Compressible fluids in time-dependent domains: Existence by a Brinkman-type penalization

Eduard Feireisl Jiří Neustupa Jan Stebel

Institute of Mathematics, Academy of Sciences of the Czech Republic

Workshop Heidelberg-Prague, 27th February 2010

Equations of motion

Navier-Stokes system for compressible fluid

$$\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0,$$
$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\varrho) = \operatorname{div} \mathbb{S} + \varrho \mathbf{f},$$

Equations of motion

Navier-Stokes system for compressible fluid

$$\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0,$$
$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\varrho) = \operatorname{div} \mathbb{S} + \varrho \mathbf{f},$$

Newton's rheological law ($\mu > 0$, $\eta \geq 0$)

$$\mathbb{S} = \mu \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^\top - \frac{2}{3} \operatorname{div} \mathbf{u} \mathbb{I} \right) + \eta \operatorname{div} \mathbf{u} \mathbb{I},$$

Equations of motion

Navier-Stokes system for compressible fluid

$$\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0,$$
$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\varrho) = \operatorname{div} \mathbb{S} + \varrho \mathbf{f},$$

Newton's rheological law ($\mu > 0$, $\eta \geq 0$)

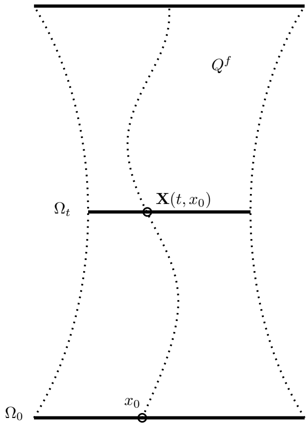
$$\mathbb{S} = \mu \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^\top - \frac{2}{3} \operatorname{div} \mathbf{u} \mathbb{I} \right) + \eta \operatorname{div} \mathbf{u} \mathbb{I},$$

Isentropic equation of state ($a > 0$, $\gamma > 1$)

$$p(\varrho) = a \varrho^\gamma.$$

Time-dependent domain

Let $\Omega_0 \subset \mathbb{R}^3$ be a given domain occupied by the fluid at $t = 0$ and a vector field $\mathbf{v}_s$ defined in $[0, T] \times \mathbb{R}^3$.



Time-dependent domain

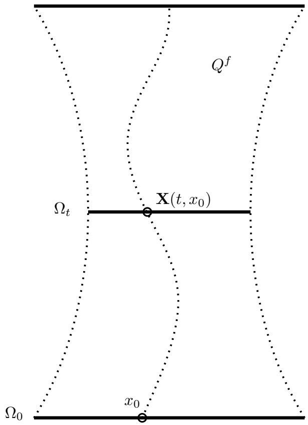
Let $\Omega_0 \subset \mathbb{R}^3$ be a given domain occupied by the fluid at $t = 0$ and a vector field $\mathbf{v}_s$ defined in $[0, T] \times \mathbb{R}^3$.

For $t > 0$ we define

$$\Omega_t := \{\mathbf{X}(t, \mathbf{x}_0);\ \mathbf{x}_0 \in \Omega_0\}$$

where $\mathbf{X}$ solves the problem

$$\partial_t \mathbf{X}(t, \mathbf{x}_0) = \mathbf{v}_s(t, \mathbf{X}(t, \mathbf{x}_0)),$$
$$\mathbf{X}(0, \mathbf{x}_0) = \mathbf{x}_0.$$



Time-dependent domain

Let $\Omega_0 \subset \mathbb{R}^3$ be a given domain occupied by the fluid at $t = 0$ and a vector field $\mathbf{v}_s$ defined in $[0, T] \times \mathbb{R}^3$.

For $t > 0$ we define

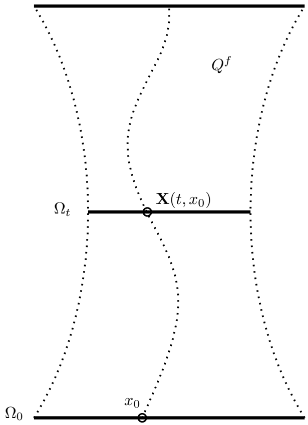
$$\Omega_t := \{\mathbf{X}(t, \mathbf{x}_0);\ \mathbf{x}_0 \in \Omega_0\}$$

where $\mathbf{X}$ solves the problem

$$\partial_t \mathbf{X}(t, \mathbf{x}_0) = \mathbf{v}_s(t, \mathbf{X}(t, \mathbf{x}_0)),$$
$$\mathbf{X}(0, \mathbf{x}_0) = \mathbf{x}_0.$$

The space-time domain occupied by the fluid is denoted

$$Q^f := \{(t, \mathbf{x});\ t \in (0, T);\ \mathbf{x} \in \Omega_t\}.$$



Boundary and initial conditions

On the lateral boundary of Q^f we assume no-slip:

$$\mathbf{u}(t,\cdot) = \mathbf{v}_s(t,\cdot) \text{ on } \partial\Omega_t.$$

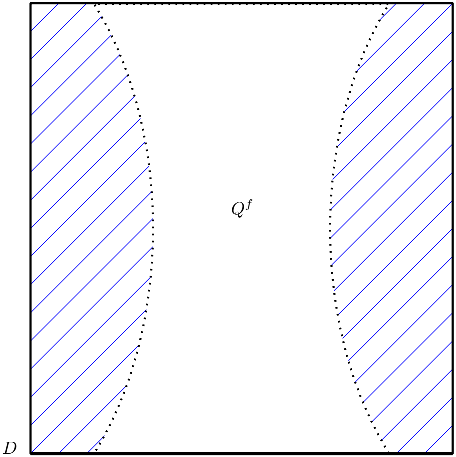
Initial conditions:

$$\varrho(0,\cdot) = \varrho_0,$$
$$(\varrho\mathbf{u})(0,\cdot) = (\varrho\mathbf{u})_0.$$

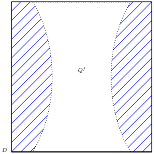
Penalized problem

We fix a reference domain D containing Ω_0 and such that

$$\mathbf{v}_{s|\partial D} = 0.$$



Penalized problem



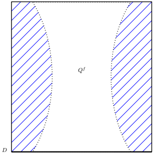
We fix a reference domain D containing Ω_0 and such that

$$\mathbf{v}_{s|\partial D} = 0.$$

Penalized problem

We fix a reference domain D containing Ω_0 and such that

$$\mathbf{v}_{s|\partial D} = 0.$$



The original system is replaced by a penalized problem (P_ε)

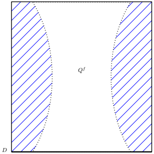
$$\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0,$$

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\varrho) = \operatorname{div} \mathbb{S} + \varrho \mathbf{f} - \frac{1}{\varepsilon}\chi(\mathbf{u} - \mathbf{v}_s),$$

considered in $(0, T) \times D$ where

$$\chi(t, \mathbf{x}) = \begin{cases} 0 & \text{if } (t, \mathbf{x}) \in Q^f, \\ 1 & \text{if } (t, \mathbf{x}) \in Q^s := ((0, T) \times D) \setminus Q^f. \end{cases}$$

Penalized problem



We fix a reference domain D containing Ω_0 and such that

$$\mathbf{v}_{s|\partial D} = 0.$$

The original system is replaced by a penalized problem (P_ε)

$$\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0,$$

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\varrho) = \operatorname{div} \mathbb{S} + \varrho \mathbf{f} - \frac{1}{\varepsilon}\chi(\mathbf{u} - \mathbf{v}_s),$$

considered in $(0, T) \times D$ where

$$\chi(t, \mathbf{x}) = \begin{cases} 0 & \text{if } (t, \mathbf{x}) \in Q^f, \\ 1 & \text{if } (t, \mathbf{x}) \in Q^s := ((0, T) \times D) \setminus Q^f. \end{cases}$$

Boundary and initial conditions

$$\mathbf{u}_{|\partial D} = 0, \qquad \varrho(0, \cdot) = \varrho_{0,\varepsilon}, \qquad (\varrho \mathbf{u})(0, \cdot) = (\varrho \mathbf{u})_{0,\varepsilon}.$$

Remarks

- The aim is to prove for $\varepsilon \to 0$ convergence to solutions of NS in a general time-dependent domain.
- Physical motivation: In the penalized problem, Q^s represents a porous media with permeability ε (Brinkman's method). It has been used for incompressible fluids e.g. by Angot et al. (1999).
- Applications: Easy numerical implementation of complicated geometries, possible extension to fluid-structure interaction problems.
- Existence of solutions to penalized problem is covered by the results of Lions (1998) and Feireisl (2001).

Finite energy weak solutions

We say that $(\varrho, \mathbf{u})$ is a finite energy weak solution to (P_ε) if

- $\varrho \in L^\infty(0,T; L^\gamma(D))$, $\mathbf{u} \in L^2(0,T; W_0^{1,2}(D,\mathbb{R}^3))$,
- for any $\varphi \in \mathcal{C}_c^\infty([0,T) \times \overline{D})$:

$$\int_0^T \int_D \Big(b(\varrho)\partial_t \varphi + b(\varrho)\mathbf{u} \cdot \nabla \varphi + (b(\varrho) - b'(\varrho)\varrho) \mathrm{div}\, \mathbf{u}\, \varphi \Big) = - \int_D b(\varrho_{0,\varepsilon}) \varphi(0,\cdot)$$

- for any $\boldsymbol{\varphi} \in \mathcal{C}_c^\infty([0,T) \times D; \mathbb{R}^3)$:

$$\int_0^T \int_D \Big(\varrho \mathbf{u} \cdot \partial_t \boldsymbol{\varphi} + \varrho(\mathbf{u} \otimes \mathbf{u}) : \nabla \boldsymbol{\varphi} + p \,\mathrm{div}\, \boldsymbol{\varphi} \Big)$$
$$= \int_0^T \int_D \Big(\mathbb{S} : \nabla \boldsymbol{\varphi} - \varrho \mathbf{f} \cdot \boldsymbol{\varphi} + \frac{\chi}{\varepsilon}(\mathbf{u} - \mathbf{v}_s) \cdot \boldsymbol{\varphi} \Big) - \int_D (\varrho \mathbf{u})_{0,\varepsilon} \cdot \boldsymbol{\varphi}(0,\cdot)$$

Finite energy weak solutions II.

- the energy inequality holds for a.a. $\tau \in (0, T)$:

$$\int_D \Big(\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)\Big)(\tau,\cdot) + \int_0^\tau \int_D \mathbb{S} : \nabla \mathbf{u}$$
$$\leq \int_0^\tau \int_D \Big(\varrho \mathbf{f} \cdot \mathbf{u} - \frac{\chi}{\varepsilon}(\mathbf{u} - \mathbf{v}_s) \cdot \mathbf{u}\Big)$$
$$+ \int_D \Big(\frac{1}{2\varrho_{0,\varepsilon}}|(\varrho\mathbf{u})_{0,\varepsilon}|^2 + P(\varrho_{0,\varepsilon})\Big)$$

Modified energy inequality

If $\mathbf{v}_s$ is sufficiently regular then one can test by $\psi(t)\mathbf{v}_s$, where $\psi \in \mathcal{C}_c^\infty[0,T)$. The resulting expression gives rise to the following modified energy inequality:

$$\int_D \Big(\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)\Big)(\tau,\cdot) + \int_0^\tau\int_D \mathbb{S}:\nabla\mathbf{u} + \int_0^\tau\int_D \frac{\chi}{\varepsilon}|\mathbf{u}-\mathbf{v}_s|^2$$
$$\leq \int_D \Big(\frac{1}{2\varrho_{0,\varepsilon}}|(\varrho\mathbf{u})_{0,\varepsilon}|^2 + P(\varrho_{0,\varepsilon}) + (\varrho\mathbf{u}\cdot\mathbf{v}_s)(\tau,\cdot) - (\varrho\mathbf{u})_{0,\varepsilon}\cdot\mathbf{v}_s(0,\cdot)\Big)$$
$$+\int_0^\tau\int_D \Big(\varrho\mathbf{f}\cdot(\mathbf{u}-\mathbf{v}_s)+\mathbb{S}:\nabla\mathbf{v}_s-\varrho\mathbf{u}\cdot\partial_t\mathbf{v}_s-\varrho(\mathbf{u}\otimes\mathbf{u}):\nabla\mathbf{v}_s-p\,\mathrm{div}\,\mathbf{v}_s\Big)$$

Main result

Assumptions:

(A1) $\Omega_0 \subset \overline{\Omega}_0 \subset D$ are bounded domains of class $C^{2+\nu}$;

(A2) $\gamma > 3/2$, $\mathbf{f} \in L^\infty((0, T) \times D; \mathbb{R}^3)$;

(A3) $\mathbf{v}_s \in \mathcal{C}^{2+\nu}([0, T] \times \overline{D}; \mathbb{R}^3)$, $\mathbf{v}_s|_{\partial D} = 0$;

(A4) the initial data satisfy

$$\varrho_{0,\varepsilon} \to \varrho_0 \text{ in } L^\gamma(D),\ \varrho_0|_{\Omega_0} \geq 0,\ \varrho_0|_{D \setminus \Omega_0} = 0,$$
$$(\varrho \mathbf{u})_{0,\varepsilon} \to (\varrho \mathbf{u})_0 \text{ in } L^1(D; \mathbb{R}^3),\ (\varrho \mathbf{u})_0|_{D \setminus \Omega_0} = 0,$$
$$\int_D \frac{|(\varrho \mathbf{u})_{0,\varepsilon}|^2}{\varrho_{0,\varepsilon}} < c,$$

where c is independent of $\varepsilon \to 0$.

Main result II.

Theorem

Let the assumptions (A1)–(A4) be satisfied.
Then any sequence $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon\}_{\varepsilon>0}$ *of finite energy weak solutions of problem* (P_ε) *contains a subsequence such that*

$$\varrho_\varepsilon \to \varrho \text{ in } C_{\rm weak}([0,T]; L^\gamma(D)) \cap L^\gamma(Q^f),$$
$$\mathbf{u}_\varepsilon \to \mathbf{u} \text{ in } L^2(0,T; W_0^{1,2}(D;\mathbb{R}^3)),\ \mathbf{u} = \mathbf{v}_s \text{ in } Q^s,$$

where the limit functions ϱ, $\mathbf{u}$ *are distributional solutions of the equation of continuity in* $(0,T)\times D$ *and of the original momentum equation in* Q^f.

Idea of the proof

- Uniform bounds following from the energy inequality
- Estimates of the pressure in Q^f:
  - local
  - up to the boundary
- Strong convergence of ϱ_ε in Q^s
- Renormalized continuity equation:
  - Weak continuity of effective viscous flux
  - Bounds on oscillations defect measure
- Strong convergence of ϱ_ε in Q^f

Uniform bounds

From the energy inequality we obtain:

$$\{\varrho_\varepsilon\}_{\varepsilon>0} \text{ bounded in } L^\infty(0,T;L^\gamma(D));$$

$$\{\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon\}_{\varepsilon>0} \text{ bounded in } L^\infty(0,T;L^2(D,\mathbb{R}^3));$$

$$\left\{\nabla\mathbf{u}_\varepsilon + (\nabla\mathbf{u}_\varepsilon)^\top - \frac{2}{3}\mathrm{div}\,\mathbf{u}\mathbb{I}\right\}_{\varepsilon>0} \text{ bounded in } L^2(0,T;L^2(D,\mathbb{R}^{3\times 3})).$$

Korn's inequality yields:

$$\{\mathbf{u}_\varepsilon\}_{\varepsilon>0} \text{ bounded in } L^2(0,T;W_0^{1,2}(D,\mathbb{R}^3))$$

and finally

$$\left\{\frac{\mathbf{u}_\varepsilon - \mathbf{v}_s}{\sqrt{\varepsilon}}\right\}_{\varepsilon>0} \text{ bounded in } L^2(Q^s).$$

Uniform bounds II.

From the uniform bounds we infer that

$$\begin{aligned}
\varrho_\varepsilon &\to \varrho && \text{in } \mathcal{C}_{\rm weak}([0,T]; L^\gamma(D)),\\
\mathbf{u}_\varepsilon &\to \mathbf{u} && \text{weakly in } L^2(0,T; W^{1,2}_0(D,\mathbb{R}^3)),\\
\mathbf{u}_\varepsilon &\to \mathbf{v}_s && \text{strongly in } L^2(Q^s).
\end{aligned}$$

Pressure estimates

Local estimates in Q^f

Taking a test function

$$\varphi(t,\mathbf{x}) := \psi(t,\mathbf{x})\nabla\Delta^{-1}[1_D \varrho_\varepsilon^\nu],$$

where $\nu > 0$ is a small positive number and ψ is a smooth cut-off function with $\operatorname{supp}\psi \subset Q^f$, we obtain from the momentum equation that

$$a\int_K \varrho_\varepsilon^{\gamma+\nu} = \int_K p(\varrho_\varepsilon)\varrho_\varepsilon^\nu \leq c(K)$$

for any compact $K \subset Q^f$.

Pressure estimates II.

Estimates up to the boundary

Assume that there exists a test function φ with the following properties:

- $\partial_t \varphi$, $\nabla \varphi \in L^q(Q^f)$ for a given $q \gg 1$;
- $\varphi(t, \cdot) \in W_0^{1,q}(\Omega_t, \mathbb{R}^3)$ for any $t \in (0, T)$;
- $\varphi(T, \cdot) = 0$;
- $\operatorname{div} \varphi(t, \mathbf{x}) \to \infty$ for $\mathbf{x} \to \partial\Omega_t$ uniformly for t in compact subsets of $(0, T)$.

Pressure estimates II.

Estimates up to the boundary

Assume that there exists a test function φ with the following properties:

- $\partial_t \varphi$, $\nabla \varphi \in L^q(Q^f)$ for a given $q \gg 1$;
- $\varphi(t, \cdot) \in W_0^{1,q}(\Omega_t, \mathbb{R}^3)$ for any $t \in (0, T)$;
- $\varphi(T, \cdot) = 0$;
- $\operatorname{div} \varphi(t, \mathbf{x}) \to \infty$ for $\mathbf{x} \to \partial\Omega_t$ uniformly for t in compact subsets of $(0, T)$.

Then for any $\tau < T$,

$$\int_{Q^f \cap ([0,\tau] \times D)} p(\varrho_\varepsilon) \operatorname{div} \varphi \leq c(\tau)$$

and consequently $\{p(\varrho_\varepsilon)\}_{\varepsilon > 0}$ is equi-integrable in $L^1(Q^f)$.

Pressure estimates III.

Estimates up to the boundary

In order to construct the test function φ, we denote

$$d(t, \mathbf{x}) = \mathrm{dist}\,(\mathbf{x}, \partial\Omega_t).$$

Pressure estimates III.

Estimates up to the boundary

In order to construct the test function φ, we denote

$$d(t, \mathbf{x}) = \operatorname{dist}(\mathbf{x}, \partial\Omega_t).$$

Since the boundary of Q^f is of class $\mathcal{C}^2$, there exists a neighborhood U of $\partial Q^f \cap ((\tau_1, \tau_2) \times D)$, $\tau_1, \tau_2 \in (0, T)$, such that

$$d \in \mathcal{C}^2(U).$$

Pressure estimates III.

Estimates up to the boundary

In order to construct the test function $\boldsymbol{\varphi}$, we denote

$$d(t, \mathbf{x}) = \mathrm{dist}\,(\mathbf{x}, \partial\Omega_t).$$

Since the boundary of Q^f is of class $\mathcal{C}^2$, there exists a neighborhood U of $\partial Q^f \cap ((\tau_1, \tau_2) \times D)$, $\tau_1, \tau_2 \in (0, T)$, such that

$$d \in \mathcal{C}^2(U).$$

Let us choose $h \in C([0,\infty)) \cap \mathcal{C}^\infty((0,\infty))$ s.t.

$$h(z) := \begin{cases} z^\alpha & \text{for } z \in ([0, \delta/2), \\ 0 & \text{for } z \geq \delta, \end{cases}$$

$\boldsymbol{\psi} \in \mathcal{C}_c^\infty(0, T)$, $\boldsymbol{\psi} \geq 0$, $\psi = 1$ in $[\tau_1, \tau_2]$, and define

$$\boldsymbol{\varphi}(t, \mathbf{x}) := \boldsymbol{\psi}(t) h(d(t, \mathbf{x})) \nabla d(t, \mathbf{x}).$$

Pressure estimates III.

Estimates up to the boundary

In order to construct the test function $\boldsymbol{\varphi}$, we denote

$$d(t, \mathbf{x}) = \mathrm{dist}\,(\mathbf{x}, \partial\Omega_t).$$

Since the boundary of Q^f is of class $\mathcal{C}^2$, there exists a neighborhood U of $\partial Q^f \cap ((\tau_1, \tau_2) \times D)$, $\tau_1, \tau_2 \in (0, T)$, such that

$$d \in \mathcal{C}^2(U).$$

Let us choose $h \in C([0, \infty)) \cap \mathcal{C}^\infty((0, \infty))$ s.t.

$$h(z) := \begin{cases} z^\alpha & \text{for } z \in ([0, \delta/2), \\ 0 & \text{for } z \geq \delta, \end{cases}$$

$\boldsymbol{\psi} \in \mathcal{C}_c^\infty(0, T)$, $\boldsymbol{\psi} \geq 0$, $\psi = 1$ in $[\tau_1, \tau_2]$, and define

$$\boldsymbol{\varphi}(t, \mathbf{x}) := \boldsymbol{\psi}(t) h(d(t, \mathbf{x})) \nabla d(t, \mathbf{x}).$$

For $\delta > 0$ and $\alpha \in (0, 1)$ small enough, $\boldsymbol{\varphi}$ meets the requirements:

$$\mathrm{div}\,\boldsymbol{\varphi} = \boldsymbol{\psi} h'(d) |\nabla d|^2 + \psi h(d) \Delta d.$$

Convergence of ϱ_ε in Q^s

The available convergence information allows to check that

$$\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0$$

in the sense of distributions in $(0, T) \times D$ and, since $\mathbf{u} = \mathbf{v}_s$ in Q^s, also in $(0, T) \times \mathbb{R}^3$.

Convergence of ϱ_ε in Q^s

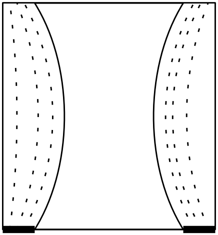
The available convergence information allows to check that

$$\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0$$

in the sense of distributions in $(0, T) \times D$ and, since $\mathbf{u} = \mathbf{v}_s$ in Q^s, also in $(0, T) \times \mathbb{R}^3$.

From the fact that the motion of $\partial\Omega_t$ is governed by $\mathbf{v}_s$ and that $\varrho_0 = 0$ outside Ω_0 we conclude that

$$\varrho = 0 \quad \text{in } Q^s.$$



Consequently, for any $q \in [1, \gamma)$,

$$\varrho_\varepsilon \to \varrho \text{ in } L^q(Q^s).$$

Renormalized continuity equation

Effective viscous flux

We are able to pass in the momentum equation:

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}\,(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla \overline{p(\varrho)} = \mathrm{div}\,\mathbb{S} \text{ in } Q^f$$

in the sense of distributions.

Renormalized continuity equation

Effective viscous flux

We are able to pass in the momentum equation:

$$\partial_t(\varrho\mathbf{u}) + \mathrm{div}\,(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla\overline{p(\varrho)} = \mathrm{div}\,\mathbb{S} \text{ in } Q^f$$

in the sense of distributions.
Next, introducing the operator $T_k(f) := \min\{f,k\}$, it is possible to show that the identity

$$(4/3\mu+\eta)\left(\overline{T_k(\varrho)\mathrm{div}\,\mathbf{u}} - \overline{T_k(\varrho)}\mathrm{div}\,\mathbf{u}\right) = \overline{p(\varrho)\,T_k(\varrho)} - \overline{p(\varrho)}\;\overline{T_k(\varrho)}$$

holds locally in Q^f. In particular, as p is a monotone function of ϱ, the expression on the left is non-negative.

Renormalized continuity equation

Oscillations defect measure

Using the property of the effective viscous flux we can prove that

$$\mathrm{osc}_{\gamma+1}[\varrho_\varepsilon \to \varrho](O) := \sup_{k\geq 1} \limsup_{\varepsilon\to 0} \int_O |T_k(\varrho_\varepsilon) - T_k(\varrho)|^{\gamma+1} \leq c(|O|)$$

for every compact $O \subset (0, T) \times D$

Renormalized continuity equation

Oscillations defect measure

Using the property of the effective viscous flux we can prove that

$$\operatorname{osc}_{\gamma+1}[\varrho_\varepsilon \to \varrho](O) := \sup_{k\geq 1} \limsup_{\varepsilon\to 0} \int_O |T_k(\varrho_\varepsilon) - T_k(\varrho)|^{\gamma+1} \leq c(|O|)$$

for every compact $O \subset (0,T)\times D$, and consequently

$$\operatorname{osc}_{\gamma+1}[\varrho_\varepsilon \to \varrho]((0,T)\times D) \leq c.$$

This, together with the uniform bounds, implies that ϱ, $\mathbf{u}$ satisfy the renormalized equation of continuity in $(0,T)\times D$.

Convergence of ϱ_ε in Q^f

From the renormalized equation of continuity we can derive that

$$\int_D \left(\overline{L_k(\varrho)} - L_k(\varrho) \right)(\tau) + \int_0^\tau \int_D \left(\overline{T_k(\varrho) \mathrm{div}\, \mathbf{u}} - \overline{T_k(\varrho)} \mathrm{div}\, \mathbf{u} \right)$$
$$= \int_D \left(\overline{L_k(\varrho_0)} - L_k(\varrho_0) \right) + \int_0^\tau \int_D \left(T_k(\varrho) \mathrm{div}\, \mathbf{u} - \overline{T_k(\varrho)} \mathrm{div}\, \mathbf{u} \right),$$

for any $\tau \in (0, T)$, where

$$L_k(\varrho) := \varrho \int_1^\varrho \frac{T_k(z)}{z^2} \mathrm{d}z.$$

Convergence of ϱ_ε in Q^f

From the renormalized equation of continuity we can derive that

$$\int_D \left(\overline{L_k(\varrho)} - L_k(\varrho) \right)(\tau) + \int_0^\tau \int_D \Big(\underbrace{\overline{T_k(\varrho) \mathrm{div}\, \mathbf{u}} - \overline{T_k(\varrho)} \mathrm{div}\, \mathbf{u}}_{\geq 0} \Big)$$

$$= \int_D \left(\overline{L_k(\varrho_0)} - L_k(\varrho_0) \right) + \int_0^\tau \int_D \left(T_k(\varrho) \mathrm{div}\, \mathbf{u} - \overline{T_k(\varrho)} \mathrm{div}\, \mathbf{u} \right),$$

for any $\tau \in (0, T)$, where

$$L_k(\varrho) := \varrho \int_1^\varrho \frac{T_k(z)}{z^2} \mathrm{d}z.$$

Convergence of ϱ_ε in Q^f

From the renormalized equation of continuity we can derive that

$$\int_D \left(\overline{L_k(\varrho)} - L_k(\varrho) \right)(\tau) + \int_0^\tau \int_D \Big(\underbrace{\overline{T_k(\varrho)\mathrm{div}\,\mathbf{u}} - \overline{T_k(\varrho)}\mathrm{div}\,\mathbf{u}}_{\geq 0} \Big)$$

$$= \int_D \Big(\underbrace{\overline{L_k(\varrho_0)} - L_k(\varrho_0)}_{=0} \Big) + \int_0^\tau \int_D \Big(T_k(\varrho)\mathrm{div}\,\mathbf{u} - \overline{T_k(\varrho)}\mathrm{div}\,\mathbf{u} \Big),$$

for any $\tau \in (0, T)$, where

$$L_k(\varrho) := \varrho \int_1^\varrho \frac{T_k(z)}{z^2} \mathrm{d}z.$$

Convergence of ϱ_ε in Q^f

From the renormalized equation of continuity we can derive that

$$\int_D \Big(\overline{L_k(\varrho)} - L_k(\varrho) \Big)(\tau) + \int_0^\tau \int_D \Big(\underbrace{\overline{T_k(\varrho)\mathrm{div}\,\mathbf{u}} - \overline{T_k(\varrho)}\mathrm{div}\,\mathbf{u}}_{\geq 0} \Big)$$

$$= \int_D \Big(\underbrace{\overline{L_k(\varrho_0)} - L_k(\varrho_0)}_{=0} \Big) + \underbrace{\int_0^\tau \int_D \Big(T_k(\varrho)\mathrm{div}\,\mathbf{u} - \overline{T_k(\varrho)}\mathrm{div}\,\mathbf{u} \Big)}_{\to 0},$$

for any $\tau \in (0, T)$, where

$$L_k(\varrho) := \varrho \int_1^\varrho \frac{T_k(z)}{z^2}\mathrm{d}z.$$

Convergence of ϱ_ε in Q^f

From the renormalized equation of continuity we can derive that

$$\int_D \left(\overline{L_k(\varrho)} - L_k(\varrho)\right)(\tau) + \int_0^\tau \int_D \Big(\underbrace{\overline{T_k(\varrho)\mathrm{div}\,\mathbf{u}} - \overline{T_k(\varrho)}\mathrm{div}\,\mathbf{u}}_{\geq 0} \Big)$$

$$= \int_D \Big(\underbrace{\overline{L_k(\varrho_0)} - L_k(\varrho_0)}_{=0} \Big) + \underbrace{\int_0^\tau \int_D \Big(T_k(\varrho)\mathrm{div}\,\mathbf{u} - \overline{T_k(\varrho)}\mathrm{div}\,\mathbf{u} \Big)}_{\to 0},$$

for any $\tau \in (0, T)$, where

$$L_k(\varrho) := \varrho \int_1^\varrho \frac{T_k(z)}{z^2} \mathrm{d}z.$$

Letting $k \to \infty$ we conclude that

$$\int_D \left(\overline{\varrho \log \varrho} - \varrho \log \varrho \right)(\tau) = 0$$

for any $\tau \geq 0$, which implies for any $q \in [1, \gamma)$:

$$\varrho_\varepsilon \to \varrho \text{ in } L^q((0, T) \times D).$$

Conclusion

- We proved:
  - Convergence of the Brinkman penalization for compressible isentropic fluids
  - Existence of solutions to NSE in time-dependent domains
- The limit passage requires only local pressure estimates
- The assumptions on smoothness of $\mathbf{v}_s$ were not optimal and can be possibly relaxed