Shape stability of incompressible fluids subject to Navier's slip

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Introduction

We consider the time evolution of an incompressible fluid with shear-rate–dependent viscosity:

$$\operatorname{div} \vec{v} = 0, \tag{1a}$$

$$\partial_t \vec{v} + \operatorname{div}(\vec{v} \otimes \vec{v}) = -\nabla p + \operatorname{div}\left[\nu(|\mathbb{D}(\vec{v})|^2)\mathbb{D}(\vec{v})\right] + \vec{f}, \tag{1b}$$

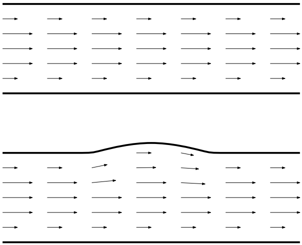
in a bounded 3D domain, completed by the Navier slip boundary condition

$$(\mathbb{T}\vec{n})_\tau = -a\vec{v}_\tau, \;\; \vec{v} \cdot \vec{n} = 0. \tag{1c}$$

Our primary aim is to study the domain dependence of solutions $(\vec{v}, p)$.

What is shape stability

Problem is shape-stable, if small change in the geometry of the domain leads to small change in the solution(s).



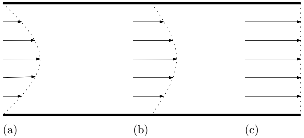
In other words: If $\Omega_n \to \Omega$, do solutions $(\vec{v}_n, p_n)$ converge to a solution of the same problem on Ω?

- Indicator of model robustness
- Keystone for existence of optimal shapes

Slip vs. no slip

In general it is not clear what is the right boundary condition for walls.

- Often it is reasonable to prescribe no slip: $\vec{v}_{|\partial\Omega} = 0$. This condition is known to be stable under quite general boundary perturbations (e.g. equi-Lipschitz setting).
- In case of e.g. rough or chemically patterned surfaces some kind of slip condition is more suitable.



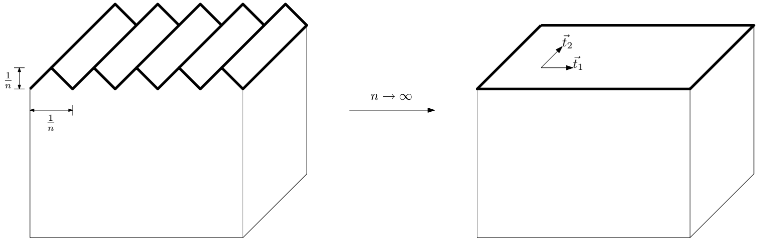
Velocity profiles: (a) no-slip, (b) partial slip, (c) complete slip.

We study the shape stability problem for Navier's slip condition

$$(\mathbb{T}\vec{n})_\tau = -a\vec{v}_\tau,\ \ \vec{v} \cdot \vec{n} = 0,\ \ a \geq 0.$$

Examples of instability of Navier's slip (1/2)

Equi-Lipschitz domains, stationary problem



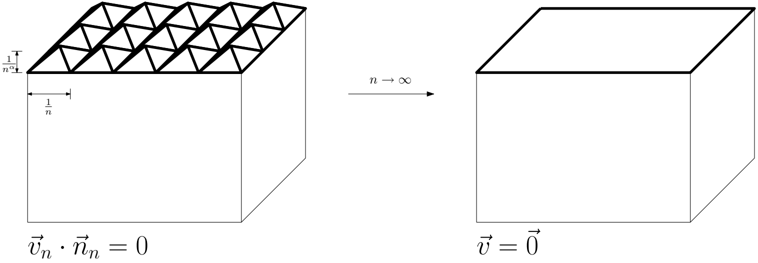
$$\vec{v}_n \cdot \vec{n}_n = 0$$
$$(\mathbb{T}\vec{n}_n)_{\tau_n} = -a(\vec{v}_n)_{\tau_n}$$

$$\vec{v} \cdot \vec{n} = 0$$
$$\vec{v} \cdot \vec{t}_1 = 0$$
$$\mathbb{T}\vec{n} \cdot \vec{t}_2 = -\tilde{a}\vec{v} \cdot \vec{t}_2, \ \tilde{a} > a$$

(Bucur, Feireisl, and Nečasová [2008])

Examples of instability of Navier's slip (2/2)

Domains in $\mathcal{C}^{1,\alpha-1}$, $\alpha < \frac{2r}{r+1}$
arbitrary $\{\vec{v}_n\}$ s.t. $\vec{v}_n \rightharpoonup \vec{v}$ in $W^{1,r}$



(Březina [2009])

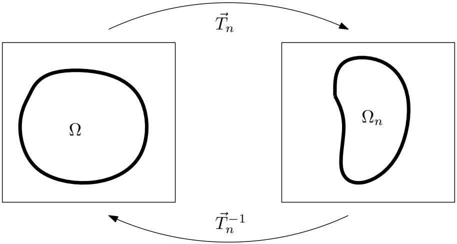
Convergence of domains

Let $D \subset \mathbb{R}^3$ be bounded and $\mathcal{O} := \{\Omega \in \mathcal{C}^{1,1};\ \Omega \subset D\}$.

Definition

We say that $\Omega_n \to \Omega$ in $\mathcal{O}$ if there exist $\mathcal{C}^{1,1}$-diffeomorphisms $\vec{T}_n : \overline{D} \to \overline{D}$ such that

- $\vec{T}_n(\Omega) = \Omega_n$,
- $\vec{T}_n \to \vec{Id}$ and $\vec{T}_n^{-1} \to \vec{Id}$ in $\mathcal{C}^{1,1}(\overline{D}, \overline{D})$,
- $|\nabla^2 \vec{T}_n|$ and $|\nabla^2 \vec{T}_n^{-1}|$ are bounded independently of n.



Convergence of functions

Let

$$W^{1,q}_N(\Omega) := \{\vec{\varphi} \in W^{1,q}(\Omega);\ \vec{\varphi}\cdot\vec{n} = 0 \text{ on } \partial\Omega\},$$
$$\vec{X}^q := L^q(D,\mathbb{R}^3) \times L^q(D,\mathbb{R}^{3\times 3}).$$

For $\Omega \in \mathcal{O}$ we define

$$E_\Omega : W^{1,q}(\Omega,\mathbb{R}^3) \to \vec{X}^q;\ \vec{\varphi} \mapsto (\chi_\Omega\vec{\varphi}, \chi_\Omega\nabla\vec{\varphi}).$$

Definition

Let $\Omega_n \to \Omega$ in $\mathcal{O}$. We say that

$$\vec{\varphi}_n \to \vec{\varphi} \text{ in } \vec{X}^q \text{ iff } E_{\Omega_n}\vec{\varphi}_n \to E_\Omega\vec{\varphi};$$

$$\vec{\varphi}_n \rightharpoonup \vec{\varphi} \text{ in } \vec{X}^q \text{ iff } E_{\Omega_n}\vec{\varphi}_n \rightharpoonup E_\Omega\vec{\varphi}.$$

If $\vec{\varphi}_n \in W^{1,q}_0(\Omega_n)$ then

$$\vec{\varphi}_n \to \vec{\varphi} \text{ in } \vec{X}^q \qquad \Leftrightarrow \qquad \chi_{\Omega_n}\vec{\varphi}_n \to \chi_\Omega\vec{\varphi} \text{ in } W^{1,q}_0(D).$$

Structural assumptions

Assumptions on the extra stress $\mathbb{S}(\mathbb{D}(\vec{v})) := \nu(|\mathbb{D}(\vec{v})|^2)\mathbb{D}(\vec{v})$:

1. $\mathbb{S} \in \mathcal{C}^1(\mathbb{R}^{3\times 3}, \mathbb{R}^{3\times 3})$, $\mathbb{S}(0) = 0$;
2. There exist constants $C_1, C_2 > 0$, $\kappa \in \{0, 1\}$ and $r > 1$ s.t.

$$C_1(\kappa + |\mathbb{A}|^{r-2})|\mathbb{B}|^2 \leq \frac{\partial \mathbb{S}(\mathbb{A})}{\partial \mathbb{A}} :: (\mathbb{B} \otimes \mathbb{B}) \leq C_2(\kappa + |\mathbb{A}|^{r-2})|\mathbb{B}|^2$$

   for any $0 \neq \mathbb{A}, \mathbb{B} \in \mathbb{R}^{3\times 3}$.

Assumptions on the body force and initial datum:

3. $\vec{f} \in L^{r'}(0, T; \vec{X}^{r'})$;
4. $\vec{v}_0 \in L^2(D)^3$, div $\vec{v}_0 = 0$.

Definition of weak solution

We say that $(\vec{v}, p)$ is a weak solution of problem $(P(\Omega))$, iff

- $\vec{v} \in L^\infty(0, T; L^2(\Omega)) \cap L^r(0, T; W^{1,r}_{N,div}(\Omega))$; $\partial_t \vec{v} \in L^\sigma(0, T; W_N^{-1,\sigma}(\Omega))$ and $p \in L^\sigma(0, T; L^\sigma_0(\Omega))$;
- for every $\vec{\varphi} \in L^{\sigma'}(0, T; W_N^{1,\sigma'}(\Omega))$, $\sigma = \begin{cases} r' & \text{if } r \geq \frac{11}{5} \\ \frac{5r}{6} & \text{if } r < \frac{11}{5} \end{cases}$:

$$\int_0^T \Big[\langle \partial_t \vec{v}, \vec{\varphi} \rangle - (\vec{v} \otimes \vec{v}, \nabla \vec{\varphi}) + (\mathbb{S}(\mathbb{D}(\vec{v})), \mathbb{D}(\vec{\varphi})) - (p, \operatorname{div} \vec{\varphi}) + a \int_{\partial\Omega} \vec{v} \cdot \vec{\varphi} \Big] = \int_0^T \langle \vec{f}, \vec{\varphi} \rangle;$$

- for a.a. $t \in (0, T)$ the energy inequality holds:

$$\frac{1}{2} \|\vec{v}(t)\|_2^2 + \int_0^t (\mathbb{S}(\mathbb{D}(\vec{v})), \mathbb{D}(\vec{v})) + a \int_0^t \|\vec{v}\|^2_{2,\partial\Omega} \leq \frac{1}{2} \|\vec{v}_0\|_2^2 + \int_0^t \langle \vec{f}, \vec{v} \rangle;$$

- $\lim_{t \to 0+} \|\vec{v}(t) - \vec{v}_0\|_2^2 = 0$.

Existence of solutions

Theorem (Bulíček, Málek, and Rajagopal [2007])

Let $T > 0$, $\Omega \in \mathcal{C}^{1,1}$, $\Omega \subset \mathbb{R}^3$ *and* $r > \frac{8}{5}$. *Then*

- *there exists a weak solution* $(\vec{v}, p)$ *to* $(P(\Omega))$;
- *if moreover* $r > \frac{5}{2}$, *then the weak solution is unique.*

Shape stability

Theorem (Main result)

- *Let* $r > \frac{8}{5}$, $\Omega_n \to \Omega$ *and* $(\vec{v}_n, p_n)$ *be solutions to* $(P(\Omega_n))$. *Then*

$$
\begin{aligned}
\vec{v}_n &\rightharpoonup \vec{v} && \text{in } L^r(0,T;\vec{X}^r),\\
\chi_{\Omega_n}\vec{v}_n &\rightharpoonup^* \chi_\Omega \vec{v} && \text{in } L^\infty(0,T;L^2(D)),\\
\chi_{\Omega_n} p_n &\rightharpoonup \chi_\Omega p && \text{in } L^\sigma(0,T;L^\sigma_0(D)).
\end{aligned}
$$

where $(\vec{v}, p)$ *is a solution to* $(P(\Omega))$.

Outline of the proof

1. Uniform estimate:
   - Korn's inequality for $\Omega \in \mathcal{C}^{0,1}$
   - L^q-regularity for Laplace eq. with Neumann b.c. ($\Omega \in \mathcal{C}^{1,1}$)
2. Limit passage $\Omega_n \to \Omega$:
   - Mosco's conditions:

$$\begin{array}{rcl} \vec{\varphi}_n \cdot \vec{n} = 0 \text{ and } \vec{\varphi}_n \rightharpoonup \vec{\varphi} & \Rightarrow & \vec{\varphi} \cdot \vec{n} = 0 \\ \exists \{\vec{\varphi}_n\},\ \vec{\varphi}_n \cdot \vec{n} = 0 \text{ and } \vec{\varphi}_n \to \vec{\varphi} & \Leftarrow & \vec{\varphi} \cdot \vec{n} = 0 \end{array}$$

   - Weak convergence of $(\vec{v}_n, p_n)$ satisfies to pass in the terms

$$\int_0^T \left[\langle \partial_t \vec{v}_n, \vec{\varphi}_n \rangle_{\Omega_n} - (p_n, \operatorname{div} \vec{\varphi}_n)_{\Omega_n} + a \int_{\partial \Omega_n} \vec{v}_n \cdot \vec{\varphi}_n \right];$$

   - Aubin-Lions lemma gives strong convergence enabling limit in convective term;
   - Nonlinear term $\mathbb{S}(\mathbb{D}(\vec{v}_n))$ handled by strong monotonicity and Vitali's lemma ($r > \frac{11}{5}$) or by L^∞-truncation method ($r < \frac{11}{5}$).

Mosco's conditions (1/4)

Lemma (Mosco's conditions)

Let $\Omega_n \to \Omega$.

1. *For every* $\{\vec{\varphi}_n\}$ *s.t.* $\vec{\varphi}_n \in L^q(0,T;W^{1,s}_N(\Omega_n))$ *and*

$$\vec{\varphi}_n \rightharpoonup \vec{\varphi} \text{ in } L^q(0,T;\vec{X}^s)$$

   it holds that $\vec{\varphi} \in L^q(0,T;W^{1,s}_N(\Omega))$;

2. *For any* $\vec{\varphi} \in L^q(0,T;W^{1,s}_N(\Omega))$ *there exists a sequence* $\{\vec{\varphi}_n\}$, $\vec{\varphi}_n \in L^q(0,T;W^{1,s}_N(\Omega_n))$ *such that*

$$\vec{\varphi}_n \to \vec{\varphi} \text{ in } L^q(0,T;\vec{X}^s).$$

Mosco's conditions (2/4)

Proof of (i): We rewrite the impermeability condition as follows:

$$\int_{\Omega_n} (\vec{\varphi}_n(t) \cdot \nabla\psi - \psi \operatorname{div} \vec{\varphi}_n(t)) = 0 \qquad \forall \psi \in \mathcal{C}^\infty(\overline{D})$$

and pass to the limit with $n \to \infty$ to conclude that

$$\int_{\Omega} (\vec{\varphi} \cdot \nabla\psi - \psi \operatorname{div} \vec{\varphi}) = 0$$

a.a. in $(0, T)$, thus $\vec{\varphi} \cdot \vec{n}_\Omega = 0$.

Mosco's conditions (3/4)

Proof of (ii): (idea)
Let $\vec{T}_n$ be the transformation which maps Ω onto Ω_n and assume first that $\vec{T}_n, \vec{T}_n^{-1} \to \vec{Id}$ in $\mathcal{C}^2$. If we define

$$\vec{\varphi}_n := \left[(\nabla T_n)\vec{\varphi}\right] \circ T_n^{-1}$$

then $\vec{\varphi}_n \cdot \vec{n}_{\Omega_n} = \vec{\varphi} \cdot \vec{n}_\Omega = 0$, so that $\vec{\varphi}_n \in L^q(0,T;W_N^{1,s}(\Omega_n))$.
The convergence of $\vec{T}_n$ and $\vec{T}_n^{-1}$ is then sufficient to prove

$$E_{\Omega_n}\vec{\varphi}_n \to E_\Omega\vec{\varphi} \text{ in } L^q(0,T;\vec{X}^s).$$

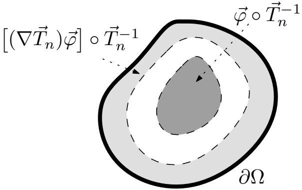
Roughly speaking,

$$\nabla\left[(\nabla\vec{T}_n)\vec{\varphi}\right] \approx \underbrace{(\nabla\vec{T}_n)}_{\to\mathbb{I}}\nabla\vec{\varphi} + \underbrace{(\nabla^2\vec{T}_n)}_{\to 0}\vec{\varphi}.$$

Mosco's conditions (4/4)

Proof of (ii): (continued)
If $\vec{T}_n$, $\vec{T}_n^{-1}$ converge in $\mathcal{C}^{1,1}$ only, then we modify the construction using a cut-off function $\eta_\epsilon = \begin{cases} 1 & \text{near } \partial\Omega \\ 0 & \text{if dist}_{\partial\Omega} > \epsilon \end{cases}$:

$$\vec{\varphi}_n := \left\{ \left[\eta_\epsilon (\nabla T_n) + (1-\eta_\epsilon)\mathbb{I} \right] \vec{\varphi} \right\} \circ T_n^{-1}.$$

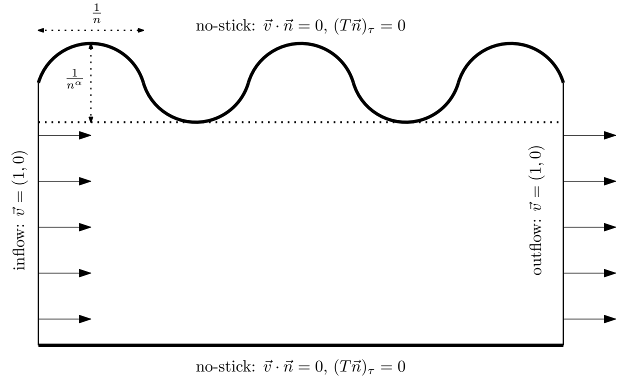


Then

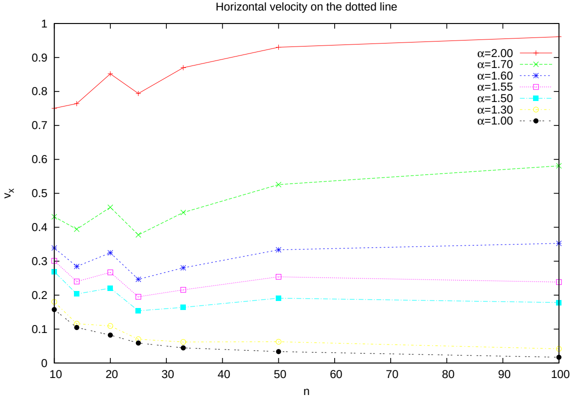
$$\nabla \left[\eta_\epsilon (\nabla \vec{T}_n - \mathbb{I})\vec{\varphi} + \vec{\varphi} \right] \approx \underbrace{(\nabla^2 \vec{T}_n)}_{\text{bdd.}} \underbrace{\eta_\epsilon \vec{\varphi}}_{\text{small}} + \underbrace{(\nabla \vec{T}_n - \mathbb{I})}_{\to 0} (\eta_\epsilon \nabla \vec{\varphi} + \nabla \eta_\epsilon \vec{\varphi}) + \nabla \vec{\varphi}.$$

Numerical example

Stationary Stokes problem in a chanel with curved top:

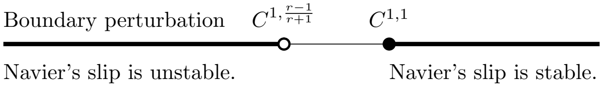


Numerical example - results



Limit boundary condition: no-slip

Conclusion



- Numerical and rigorous arguments show that Navier's slip condition is unstable under boundary perturbations of low regularity, while for smooth deformations it remains stable.
- The minimal regularity of boundary that preserves slip is optimal for unsteady flows of power law fluids.

J. Březina. Asymptotic properties of solutions to the equations of incompressible fluid mechanics. Preprint 1, Nečas Center for Mathematical Modeling, 2009.

D. Bucur, E. Feireisl, and Š. Nečasová. Influence of wall roughness on the slip behaviour of viscous fluids. *Proc. Roy. Soc. Edinburgh Sect. A*, 138(5):957–973, 2008.

M. Bulíček, J. Málek, and K. R. Rajagopal. Navier's slip and evolutionary Navier-Stokes-like systems with pressure and shear-rate dependent viscosity. *Indiana Univ. Math. J.*, 56(1): 51–85, 2007. ISSN 0022-2518.