\|_{L^2} \geq c(\varepsilon)^2. \quad (3.1)$$

Suppose u is a connecting orbit from $x_i(t)$ to $x_j(t)$. Since $\lim_{s \rightarrow \pm\infty} u(s, \cdot)$ (in the C^0 -sense) are periodic solutions $x_i(t)$ and $x_j(t)$, we obtain:

CLAIM 3.6. *There exist numbers R^- and R^+ such that*

- (i) $u(R^-, t) \in B_i(\varepsilon)$ and $u(s, t) \notin B_i(\varepsilon)$ for all $s > R^-$,
- (ii) $u(R^+, t) \in B_j(\varepsilon)$ and $u(s, t) \notin B_j(\varepsilon)$ for all $s < R^+$.

We observe that

$$E(u) \geq \int_{R^-}^{R^+} \int_0^1 \left| \frac{\partial u}{\partial s} \right|^2 dt ds = \int_{R^-}^{R^+} \left(\sqrt{\int_0^1 |\dot{u}(s, t) - X_{\theta_i}(u(s, t))|^2 dt} \right)^2 ds.$$

Applying Cauchy's inequality we get

$$E(u) \geq \frac{1}{R^+ - R^-} \left(\int_{R^-}^{R^+} \sqrt{\int_0^1 |\dot{u}(s, t) - X_{\theta_i}(u(s, t))|^2 dt} ds \right)^2.$$

Taking (3.1) and Claim 3.6 into account we get

$$E(u) \geq c(\varepsilon) \int_{R^-}^{R^+} \sqrt{\int_0^1 \left| \frac{\partial u}{\partial s} \right|^2 dt} ds.$$

Once again applying Cauchy's inequality, we get

$$E(u) \geq c(\varepsilon) \int_{R^-}^{R^+} \int_0^1 \left| \frac{\partial u}{\partial s} \right|^2 dt ds \geq c(\varepsilon)(\rho(x_i(t), x_j(t)) - 2\varepsilon). \quad \square$$

4. FLOER HOMOLOGY

In this section we define Floer homology of fixed points of a symplectomorphism isotopic to the identity and prove its invariance under exact deformations. As a result, if two symplectomorphisms have the same Calabi invariant then the associated Floer homology groups are isomorphic. We will work on the covering space $\tilde{\mathcal{L}}\tilde{M}$.

In this paper [14], Salamon and Zehnder used the Conley–Zehnder index (which they also call the Maslov index) of a non-degenerate contractible periodic solution as a natural grading for the Floer complex. Recall that the Conley–Zehnder index of a contractible periodic solution $x(t)$ bounding a disk u depends only on the trivialization of the induced complex bundle u^*TM and the linearized flow along $x(t)$. Therefore, the Conley–Zehnder index of a periodic solution $[\tilde{x}, \tilde{u}] \in \tilde{\mathcal{L}}\tilde{M}$ is Γ_1 -invariant, that is $\mu([\tilde{x}, \tilde{u}]) = \mu(g \cdot [\tilde{x}, \tilde{u}])$ for any $g \in \Gamma_1$, where Γ_1 is the covering transformation group of \tilde{M} . This Conley–Zehnder index satisfies the following identities (cf. [8, 14]).

$$\mu([\tilde{x}, A \# \tilde{u}]) - \mu([\tilde{x}, \tilde{u}]) = -2c_1(A) \quad \text{for } A \in \pi_2(M) \tag{4.1}$$

$$\dim \mathcal{M}([\tilde{x}^-, \tilde{u}^-], [\tilde{x}^+, \tilde{u}^+]; \tilde{H}, J) = \mu([\tilde{x}^-, \tilde{u}^-]) - \mu([\tilde{x}^+, \tilde{u}^+]). \tag{4.2}$$

The Conley–Zehnder index $\mu(x(t))$ of a periodic solution $x(t)$ is well-defined modulo $2N$, where N is the minimal Chern number. However, we will write $\mu(x(t)) = k \in \mathbf{Z}$ if there is a bounding disk u_x such that $\mu([x, u_x]) = k$.

Denote by $\tilde{\mathcal{P}}_k(\tilde{H})$ the subset of all periodic solutions $[\tilde{x}, \tilde{u}]$ with $\mu([\tilde{x}, \tilde{u}]) = k$. Consider the chain complex whose k th chain group $C_k(\tilde{H})$ consists of all the formal sums $\sum \xi_{[\tilde{x}, \tilde{u}]} \cdot [\tilde{x}, \tilde{u}]$, $[\tilde{x}, \tilde{u}] \in \tilde{\mathcal{P}}_k(\tilde{H})$, $\xi_{[\tilde{x}, \tilde{u}]} \in \mathbf{Z}_2$, such that the set $\{[\tilde{x}, \tilde{u}] \mid \xi_{[\tilde{x}, \tilde{u}]} \neq 0, \mathcal{A}_{\tilde{H}}([\tilde{x}, \tilde{u}]) > c\}$ is finite for all $c \in \mathbf{R}$.

Let Γ_0 be the following subgroup of Γ_2 ,

$$\frac{\ker \phi_{c_1}}{\ker \phi_{c_1} \cap \ker \phi_\omega}$$

and define Γ' to be $\Gamma_1 \oplus \Gamma_0 \subset \Gamma$.

Denote by $\Lambda_{\theta, \omega}$ the completion of the group ring of Γ' over the field \mathbf{Z}_2 with respect to the weight homomorphism $\Psi_{\theta, \omega} = I_\theta \oplus -\phi_\omega : \Gamma' \rightarrow \mathbf{R}$, i.e. the set of all the formal sums $\sum \lambda_A \cdot \delta_A$, such that $\lambda_A \in \mathbf{Z}_2$, and the set $\{A \in \Gamma' \mid \lambda_A \neq 0, \Psi_{\theta, \omega}(A) > c\}$ is finite for all $c \in \mathbf{R}$. In fact, $\Lambda_{\theta, \omega}$ is a commutative algebra over \mathbf{Z}_2 without zero divisors.

Remark 4.1. If M satisfies the condition of the Main Theorem, then Γ_0 is trivial and $\Lambda_{\theta, \omega} \cong \Lambda_\theta$, where Λ_θ is the Novikov ring associated to the closed 1-form θ (see Appendix C). It is easy to see that in this case the ambiguity of the Conley–Zehnder index of a periodic solution $x(t)$ can be controlled. More precisely, if $\mu(x(t)) = k \pmod{2N}$ then there exists a bounding disk u_x , which is unique up to the connected sum of an element of $\ker \phi_{c_1}$, such that $\mu([x(t), u_x]) = k$.

The algebra $\Lambda_{\theta, \omega}$ acts on $\tilde{\mathcal{P}}(\tilde{H})$ in the following way:

$$(\lambda * \xi)_{[\tilde{x}, \tilde{u}]} = \sum_{g \in \Gamma'} \lambda_g \xi_{g \cdot [\tilde{x}, \tilde{u}]}$$

We easily deduce the following lemma.

LEMMA 4.2. *The chain group $C_k(\tilde{H})$ is a torsion-free module over the algebra $\Lambda_{\theta, \omega}$. The rank of this module is the number of contractible 1-periodic solutions $x \in \mathcal{P}(\theta_1)$ with Conley–Zehnder index $\mu(x) = k$.*

For a generator $[\tilde{x}, \tilde{u}]$ in $C_k(\tilde{H})$, we define the boundary operator ∂_k as follows:

$$\partial_k([\tilde{x}, \tilde{u}]) = \sum_{\mu([\tilde{y}, \tilde{v}]) = k-1} n_2([\tilde{x}, \tilde{u}], [\tilde{y}, \tilde{v}]) [\tilde{y}, \tilde{v}] \tag{4.3}$$

where $n_2([\tilde{x}, \tilde{u}], [\tilde{y}, \tilde{v}])$ denotes the modulo-2 reduction of the number of elements in the

space $\mathcal{M}([\tilde{x}, \tilde{u}], [\tilde{y}, \tilde{v}]; \tilde{H}, J)/\mathbf{R}$. The weak compactness argument gives that ∂_k is well-defined. To show that $\partial_k([\tilde{x}, \tilde{u}]) \in C_{k-1}$ we can use a compactness argument as in the proof of Lemma 5.5 below or combine the weak-compactness result with Lemma 3.5. In fact, write $\partial_k([\tilde{x}, \tilde{u}]) = \sum_{g_{p,i} \in \Gamma} g_{p,i}[\tilde{x}_i, \tilde{u}_i]$, where $\tilde{x}_i, i = \{1, \dots, K\}$, is an arbitrarily chosen lift to \tilde{M} of a periodic solution $x_i \in \mathcal{P}(\theta_i)$ and \tilde{u}_i is an arbitrarily chosen bounding disk of \tilde{x}_i . We need to show that for any given c and i the set $S_c = \{g_{p,i} | \mathcal{A}_{\tilde{H}}(g_{p,i}[\tilde{x}_i, \tilde{u}_i]) > c\}$ is finite. Taking into account the energy identity (2.10) and Lemma 3.5 we get that the set of distances $\rho(x(t), g \cdot x_i(t))$ for $g \in S_c$ is bounded. Hence for each i , there is only a finite number of $g \in \Gamma_1$ such that the coefficient of $[g \cdot \tilde{x}_i, \tilde{u}_i]$ in $\partial_k([\tilde{x}, \tilde{u}])$ is not zero and $\mathcal{A}_{\tilde{H}}([g \cdot x_i(t), \tilde{u}]) > c$. If Γ_0 is trivial (e.g. M satisfies the condition in Main Theorem), then we are done. If not, we use Gromov's compactness theorem to show that there is only finite number of homotopy types of connecting orbits between $x(t)$ and $x_i(t)$ with bounded energy.

Since ∂_k is invariant under the action of Γ' , we extend ∂_k as a $\Lambda_{\theta, \omega}$ -linear map from $C_k(\tilde{H})$ to $C_{k-1}(\tilde{H})$. Using a gluing argument and the weak compactness argument, we also deduce that $\partial^2 = 0$. The homology groups

$$HF_k(M, \omega, \theta_i, J; \mathbf{Z}_2) = \frac{\ker \partial_k}{\text{im } \partial_{k+1}}$$

are called the Floer homology groups of the quadruple (M, ω, θ_i, J) with coefficients in \mathbf{Z}_2 . Obviously, they are a graded $\Lambda_{\theta, \omega}$ -modules. The following theorem shows that the Floer homology groups are invariant under exact deformations.

THEOREM 4.3. *For generic pairs $(\theta_i^\alpha, J^\alpha), (\theta_i^\beta, J^\beta)$ such that $\theta_i^\alpha = \theta_i^\beta + df_i$, there exists a natural $\Lambda_{\theta, \omega}$ -module homomorphism*

$$HF^{\beta, \alpha} : HF_*(\theta_i^\alpha, J^\alpha) \rightarrow HF_*(\theta_i^\beta, J^\beta)$$

which preserves the grading by the Conley–Zehnder index. If $(\theta_i^\gamma, J^\gamma)$ is any other such pair then

$$HF^{\gamma\beta} \circ HF^{\beta\alpha} = HF^{\gamma\alpha}, \quad HF^{\alpha\alpha} = Id.$$

In particular, $HF^{\alpha\beta}$ is a $\Lambda_{\theta, \omega}$ -module isomorphism.

Proof. The proof of Theorem 4.3 is carried out in the same way as in [5, 8] (see also the Section 5 below). Namely, we construct a chain homomorphism with the help of the “chain homomorphism equation” which is a s -dependent analogue of the connecting orbit equation (2.9.1), (2.9.2), (2.9.3). Let $\theta_{s,t}$ denote a generic path connecting θ_i^α and θ_i^β in the fixed cohomology class. More precisely, there is a two-parameter family of functions $H_{s,t}$ on M , such that

$$\theta_{s,t} = \theta_i^\alpha + dH_{s,t}$$

and for a sufficiently large R ,

$$H_{s,t} = 0 \quad \text{for } s < -R$$

$$H_{s,t} = f_i \quad \text{for } s > R.$$

To construct a chain homomorphism $\phi^{\alpha, \beta} : C_*(\theta_i^\alpha, J^\alpha) \rightarrow C_*(\theta_i^\beta, J^\beta)$, we consider the following equation

$$\frac{\partial u}{\partial s} + J(u) \left\{ \frac{\partial u}{\partial t} - X_{\theta_t} - X_{H_{s,t}} \right\} = 0 \tag{4.4}$$

Here, the key problem is to control the energy of solutions $\tilde{u}^{\beta, \alpha}(s, t)$ “connecting” two

periodic solutions $[\tilde{x}^\alpha, \tilde{u}^\alpha]$ and $[\tilde{x}^\beta, \tilde{u}^\beta]$. Let \tilde{H} be a Hamiltonian function on \tilde{M} such that $d\tilde{H} = \pi^*\theta_t^\alpha$. Then $\tilde{H}' = \tilde{H} + \pi^*f_t$ satisfies $d\tilde{H}' = \pi^*\theta_t^\beta$. Let $[\tilde{x}, u^-] \in \tilde{\mathcal{P}}(\tilde{H})$ and $[\tilde{y}, u^+] \in \tilde{\mathcal{P}}(\tilde{H}')$. We then have the following inequality:

$$|E(u) - \{\mathcal{A}_{\tilde{H}}([\tilde{x}, u^-]) - \mathcal{A}_{\tilde{H}'}([\tilde{y}, u^+])\}| \leq \int_{-\infty}^{\infty} \max_{x \in M, t \in S^1} \left| \frac{\partial H_{s,t}}{\partial s} \right| ds$$

for a solution u of (4.4) with $\lim_{s \rightarrow -\infty} u(s, t) = \pi(\tilde{x}) \in \mathcal{P}(\theta_t^\alpha)$, $\lim_{s \rightarrow \infty} u(s, t) = \pi(\tilde{y}) \in \mathcal{P}(\theta_t^\beta)$ and $[\tilde{y}, u^- \# u] = [\tilde{y}, u^+]$. Hence we have the weak-compactness of $\mathcal{M}(\tilde{x}, \tilde{y}; \theta_t + dH_{s,t}, J_s)$ and the argument in [8] yields Theorem 4.3. \square

5. A VARIANT OF THE PALAIS-SMALE CONDITION AND CONTINUATION

To compute the Floer homology groups we need to use deformations which change the Calabi invariant. There are two difficulties which arise in proving the chain homomorphism between the Floer homologies associated to different Calabi invariants: the first is to control the energy of solutions of the “chain homomorphism equation”, and the second to make sure that the chain homomorphism preserves the “finiteness condition” which arises in the definition of the chain complexes. We overcome the first one by using a variant of the Palais-Smale condition. We have to restrict ourselves to the case of symplectic manifolds satisfying the condition in Main Theorem in order to avoid the second difficulty. We start by proving a variant of the Palais-Smale condition as follows:

LEMMA 5.1. *Let $x_j : S^1 \rightarrow M$ be a sequence of contractible $W^{1,2}$ -loops in M . If $\| \dot{x}_j - X_{\theta_t}(x_j) \|_{L^2}$ tends to 0 as $j \rightarrow +\infty$, then there exists a subsequence, which we also denote by $\{x_j\}$, such that x_j converges to a contractible periodic solution x_∞ in the C^0 -sense.*

Proof. Without loss of generality, M is assumed to be embedded in \mathbf{R}^{N_0} for some sufficiently large N_0 and $x_j \in W^{1,2}(S^1, \mathbf{R}^{N_0})$ such that $\text{Im}(x_j) \subset M$. Since M is compact and $\lim_{j \rightarrow \infty} \| \dot{x}_j(t) - X_{\theta_t}(x_j(t)) \|_{L^2} = 0$, there exists a constant $C > 0$ such that $\| x_j \|_{W^{1,2}} < C$. By the Rellich lemma, x_j converges to x_∞ in the C^α -topology for $0 \leq \alpha < 1/2$, and x_∞ is the weak-limit of x_j in $W^{1,2}(S^1, \mathbf{R}^{N_0})$. In particular, x_∞ is in $W^{1,2}(S^1, \mathbf{R}^{N_0})$.

Since x_j converges to x_∞ in the C^0 -topology, $X_{\theta_t}(x_j)$ also converges to $X_{\theta_t}(x_\infty)$ in the C^0 -topology. Thus it is easy to see that

$$\int_0^1 \langle x_\infty(t), \dot{\phi}(t) \rangle dt = - \int_0^1 \langle X_{\theta_t}(x_\infty(t)), \phi(t) \rangle dt$$

for any $\phi \in C^\infty(S^1, \mathbf{R}^{N_0})$, i.e. $\dot{x}_\infty(t) - X_{\theta_t}(x_\infty(t)) = 0$ in $L^2(S^1, \mathbf{R}^{N_0})$. By the regularity argument, x_∞ satisfies $\dot{x}_\infty(t) - X_{\theta_t}(x_\infty(t)) = 0$ in the classical sense. Contractible loops x_j converges to x_∞ in the C^0 -topology, therefore x_∞ is a contractible loop. \square

For a generic periodic time-dependent symplectic vector field X_{θ_t} , the set of periodic solutions is finite. Let x_1, \dots, x_l be all the contractible periodic solutions and U_1, \dots, U_l tubular neighborhoods of the graphs of x_j in $M \times S^1$. The following lemma is a direct consequence of Lemma 5.1.

LEMMA 5.2. *There exists $c > 0$ such that $\| \dot{x} - X_{\theta_t} \|_{L^2} > c$ for any contractible loop x in M whose graph is not contained in any of the U_j 's.*

Let η be a closed 1-form on M , and $p : M \times S^1 \rightarrow M$ the projection to the second factor. Since each x_j is contractible, the restriction of $p^*\eta$ to U_j is exact. Hence we can find

a periodic family $\{\eta_t\}$ of closed 1-forms on M which are cohomologous to η and vanish on U_j for all j . Let $\{\theta_t\}$ be a regular periodic family of closed 1-forms on M (see Theorems 3.1 and 3.2). Using a perturbation of η_t we may also assume that $\theta_t + \varepsilon \cdot \eta_t$ is regular. Then we can show the following theorem.

THEOREM 5.3. *Suppose that M satisfies the condition of the Main Theorem. For a periodic family $\{\eta_t\}$ in the cohomology class $\eta = \text{Cal}(\theta_t)$, there exists $\hbar > 0$ such that*

$$HF_*(\theta_t, J) \cong HF_*(\theta_t + \varepsilon \cdot \eta_t, J')$$

for $\varepsilon < \hbar$. Moreover the above isomorphism is Λ_θ -linear.

To construct a chain homomorphism, we consider the following equation:

$$\frac{\partial u}{\partial s} + J_s(u) \left(\frac{\partial u}{\partial t} - X_{\theta_{s,t}} \right) = 0 \tag{5.1}$$

with

$$\begin{aligned} (J_s, \theta_{s,t}) &= (J, \theta_t) \quad \text{for } s < -R \\ (J_s, \theta_{s,t}) &= (J', \theta_t + \varepsilon \eta_t) \quad \text{for } s > R \end{aligned}$$

and

$$\theta_{s,t} = \theta_t + \phi(s)\varepsilon \cdot \eta_t.$$

Here $\phi(s)$ is a monotone increasing smooth function on $[-R, R]$ which vanishes near $-R$ and equals 1 near R .

Define the energy by

$$E(u) = \int_{-\infty}^{\infty} \int_0^1 \left| \frac{\partial u}{\partial s} \right|^2 dt ds.$$

If a solution u of (5.1) has finite energy, then $\lim_{s \rightarrow \pm\infty} u(s, t)$ exists and

$$\left. \begin{aligned} z^- &= \lim_{s \rightarrow -\infty} u(s, t) \in \mathcal{P}(\theta_t) \\ z^+ &= \lim_{s \rightarrow \infty} u(s, t) \in \mathcal{P}(\theta_t + \varepsilon \eta_t) \end{aligned} \right\} \tag{5.2}$$

Using a perturbation of J , we assume that the path $(J_s, \theta_{s,t})$ is regular (in the sense of Section 3), and moreover, we may assume that J' is sufficiently close to J so that $|J_s - J|(x) < \delta$ for all s, x for some small $\delta > 0$ to be specified later. The following lemma contains a key estimate needed in our compactness argument.

LEMMA 5.4. *Let ε be a real number such that $|\varepsilon \cdot \eta_t| < c/3$ for all t . For a solution \tilde{u} of (5.1) satisfying (5.2) and the boundary condition $[\tilde{z}^+, u^-, \# \tilde{u}] = [\tilde{z}^+, u^+]$, we have $E(\tilde{u}) \leq 3(\mathcal{A}_{\tilde{H}_t}([\tilde{z}^-, u^-]) - \mathcal{A}_{\tilde{H}_t}([\tilde{z}^+, u^+]))$, where \tilde{H}_t is a Hamiltonian function on \tilde{M} such that $d\tilde{H}_t = \pi^* \theta_t$.*

Proof. Since the formal L^2 -gradient of $\mathcal{A}_{\tilde{H}_t}$ is $J(\partial \tilde{u} / \partial t - X_{\theta_t})$, we have

$$\frac{\partial \mathcal{A}_{\tilde{H}_t}}{\partial s}(\tilde{u}_s) = \int_0^1 \left\langle \frac{\partial \tilde{u}}{\partial s}, J \left(\frac{\partial \tilde{u}}{\partial t} - X_{\theta_t} \right) \right\rangle dt.$$

If $u_s = \pi \circ \tilde{u}_s : S^1 \rightarrow M$ factors through $U_j \rightarrow M$ for some j , then $X_{\eta_t} = 0$ along u_s , and hence

we get

$$\frac{\partial \mathcal{A}_{\tilde{H}_t}}{\partial s}(\tilde{u}_s) = \int_0^1 \left\langle \frac{\partial \tilde{u}}{\partial s}, J \left(\frac{\partial u}{\partial t} - X_{\theta_t} \right) \right\rangle dt \leq -(1-\delta) \left\| \frac{\partial u}{\partial s} \right\|_{L^2}^2. \quad (5.3)$$

If not, we have

$$-\left\langle \frac{\partial u}{\partial s}, J \left(\frac{\partial u}{\partial t} - X_{\theta_t} \right) \right\rangle \geq \left| \frac{\partial u}{\partial s} \right|^2 - \left| \frac{\partial u}{\partial s} \right| \cdot \left| \frac{\partial u}{\partial s} + J \left(\frac{\partial u}{\partial t} - X_{\theta_t} \right) \right| \quad (5.4)$$

Since u satisfies (5.1) we get from (5.4)

$$-\left\langle \frac{\partial u}{\partial s}, J \left(\frac{\partial u}{\partial t} - X_{\theta_t} \right) \right\rangle \geq \left| \frac{\partial u}{\partial s} \right|^2 - \left| \frac{\partial u}{\partial s} \right| \cdot \left((1+\delta) |\phi(s) \cdot \varepsilon \eta_t| + |J_s - J| \cdot \left| \frac{\partial u}{\partial t} - X_{\theta_t} \right| \right) \quad (5.5)$$

Our assumption that $|\phi(s) \cdot \varepsilon \eta_t| \leq c/3$ yields

$$|J_s - J| \cdot \left| \frac{\partial u}{\partial t} - X_{\theta_t} \right| = |J_s - J| \cdot \left| J_s \frac{\partial u}{\partial s} + \phi(s) \cdot \varepsilon \eta_t \right| \leq \delta \left((1+\delta) \left| \frac{\partial u}{\partial s} \right| + c/3 \right). \quad (5.6)$$

Applying the Cauchy inequality to $\int_0^1 \left| \frac{\partial u}{\partial s} \right| dt$, we obtain from (5.5) and (5.6)

$$-\frac{\partial \mathcal{A}_{\tilde{H}_t}}{\partial s}(\tilde{u}_s) \geq (1-\delta) \left\| \frac{\partial u}{\partial s} \right\|_{L^2}^2 - \left\| \frac{\partial u}{\partial s} \right\|_{L^2} \cdot |(1+2\delta)c/3|. \quad (5.7)$$

Our assumptions and Lemma 5.2 imply $\|\partial u/\partial s\|_{L^2} \geq 2c/3$. Now it is easy to verify the following inequality

$$\left\| \frac{\partial u}{\partial s} \right\|_{L^2} \left(\left\| \frac{\partial u}{\partial s} \right\|_{L^2} - \frac{7c}{12} \right) > 0.$$

Therefore, for δ small enough (e.g. $\delta = 1/100$), we obtain

$$(1-\delta) \left\| \frac{\partial u}{\partial s} \right\|_{L^2}^2 - \left\| \frac{\partial u}{\partial s} \right\|_{L^2} \cdot |(1+\delta)c/3| \geq \frac{1}{3} \left\| \frac{\partial u}{\partial s} \right\|_{L^2}^2. \quad (5.8)$$

Combining (5.3), (5.7) and (5.8) yields the desired estimate. \square

Once we have the uniform bound of the energy functional, the weak-compactness property holds. In particular, $\mathcal{M}(\tilde{z}^-, \tilde{z}^+; \theta_{s,t}, J_s)$ is a finite set if $\mu_{\theta_t}(\tilde{z}^-) = \mu_{\theta_t + \varepsilon \eta_t}(\tilde{z}^+)$. We define a chain homomorphism $\phi: C_*(\theta_t, J) \rightarrow C_*(\theta_t + \varepsilon \eta_t, J')$ as follows:

$$\phi(\tilde{x}) = \sum_{\mu_{\theta_t + \varepsilon \eta_t}(\tilde{y}) = \mu_{\theta_t}(\tilde{x})} m_2(\tilde{x}, \tilde{y}) \cdot \tilde{y}$$

where $m_2(\tilde{x}, \tilde{y})$ denotes the modulo-2 reduction of the cardinality of $\mathcal{M}(\tilde{x}, \tilde{y}; \theta_{s,t}, J_s)$. Now we show that $\phi(\tilde{x}) \in C_*(\tilde{H}_t)$.

LEMMA 5.5. *For each c and fixed periodic solution $\tilde{y} \in \mathcal{P}(\tilde{H}_t)$ there is only a finite number of elements $g \in \Gamma_1$ such that the coefficient of $g \cdot \tilde{y}$ in $\phi(\tilde{x})$ is not zero and $\mathcal{A}_{\tilde{H}_t}(g \cdot \tilde{y}) > c$.*

Proof. The same argument as in the proof of Lemma 5.4 gives that $\mathcal{A}_{\tilde{H}_t}(\tilde{z}^+) \leq \mathcal{A}_{\tilde{H}_t}(\tilde{z}^-) - E(\tilde{u})/3$, where \tilde{H}_t is a Hamiltonian function for $\pi^*(\theta_t + \varepsilon \eta_t)$. In other words, $\mathcal{A}_{\tilde{H}_t}$ is also a Liapunov function for the “flow” defined by eq. (5.1). This inequality implies that if g satisfies the condition of Lemma 5.5 then the energy of a solution u of (5.1),

whose lift to \tilde{M} joins \tilde{x} and $g \cdot \tilde{y}$, is uniformly bounded by some c' . Assume the contrary, i.e. there exists an infinite number of $g_1 \in \Gamma_1$ satisfying the condition in Lemma 5.5. Then there exists an infinite number of solutions \tilde{u}_i of (5.1) on \tilde{M} such that $\lim_{s \rightarrow -\infty} \tilde{u}_i = \tilde{x}$, $\lim_{s \rightarrow \infty} \tilde{u}_i = g_i \cdot \tilde{y}$ and $E(u_i) < c'$. By the compactness result, there exists a limit u_∞ . Let g_∞ be an element in Γ_1 such that $\lim_{s \rightarrow \infty} \tilde{u}_\infty = g_\infty \cdot \tilde{y}$. Note that all $\tilde{u}_i, \tilde{u}_\infty$ have \tilde{x} as one of the ends. Then the other end of \tilde{u}_i , when l is sufficient large, is also $g_\infty \cdot \tilde{y}$. We arrive at a contradiction. \square

It is easy to see that ϕ is invariant under the action of Γ_1 . By Lemma 5.5 we can extend ϕ as a chain homomorphism of Λ_θ -modules $C_*(\tilde{H})$ and $C_*(\tilde{H}')$. The same argument now shows that ϕ is a Λ_θ -linear isomorphism. This completes the proof of Theorem 5.3. \square

6. FLOER HOMOLOGY AND NOVIKOV HOMOLOGY

First of all, we recall some fundamental facts about Novikov homology. Let X be a closed manifold and η a closed 1-form on X . Denote by $\pi: \tilde{X} \rightarrow X$ the covering space associated to the homomorphism $I_n: \pi_1(X) \rightarrow \mathbf{R}$. Then there exists a function $f: \tilde{X} \rightarrow \mathbf{R}$ such that $\pi^*\eta = df$. For a generic Riemannian metric g on X , the gradient flow of f with respect to π^*g is of Morse–Smale type and the Novikov complex $C_*^{Nov}(\eta, g)$ is defined in the same way as the Morse complex (cf. Appendix C). The complex $C_*^{Nov}(\eta, g)$ is a graded module over the Novikov ring Λ_η . The homology group $Nov_*(\eta, g)$ of $C_*^{Nov}(\eta, g)$ is called the Novikov homology associated to η . In this note we consider only Novikov rings over \mathbf{Z}_2 .

Fact 6.1: $Nov_*(\eta, g)$ does not depend on the choice of Riemannian metric g for which the gradient flow of f is of Morse–Smale type.

Fact 6.2: $Nov_*(\eta) = Nov_*(\eta, g)$ depends only on the projective class of the cohomology class of η , i.e.

$$Nov_*(\eta) \cong Nov_*(\eta') \quad \text{if } [\eta] = \lambda[\eta'] \text{ in } H^1(X; \mathbf{R}) \text{ for some } \lambda \neq 0.$$

The following Fact 6.3 tells us that the Novikov homology can be computed from the Morse complex $\tilde{C}_*(h)$ of $\pi^*h: \tilde{X} \rightarrow \mathbf{R}$.

Fact 6.3: $Nov_*(\eta) = H_*(\tilde{C}_*(h) \otimes \Lambda_\eta)$.

The goal of this section is to show the following result.

THEOREM 6.4. *Let (M, ω) be a symplectic manifold of dimension $2n$ satisfying the condition of the Main Theorem. For a generic periodic family $\{\theta_t\}$ of closed 1-forms in a fixed cohomology class $[\eta]$ there exists a natural isomorphism of graded Λ_η -modules*

$$HF_k(\theta_t, J) \cong \bigoplus_{j=k \pmod{2n}} Nov_{j+n}(\eta).$$

Proof. For generic pairs $(\{\theta_t^\alpha\}, J^\alpha)$ and $(\{\theta_t^\beta\}, J^\beta)$ with the same Calabi invariant,

$$HF_*(\theta_t^\alpha, J^\alpha) \cong HF_*(\theta_t^\beta, J^\beta)$$

by Theorem 5.1. In other words, the Floer homology does not depend on the choice of a generic pair $(\{\theta_t\}, J)$ with prescribed Calabi invariant η . We denote this homology by

$HF_*(\eta)$. Theorem 5.4 implies that

$$HF_*(\eta) \cong HF_*((1 + \varepsilon) \cdot \eta) \quad \text{for } |\varepsilon| < \hbar(\eta).$$

Let h be a Morse function on M . We consider the action functional $\mathcal{A}_h: \tilde{\mathcal{L}}M \rightarrow \mathbf{R}$. Note that this is the pull-back of the action functional on $\tilde{\mathcal{L}}M$. We shall define a chain complex $C_*(\pi^*h, J)$. The chain group consists of $\sum a_{[\tilde{x}, \tilde{u}]}[\tilde{x}, \tilde{u}]$, where the sum is taken over pairs $[\tilde{x}, \tilde{u}] \in \tilde{\mathcal{L}}M$ such that \tilde{x} is a critical point of π^*h and the set $\{[\tilde{x}, \tilde{u}] \mid a_{[\tilde{x}, \tilde{u}]} \neq 0 \text{ and } \mathcal{A}_f([\tilde{x}, \tilde{u}]) > c\}$ is finite for all c . We choose h to be a sufficiently C^2 -small function such that the 1-periodic solution $x(t)$ are precisely the critical points of h . Moreover we may assume that the critical points of h and the gradient trajectories with Hessian index difference 1 do not intersect the J -holomorphic spheres with Chern number less than or equal to 1. The following lemma shows that the boundary operator ∂ depends only on the gradient trajectories of h .

LEMMA 6.5 [10, Corollary 4.6]. *Let (M, ω) be a closed symplectic manifold of dimension $2n$ which satisfies the condition in the Main Theorem or $c_{1|\pi_2(M)} = 0$. Suppose h is a Morse function on M . Then there exists a number $\tau > 0$ such that if u is a solution of (2.9.1), (2.9.2) and (2.9.3) corresponding to the (time-independent) Hamiltonian τh and $\mu(u) \leq 1$, then u is independent of t .*

We also have that the linearization of the operator (2.9.1) is non-degenerate at the gradient trajectories of h (see Appendix B). The boundary operator ∂ is defined exactly same as in (4.3). We denote the resulting homology group by $HF_*(\pi^*h, J)$. Then as in the proof of Theorem 5.4, the fact that \mathcal{A}_f is a Liapunov function for (5.1) yields that

$$HF_*(h, J) \cong HF_*(\varepsilon \cdot \eta, J') \quad \text{for } |\varepsilon| < \hbar(0).$$

Therefore $HF_*(\eta) \cong HF_*([dh])$. Now recall the relation between the Conley–Zehnder index of the pair $[x = x(t), u_x]$, where u_x is an element in Γ_2 , and the Morse index $\text{ind}_h(x)$ of h at x : $\mu([x(t), u_x]) = \text{ind}_h(x) - n$ [14, 8]. Therefore the set of all $x(t)$ with Conley–Zehnder index k coincides with the set of all critical points $x = x(t)$ of Morse index $j = k \pmod{2N}$. Taking Fact 6.3 into account, Lemma 6.5 implies that the Floer complex $C_*(dh, J)$ is isomorphic to the Novikov complex $C_*^{\text{Nov}}(dh)$.

Theorem 6.4 implies our Main Theorem.

7. AN EXAMPLE

By Theorem C.4 the Euler number of the Novikov homology of a free Abelian covering \tilde{M} is the same as that of the original manifold M . Hence, our theorem also implies the Lefschetz fixed point formula for symplectomorphisms which are symplectically isotopic to the identity. Here we will give a non-trivial example of symplectic manifolds satisfying the condition of the Main Theorem such that the sum of the Betti numbers of the Novikov homology corresponding to any free Abelian covering of M is strictly greater than the Euler number of M . By Theorem C.2 it suffices to consider the maximal free Abelian covering \tilde{M} of M .

The following example of 3-folds with non-vanishing odd Betti number was pointed out to us by Keiji Oguiso. Let X_k be the hypersurface in CP^4 defined by the equation $\{x_0^k + \dots + x_4^k = 0\}$. Denote by h the generator of $H^2(CP^4; \mathbf{Z})$. A direct calculation yields

CLAIM 7.1. *The first Chern class of X_k satisfies $c_1 = (5 - k)h|_{X_k}$. If $k \geq 3$ the Betti number $b_3(X_k)$ is non-zero.*

Let Σ_g denote a Riemannian surface of genus $g \neq 0$. We have $\pi_2(\Sigma_g) = 0$ and any non-degenerate 2-form ω_g on Σ_g is a symplectic form.

CLAIM 7.2. *The product manifold $(X_k \times \Sigma_g, \omega_X \oplus \omega_g)$, $k \neq 5$, is a symplectic manifold which satisfies the condition in Main Theorem. Further suppose that $k \geq 3$ and $g \geq 2$. Then the sum of the Betti numbers of the maximal free Abelian covering of $X_k \times \Sigma_g$ is strictly greater than the Euler number of $X_k \times \Sigma_g$.*

Proof. The first statement is trivial. The maximal free Abelian covering of $X \times \Sigma_g$ is $X \times \tilde{\Sigma}_g$, where $\tilde{\Sigma}_g$ denotes the maximal free Abelian covering of Σ_g . Using Theorem C.4 it is easy to see that the Betti numbers of $\tilde{\Sigma}_g$ are $0, 2g - 2, 0$. With the help of Claim 7.1 and Theorem C.3 we obtain the second statement in Claim 7.2. In particular, if $k = 3$ we have the Euler number of $X_3 \times \Sigma_g = 12(g - 1)$ while the sum of its Betti numbers equals $28(g - 1)$, if $k = 4$ we have the Euler number of $X_4 \times \Sigma_g = 112(g - 1)$ while the sum of its Betti numbers equals $128(g - 1)$. \square

Thus, if $k \geq 3$, $k \neq 5$, and $g \geq 2$, the number of the fixed points of a symplectomorphism $f \in \text{Diff}_\omega^0(X_k \times \Sigma_g)$ is strictly greater than the Euler number of $X_k \times \Sigma_g$, provided that all the fixed points are non-degenerate. In particular, if $k = 3$ (or $k = 4$, resp.) this number is at least $28(g - 1)$ (or $128(g - 1)$, resp.).

8. CONCLUDING REMARKS

If $c_{1|\pi_2(M)} = 0$ the Novikov ring $\Lambda_{\omega, \theta}$ changes when the cohomology class θ changes. That is the main obstruction to the control of the “finiteness condition”, and therefore, to the construction of “chain homomorphisms” and computation of the corresponding Floer homology groups. However, if the Calabi invariant is small enough we still have the following theorem.

THEOREM 8.1. *Let (M, ω) be a closed symplectic manifold and $c_{1|\pi_2(M)} = 0$. There exists $\varepsilon > 0$ such that if $|\llbracket \theta \rrbracket| < \varepsilon$ then*

$$HF_*(\theta) \cong \text{Nov}_*(\theta) \otimes_{\Lambda_\theta} \Lambda_{\theta, \omega}.$$

Consequently, the sum of ranks of $HF_(\theta)$ equals of $\text{Nov}_*(\theta)$.*

Here we suppose that M is equipped with some Riemannian metric and that the norm $|\llbracket \theta \rrbracket|$ is defined as the infimum of the norms of 1-forms θ in the cohomology class $[\theta]$.

Proof of Theorem 8.1. By Theorem 4.3 (invariance under exact deformations) it suffices to show that for each regular periodic 1-form θ , on M there exists a positive number τ such that $HF_*(\tau \cdot \theta, J) \cong \text{Nov}_*(\theta) \otimes_{\Lambda_\theta} \Lambda_{\theta, \omega}$. First we choose a C^2 -small Morse function which satisfies the condition of Lemma 6.5. We lift this function h to the covering space \tilde{M} corresponding to the form θ . As in Section 6 we define the Floer complex $C_*(\tilde{h}, J)$ for the lifted function \tilde{h} . Further we choose τ small enough so that the energy estimate for the “chain homomorphism” solution u between $HF_*(\tilde{h}, J)$ and $HF_*(\tau \cdot \theta, J')$ holds (see Lemma 5.4). Finally we compare the Floer homology $HF_*(\tilde{h})$ with the Novikov homology $\text{Nov}_*(\theta)$. As in

the proof of Theorem 6.4, we have

$$HF_*(\tilde{h}, J) \cong \text{Nov}_{*+n}(\theta) \otimes_{\Lambda_\theta} \Lambda_{\omega, \theta}. \quad \square$$

Remark 8.2. If the dimension of a symplectic manifold (M, ω) is less than or equal to 6, then M is automatically weakly monotone and the conclusion of the Main Theorem holds without the assumption on the minimal Chern number.

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REFERENCES

1. V. I. ARNOLD: *Mathematical Methods of Classical Mechanics*, Springer Verlag (1989).
2. M. F. ATIYAH, and I. G. MACDONALD: *Introduction to Commutative Algebra*, Addison-Wesley, (1969).
3. A. BANYAGA: Sur la structure du groupe de difféomorphismes qui préservent une forme symplectique, *Com. Math. Helv.* **53** (1978), 174–227.
4. C. C. CONLEY and E. ZEHNDER: The Birkhoff–Lewis fixed point theorem and a conjecture of V. I. Arnol'd, *Invent. Math.* **73** (1983), 33–49.
5. A. FLOER: Symplectic fixed points and holomorphic spheres, *Comm. Math. Phys.* **120** (1989), 575–611.
6. A. FLOER: The unregularized gradient flow of the symplectic action, *Comm. Pure Appl. Math.* **41** (1988), 775–813.
7. A. FLOER: Witten's complex and infinite dimensional Morse theory, *J. Diff. Geom.* **30** (1989), 207–221.
8. H. HOFER and D. A. SALAMON: Floer homology and Novikov rings, preprint (1992).
9. LÊ HÔNG VÂN: Curvature estimate for the volume growth of globally minimal submanifolds, *Math. Ann.* **196** (1993), 103–118.
10. K. ONO: On the Arnold conjecture for weakly monotone symplectic manifolds, *Inv. Math.*, 1994.
11. A. V. PAZHITNOV: On the sharpness of Novikov type inequalities for manifolds with free abelian fundamental group, *Math. USSR Sbornik* **68** (1991), 351–389.
12. M. POZNIAK: The Morse complex, Novikov homology and Fredholm theory, University of Warwick, preprint (1993).
13. D. SALAMON: Morse theory, the Conley index and Floer homology, *Bull. London Math. Soc.* **22** (1990), 113–140.
14. D. SALAMON and E. ZEHNDER: Morse theory for periodic solutions of Hamiltonian systems and the Maslov index, *Comm. Pure Appl. Math.* **XLV** (1992), 1303–1360.
15. M. SCHWARZ: Morse homology, Ruhr University Bochum, preprint (1993).

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APPENDIX A: ON POINCARÉ'S INVARIANT FOR SYMPLECTOMORPHISMS

In this section, for symplectic manifolds of dimension greater than or equal to four we prove the converse of the Poincaré theorem, namely, if two embedded loops on a symplectic manifold M have the same Poincaré invariant then there exists an exact Hamiltonian flow on M which sends one loop to the other.

Originally the Poincaré theorem was stated for the symplectic manifold \mathbf{R}^{2m} and then for the cotangent bundle T^*M with the canonical 1-form $\alpha = p dq$ and the canonical symplectic form $\omega = d\alpha$ [1]. In the general case of an arbitrary symplectic manifold M we replace the Poincaré integral by the action functional \mathcal{A} defined on an appropriate covering space of the space $E\mathcal{L}(M)$ of contractible embedded loops. We define this covering space $E\tilde{\mathcal{L}}(M)$ to be the quotient of the space of pairs

$\{[\gamma, u] \mid \partial u = \gamma\}$ by the equivalence relation defined by homotopy equivalence of the bounding disk u . We have

$$\mathcal{A}([\gamma, u]) = \int u^* \omega. \tag{A1}$$

Descending to the base space, the action functional is only defined modulo the group $\omega([\pi_2(M)])$. However, the differential of the action functional is well-defined on the loop space:

$$d\mathcal{A}(\gamma)(X) = \int_{\gamma} \omega(X, \dot{\gamma}) = -\Phi(X)([\gamma]) \tag{A2}$$

where Φ is defined in (2.1). From (A2) we immediately obtain the following well-known theorem.

THEOREM A.1. *Any Hamiltonian flow f_s on M preserves the generalized Poincaré invariant, namely the action functional on the covering space $E\mathcal{L}(M)$ of the space of embedded contractible loops $\mathcal{E}\mathcal{L}(M)$*

$$\mathcal{A}([\gamma, u]) = \mathcal{A}([f_s(\gamma), f_s(u)]).$$

Note that if the symplectic form ω on M is exact, that is $\omega = d\alpha$, then

$$\mathcal{A}([\gamma, u]) = \int_{\gamma} \alpha. \tag{A3}$$

THEOREM A.2. *Let (M, ω) be a symplectic manifold of dimension $2m \geq 4$. Suppose that two embedded contractible loops $[\gamma_0, u_0]$ and $[\gamma_1, u_1]$ have the same Poincaré invariant. Then there exists a Hamiltonian flow f_s such that $f_0 = Id$ and $f_1(\gamma_0) = \gamma_1$.*

Proof. This follows from the following two propositions.

PROPOSITION A.3. *Every level surface $\mathcal{A}^{-1}(a) \subset E\mathcal{L}(M)$ is path-connected.*

PROPOSITION A.4. *Suppose a path $[\gamma_s, u_s]$, $s \in [0, 1]$, of embedded loops lies on a level surface $\mathcal{A}^{-1}(a)$. Then there exists a Hamiltonian flow f_s such that $f_s(\gamma_0) = \gamma_s$.*

Proof of Proposition A.3. It is easy to see that the space $E\mathcal{L}(M)$ is path-connected. Suppose that $[\gamma_s, u_s]$ is a path in $E\mathcal{L}(M)$ which joins two points of $[\gamma_0, u_0]$ and $[\gamma_1, u_1]$ with the same Poincaré invariant. Our aim now is to find a deformation of the path $[\gamma_s, u_s]$ to a new path with the same endpoints and constant Poincaré invariant.

Fix two distinct points $p \in \gamma_0$ and $q \in \gamma_1$. Yet $\sigma: [0, 1] \rightarrow M$ be an embedding with $\sigma(0) = p$ and $\sigma(1) = q$. By Darboux's theorem there exists a tubular neighborhood N_ε of $\sigma([0, 1])$ in M such that N_ε is symplectomorphic to $[0, 1] \times D_\varepsilon^{2m-1}$ in \mathbf{R}^{2m} with the standard symplectic structure for some $\varepsilon > 0$ (here D_ε^{2m-1} is the ε -ball in \mathbf{R}^{2m-1}). Thus we can find a smooth family of embeddings $\phi_s: D^2(\varepsilon) \times [0, 1] \rightarrow N_\varepsilon$ such that the following condition holds.

For all s and τ the restriction ϕ_s to $D^2_\tau(\varepsilon) = (D^2(\varepsilon), \tau)$ is a symplectic embedding (here the disk $D^2(\varepsilon)$ of radius ε carries the standard symplectic structure).

Let $\chi(s)$ be a smooth function on $[0, 1]$ with $\chi(0) = \chi(1) = 0$ and $0 < \chi(s) \leq 1$ for $s \in (0, 1)$. Now we can find a finite number of points $\tau_1, \dots, \tau_n \in [0, 1]$ and construct an embedding $\Phi: S^1 \times [0, 1] \rightarrow N_\varepsilon$ with the following properties.

(1a) For each s , the image $\Phi_s(S^1) = \Phi(S^1, s)$ is a connected sum of n circles

$$\phi_s(S^1_{\tau_1, s}) \# \phi_s(S^1_{\tau_2, s}) \# \dots \# \phi_s(S^1_{\tau_n, s})$$

where $S^1_{\tau_i, s}$ is the boundary of $D^2_{\tau_i}(\chi(s) \cdot \varepsilon)$,

(1b) $\Phi_0(S^1) = p$, $\Phi_1(S^1) = q$.

(2) For each s , we have $\mathcal{A}(\Phi_s(S^1)) = \int_{\Phi_s(S^1)} \alpha \geq |\mathcal{A}(\gamma_s) - \mathcal{A}(\gamma_0)|$. Here α is an 1-form on N_ε such that $\omega|_{N_\varepsilon} = d\alpha$.

Here the condition $2m \geq 4$ makes sure that the connected sum of circles $\Phi_s(S^1_{\tau_i, s})$ can be chosen as embedded loops in M . Note that each circle $S^1_{\tau_i, s}$ bounds a disk $D^2_{\tau_i}(\chi(s) \cdot \varepsilon)$. Therefore, for each s , the

connected sum $\Phi_s(S^1)$ bounds a disk, which is a boundary connected sum of the disks $\phi_s(D_i^2(\chi(s) \cdot \varepsilon))$ for $i = 1, \dots, n$. More precisely, the map Φ can be extended to $D^2 \times [0, 1]$, which is also denoted by Φ . Now we consider a new path $[\gamma'_s, u'_s]$ which is the connected sum $[\gamma_s \# \Phi_s(S^1), u_s \# \Phi_s(D^2)]$. The conditions (1b) and (2) imply that

- (3) $[\gamma'_0, u'_0] = [\gamma_0, u_0], [\gamma'_1, u'_1] = [\gamma_1, u_1]$,
- (4) for each s we have $\mathcal{A}(\gamma'_s) \geq \mathcal{A}(\gamma_0) = \mathcal{A}(\gamma_1)$.

Once again we apply the construction of connected sum to the path $[\gamma'_s, u'_s]$ with the path $[\Phi'_s(S^1), \Phi'_s(D^2)]$ which satisfies the opposite condition to (2). Namely we have

$$(2') \mathcal{A}(\Phi'_s(S^1)) \leq -|\mathcal{A}(\gamma'_s) - \mathcal{A}(\gamma'_0)|.$$

Clearly, the path $[\gamma''_s, u''_s]$ also joins $[\gamma_0, u_0]$ and $[\gamma_1, u_1]$. By our construction it is easy to find a positive function $\eta(s)$ on $[0, 1]$, $0 \leq \eta(s) \leq 1$, such that the path $[\gamma''_s, u''_s] = [\gamma'_s \# \Phi'_s(\eta(s) \cdot S^1), u'_s \# \Phi'_s(\eta(s) \cdot D^2)]$ has the properties required in Proposition A.3. Geometrically, $\eta(s) \cdot D^2$ is a boundary connected sum of shrunken disks $D_i^2(\eta(s) \cdot \varepsilon)$ for $i = 1, \dots, n'$. □

Proof of Proposition A.4. Let ϕ_s be a 1-parameter family of embeddings of S^1 into M with the same Poincaré invariant. Differentiating with respect to s , we get a 1-parameter family V_s of vector fields along ϕ_s . It is sufficient to show that V_s , which is defined on $\gamma_s = \phi_s(S^1)$ can be extended to a Hamiltonian vector field on M . Using the isomorphism Φ (cf. (2.1)), we get a cross section $\Phi(V'_s)$ of $T^*M|_{\gamma_s}$. We shall extend this section to a closed 1-form on a tubular neighborhood $N(\gamma_s)$. Since symplectic manifolds are orientable, $N(\gamma_s)$ is diffeomorphic to $S^1 \times D^{2m-1}$ in such a way that $S^1 \times \{0\}$ corresponds to γ_s . Let $\{x_1, \dots, x_{2m-1}\}$ be coordinate functions on D^{2m-1} . Then there exist functions $a(t), b_1(t), \dots, b_{2m-1}(t)$ on S^1 such that

$$\Phi(V_s) = a(t) dt + \sum_{i=1}^{2m-1} b_i(t) dx_i.$$

Put

$$\tilde{a}(t, x_i) = a(t) + \frac{db_i}{dt} \cdot x_i.$$

It is easy to see that

$$\eta = \tilde{a}(t, x_i) dt + \sum_{i=1}^{2m-1} b_i(t) dx_i$$

is the desired extension as a closed 1-form. Since the embedded loops γ_t have the same Poincaré invariant, (A2) implies that $\Phi(V_t)$ is an exact 1-form on γ_s . Note that $S^1 \times \{0\}$ is a deformation retract of $S^1 \times D^{2m-1}$, hence η is also an exact 1-form, i.e. $\eta = dh$ for some function h .

For a cut-off function φ which equals 1 near $S^1 \times \{0\}$, the Hamiltonian vector field of $\varphi \cdot h$ coincides with V_t on γ_t and vanishes near the boundary of the tubular neighborhood. Therefore it naturally extends to a Hamiltonian vector field \tilde{V}_t which vanishes outside of the tubular neighborhood. This completes the proof of Proposition A.4. □

Remark A.5. The same argument may be used to generalize Theorem A.2 to embedded loops with non-trivial homotopy class by the same line of argument. We can also show the case that $M = \mathbb{R}^2$ with the standard symplectic structure.

APPENDIX B: NON-DEGENERACY OF THE LINEARIZED OPERATOR FOR TIME INDEPENDENT HAMILTONIANS

In this appendix, we shall show the surjectivity of the linearized operator at gradient trajectories (see also [14]). Let f be a C^2 -small Morse function on a symplectic manifold M such that the Conley–Zehnder indices at critical points with trivial bounding disks coincide with the indices of the Hessian of f . We also fix a metric for which the gradient flow of f is of Morse–Smale type and which is compatible with an almost complex structure calibrated by ω .

Let $\gamma: \mathbf{R} \rightarrow M$ be a trajectory of the gradient flow joining two critical points p and q . Denote by $u_\gamma: \mathbf{R} \times S^1 \rightarrow M$ the mapping defined by $u(s, t) = \gamma(s)$. The linearization operator $D\bar{\partial}_{J,H}$ of $\bar{\partial}_{J,H}$ at u_γ is given by

$$D\bar{\partial}_{J,H}\xi = \nabla_{\frac{\partial}{\partial s}}\xi + J\nabla_{\frac{\partial}{\partial t}}\xi + \text{Hess}(f)\xi \tag{B1}$$

where $\text{Hess}(f)\xi = \nabla_\xi \nabla f$. Since the Hamiltonian f and the connecting orbit u_γ are t -independent, we have a symmetry in the t -variable. Hence $D\bar{\partial}_{J,H}$ decomposes into Fredholm operators

$$P^{(k)}: V_k \rightarrow W_k$$

where $W^{1,2}(u_\gamma^*TM) = \bigoplus_{k \in \mathbf{Z}} V_k$ and $L^2(u_\gamma^*TM) = \bigoplus_{k \in \mathbf{Z}} W_k$ are decompositions as S^1 -modules according to weights. From (B1), $P^{(k)}\xi = \nabla_{\frac{\partial}{\partial s}}\xi + (-k + \text{Hess}(f))\xi$.

For $k = 0$, $P^{(0)}$ is surjective and moreover we have $\text{index } P^{(0)} = \dim \ker P^{(0)} = \text{index } \text{Hess}_p(f) - \text{index } \text{Hess}_q(f)$, since the gradient flow of f is of Morse–Smale type. ($P^{(0)}$ coincides with the linearization operator for the gradient flow of a Morse function [13, 15].)

From now on, we assume that f is C^2 -small such that $\|\text{Hess}(f)\| < \delta < 1$ for some $\delta > 0$. Let $\langle\langle \xi, \eta \rangle\rangle(s) = \int_0^1 \langle \xi(s, t), \eta(s, t) \rangle dt$.

For $k > 0$, the solution ξ of $P^{(k)}\xi = 0$ satisfies

$$\begin{aligned} \frac{\partial}{\partial s} \|\xi\|^2 &= 2 \langle \nabla_{\frac{\partial}{\partial s}} \xi, \xi \rangle \\ &= 2 \langle k - \text{Hess}(f)\xi, \xi \rangle \\ &> (1 - \delta) \|\xi\|^2. \end{aligned}$$

Hence $\|\xi\|^2$ grows exponentially as s tends to $+\infty$, unless $\xi \equiv 0$. Since ξ is square integrable, we get $\xi \equiv 0$, i.e. $\ker P^{(k)} = 0$ for $k > 0$. The same conclusion holds for $k < 0$. Therefore we get $\text{index } D\bar{\partial}_{J,H} = \dim \ker P^{(0)} - \sum_{k \neq 0} \dim \text{coker } P^{(k)}$. On the other hand, $\text{index } D\bar{\partial}_{J,H} = \dim \ker P^{(0)}$, hence $P^{(k)}$ is surjective for all k , i.e. $D\bar{\partial}_{J,H}$ is surjective.

APPENDIX C: A NOTE ON NOVIKOV HOMOLOGY THEORY¹

In this appendix, we shall give simple proofs of some fundamental facts on the Novikov homology theory, which seem to be folklore to specialists (see also [11, 12]). Floer [7] interpreted the Morse complex in terms of gradient trajectories for generic Morse functions. Details were carried out by Matthias Schwarz [15] (see also [13]). For a closed 1-form η on M , there is a smallest Abelian covering space on which the pull-back of η is exact. We denote this covering space by $\pi: \tilde{M} \rightarrow M$. Let f be a function on \tilde{M} such that $\pi^*\eta = df$. For a generic Riemannian metric on M , the gradient flow of f with respect to the pull-back metric is of Morse–Smale type. An element of the k th Novikov chain group of η is $\sum a_{\tilde{x}} \tilde{x}$, where the index of Hessian of f at \tilde{x} is k and $\{\tilde{x} \mid a_{\tilde{x}} \neq 0 \text{ and } f(\tilde{x}) > c\}$ is a finite set for all c . The Novikov ring Λ_ξ is defined as the completion of the group ring of the covering transformation group of $\pi: \tilde{M} \rightarrow M$ with respect to the weight homomorphism I_ξ . The argument in Lemma 3.5 implies that there are at most finitely many trajectories joining two critical points \tilde{x}, \tilde{y} and we can define the boundary operator, which is linear over the Novikov ring, exactly same as (4.3). The Novikov chain group, and hence the Novikov homology groups, are finitely generated modules over the Novikov ring. Note that the Novikov complexes of η and $\lambda\eta$, with the same Riemannian metric, are same if $\lambda \neq 0$. The argument in Theorem 4.3 gives Facts 6.1 and 6.2.

We shall prove Fact 6.3. Our argument is a finite-dimensional analogue of the proof of a critical points p_i of h . Since M is compact, there exists a number $\varepsilon > 0$ such that the norm of the gradient vector field ∇h satisfies $\|\nabla h\| > \varepsilon$ outside of $\bigcup U_i$. Since each U_i is contractible, we can find a closed 1-form η' which is cohomologous to η and vanishes identically on $\bigcup U_i$. Then we can find $\lambda > 0$ such that $\|\lambda \cdot \eta'^*\| < \varepsilon/3$, where η'^* is the vector field associated to η' with respect to the given Riemannian metric. From now on η denotes $\lambda \cdot \eta'$ defined as above. This implies that f and $f + \pi^*h$ are both Liapunov functions for the gradient flow of f and for the gradient flow of $f + \pi^*h$. Note that the critical point sets of f and $f + \pi^*h$ coincide. We can define chain homomorphisms between the Morse

¹This appendix was written in collaboration with Lê Tu Quốc Thang.

complex of f and $f + \pi^*h$ by using the following ordinary differential equation.

$$\frac{d\gamma}{ds} + \nabla\{f + \pi^*h_s\}(\gamma(s)) = 0 \tag{C1}$$

where h_s is a 1-parameter family of functions on M such that

$$h_s = 0, \text{ for } s < -R \text{ and } h_s = h, \text{ for } s > R$$

for some $R > 0$. The energy of a solution γ of (C1) is defined by

$$E(\gamma) = \int_{-\infty}^{\infty} \left| \frac{d\gamma}{ds} \right|^2 ds.$$

The set of solutions of (C1) with bounded energy coincides with the set of solutions of (C1) such that

$$\lim_{s \rightarrow -\infty} \gamma(s) = \tilde{x} \text{ and } \lim_{s \rightarrow \infty} \gamma(s) = \tilde{y} \tag{C2}$$

for some critical points \tilde{x}, \tilde{y} of f . Moreover, we can estimate the energy in terms of \tilde{x}, \tilde{y} as follows:

$$E(\gamma) < 3(f(\tilde{x}) - f(\tilde{y}))$$

and

$$E(\gamma) < 3((f + \pi^*h)(\tilde{x}) - (f + \pi^*h)(\tilde{y})).$$

The bound on energy implies the weak compactness of the set of solutions and no bubbling phenomenon occurs. We define $\phi(\tilde{x}) = \sum m_2(\tilde{x}, \tilde{y}) \cdot \tilde{y}$, where $m_2(\tilde{x}, \tilde{y})$ denotes the modulo-2 reduction of the number of solutions of (C1) satisfying the asymptotic condition (C2). To get a chain homomorphism from ϕ , we have to show that ϕ preserves the finiteness condition with respect to $f + \pi^*h$. However this can be derived in the same way as in Lemma 3.5. We can also define a chain homomorphism in the other direction and they are inverses each other on homology groups.

We define the rank of a module L over a commutative algebra A with unit to be the dimension of the vector space $L \otimes F(A)$ over the field of fractions $F(A)$ of A . The rank of the Novikov homology is defined to be the rank of it over the Novikov ring. Now we shall show the following:

THEOREM C.1. *Let ξ and ξ' be closed 1-forms on M whose corresponding Abelian coverings are same. Then ranks of the Novikov homology of ξ and the Novikov homology of ξ' are same.*

THEOREM C.2. *Let ξ and ξ' be closed 1-forms such that ξ' vanishes on the kernel of ξ . Then the rank of $Nov_*(\xi)$ is less than or equal to the rank of $Nov_*(\xi')$.*

Proof of Theorem C.1. Let $\tilde{M} \rightarrow M$ be the Abelian covering of M corresponding to the 1-form ξ . Let h be a Morse function on M and $\tilde{C}_*(h) = C_*(\pi^*h)$ the Morse complex of π^*h . Let Λ be the group ring of the covering transformation group and Λ_ξ the Novikov ring of ξ . Then Fact 6.3 implies that $Nov_*(\xi) \cong H_*(\tilde{C}_*(h) \otimes_\Lambda \Lambda_\xi)$. Since Λ_ξ is flat over Λ (see [2, Chap. 10]), we get $Nov_*(\xi) \cong H_*(\tilde{C}_*(h)) \otimes_\Lambda \Lambda_\xi$ (in fact, Sikorav has shown that Λ_ξ is faithfully flat [11]). Now we have $rank_{\Lambda_\xi} Nov_*(\xi) = rank_{F(\Lambda_\xi)} H_*(\tilde{C}_*(h)) \otimes_\Lambda F(\Lambda_\xi) = rank_{F(\Lambda_\xi)} \{H_*(\tilde{C}_*(h)) \otimes_\Lambda F(\Lambda)\} \otimes_{F(\Lambda)} F(\Lambda_\xi) = rank_{F(\Lambda)} H_*(\tilde{C}_*(h)) \otimes_\Lambda F(\Lambda)$. Therefore the rank of the Novikov homology depends only on the covering space \tilde{M} , i.e. the ranks of $Nov_*(\xi)$ and $Nov_*(\xi')$ are same. \square

Proof of Theorem C.2. Let $\tilde{M} \rightarrow M$ and $\bar{M} \rightarrow M$ be the covering spaces corresponding to ξ and ξ' with covering transformation groups Γ_1 and Γ_2 , respectively. Let t_1, \dots, t_k and t'_1, \dots, t'_l ($l < k$) be generators of Γ_1 and Γ_2 , respectively, such that t'_i is the image of t_i by the quotient homomorphism $\Gamma_1 \rightarrow \Gamma_2$. Let Λ_1 and Λ_2 be the group rings of Γ_1 and Γ_2 , respectively. Then there is a natural homomorphism $\phi: \Lambda_1 \rightarrow \Lambda_2$ which maps t_i to 1 for $l + 1 \leq i \leq k$. We may assume that $l = k - 1$. By Theorem C.1 and Fact 6.3, it is enough to show that $rank_{\Lambda_1} H_*(\tilde{C}_*(h)) \leq rank_{\Lambda_2} H_*(\tilde{C}_*(h) \otimes_{\Lambda_1} \Lambda_2)$.

CLAIM. *The natural map $H_*(\tilde{C}_*(h)) \otimes_{\Lambda_1} \Lambda_2 \rightarrow H_*(\tilde{C}_*(h) \otimes_{\Lambda_1} \Lambda_2)$ is injective. Therefore we get $rank_{F(\Lambda_2)} \{H_*(\tilde{C}_*(h) \otimes_{\Lambda_1} \Lambda)\} \otimes_{\Lambda_2} F(\Lambda_2) \geq rank_{F(\Lambda_2)} \{H_*(\tilde{C}_*(h)) \otimes_{\Lambda_1} \Lambda_2\} \otimes_{\Lambda_2} F(\Lambda_2)$.*

This Claim follows from the observation that $B_*(\tilde{C}_*(h)) \otimes \Lambda_2 = B_*(\tilde{C}_*(h) \otimes \Lambda_2)$, where $B_*(*)$ denotes the submodule of boundary cycles.

In order to complete the proof of Theorem C.2, it is sufficient to show that

$$\text{rank}_{F(\Lambda_2)} H_*(\tilde{C}_*(h)) \otimes_{\Lambda_2} F(\Lambda_2) \geq \text{rank}_{\Lambda_1} H_*(\tilde{C}_*(h)).$$

Put $R = F(\Lambda_1)[[t_k^{\pm}]]$. Write $L = H_*(\tilde{C}_*(h))$. Then R is a principal ideal domain and has the same fractional field as Λ_1 . Hence for the Λ_1 -module L , we have $\text{rank}_{\Lambda_1} L = \text{rank}_R L \otimes_{\Lambda_1} R$. Since R is a principal ideal domain, we have $L \otimes_{\Lambda_1} R \supset R^p$, gives that $\{L \otimes_{\Lambda_1} R\} \otimes_R F(\Lambda_2) \supset (F(\Lambda_2))^p$. Hence we get $\text{rank}_{F(\Lambda_2)} L \otimes_{\Lambda_2} F(\Lambda_2) \geq \text{rank}_{\Lambda_1} L$ and the desired inequality. \square

We have Künneth's formula in Novikov homology theory.

THEOREM C.3. *Let M_1 and M_2 be closed manifolds and ξ and η closed 1-forms on M_1 and M_2 respectively. If the kernel of the weight homomorphism for $\pi_1^*\xi + \pi_2^*\eta$ is the direct sum of the kernels of the weight homomorphism for ξ and η , we have*

$$\text{Nov}_*(M_1 \times M_2; \pi_1^*\xi + \pi_2^*\eta) \cong (\text{Nov}_*(M_1; \xi) \otimes_{\mathbb{Z}_2} \text{Nov}_*(M_2; \eta)) \otimes_{\Lambda_1 \otimes \Lambda_2} \Lambda_{\pi_1^*\xi + \pi_2^*\eta}.$$

Proof. We denote by Γ_1 and Γ_2 the group ring of the covering transformation groups corresponding to ξ and η . The Künneth formula for the corresponding covering spaces gives

$$H_*(\tilde{C}_*(\pi_1^*h_1 + \pi_2^*h_2)) \cong H_*(\tilde{C}_*(\pi_1^*h_1)) \otimes H_*(\tilde{C}_*(\pi_2^*h_2))$$

where h_1 and h_2 are Morse functions on M_1 and M_2 , respectively. Since the kernel of the weight homomorphism for $\pi_1^*\xi + \pi_2^*\eta$ is the direct sum of the kernels of the weight homomorphisms for ξ and η , the Novikov ring for $\pi_1^*\xi + \pi_2^*\eta$ is the completion of the tensor product of the Novikov rings for ξ and η . Note that $\text{Nov}_*(M_1 \times M_2; \pi_1^*\xi + \pi_2^*\eta)$ is isomorphic to $H_*(\tilde{C}_*(\pi_1^*h_1 + \pi_2^*h_2)) \otimes_{\mathbb{Z}_2[\Gamma_1 \otimes \Gamma_2]} \Lambda_{\pi_1^*\xi + \pi_2^*\eta}$. Combining these facts, we get that $\text{Nov}_*(M_1 \times M_2; \pi_1^*\xi + \pi_2^*\eta)$ is isomorphic to the completion of the tensor product of

$$\text{Nov}_*(M_1; \xi) \cong H_*(\tilde{C}_*(h_1)) \otimes_{\mathbb{Z}_2[\Gamma_1]} \Lambda_\xi$$

and

$$\text{Nov}_*(M_2; \eta) \cong H_*(\tilde{C}_*(h_2)) \otimes_{\mathbb{Z}_2[\Gamma_2]} \Lambda_\eta$$

with respect to the weight homomorphism for $\pi_1^*\xi + \pi_2^*\eta$, i.e.

$$\{ \text{Nov}_*(M_1; \xi) \otimes \text{Nov}_*(M_2; \eta) \} \otimes_{\Lambda_1 \otimes \Lambda_2} \Lambda_{\pi_1^*\xi + \pi_2^*\eta} \quad \square$$

Since the Euler number of the homology equals the alternating sum of ranks of the complex, we get

THEOREM C.4. *The Euler number of the Novikov homology, i.e. the alternating sum of the ranks of the Novikov homology groups, equals to the Euler number of the ordinary homology of M .*