

An Introduction to Sign Pattern Matrices  
and Some Connections Between Boolean and  
Nonnegative Sign Patterns

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## 1. Introduction

The origins of sign pattern matrices are in the 1947 book *Foundations of Economic Analysis* by the Nobel Economics Prize winner P. Samuelson, who pointed to the need to solve certain problems in economics and other areas based only on the signs of the entries of the matrices. (the exact values of the entries of the matrices may not always be known)

The study of sign pattern matrices has become somewhat synonymous with qualitative matrix analysis. Because of the interplay between sign pattern matrices and graph theory, the study of sign patterns is regarded as a part of combinatorial matrix theory.

The 1987 dissertation of C. Eschenbach, directed by C.R. Johnson, studied sign patterns that “require” or “allow” certain properties and summarized the work on sign patterns up to that point.

In 1995, Richard Brualdi and Bryan Shader produced a thorough treatment *Matrices of Sign-Solvable Linear Systems* on sign pattern matrices from the sign-solvability vantage point.

Since 1995 there has been a considerable number of papers on sign patterns and some generalized notions such as ray patterns.

*Handbook of Linear Algebra*, 2007, CRC Press, Chapter 33 Sign Pattern Matrices (Hall/Li)

A matrix whose entries are from the set  $\{+, -, 0\}$  is called a *sign pattern matrix* (or sign pattern, or pattern). For a real matrix  $B$ ,  $\text{sgn}(B)$  is the sign pattern matrix obtained by replacing each positive (resp, negative) entry of  $B$  by  $+$  (resp,  $-$ ). For a sign pattern matrix  $A$ , the *sign pattern class of  $A$*  is defined by

$$Q(A) = \{ B : \text{sgn}(B) = A \}.$$

EG

If

$$B = \begin{bmatrix} 7 & -2 & 0 \\ 0 & 5 & -9 \end{bmatrix},$$

then

$$B \in Q(A)$$

where

$$A = \begin{bmatrix} + & - & 0 \\ 0 & + & - \end{bmatrix}.$$

A sign pattern matrix  $S$  is called a *permutation pattern* if exactly one entry in each row and column is equal to  $+$ , and all other entries are  $0$ . A product of the form  $S^T AS$ , where  $S$  is a permutation pattern, is called a *permutational similarity*.

Two sign pattern matrices  $A_1$  and  $A_2$  are said to be *permutationally equivalent* if there are permutation patterns  $S_1$  and  $S_2$  such that  $A_1 = S_1 A_2 S_2$ .

A sign pattern  $A$  *requires* property  $P$  if every matrix in  $Q(A)$  has property  $P$ .

A sign pattern  $A$  *allows* property  $P$  if some matrix in  $Q(A)$  has property  $P$ .

An  $n \times n$  sign pattern  $A$  is said to be *sign nonsingular* (SNS) if every matrix  $B \in Q(A)$  is nonsingular, ie,  $A$  requires nonsingularity. It is well known that  $A$  is sign nonsingular if and only if  $\det A = +$  or  $\det A = -$ , that is, in the standard expansion of  $\det A$  into  $n!$  terms, there is at least one nonzero term, and all the nonzero terms have the same sign.

SNS was one of the earliest notions studied in QMA. (John Maybee and others in the 60's)

#### NOTE

Each nonzero term in  $\det A$  is a (composite) cycle of length  $n$  properly signed.

A composite cycle is a product of simple cycles.

A simple cycle in  $A$  corresponds to a (simple) cycle in  $D(A)$ , the directed graph of  $A$ .

EGS of SNS patterns (just consider the cycle structure):  
 any diagonal pattern with all nonzero diagonal entries;  
 any triangular pattern with all nonzero diagonal entries;

$$\begin{bmatrix} 0 & + & 0 \\ 0 & 0 & + \\ + & 0 & 0 \end{bmatrix} \quad (\text{any } + \text{ could be } - \text{ here})$$

$$\begin{bmatrix} 0 & + & 0 & - \\ + & 0 & + & 0 \\ 0 & + & 0 & + \\ - & 0 & + & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & - & 0 & + \\ 0 & 0 & - & 0 \end{bmatrix} \quad (\text{diagonal could be: } +, +, +, +; -, -, +, +)$$

$$\begin{bmatrix} + & + & + & 0 & 0 \\ - & + & + & 0 & 0 \\ 0 & - & + & + & 0 \\ 0 & + & - & + & + \\ 0 & - & + & - & + \end{bmatrix}$$

(“complementary zig-zag shape”, M. Fiedler LAA 2008)

Which sign patterns allow orthogonality? In other words, given an  $n \times n$  sign pattern  $A$ , is there a  $B$  in  $Q(A)$  such that  $BB^T = I$ ? This question was originally raised by M. Fiedler in Proceedings: Theory of Graphs and Its Applications, Publishing House of the Czechoslovakia Academy of Sciences, Prague, 1964. Because of this question, there has been much research related to this topic since that time. In particular, there is one whole chapter in the HLA.



A square sign pattern  $A$  is *potentially orthogonal* (PO) if  $A$  allows orthogonality.

A square sign pattern  $A$  that does not have a zero row or zero column is *sign potentially orthogonal* (SPO) if every pair of rows and every pair of columns allows orthogonality.

Every PO pattern is SPO.

For  $n \leq 4$ , every  $n \times n$  SPO pattern is PO.

There is a  $5 \times 5$  fully indecomposable SPO pattern that is not PO.

There is a  $6 \times 6$   $(+, -)$  pattern that is SPO but not PO.

$\begin{bmatrix} + & + & + \\ + & + & - \\ + & - & + \end{bmatrix}$  is both PO and SPO.

Recall that the *rank* of a real matrix is the maximum number of linearly independent rows (or columns).

For an  $m \times n$  sign pattern matrix  $A$ , the *minimum rank* of  $A$ , denoted  $\text{mr}(A)$ , is defined as

$$\text{mr}(A) = \min_{B \in Q(A)} \{\text{rank } B\},$$

while the *maximum rank* of  $A$ , denoted  $\text{MR}(A)$ , is defined as

$$\text{MR}(A) = \max_{B \in Q(A)} \{\text{rank } B\}.$$

The maximum rank of a sign pattern  $A$  is the same as the term rank of  $A$ , which is the maximum number of nonzero entries which lie in distinct rows and in distinct columns of  $A$ . However, determination of the minimum rank of a sign pattern matrix in general is a longstanding open problem in combinatorial matrix theory.

The all  $+$  pattern  $J$  has  $\text{mr } 1$ .

Any  $n \times n$  SNS pattern has  $\text{mr } n$ .

$$\begin{bmatrix} + & + & + \\ + & + & - \\ + & + & + \end{bmatrix} \text{ has mr } 2.$$

A *subpattern* of a sign pattern  $A$  is a sign pattern matrix obtained by replacing some (possibly none) of the  $+$  or  $-$  entries in  $A$  with  $0$ . The sign pattern  $I_n \in Q_n$  is the diagonal pattern of order  $n$  with  $+$  diagonal entries.

An  $m \times n$  sign pattern matrix  $A$  is said to be an *L-matrix* if every real matrix  $B \in Q(A)$  has linearly independent rows (so  $m \leq n$ ). It is known that  $A$  is an L-matrix iff for every nonzero diagonal pattern  $D$ ,  $DA$  has a unsigned column (that is, a nonzero column that is nonnegative or nonpositive). (see B/S book)

From P. Rao/B. Rao, On Generalized Inverses of Boolean Matrices, LAA(1975); also, book by Kim:

Let  $\mathcal{B}$  be the  $(0, 1)$  Boolean algebra ( $1 + 1 = 1$ ). A Boolean matrix (or vector) has entries (or components) in  $\mathcal{B}$ . Let  $\mathcal{B}^n$  be the set of all Boolean vectors with  $n$  components. For Boolean vectors  $x_1, x_2, \dots, x_k \in \mathcal{B}^n$ , the linear manifold  $\mathcal{M}(x_1, x_2, \dots, x_k)$  is the set of all vectors of the form  $\sum_{i=1}^k c_i x_i$ , where  $c_i \in \mathcal{B}$ . A set of Boolean vectors  $\{x_1, x_2, \dots, x_k\} \subseteq \mathcal{B}^n$  is said to be *dependent* if one vector in the set is the sum of some of the remaining vectors or the zero vector is in the set. Otherwise, the set is said to be *independent*.

$\{x_1, x_2, \dots, x_k\}$  dependent means:  
 $\{0\}$  is dependent.

For  $k \geq 2$ , some  $x_i$  is in the linear manifold spanned by the remaining vectors.

The vectors

1 1 1

1 1 0

0 1 1

are Boolean dependent but independent over the reals.

1 0 0

0 1 1

0 0 1

1 1 0

are Boolean independent but dependent over the reals.

1 0 0

0 1 0

are independent in both ways.

1 0 0

0 1 0

0 0 1

1 1 1

are dependent in both ways.

Let  $T = \{x_1, x_2, \dots, x_k\}$ , where  $x_i \in \mathcal{B}^n$ . A set  $S \subseteq T$  is said to be a *basis* of  $T$  if  $S$  is independent and  $T \subseteq \mathcal{M}(S)$ . It is known that every  $T \subseteq \mathcal{B}^n$ ,  $T \neq \{0\}$ , has a unique basis. The cardinality of the basis for  $T$  is called the *rank* of  $T$ .

Note:

1) In  $T \subseteq \mathcal{M}(S)$  we could have proper set inclusion, eg,  $S = T = \{(1, 0, 1), (0, 1, 0)\}$ ;  $\mathcal{M}(S)$  also includes  $(1, 1, 1)$ .

But, if  $T$  is a “subspace” of  $\mathcal{B}^n$ , then  $T = \mathcal{M}(S)$ .

2)  $\text{rank } T \leq k$  since  $S \subseteq T$ .

3)  $\text{rank } T \leq n$  may not hold, eg,

$\text{rank } T = \{(1, 0, 0), (0, 1, 1), (0, 0, 1), (1, 1, 0)\} = 4!$   
 ( $T$  is a basis of  $T$ )

Let  $A$  be a Boolean matrix. The *Boolean row (column) rank* of  $A$  is defined to be the rank of the set of row (column) vectors of  $A$ . Since a nonnegative sign pattern matrix (namely, a matrix whose entries are from the set  $\{+, 0\}$ ) may be viewed as a Boolean matrix (by identifying each  $+$  entry with 1), Boolean row (column) rank is now defined for a nonnegative sign pattern matrix. Note that for a nonnegative sign pattern matrix  $A$ , the Boolean row rank of  $A$  and the Boolean column rank of  $A$  may be different. When these are the same, this common value is called the *Boolean rank* of  $A$ .

**Example** Let

$$A = \begin{bmatrix} + & 0 & 0 & + \\ 0 & + & 0 & + \\ + & + & 0 & + \\ + & + & + & 0 \end{bmatrix} .$$

Rows 1, 2, and 4 form a basis for the set of rows of  $A$ . Boolean row rank of  $A$  is 3.

The 4 columns of  $A$  form a basis for the set of columns of  $A$ . Boolean column rank of  $A$  is 4.



Two new notions:

Let  $T = \{x_1, x_2, \dots, x_k\}$ , where  $x_i \in \mathcal{B}^n$ .  $T$  is said to be *weakly dependent* if there exist two disjoint subsets  $S_1$  and  $S_2$  of  $\{1, 2, \dots, k\}$ , not both empty (by convention, an empty sum is equal to 0), such that

$$\sum_{i \in S_1} x_i = \sum_{j \in S_2} x_j.$$

Otherwise,  $T$  is said to be *strongly independent*.

Note:

1) dependent  $\rightarrow$  weakly dependent, equivalently,  
strongly independent  $\rightarrow$  independent, (Boolean)

2)  $\{1, 0, 0), (0, 1, 1), (0, 0, 1), (1, 1, 0)\}$  is weakly dependent,  
but independent!

$(S_1 = \{1, 2\}, S_2 = \{3, 4\})$

It can be seen that for row vectors  $x_1, x_2, \dots, x_k$  in  $\mathcal{B}^n$ ,  $\{x_1, x_2, \dots, x_k\}$  is weakly dependent iff the matrix  $\begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix}$  is not an L-matrix. In other words,  $\{x_1, x_2, \dots, x_k\}$

is strongly independent iff the matrix  $\begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix}$  is an L-matrix.

Note that for  $k \leq 3$ ,  $\{x_1, \dots, x_k\}$  is independent iff  $\{x_1, \dots, x_k\}$  is strongly independent, iff the matrix  $\begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix}$  is an L-matrix.

## 2. BOOLEAN ROW (COLUMN) RANK AND MINIMUM RANK

**Observation 2.1** Let  $A$  be an  $m \times n$  nonnegative sign pattern matrix. Then

$$\text{mr}(A) \leq \min \{ \text{Bool col rank of } A, \text{ Bool row rank of } A \}.$$

This observation follows from the fact that a Boolean basis for the columns (rows) of  $A$  can serve as a spanning set for the columns (rows) of some real matrix  $B \in Q(A)$ .

For nonnegative sign patterns that have fewer than four rows (or fewer than four columns), we have a Boolean basis for the rows (cols) of  $A$  with fewer than four rows (cols). The rows (cols) in the basis then form an L-matrix. Hence, we have equality in the above inequality. However, equality does not hold in general, as can be seen from the following example.

**Example 2.2** Let

$$A = \begin{bmatrix} + & + & + & 0 \\ + & + & 0 & + \\ + & 0 & + & + \\ 0 & + & + & + \end{bmatrix}.$$

Then  $\text{mr}(A) = 3 < 4 = \text{Boolean rank of } A$ .

That  $A$  has Boolean column and row rank 4 should be clear. Now, the upper-right  $3 \times 3$  submatrix  $A_1$  of  $A$  is sign nonsingular, with  $\det(A_1) = -$ . So,  $\text{mr}(A) \geq 3$ . However,  $A$  is not sign nonsingular as the matrix

$$B = \begin{bmatrix} 1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \end{bmatrix}$$

in  $Q(A)$  is singular. Indeed, for  $B$ , row 1 + row 2 = row 3 + row 4. Hence,  $\text{mr}(A) = 3$ . Note that the rows of  $A$  are independent, but weakly dependent. ■

In the following theorem we determine exactly when we can have equality in Observation 2.1.

**Theorem 2.3** Let  $A$  be an  $m \times n$  nonnegative sign pattern matrix and let  $F$  be a submatrix of  $A$  whose rows form a Boolean basis for the rows of  $A$ . Then

$$\text{mr}(A) = \text{Boolean row rank of } A$$

if and only if  $F$  is an L-matrix.

We next discuss rank factorizations. In general, a nonnegative real matrix may not have a nonnegative full-rank factorization. For nonnegative sign pattern matrices, minimum rank factorizations are crucial and we make the following definition.

Let  $A$  be an  $m \times n$  nonnegative sign pattern matrix, with  $\text{mr}(A) = r$ . We say that  $A$  has a *nonnegative minimum rank factorization* if  $A = HK$  for some  $m \times r$  ( $r \times n$ ) nonnegative sign pattern matrices  $H$  ( $K$ ) where  $\text{mr}(A) = \text{mr}(H) = \text{mr}(K) = r$ .

If  $A$  has such a factorization, then since  $r = \text{mr}(K) \leq \text{Boolean row rank of } K \leq r$ ,  $\text{mr}(K) = \text{Boolean row rank of } K$ ; similarly,  $\text{mr}(H) = \text{Boolean column rank of } H$ . Further,  $H$  ( $K$ ) has strongly independent columns (rows). However, nonnegative minimum rank factorization is not always possible.

**Example 2.4** As in Example 2.2, let

$$A = \begin{bmatrix} + & + & + & 0 \\ + & + & 0 & + \\ + & 0 & + & + \\ 0 & + & + & + \end{bmatrix}.$$

It can be shown by discussing various cases that the columns of  $A$  cannot be generated (as Boolean combinations) by any three nonnegative vectors. Therefore,  $A$  does not have a nonnegative minimum rank factorization.

■

Let  $A$  be an  $m \times n$  nonnegative sign pattern matrix. Then the *Schein rank* of  $A$  is the smallest positive integer  $k$  such that  $A = HK$  for some  $m \times k$  ( $k \times n$ ) nonnegative sign pattern matrices  $H$  ( $K$ ). (possible applics in biology, etc)

The following are easily established:

$$\text{mr}(A) \leq \text{Schein rank}(A)$$

$$\leq \min \{ \text{Bool col rank of } A, \text{ Bool row rank of } A \}.$$

$A$  has a nonnegative minimum rank factorization if and only if

$$\text{mr}(A) = \text{Schein rank}(A).$$

When does  $A$  have a nonnegative minimum rank factorization??

Also recall: (Theorem 2.3) Let  $F$  be a submatrix of  $A$  whose rows form a Boolean basis for the rows of  $A$ . Then

$$\text{mr}(A) = \text{Boolean row rank of } A$$

if and only if  $F$  is an L-matrix. (this tells us when we have equalities in both of the above inequalities)



Even when a nonnegative sign pattern matrix  $A$  has a nonnegative minimum rank factorization,  $\text{mr}(A)$  may not be equal to the Boolean row (column) rank of  $A$ .

**Example 2.5** Let

$$A = \begin{bmatrix} + & + & 0 & 0 \\ + & + & 0 & + \\ 0 & 0 & + & + \\ 0 & + & + & + \end{bmatrix} = \begin{bmatrix} + & 0 & 0 \\ + & + & 0 \\ 0 & 0 & + \\ 0 & + & + \end{bmatrix} \begin{bmatrix} + & + & 0 & 0 \\ 0 & + & 0 & + \\ 0 & 0 & + & + \end{bmatrix} = HK.$$

Clearly,  $A$  is not sign nonsingular. Since the upper-right  $3 \times 3$  submatrix of  $A$  is sign nonsingular,  $\text{mr}(A) = 3$ , and  $HK$  is a nonnegative minimum rank factorization of  $A$ . However, both the Boolean row and column ranks of  $A$  are 4. ■

It is worth mentioning that if  $m_r(A)$  ( $m_c(A)$ ) denotes the maximum number of strongly independent rows (columns) of a nonnegative sign pattern  $A$ , then clearly we have

**Proposition 2.6** For every nonnegative sign pattern  $A$ ,  $\max\{m_r(A), m_c(A)\} \leq \text{mr}(A)$ .

Hence,  
 $\max\{m_r(A), m_c(A)\} \leq \text{mr}(A) \leq \text{Schein rank}(A)$   
 $\leq \min \{\text{Bool col rank } A, \text{ Bool row rank } A\}.$

Strict inequality in Prop 2.6 is possible, as the following example shows.

**Example 2.7** Let  $G$  be the  $5 \times 10$  sign pattern corresponding to the matrix  $\Gamma_2$  as defined on page 20 of the Brualdi/Shader book. That is,

$$G = \begin{bmatrix} 0 & 0 & 0 & 0 & + & + & + & + & + & + \\ 0 & + & + & + & 0 & 0 & 0 & + & + & + \\ + & 0 & + & + & 0 & + & + & 0 & 0 & + \\ + & + & 0 & + & + & 0 & + & 0 & + & 0 \\ + & + & + & 0 & + & + & 0 & + & 0 & 0 \end{bmatrix}$$

which is the  $5 \times 10$  nonnegative sign pattern consisting of all possible columns with exactly three positive entries in each column. Now,  $G$  is a barely L-matrix, that is to say,  $G$  is an L-matrix and if one or more columns are deleted from  $G$ , then the resulting matrix is not an L-matrix. The fact that  $G$  is an L-matrix means that the 5 rows of  $G$  are strongly independent, so that  $m_r(G) = 5$ . However,  $G$  does not have 5 strongly independent columns. In fact, if  $G$  had 5 strongly independent columns, then such 5 columns would form a  $5 \times 5$  sign nonsingular matrix, and thus we obtain a  $5 \times 5$  submatrix of  $G$  that is an L-matrix, contradicting the fact that  $G$  is a barely L-matrix. On the other hand, the columns  $c_1, c_2, c_3, c_5$  of  $G$  can be seen to be strongly independent. Thus we have  $m_c(G) = 4 < m_r(G) = 5$ . Furthermore, for  $A = \begin{bmatrix} 0 & G \\ G^T & 0 \end{bmatrix}$ , we have  $mr(A) = 2 \, mr(G) = 10$ , while  $m_r(A) = m_c(A) = 9$ . Thus  $\max\{m_r(A), m_c(A)\} < mr(A)$ . ■

In the above example,

$$\max\{m_r(A), m_c(A)\} = 9$$

$$\text{mr}(A) = 10 = \text{Schein rank}(A)$$

$$\min \{\text{Bool col rank of } A, \text{ Bool row rank of } A\} = 15.$$

The reason  $\text{mr}(A) = \text{Schein rank}(A)$  is that  $A$  has the nonnegative minimum rank factorization

$$A = \begin{bmatrix} 0 & G \\ G^T & 0 \end{bmatrix} = \begin{bmatrix} I_5 & 0 \\ 0 & G^T \end{bmatrix} \begin{bmatrix} 0 & G \\ I_5 & 0 \end{bmatrix}.$$

What is an example where

$$\max\{m_r(A), m_c(A)\} < \text{mr}(A) < \text{Schein rank}(A)$$

$$< \min \{\text{Bool col rank of } A, \text{ Bool row rank of } A\}?$$

## Related ideas

Recall: Let  $A$  be an  $m \times n$  Boolean (nonnegative sign pattern) matrix. Then the *Schein rank* of  $A$  is the minimum  $k$  such that  $A = HK$  for some  $m \times k$  ( $k \times n$ ) Boolean (nonnegative sign pattern) matrices  $H$  ( $K$ ).

More generally: In “Real Rank Versus Nonnegative Rank”, LAA, 2009, L. Beasley and T. Laffey define the *factor rank* for matrices over a general semiring  $\mathcal{S}$ .

The factor rank of an  $m \times n$  matrix  $A$  over  $\mathcal{S}$  is the minimum  $k$  such that  $A = HK$  for some  $m \times k$  ( $k \times n$ ) matrices  $H$  ( $K$ ) over  $\mathcal{S}$ .

When  $\mathcal{S}$  is the set of real numbers, the factor rank is the same as the usual rank ( $A$ ). (in full-rank factorization)

For some applications, in actual practice, we try to get as “close as possible” to  $A$  with  $HK$ .

For Boolean matrices, determining the Schein rank is NP-hard (at least as hard as NP-complete) - G. Markowsky, 1992.

In the study of human olfactory perception (relating to the sense of smell), we are using the receptor-ligand recognition model. Our algorithm stops when a required percentage of 1's in  $A$  are covered.

In nonnegative data analysis, the *nonnegative matrix factorization* (NNMF) problem, probably due to Paatero and Tapper, 1994:

Given a real  $m \times n$  nonnegative matrix  $Y$  and a positive integer  $p < \min\{m, n\}$ , find nonnegative  $m \times p$  ( $p \times n$ ) matrices  $U$  ( $V$ ) so as to minimize the functional

$$\|Y - UV\|_F,$$

where  $F$  denotes the Frobenius norm. An appropriate decision on the value of  $p$  is critical in practice, but the choice of  $p$  is very often problem dependent.

(papers by R. Plemmons, etc)

We remark that in the characterizations in the next two sections the sign patterns  $A$  have a nonnegative minimum rank factorization and also  $\text{mr}(A) = \text{Boolean rank of } A$ .

### 3. IDEMPOTENTS

Clearly, if a square nonnegative pattern  $A$  allows a real idempotent, that is, there is an idempotent matrix  $B \in Q(A)$ , then  $A$  is idempotent. The converse does not hold. For example, the pattern  $A = \begin{bmatrix} + & + \\ 0 & + \end{bmatrix}$  is idempotent, but does not allow a real idempotent. The following result from Eschenbach/Hall/Li, Sign pattern Matrices and Generalized Matrices, LAA(1994) determines when a nonnegative pattern allows a real idempotent.

**Proposition 3.1** Let  $A$  be a square nonnegative sign pattern matrix, with  $\text{mr}(A) = r$ . Then  $A$  allows a real idempotent if and only if  $A$  is permutationally similar to a pattern of the form

$$\begin{bmatrix} I_r & A_2 \\ A_3 & A_3 A_2 \end{bmatrix}$$

where  $A_2 A_3$  is a subpattern of  $I_r$  (that is,  $A_2 A_3$  is a diagonal pattern).

**Theorem 3.2** Let  $A$  be a square nonnegative sign pattern matrix, with  $\text{mr}(A) = r$ . Then  $A$  is idempotent if and only if  $A$  is permutationally similar to a sign pattern of the form

$$\begin{bmatrix} A_1 & A_1A_2 \\ A_3A_1 & A_3A_1A_2 \end{bmatrix}$$

where  $A_1$  is  $r \times r$  sign nonsingular and idempotent, and  $A_2A_3$  is a subpattern of  $A_1$ .

We note that when  $A$  is idempotent as in the above theorem that (a permutational similarity of)

$$\begin{bmatrix} A_1 \\ A_3 \end{bmatrix} \begin{bmatrix} A_1 & A_1A_2 \end{bmatrix}$$

is a nonnegative minimum rank factorization of  $A$ .

**Theorem 3.3** Let  $A$  be a nonnegative idempotent sign pattern matrix. Then

$$\text{mr}(A) = \text{Boolean rank of } A.$$

Proof: Let  $\text{mr}(A) = r$ . By Theorem 3.2,  $A$  is permutationally similar to a sign pattern of the form

$$\begin{bmatrix} A_1 & A_1A_2 \\ A_3A_1 & A_3A_1A_2 \end{bmatrix}$$

where  $A_1$  is  $r \times r$  sign nonsingular and idempotent. Hence,

$$\text{mr}(A) = r = \text{mr}(A_1) \leq \text{Boolean rank of } A_1.$$



So, Boolean rank of  $A_1 = r$ . Since Boolean rank of  $A =$  Boolean rank of  $A_1$ , the result follows. ■

**Corollary 3.4** If  $A$  is an  $n \times n$  nonnegative idempotent sign pattern matrix with Boolean rank  $n$ , then  $\text{mr}(A) = n$ , that is,  $A$  is sign nonsingular.

For symmetric patterns, the blocks in the strictly upper triangular part of the Frobenius normal form are zero. The proof of the following theorem is parallel to the proof of Theorem 3.2.

**Theorem 3.5** Let  $A$  be a symmetric nonnegative sign pattern matrix, with  $\text{mr}(A) = r$ . Then  $A$  is idempotent if and only if  $A$  is permutationally similar to a pattern of the form

$$\begin{bmatrix} I_r & A_2 \\ A_2^T & A_2^T A_2 \end{bmatrix}$$

where  $A_2 A_2^T$  is a subpattern of  $I_r$ .

Proposition 3.1 and Theorem 3.5 immediately yield the following.

**Theorem 3.6** Let  $A$  be a symmetric nonnegative sign pattern matrix. Then  $A$  is idempotent if and only if  $A$  allows a real idempotent.

## 4. PATTERNS THAT ALLOW NONNEGATIVE GENERALIZED INVERSES

Let  $B, X$  be real (or Boolean) matrices. Consider the following conditions.

- (1)  $BXB = B$ .
- (2)  $XBX = X$ .
- (3)  $BX$  is symmetric.
- (4)  $XB$  is symmetric.

If (1) holds,  $X$  is called a (1)-inverse of  $B$ ; if (1) and (2) hold,  $X$  is called a (1, 2)-inverse of  $B$ ; if (1) and (3) hold,  $X$  is called a (1, 3)-inverse of  $B$ ; if (1) and (4) hold,  $X$  is called a (1, 4)-inverse of  $B$ ; if (1)–(4) hold, then  $X$  is unique and is called the Moore-Penrose inverse of  $B$ .

**Lemma 4.1** Let  $A$  be an  $m \times n$  nonnegative sign pattern matrix. If  $A$  allows a nonnegative (1)-inverse, then  $A$  has a nonnegative (1)-inverse.

The same result holds for other generalized inverses such as (1, 3)- and Moore-Penrose inverse. For the proof just replace positive entries by  $+$ .

The converse of Lemma 4.1 does not hold. For example, the pattern  $A = \begin{bmatrix} + & + \\ 0 & + \end{bmatrix}$  is in fact a (1, 2)-inverse of itself, but  $A$  does not allow a nonnegative (1)-inverse. This pattern  $A$  is sign nonsingular with inverse pattern  $\begin{bmatrix} + & - \\ 0 & + \end{bmatrix}$ .

Using two results from Rao/Rao:

**Theorem 4.2** Let  $A$  be an  $m \times n$  nonnegative sign pattern matrix, with  $\text{mr}(A) = r$ . Then the following are equivalent:

- (i)  $A$  has a nonnegative (1)-inverse.
- (ii)  $A$  is permutationally equivalent to a sign pattern of the form

$$\begin{bmatrix} A_1 & A_1A_2 \\ A_3A_1 & A_3A_1A_2 \end{bmatrix}$$

where  $A_1$  is  $r \times r$  sign nonsingular and idempotent.

- (iii)  $A = HK, H = AN, K = SA$  for some  $m \times k, k \times n, n \times k, k \times m$  nonnegative patterns  $H, K, N, S$ , respectively.

**Corollary 4.3** Let  $A$  be an  $m \times n$  nonnegative sign pattern matrix that has a nonnegative (1)-inverse. Then  $A$  has a nonnegative minimum rank factorization and also  $\text{mr}(A) = \text{Boolean rank of } A$ .

It follows from Lemma 4.1 and Corollary 4.3 that if a nonnegative pattern allows a nonnegative (1)-inverse (in particular say a (1, 3)-inverse), then  $A$  has a nonnegative minimum rank factorization and also  $\text{mr}(A) = \text{Boolean rank of } A$ .

We recall that a real nonnegative matrix  $B$  is said to be *monomial* if and only if  $B$  has exactly one nonzero entry in each row and each column, that is,  $B$  can be expressed as a product of a nonsingular diagonal matrix and a permutation matrix. It is well-known that an  $m \times n$  rank  $r$ , real, nonnegative matrix  $B$  has a nonnegative (1)-inverse if and only if  $B$  has a monomial submatrix of order  $r$  (see Th 4 in Berman/Plemmons, *Inverses of Nonnegative Matrices*, LAMA(1974)).

**Theorem 4.4** Let  $A$  be an  $m \times n$  nonnegative sign pattern matrix, with  $\text{mr}(A) = r$ . Then the following are equivalent:

- (i)  $A$  is permutationally equivalent to a sign pattern of the form

$$\begin{bmatrix} I_r & A_2 \\ A_3 & A_3 A_2 \end{bmatrix}.$$

- (ii)  $A$  allows a nonnegative (1)-inverse.
- (iii)  $A$  allows a nonnegative (1, 2)-inverse.
- (iv)  $A = HK$  where  $H$  ( $K$ ) is an  $m \times r$  ( $r \times n$ ) nonnegative pattern and both  $H$  and  $K$  contain some row-permutation of  $I_r$  as a submatrix.

From the proof of Theorem 4.4, the only matrices  $B \in Q(A)$  that can have a nonnegative (1)-inverse are of rank equal to  $\text{mr}(A)$ . In fact, if  $A$  allows a nonnegative (1)-inverse, then all matrices  $B \in Q(A)$  of rank equal to  $\text{mr}(A)$  have a nonnegative (1)-inverse. Indeed, such a  $B$  is permutationally equivalent to a matrix of the form

$$\begin{bmatrix} D_r & C \\ D & DD_r^{-1}C \end{bmatrix},$$

where  $D_r$  is a diagonal matrix with positive diagonal entries, and so  $B$  has a monomial submatrix of order  $r$ . Furthermore, since these matrices  $B$  have a nonnegative (1)-inverse, they then have a nonnegative full rank factorization.

We will now show that if a nonnegative sign pattern  $A$  has a nonnegative (1, 4)-inverse, then  $A$  allows a nonnegative (1, 4)-inverse. The same is true for (1, 3)-inverse and the Moore-Penrose inverse. As was seen earlier, this is not the case in general for (1)-inverse.

**Theorem 4.5** Let  $A$  be an  $m \times n$  nonnegative sign pattern matrix, with  $\text{mr}(A) = r$ . Then TFAE:

(i)  $A$  has a nonnegative  $(1, 4)$ -inverse.

(ii)  $A$  is permutat equiv to a pattern of the form  $\begin{bmatrix} I_r & A_2 \\ A_3 & A_3 A_2 \end{bmatrix}$   
where  $A_2 A_2^T$  is a subpattern of  $I_r$ .

(iii)  $A$  is permutat equiv to a pattern of the form  $\begin{bmatrix} F \\ GF \end{bmatrix}$   
where  $F$  is  $r \times n$  and has orthogonal rows, and  $G$  is nonnegative.

(iii)'  $A$  is permutat equiv to a pattern of the form  $\begin{bmatrix} J & 0 \\ GJ & 0 \end{bmatrix}$ ,  
where  $G$  is nonnegative and  $J = \begin{bmatrix} J_1 & & 0 \\ & \cdots & \\ 0 & & J_r \end{bmatrix}$ , with  
each  $J_i$  an all + row.

(iv)  $A$  allows a nonnegative  $(1, 4)$ -inverse.

(v)  $A$  allows a nonnegative  $(1, 2, 4)$ -inverse.

(vi)  $A = HK$ , where  $H(K)$  is an  $m \times r$  ( $r \times n$ ) nonnegative pattern,  $H$  contains some row-permutation of  $I_r$  as a submatrix,  $\text{mr}(K) = r$ , and the rows of  $K$  are orthogonal.

(i)  $\Leftrightarrow$  (ii) follows from Th 4.1 in Rao/Rao. The proofs of (iii)  $\Rightarrow$  (iv)  $\Rightarrow$  (v) follow from Berman/Plemmons, but we have given self-contained proofs. Th 4.6 in our paper is a parallel theorem for  $(1, 3)$ -inverses.

Finally, we consider the Moore-Penrose inverse.

**Theorem 4.7** Let  $A$  be an  $m \times n$  nonnegative sign pattern matrix, with  $\text{mr}(A) = r$ . Then TFAE:

- (i)  $A$  has a nonnegative Moore-Penrose inverse.
- (ii)  $A$  is permutat equiv to a pattern of the form  $\begin{bmatrix} I_r & A_2 \\ A_3 & A_3 A_2 \end{bmatrix}$  where  $A_2 A_2^T$  and  $A_3^T A_3$  are subpatterns of  $I_r$ .
- (iii)  $A$  is permutat equiv to a pattern of the form  $\begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix}$ , where  $J = \begin{bmatrix} J_1 & & 0 \\ & \cdots & \\ 0 & & J_r \end{bmatrix}$ , with each  $J_i$  an all + (not necessarily square) block.
- (iv)  $A$  allows a nonnegative Moore-Penrose inverse (in  $Q(A^T)$ ).
- (v)  $A = HK$ , where  $H$  ( $K$ ) is an  $m \times r$  ( $r \times n$ ) nonnegative pattern,  $\text{mr}(H) = \text{mr}(K) = r$ , and the columns (rows) of  $H$  ( $K$ ) are orthogonal.



(i)  $\Leftrightarrow$  (ii) follows from Theorem 4.3 in Rao/Rao. Using a result in Plemmons/Cline, The Generalized Inverse of a Nonnegative Matrix, AMS Proc (1972), one can show that (iii)  $\Leftrightarrow$  (iv). When  $A$  allows a nonnegative Moore-Penrose inverse, then all matrices  $B \in Q(A)$  of rank equal to  $r = \text{mr}(A)$  have a nonnegative Moore-Penrose inverse. This follows since such a matrix  $B$  is of the form in Plemmons/Cline.

## 5. NONNEGATIVE PATTERNS THAT ALLOW POSITIVE GENERALIZED INVERSES

**Proposition 5.1** Let  $A$  be an  $m \times n$  nonnegative sign pattern matrix. Then the following are equivalent:

- (i)  $A$  allows a positive (1)-inverse.
- (ii)  $A$  has a positive (1)-inverse.
- (iii)  $A$  is permutationally equivalent to a pattern of the form

$$\begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix},$$

where  $J$  is an all + (possibly empty) pattern.

In contrast, we have the following result on (2)-inverses.

**Proposition 5.2** Let  $A$  be an  $m \times n$  sign pattern matrix. Then the following are equivalent:

- (i)  $A$  allows a positive (2)-inverse.
- (ii)  $A$  allows a nontrivial nonnegative (2)-inverse.
- (iii)  $A$  has at least one  $+$  entry.

**Corollary 5.3** Let  $A$  be an  $m \times n$  nonnegative sign pattern matrix. Then the following are equivalent:

- (i)  $A$  allows a positive (2)-inverse.
- (ii)  $A$  allows a nontrivial nonnegative (2)-inverse.
- (iii)  $A$  has a positive (2)-inverse.
- (iv)  $A$  has a nontrivial nonnegative (2)-inverse.
- (v)  $A$  has at least one  $+$  entry.

We now consider nonnegative sign patterns that allow positive Moore-Penrose Inverses.

**Proposition 5.4** Let  $A$  be an  $m \times n$  nonnegative sign pattern matrix. Then the following are equivalent:

- (i)  $A$  allows a positive Moore-Penrose inverse.
- (ii)  $A$  has a positive Moore-Penrose inverse.
- (iii)  $A$  is the all  $+$  pattern.