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\mathcal{PT} -symmetric models in curved manifolds

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- 1 Introduction
 - Geometrical effects in quantum models
 - Hamiltonian
 - $\circ \mathcal{PT}\text{-symmetric models in 2D curved manifolds}$
- **2** Definitions of Models
 - Cylinder K = 0
 - Sphere K = 1
 - Pseudosphere K = -1
- **3** Spectral results
 - General properties
 - Cylinder $K = 0, \alpha_0$
 - Sphere K = 1
 - Pseudosphere K = -1
- **4** Conclusions

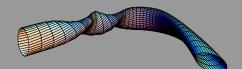


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Geometrical effects in quantum models

Waveguides

- bending acts as an attractive interaction
 - Exner, Šeba 1989, Goldstone, Jaffe 1992, etc
- twisting acts as a repulsive interaction
 - Ekholm, Krejčiřík, Kovařík 2005



Quantum strips on surfaces

- positive curvature acts as an attractive interaction
- negative curvature acts as a repulsive interaction
 - Krejčiřík 2002



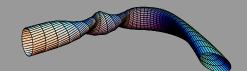
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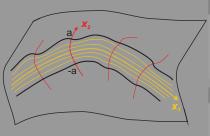
Quantum strips on surfaces

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Fermi coordinates and Hamiltonian



Metric tensor
$$g$$

 $g_{ij} = \begin{pmatrix} f(x_1, x_2) & 0 \\ 0 & 1 \end{pmatrix}$
 $g = det(g_{ij})$
Jacobi equation
 $\partial_2^2 f + Kf = 0$
 $f(\cdot, 0) = 1, \partial_2 f(\cdot, 0) = k$

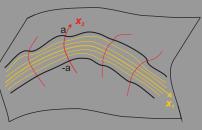
Laplace-Beltrami operator

$$\begin{split} H &= -g^{-1/2}\partial_i g^{1/2} g^{ij} \partial_j \\ \mathrm{Dom}(H) &= W^{2,2} + \mathrm{boundary\ conditions} \\ d\Omega &= g^{1/2} dx_1 dx_2 \end{split}$$

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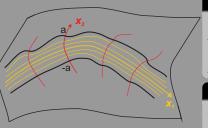
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Fermi coordinates and Hamiltonian



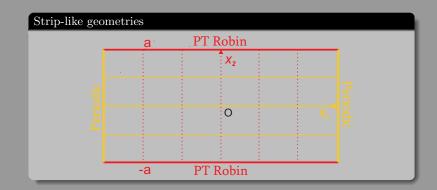
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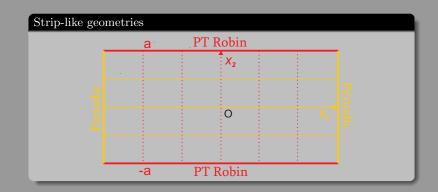




$$\mathcal{PT}$$
 Robin boundary conditions
 $\partial_2 \Psi(x_1, a) + i\alpha(x_1)\Psi(x_1, a) = 0$
 $\partial_2 \Psi(x_1, -a) + i\alpha(x_1)\Psi(x_1, -a) = 0$

$$\begin{array}{lll} \mathcal{PT} \text{ symmetry} \\ (\mathcal{P}\Psi)(x_1, x_2) &= & \Psi(x_1, -x_2) \\ (\mathcal{T}\Psi)(x_1, x_2) &= & \overline{\Psi(x_1, x_2)} \end{array}$$

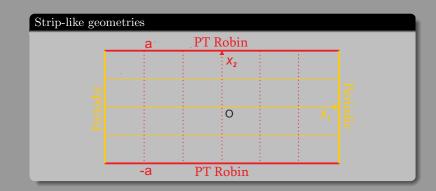




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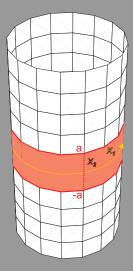
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Outline		Definitions of Models	
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Cylind	$\mathbf{er} \ K = 0$		
Oyimu	$\mathbf{CI} \mathbf{I} \mathbf{I} = 0$		

Hamiltonian for cylinder

 $H_0^{\alpha} = -\partial_1^2 - \partial_2^2$ \mathcal{PT} Robin BC

$$g_{ij} \text{ for cylinder}$$
$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$d\Omega = dx_1 dx_2$$



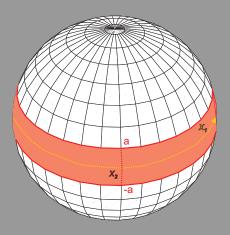
Outline		Definitions of Models	Conclusions
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Sphere	K = 1		

Hamiltonian for sphere

$$H_1^{\alpha} = -\frac{1}{\cos^2 x_2} \partial_1^2 - \partial_2^2 + \tan x_2 \partial_2$$

$$\mathcal{PT} \text{ Robin BC}$$

g_{ij} for sphere
$g_{ij} = \begin{pmatrix} \cos^2 x_2 & 0\\ 0 & 1 \end{pmatrix}$ $d\Omega = \cos x_2 dx_1 dx_2$



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Definitions of Models $\circ \circ \bullet$

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Pseudosphere K = -1

Hamiltonian for pseudosphere

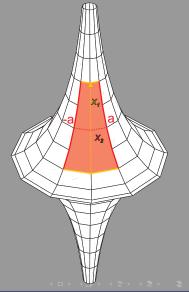
$$H^{\alpha}_{-1} = -\frac{1}{\cosh^2 x_2} \partial_1^2 - \partial_2^2 - \tanh x_2 \partial_2$$

 \mathcal{PT} Robin BC

 g_{ij} for pseudosphere

$$g_{ij} = \begin{pmatrix} \cosh^2 x_2 & 0 \\ 0 & 1 \end{pmatrix}$$

 $d\Omega = \cosh x_2 dx_1 dx_2$



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General properties

	-	
α	Lii	3
α)

m-sect	orial
m-secu	Joriar

H_K^{α} are	\mathcal{PT} -symmetric:	$\mathcal{PT}H_K^\alpha = \mathcal{PT}H_K^\alpha$
Π_K are	\mathcal{P} -pseudo-Hermitian:	$(H_K^\alpha)^* = \mathcal{P} H_K^\alpha \mathcal{P}$
	\mathcal{T} -selfadjoint:	$(H_K^\alpha)^* = \mathcal{T} H_K^\alpha \mathcal{T}$
$\sigma_r(H_K^\alpha)$	$= \emptyset$	
$\sigma(H_K^\alpha) =$	$= \sigma_d(H_K^{lpha}),$	$K \in \{-1, 0, 1\}$

 α is a real constant function, $\alpha(x_1) = \alpha_0$

$$H_0^{\alpha_0} = \bigoplus_{m \in \mathbb{N}_0} \left(-\partial_2^2 + m^2 \right)$$

$$\prod_{m \in \mathbb{N}_0} \left(-\partial_2^2 + \tanh x_2 \partial_2 + \frac{m^2}{\cos^2 x_2} \right)$$

Outline		Definitions of Models 000
Genera	d properties	
$\alpha \in I$	Lip	
	m-sectorial	

 $\begin{array}{ll} H_{K}^{\alpha} \mbox{ are } & \begin{array}{ll} \mathcal{P}\mathcal{T}\mbox{-symmetric:} & \mathcal{P}\mathcal{T}H_{K}^{\alpha}=\mathcal{P}\mathcal{T}H_{K}^{\alpha}\\ & \begin{array}{ll} \mathcal{P}\mbox{-pseudo-Hermitian:} & (H_{K}^{\alpha})^{*}=\mathcal{P}H_{K}^{\alpha}\mathcal{P}\\ & \begin{array}{ll} \mathcal{T}\mbox{-selfadjoint:} & (H_{K}^{\alpha})^{*}=\mathcal{T}H_{K}^{\alpha}\mathcal{T} \end{array} \\ & \sigma_{r}(H_{K}^{\alpha})=\emptyset\\ & \sigma(H_{K}^{\alpha})=\sigma_{d}(H_{K}^{\alpha}), & K\in\{-1,0,1\} \end{array}$

 α is a real constant function, $\alpha(x_1) = \alpha_0$

$$H_0^{\alpha_0} = \bigoplus_{m \in \mathbb{N}_0} (-\partial_2^2 + m^2)$$

$$\begin{aligned} H_1^{\alpha_0} &= \bigoplus_{m \in \mathbb{N}_0} \left(-\partial_2^2 + \tan x_2 \partial_2 + \frac{m^2}{\cos^2 x_2} \right) \\ H_{-1}^{\alpha_0} &= \bigoplus_{m \in \mathbb{N}_0} \left(-\partial_2^2 - \tanh x_2 \partial_2 + \frac{m^2}{\cosh^2 x_2} \right) \end{aligned}$$

Petr Siegl *PT*-symmetric models in curved manifolds

Spectral results

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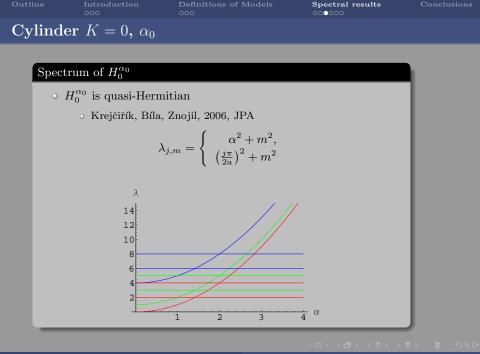
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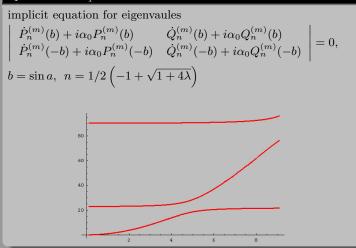
Width of the strips







Spectrum of $H_1^{\alpha_0}$



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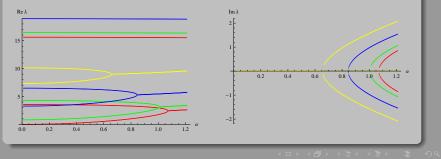
Pseudosphere $K = -1, \alpha_0$

Spectrum of $H_{-1}^{\alpha_0}$

implicit equation for eigenvaules

$$\begin{vmatrix} \left(\frac{P_k^{(l)}}{\sqrt{\cosh}}\right)(c) + i\alpha_0 \frac{P_k^{(l)}}{\sqrt{\cosh}}(c) & \left(\frac{Q_k^{(l)}}{\sqrt{\cosh}}\right)(c) + i\alpha_0 \frac{Q_k^{(l)}}{\sqrt{\cosh}}(c) \\ \vdots \\ \left(\frac{P_k^{(l)}}{\sqrt{\cosh}}\right)(-c) + i\alpha_0 \frac{Q_k^{(l)}}{\sqrt{\cosh}}(-c) & \left(\frac{Q_k^{(l)}}{\sqrt{\cosh}}\right)(-c) + i\alpha_0 Q_k^{(l)}(-c) \end{vmatrix} = 0,$$

$$c = \tanh a, \ k = \sqrt{1 - 4\lambda}, \ l = mi - 1/2$$



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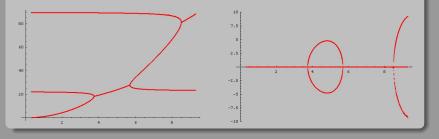
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Conclus	sions			

Presented models

- three strip-like models were presented
 - strip on a cylinder, sphere and pseudosphere
 - $\circ\,$ all models possess only discrete spectrum
 - $\circ\,$ all models are exactly solvable for constant α
- $\circ~$ curvature essentially influences the spectrum of $\mathcal{PT}\text{-symmetric}$ models

Curvature effects

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Presented models

- three strip-like models were presented
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- $\circ~$ curvature essentially influences the spectrum of $\mathcal{PT}\text{-symmetric}$ models

Curvature effects

K	realization	$\sigma(H_K^{\alpha_0})$	eigenvalues
0	cylinder	\mathbb{R}	only some α -dependent, crossing
1	sphere	\mathbb{R}	all α -dependent, no crossing
-1	pseudosphere	\mathbb{C}	all α -dependent, crossing