# $\mathcal{PT}$-symmetric models in curved manifolds

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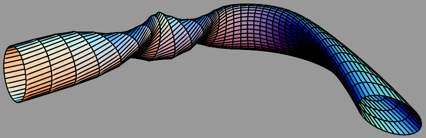
# Outline

# Geometrical effects in quantum models

## Waveguides

- bending - acts as an attractive interaction
  - Exner, Šeba 1989, Goldstone, Jaffe 1992, etc
- twisting - acts as a repulsive interaction
  - Ekholm, Krejčiřík, Kovařík 2005



## Quantum strips on surfaces

- positive curvature - acts as an attractive interaction
- negative curvature - acts as a repulsive interaction
  - Krejčiřík 2002

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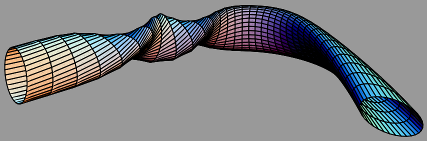

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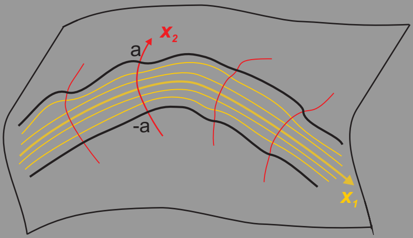
# Fermi coordinates and Hamiltonian




**Metric tensor $g$**

$$g_{ij} = \begin{pmatrix} f(x_1,x_2) & 0 \\ 0 & 1 \end{pmatrix}$$

$$g = det(g_{ij})$$

**Jacobi equation**

$$\partial_2^2 f + Kf = 0$$

$$f(\cdot,0) = 1,\ \partial_2 f(\cdot,0) = k$$

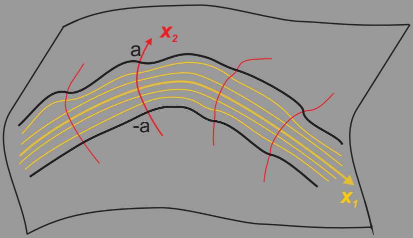
**Laplace-Beltrami operator**

$$H = -g^{-1/2}\partial_i g^{1/2} g^{ij} \partial_j$$

$$\text{Dom}(H) = W^{2,2} + \text{boundary conditions}$$

$$d\Omega = g^{1/2} dx_1 dx_2$$

# Fermi coordinates and Hamiltonian

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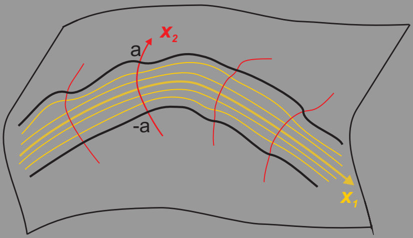


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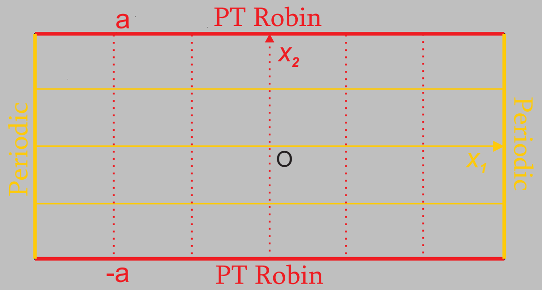
# $\mathcal{PT}$-symmetric models in 2D curved manifolds

## Strip-like geometries




### $\mathcal{PT}$ Robin boundary conditions

$$\partial_2\Psi(x_1,a) + i\alpha(x_1)\Psi(x_1,a) = 0$$
$$\partial_2\Psi(x_1,-a) + i\alpha(x_1)\Psi(x_1,-a) = 0$$

### $\mathcal{PT}$ symmetry

$$(\mathcal{P}\Psi)(x_1,x_2) = \Psi(x_1,-x_2)$$
$$(\mathcal{T}\Psi)(x_1,x_2) = \overline{\Psi(x_1,x_2)}$$

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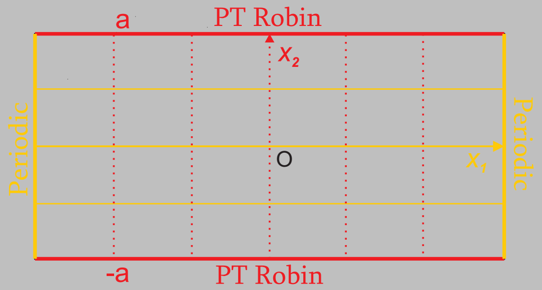


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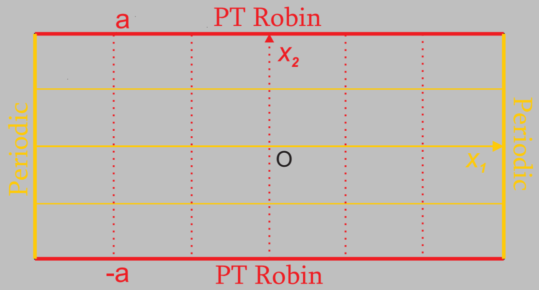


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# Cylinder $K = 0$

**Hamiltonian for cylinder**

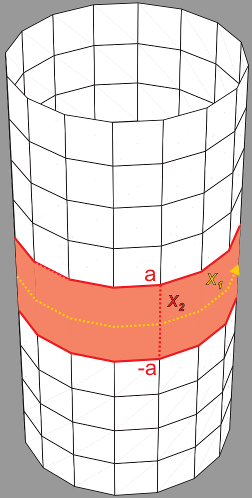
$H_0^\alpha = -\partial_1^2 - \partial_2^2$

$\mathcal{PT}$ Robin BC

**$g_{ij}$ for cylinder**

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$d\Omega = dx_1 dx_2$



# Sphere $K = 1$

**Hamiltonian for sphere**

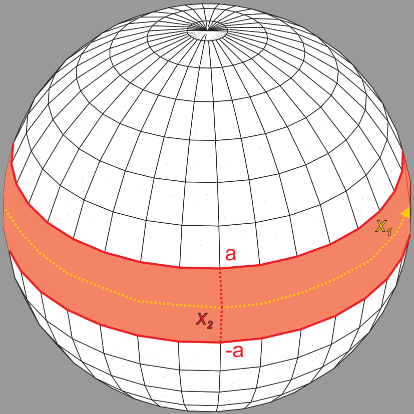
$$H_1^\alpha = -\frac{1}{\cos^2 x_2}\partial_1^2 - \partial_2^2 + \tan x_2 \partial_2$$

$\mathcal{PT}$ Robin BC

**$g_{ij}$ for sphere**

$$g_{ij} = \begin{pmatrix} \cos^2 x_2 & 0 \\ 0 & 1 \end{pmatrix}$$

$d\Omega = \cos x_2 dx_1 dx_2$



# Pseudosphere $K = -1$

## Hamiltonian for pseudosphere

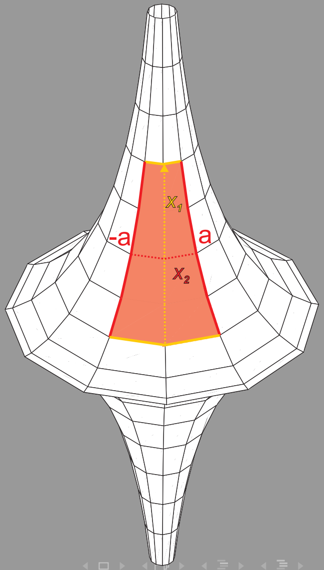
$$H^{\alpha}_{-1} = -\frac{1}{\cosh^2 x_2}\partial_1^2 - \partial_2^2 - \tanh x_2 \partial_2$$

$\mathcal{PT}$ Robin BC

## $g_{ij}$ for pseudosphere

$$g_{ij} = \begin{pmatrix} \cosh^2 x_2 & 0 \\ 0 & 1 \end{pmatrix}$$

$d\Omega = \cosh x_2 dx_1 dx_2$



# General properties

**$\alpha \in \mathrm{Lip}$**

$H_K^\alpha$ are
- m-sectorial
- $\mathcal{PT}$-symmetric: $\mathcal{PT}H_K^\alpha = \mathcal{PT}H_K^\alpha$
- $\mathcal{P}$-pseudo-Hermitian: $(H_K^\alpha)^* = \mathcal{P}H_K^\alpha\mathcal{P}$
- $\mathcal{T}$-selfadjoint: $(H_K^\alpha)^* = \mathcal{T}H_K^\alpha\mathcal{T}$

$\sigma_r(H_K^\alpha) = \emptyset$

$\sigma(H_K^\alpha) = \sigma_d(H_K^\alpha), \qquad K \in \{-1, 0, 1\}$

**$\alpha$ is a real constant function, $\alpha(x_1) = \alpha_0$**

$$H_0^{\alpha_0} = \bigoplus_{m\in\mathbb{N}_0} (-\partial_2^2 + m^2)$$

$$H_1^{\alpha_0} = \bigoplus_{m\in\mathbb{N}_0} \left(-\partial_2^2 + \tan x_2 \partial_2 + \frac{m^2}{\cos^2 x_2}\right)$$

$$H_{-1}^{\alpha_0} = \bigoplus_{m\in\mathbb{N}_0} \left(-\partial_2^2 - \tanh x_2 \partial_2 + \frac{m^2}{\cosh^2 x_2}\right)$$

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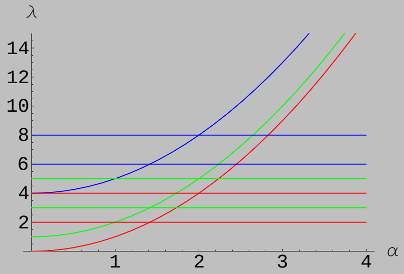
# Width of the strips

# Cylinder $K = 0$, $\alpha_0$

## Spectrum of $H_0^{\alpha_0}$

- $H_0^{\alpha_0}$ is quasi-Hermitian
  - Krejčiřík, Bíla, Znojil, 2006, JPA

$$\lambda_{j,m} = \begin{cases} \alpha^2 + m^2, \\ \left(\frac{j\pi}{2a}\right)^2 + m^2 \end{cases}$$



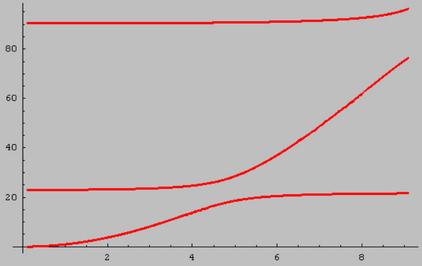
# Sphere $K = 1$, $\alpha_0$

## Spectrum of $H_1^{\alpha_0}$

implicit equation for eigenvaules

$$\begin{vmatrix} \dot{P}_n^{(m)}(b) + i\alpha_0 P_n^{(m)}(b) & \dot{Q}_n^{(m)}(b) + i\alpha_0 Q_n^{(m)}(b) \\ \dot{P}_n^{(m)}(-b) + i\alpha_0 P_n^{(m)}(-b) & \dot{Q}_n^{(m)}(-b) + i\alpha_0 Q_n^{(m)}(-b) \end{vmatrix} = 0,$$

$$b = \sin a, \quad n = 1/2\left(-1 + \sqrt{1 + 4\lambda}\right)$$



# Pseudosphere $K = -1, \alpha_0$

## Spectrum of $H_{-1}^{\alpha_0}$

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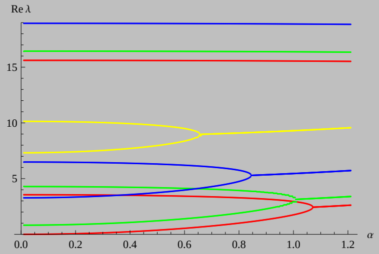
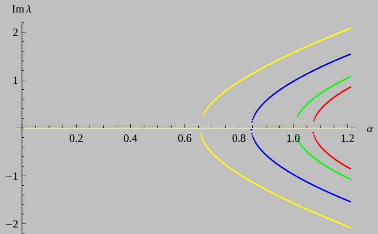
$$\begin{vmatrix} \left(\frac{\dot{P}_k^{(l)}}{\sqrt{\cosh}}\right)(c) + i\alpha_0 \frac{P_k^{(l)}}{\sqrt{\cosh}}(c) & \left(\frac{\dot{Q}_k^{(l)}}{\sqrt{\cosh}}\right)(c) + i\alpha_0 \frac{Q_k^{(l)}}{\sqrt{\cosh}}(c) \\ \left(\frac{\dot{P}_k^{(l)}}{\sqrt{\cosh}}\right)(-c) + i\alpha_0 \frac{Q_k^{(l)}}{\sqrt{\cosh}}(-c) & \left(\frac{\dot{Q}_k^{(l)}}{\sqrt{\cosh}}\right)(-c) + i\alpha_0 Q_k^{(l)}(-c) \end{vmatrix} = 0,$$

$c = \tanh a, \quad k = \sqrt{1 - 4\lambda}, \quad l = mi - 1/2$



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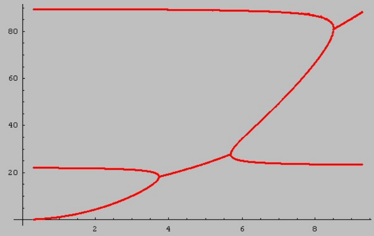


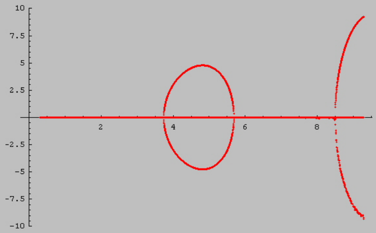

# Conclusions

## Presented models

- three strip-like models were presented
  - strip on a cylinder, sphere and pseudosphere
  - all models possess only discrete spectrum
  - all models are exactly solvable for constant $\alpha$
- curvature essentially influences the spectrum of $\mathcal{PT}$-symmetric models

## Curvature effects

| $K$ | realization | $\sigma(H_K^{\alpha_0})$ | eigenvalues |
|---|---|---|---|
| 0 | cylinder | $\mathbb{R}$ | only some $\alpha$-dependent, crossing |
| 1 | sphere | $\mathbb{R}$ | all $\alpha$-dependent, no crossing |
| -1 | pseudosphere | $\mathbb{C}$ | all $\alpha$-dependent, crossing |

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