Štefan Schwabik; Milan Tvrdý
Boundary value problems for generalized linear differential equations

Czechoslovak Mathematical Journal, Vol. 29 (1979), No. 3, 451--477

Persistent URL: http://dml.cz/dmlcz/101627

Czechoslovak Mathematical Journal, 29 (104) 1979, Praha

BOUNDARY VALUE PROBLEMS FOR GENERALIZED LINEAR DIFFERENTIAL EQUATIONS

ŠTEFAN SCHWABIK, MILAN TVRDÝ, Praha

(Received November 12, 1977)

The paper is devoted to linear boundary value (b.v.) problems for generalized linear differential equations

$$\mathrm{d}\boldsymbol{x} = \mathrm{d}[\boldsymbol{A}]\,\boldsymbol{x} + \mathrm{d}\boldsymbol{f} \quad \left(\boldsymbol{x}(t) = \boldsymbol{x}(0) + \int_0^t \mathrm{d}[\boldsymbol{A}(s)]\,\boldsymbol{x}(s) + \boldsymbol{f}(t) - \boldsymbol{f}(0)\right), \tag{0,1}$$

$$\int_0^1 \mathrm{d}[\boldsymbol{K}]\,\boldsymbol{x} = \boldsymbol{r} \in R_m\,. \tag{0,2}$$

In Section 1 a survey of basic properties of generalized linear differential equations is given. The properties of fundamental matrix solutions to $\mathrm{d}\boldsymbol{x} = \mathrm{d}[\boldsymbol{A}]\,\boldsymbol{x}$ imply a close relationship between the equations (0,1) and

$$\boldsymbol{y}^*(s) + \boldsymbol{y}^*(s)\,\boldsymbol{A}(s) - \boldsymbol{y}^*(t) - \boldsymbol{y}^*(t)\,\boldsymbol{A}(t) + \int_s^t \mathrm{d}[\boldsymbol{y}^*(\tau)]\,\boldsymbol{A}(\tau) = \tag{0,3}$$

$$= \boldsymbol{\varphi}^*(t) - \boldsymbol{\varphi}^*(s)\,, \quad t, s \in [0, 1]\,.$$

Section 2 provides the underlying theory for equations of the type (0,3). In Section 3 and 4 the b.v. problem (0,1), (0,2) is dealt with. The adjoint problem is found in such a way that the usual Fredholm theorem on the existence of a solution holds. Furthermore, the Green matrix is defined and its basic properties are discussed. The results obtained here are generalizations of those given in [2], [7]–[9] and [11].

1. GENERALIZED LINEAR DIFFERENTIAL EQUATIONS

In this section we give a short survey of the basic properties of generalized linear differential equations. More details and the proofs can be found in [7].

1.1. Notation. R_n is the space of real column n-vectors, R_n^* is the space of real row n-vectors, $L(R_n, R_m)$ denotes the space of real $m \times n$-matrices, $L(R_n, R_n) =$

$= L(R_n)$. $\boldsymbol{I}$ is the identity matrix. Given $\boldsymbol{M} = (M_{i,j})_{\substack{i=1,\dots,m \\ j=1,\dots,n}} \in L(R_n, R_m)$, $\boldsymbol{M}^*$ denotes its transpose and $|M| = \max_{i=1,\dots,m} \sum_{j=1}^{n} |M_{i,j}|$. Zero matrices of an arbitrary type are denoted by $\boldsymbol{0}$. In particular, both zero row vectors and zero column vectors are denoted by $\boldsymbol{0}$. The value of the determinant of $\boldsymbol{M} \in L(R_n)$ is denoted by $\det(\boldsymbol{M})$. Given $\alpha, \beta \in R_1$, $\alpha < \beta$, the symbols $[\alpha, \beta]$, (α, β), $[\alpha, \beta)$ and $(\alpha, \beta]$ stand for the closed, open and half-open intervals $\alpha \leqq t \leqq \beta$, $\alpha < t < \beta$, $\alpha \leqq t < \beta$ and $\alpha < t \leqq \leqq \beta$, respectively.

If a matrix valued function $\boldsymbol{F} : [0, 1] \to L(R_n, R_m)$ is of bounded variation on $[0, 1]$ (i.e. any component $f_{i,j}$ of $\boldsymbol{F}$, $\boldsymbol{F}(t) = (f_{i,j}(t))_{\substack{i=1,2,\dots,m \\ j=1,2,\dots,n}}$, is of bounded variation on $[0, 1]$) we write $\boldsymbol{F} \in BV$. The space of functions $\boldsymbol{f} : [0, 1] \to R_n$ of bounded variation on $[0, 1]$ is denoted by $\boldsymbol{BV}_n$.

Given $\boldsymbol{F} \in BV$, $t \in (0, 1]$ and $s \in [0, 1)$, then $\Delta^+ \boldsymbol{F}(s) = \boldsymbol{F}(s+) - \boldsymbol{F}(s)$ and $\Delta^- \boldsymbol{F}(t) = = \boldsymbol{F}(t) - \boldsymbol{F}(t-)$.

1.2. Generalized linear differential equations. Let $\boldsymbol{A} : [0, 1] \to L(R_n)$, $\boldsymbol{A} \in BV$ and $\boldsymbol{f} \in BV_n$. The equation

$$\boldsymbol{x}(t) = \boldsymbol{x}(s) + \int_s^t \mathrm{d}[\boldsymbol{A}(r)]\, \boldsymbol{x}(r) + \boldsymbol{f}(t) - \boldsymbol{f}(s) \tag{1,1}$$

will be called *the generalized linear differential equation*. Let $[a, b] \subset [0, 1]$. A function $\boldsymbol{x} : [a, b] \to R_n$ is said to be *a solution of* (1,1) on $[a, b]$ if (1,1) holds for every $t, s \in [a, b]$. The integral occuring in (1,1) is the Perron-Stieltjes integral,

$$\int_s^t \mathrm{d}[\boldsymbol{A}(r)]\, \boldsymbol{x}(r) = \boldsymbol{v} = (v_i)_{i=1,2,\dots,n} \in R_n\,,$$

$$v_i = \sum_{j=1}^{n} \int_s^t \mathrm{d}[A_{i,j}(r)]\, x_j(r)\,, \quad i = 1, 2, \dots, n\,.$$

We shall use the symbolic transcription

$$\mathrm{d}\boldsymbol{x} = \mathrm{d}[\boldsymbol{A}]\, \boldsymbol{x} + \mathrm{d}\boldsymbol{f} \tag{1,2}$$

for the equation (1,1).

The properties of the initial value problem

$$\mathrm{d}\boldsymbol{x} = \mathrm{d}[\boldsymbol{A}]\, \boldsymbol{x} + \mathrm{d}\boldsymbol{f}\,, \quad \boldsymbol{x}(s) = \boldsymbol{x}_0 \in R_n \tag{1,3}$$

are of a great importance for our purposes. The following existence and uniqueness theorem holds.

1.3. Theorem. *Let* $\boldsymbol{A} : [0, 1] \to L(R_n)$ *be of bounded variation on* $[0, 1]$. *The initial value problem* (1,3) *has a unique solution* $\boldsymbol{x} : [0, 1] \to R_n$ *on* $[0, 1]$ *for any* *given* $s \in [0, 1]$, $\mathbf{x}_0 \in R_n$ *and* $\mathbf{f} \in BV_n$ *if and only if*

$$\det(\mathbf{I} - \Delta^- \mathbf{A}(t)) \neq 0 \quad \textit{for} \quad t \in (0, 1] \quad \textit{and} \tag{1,4}$$
$$\det(\mathbf{I} + \Delta^+ \mathbf{A}(t)) \neq 0 \quad \textit{for} \quad t \in [0, 1) .$$

Let $\mathbf{A} : [0, 1] \to L(R_n)$ *and* $\mathbf{f} : [0, 1] \to R_n$ *be of bounded variation on* $[0, 1]$. *Then the conditions on the regularity of* $\mathbf{I} - \Delta^- \mathbf{A}(t)$, $\mathbf{I} + \Delta^+ \mathbf{A}(t)$ *can be violated only at a finite number of points in* $[0, 1]$. *This is an immediate consequence of the fact that* $|\Delta^- \mathbf{A}(t)| \geqq 1$ *or* $|\Delta^+ \mathbf{A}(t)| \geqq 1$ *may hold only for a finite number of points in* $(0, 1]$ *or* $[0, 1)$, *respectively.*

Given a solution $\mathbf{x}$ *of* (1,2) *on* $[a, b] \subset [0, 1]$, *all the onesided limits* $\mathbf{x}(a+)$, $\mathbf{x}(b-)$, $\mathbf{x}(t+)$, $\mathbf{x}(t-)$, $t \in (a, b)$ *exist and*

$$\mathbf{x}(t+) = [\mathbf{I} + \Delta^+ \mathbf{A}(t)]\, \mathbf{x}(t) + \Delta^+ \mathbf{f}(t) \quad \textit{for} \quad t \in [a, b) , \tag{1,5}$$
$$\mathbf{x}(t-) = [\mathbf{I} - \Delta^- \mathbf{A}(t)]\, \mathbf{x}(t) - \Delta^- \mathbf{f}(t) \quad \textit{for} \quad t \in (a, b] .$$

Moreover, any solution of (1,2) *on* $[a, b]$ *has a bounded variation on* $[a, b]$.

Let us notice that the condition $\det(\mathbf{I} - \Delta^- \mathbf{A}(t)) \neq 0$ enables us to define the solution at the point t if it is known on an interval $[s, t)$ (cf. (1,5)). Similarly, the condition $\det(\mathbf{I} + \Delta^+ \mathbf{A}(t)) \neq 0$ enables us to continue the solution to the point t from the right.

1.4. Theorem. *Let* $\mathbf{A} : [0, 1] \to L(R_n)$ *be of bounded variation on* $[0, 1]$ *and let the conditions* (1,4) *hold. Then for every* $t_0 \in [0, 1]$ *and* $\mathbf{X}_0 \in L(R_n)$ *there exists a unique function* $\mathbf{X} : [0, 1] \to L(R_n)$ *such that* $\mathbf{X}(t_0) = \mathbf{X}_0$ *and*

$$\mathbf{X}(t) = \mathbf{X}(s) + \int_s^t \mathrm{d}[\mathbf{A}(\tau)]\, \mathbf{X}(\tau) \quad \textit{for} \quad t, s \in [0, 1] . \tag{1,6}$$

Moreover, if $\det(\mathbf{X}_0) \neq 0$, *then*

$$\det(\mathbf{X}(t)) \neq 0 \quad \textit{on} \quad [0, 1] . \tag{1,7}$$

1.5. Definition. Any matrix valued function $\mathbf{X} : [0, 1] \to L(R_n)$ fulfilling (1,6) and (1,7) is called *a fundamental matrix solution* of the homogeneous equation

$$\mathrm{d}\mathbf{x} = \mathrm{d}[\mathbf{A}]\, \mathbf{x} . \tag{1,8}$$

For any fundamental matrix solution $\mathbf{X}$ of (1,8) its inverse $\mathbf{X}^{-1}$ is defined on $[0, 1]$, has a bounded variation on $[0, 1]$ and satisfies the relation

$$\mathbf{X}^{-1}(t) = \mathbf{X}^{-1}(s) - \mathbf{X}^{-1}(t)\, \mathbf{A}(t) + \mathbf{X}^{-1}(s)\, \mathbf{A}(s) + \int_s^t \mathrm{d}[\mathbf{X}^{-1}(\tau)]\, \mathbf{A}(\tau) \tag{1,9}$$
$$\text{for} \quad t, s \in [0, 1] .$$

The following statement is important for our considerations:

1.6. Theorem. *Let* $\mathbf{A} : [0, 1] \to L(R_n)$, $\mathbf{A} \in BV$ *and* (1,4) *hold. Then for every* $\mathbf{f} \in BV_n$, $s \in [0, 1]$ *and* $\mathbf{x}_0 \in R_n$ *the unique solution* $\mathbf{x} : [0, 1] \to R_n$ *of* (1,3) *is given by the variation-of-constants formula*

$$(1,10) \qquad \mathbf{x}(t) = \mathbf{X}(t)\,\mathbf{X}^{-1}(s)\,\mathbf{x}_0 + \mathbf{f}(t) - \mathbf{f}(s) - \mathbf{X}(t)\int_s^t \mathrm{d}[\mathbf{X}^{-1}(\tau)]\,(\mathbf{f}(\tau) - \mathbf{f}(s))\,,$$

where $\mathbf{X} : [0, 1] \to L(R_n)$ *is an arbitrary fundamental matrix solution of* (1,8).

2. FORMALLY ADJOINT EQUATION

The equation (1,9) which is satisfied by the inverse $\mathbf{X}^{-1}$ to a fundamental matrix solution $\mathbf{X}$ of (1,8) is not a generalized linear differential equation of the type (1,2). This leads us to the consideration of equations of the form

$$(2,1) \qquad -\mathbf{y}^*(t) - \mathbf{y}^*(t)\,\mathbf{A}(t) + \mathbf{y}^*(s) + \mathbf{y}^*(s)\,\mathbf{A}(s) + \int_s^t \mathrm{d}[\mathbf{y}^*(r)]\,\mathbf{A}(r) = \\ = \varphi^*(t) - \varphi^*(s)$$

or in the abbreviated form,

$$(2,2) \qquad -\mathrm{d}\mathbf{y}^* - \mathrm{d}[\mathbf{y}^*\mathbf{A}] + \mathrm{d}[\mathbf{y}^*]\,\mathbf{A} = \mathrm{d}\varphi^*\,.$$

2.1. Definition. The function $\mathbf{y} : [a, b] \to R_n$ is *a solution* of (2,2) on $[a, b] \subset \subset [0, 1]$ if (2,1) holds for all $t, s \in [a, b]$.

Let us notice that if $\mathbf{A} \in BV$ is continuous on $[0, 1]$ and $\int_0^1 \mathbf{y}^*(r)\,\mathrm{d}[\mathbf{A}(r)]$ exists, the integration by parts yields

$$\int_s^t \mathrm{d}[\mathbf{y}^*(r)]\,\mathbf{A}(r) - \mathbf{y}^*(t)\,\mathbf{A}(t) + \mathbf{y}^*(s)\,\mathbf{A}(s) = -\int_s^t \mathbf{y}^*(r)\,\mathrm{d}[\mathbf{A}(r)]$$

for all $t, s \in [0, 1]$ and (2,2) becomes the transposition of an equation of the form (1,2). Of course, as the general integration-by-parts formula involves also the jumps of $\mathbf{y}$ and $\mathbf{A}$, this is not the case in general.

It is of primary interest to prove the existence and uniqueness of a solution to the equation (2,2) on the whole interval $[0, 1]$. Though it is possible to do it directly, we shall give here a proof which makes use of the properties of the fundamental matrix solution to (1,8).

We suppose $\mathbf{A} : [0, 1] \to L(R_n)$, $\mathbf{A} \in BV$, (1,4) (i.e. the hypotheses of Theorem 1.4 are satisfied) and $\varphi \in BV_n$. Let $\mathbf{X} : [0, 1] \to L(R_n)$ be an arbitrary matrix solution of (1,8) and let us put

$$(2,3) \qquad \mathbf{z}^*(s) = \int_s^1 \mathrm{d}[\varphi^*(r)]\,\mathbf{X}(r)\,\mathbf{X}^{-1}(s) \quad \text{for} \quad s \in [0, 1]\,.$$

The function $\mathbf{z} : [0, 1] \to R_n$ is of bounded variation on $[0, 1]$. Let $t, s \in [0, 1]$, $s < t$ be given arbitrarily and let us calculate

$$\int_s^t \mathrm{d}[\mathbf{z}^*(r)]\, \mathbf{A}(r) = \int_s^t \mathrm{d}_r \left[\int_r^1 \mathrm{d}[\boldsymbol{\varphi}^*(\tau)]\, \mathbf{X}(\tau)\, \mathbf{X}^{-1}(r) \right] \mathbf{A}(r) =$$

$$= \int_s^t \mathrm{d}_r \left[\int_r^t \mathrm{d}[\boldsymbol{\varphi}^*(\tau)]\, \mathbf{X}(\tau)\, \mathbf{X}^{-1}(r) \right] \mathbf{A}(r) + \int_s^t \mathrm{d}_r \left[\int_t^1 \mathrm{d}[\boldsymbol{\varphi}^*(\tau)]\, \mathbf{X}(\tau)\, \mathbf{X}^{-1}(r) \right] \mathbf{A}(r) =$$

$$= \int_s^t \mathrm{d}_r \left[\int_r^t \mathrm{d}[\boldsymbol{\varphi}^*(\tau)]\, \mathbf{X}(\tau)\, \mathbf{X}^{-1}(r) \right] \mathbf{A}(r) + \int_s^t \left(\int_t^1 \mathrm{d}[\boldsymbol{\varphi}^*(\tau)]\, \mathbf{X}(\tau) \right) \mathrm{d}_r[\mathbf{X}^{-1}(r)]\, \mathbf{A}(r) .$$

Thus

$$\text{(2,4)} \qquad \int_s^t \mathrm{d}[\mathbf{z}^*(r)]\, \mathbf{A}(r) = \int_s^t \mathrm{d}_r \left[\int_r^t \mathrm{d}[\boldsymbol{\varphi}^*(\tau)]\, \mathbf{X}(\tau)\, \mathbf{X}^{-1}(r) \right] \mathbf{A}(r) + $$

$$+ \left(\int_t^1 \mathrm{d}[\boldsymbol{\varphi}^*(\tau)]\, \mathbf{X}(\tau) \right) \left(\int_s^t \mathrm{d}[\mathbf{X}^{-1}(r)]\, \mathbf{A}(r) \right).$$

In the first integral on the right-hand side of (2,4) we shall interchange the order of integration. To this aim, let us put

$$\text{(2,5)} \qquad \mathbf{Q}(\tau, r) = \begin{cases} \mathbf{X}(\tau)\, \mathbf{X}^{-1}(r) & \text{for} \quad r \leqq \tau , \\ \mathbf{X}(\tau)\, \mathbf{X}^{-1}(\tau) = \mathbf{I} & \text{for} \quad \tau \leqq r . \end{cases}$$

Evidently $\mathbf{Q}$ is of bounded twodimensional Vitali variation on $[0, 1] \times [0, 1]$ and $\operatorname{var}_0^1 \mathbf{Q}(0, \cdot) + \operatorname{var}_0^1 \mathbf{Q}(\cdot, 0) < \infty$. (Both $\mathbf{X}$ and $\mathbf{X}^{-1}$ are of bounded variation on $[0, 1]$.) Furthermore, for any $r \in [s, t]$ we have

$$\int_s^t \mathrm{d}[\boldsymbol{\varphi}^*(\tau)]\, \mathbf{Q}(\tau, r) = \int_s^r \mathrm{d}[\boldsymbol{\varphi}^*(\tau)]\, \mathbf{Q}(\tau, r) +$$

$$+ \int_r^t \mathrm{d}[\boldsymbol{\varphi}^*(\tau)]\, \mathbf{Q}(\tau, r) = \int_s^r \mathrm{d}[\boldsymbol{\varphi}^*(\tau)] + \int_r^t \mathrm{d}[\boldsymbol{\varphi}^*(\tau)]\, \mathbf{X}(\tau)\, \mathbf{X}^{-1}(r) .$$

Hence

$$\int_r^t \mathrm{d}[\boldsymbol{\varphi}^*(\tau)]\, \mathbf{X}(\tau)\, \mathbf{X}^{-1}(r) = \int_s^t \mathrm{d}[\boldsymbol{\varphi}^*(\tau)]\, \mathbf{Q}(\tau, r) - \int_s^r \mathrm{d}[\boldsymbol{\varphi}^*(\tau)]$$

and

$$\boldsymbol{\alpha}^* = \int_s^t \mathrm{d}_r \left[\int_r^t \mathrm{d}[\boldsymbol{\varphi}^*(\tau)]\, \mathbf{X}(\tau) \mathbf{X}^{-1}(r) \right] \mathbf{A}(r) =$$

$$= \int_s^t \mathrm{d}_r \left[\int_s^t \mathrm{d}[\boldsymbol{\varphi}^*(\tau)]\, \mathbf{Q}(\tau, r) \right] \mathbf{A}(r) - \int_s^t \mathrm{d}_r \left[\int_s^r \mathrm{d}[\boldsymbol{\varphi}^*(\tau)] \right] \mathbf{A}(r) .$$

Moreover, using [7] I.6.22 (or [6] Lemma 2.2) and taking into account (2,5), we obtain

$$\alpha^* = \int_s^t \mathrm{d}[\varphi^*(\tau)] \left(\int_s^t \mathrm{d}_r[\mathbf{Q}(\tau, r)] \, \mathbf{A}(r) \right) - \int_s^t \mathrm{d}[\varphi^*(r)] \, \mathbf{A}(r) =$$

$$= \int_s^t \mathrm{d}[\varphi^*(\tau)] \left(\int_s^\tau \mathbf{X}(\tau) \, \mathrm{d}[\mathbf{X}^{-1}(r)] \, \mathbf{A}(r) \right) - \int_s^t \mathrm{d}[\varphi^*(r)] \, \mathbf{A}(r) .$$

Inserting this into (2,4) and making use of (1,9) we get

$$\int_s^t \mathrm{d}[\mathbf{z}^*(r)] \, \mathbf{A}(r) = \int_s^t \mathrm{d}[\varphi^*(\tau)] \, \mathbf{X}(\tau) \left(\int_s^\tau \mathrm{d}[\mathbf{X}^{-1}(r)] \, \mathbf{A}(r) \right) -$$

$$- \int_s^t \mathrm{d}[\varphi^*(\tau)] \, \mathbf{A}(\tau) + \left(\int_t^1 \mathrm{d}[\varphi^*(\tau)] \, \mathbf{X}(\tau) \right) \left(\int_s^t \mathrm{d}[\mathbf{X}^{-1}(r)] \, \mathbf{A}(r) \right) =$$

$$= \int_s^t \mathrm{d}[\varphi^*(\tau)] \, \mathbf{X}(\tau) \, [\mathbf{X}^{-1}(\tau) \, (\mathbf{I} + \mathbf{A}(\tau)) - \mathbf{X}^{-1}(s) \, (\mathbf{I} + \mathbf{A}(s))] - \int_s^t \mathrm{d}[\varphi^*(\tau)] \, \mathbf{A}(\tau) +$$

$$+ \left(\int_t^1 \mathrm{d}[\varphi^*(\tau)] \, \mathbf{X}(\tau) \right) [\mathbf{X}^{-1}(t) \, (\mathbf{I} + \mathbf{A}(t)) - \mathbf{X}^{-1}(s) \, (\mathbf{I} + \mathbf{A}(s))] =$$

$$= \int_s^t \mathrm{d}[\varphi^*(\tau)] - \left(\int_s^1 \mathrm{d}[\varphi^*(\tau)] \, \mathbf{X}(\tau) \, \mathbf{X}^{-1}(s) \right) (\mathbf{I} + \mathbf{A}(s)) +$$

$$+ \left(\int_t^1 \mathrm{d}[\varphi^*(\tau)] \, \mathbf{X}(\tau) \, \mathbf{X}^{-1}(t) \right) (\mathbf{I} + \mathbf{A}(t)) .$$

Consequently, for $\mathbf{z} : [0, 1] \to R_n$ given by (2,3) we have

$$\text{(2,6)} \qquad \int_s^t \mathrm{d}[\mathbf{z}^*(r)] \, \mathbf{A}(r) = \mathbf{z}^*(t) + \mathbf{z}^*(t) \, \mathbf{A}(t) - \mathbf{z}^*(s) - \mathbf{z}^*(s) \, \mathbf{A}(s) +$$
$$+ \varphi^*(t) - \varphi^*(s) \quad \text{for} \quad t, s \in [0, 1] ,$$

i.e. $\mathbf{z}$ satisfies (2,1) for every $t, s \in [0, 1]$.

2.2. Lemma. *Let $\mathbf{A} : [0, 1] \to L(R_n)$ be of bounded variation on $[0, 1]$ and fulfil* (1,4). *Then for arbitrary $\mathbf{c} \in R_n$ and $\varphi \in BV_n$ the function*

$$\text{(2,7)} \qquad \mathbf{y}^*(s) = \mathbf{c}^* \, \mathbf{X}(1) \, \mathbf{X}^{-1}(s) + \int_s^1 \mathrm{d}[\varphi^*(\tau)] \, \mathbf{X}(\tau) \, \mathbf{X}^{-1}(s)$$

is a solution to (2,2) *on* $[0, 1]$.

Proof. Let us put $\mathbf{v}^*(s) = \mathbf{c}^* \, \mathbf{X}(1) \, \mathbf{X}^{-1}(s)$ on $[0, 1]$. Then $\mathbf{y}(s) = \mathbf{z}(s) + \mathbf{v}(s)$ with $\mathbf{z} : [0, 1] \to R_n$ given by (2,3). Multiplying the matrix equation (1,9) from the left by $\mathbf{c}^* \mathbf{X}(1)$ we obtain

$$\mathbf{v}^*(t) + \mathbf{v}^*(t) \mathbf{A}(t) - \mathbf{v}^*(s) - \mathbf{v}^*(s) \mathbf{A}(s) = \int_s^t \mathrm{d}[\mathbf{V}^*(r)] \mathbf{A}(r), \quad t, s \in [0, 1],$$

wherefrom our assertion readily follows by virtue of (2,6).

In particular, given $\mathbf{c} \in R_n$ and $\varphi \in BV_n$, the function (2,7) is a solution of the initial value problem

(2,8) $$-\mathrm{d}\mathbf{y}^* - \mathrm{d}[\mathbf{y}^* \mathbf{A}] + \mathrm{d}[\mathbf{y}^*] \mathbf{A} = \mathrm{d}\varphi^*, \quad \mathbf{y}^*(1) = \mathbf{c}^*.$$

We wish to have also a unicity result. For this reason let us consider the homogeneous initial value problem

(2,9) $$\mathrm{d}\mathbf{y}^* + \mathrm{d}[\mathbf{y}^* \mathbf{A}] - \mathrm{d}[\mathbf{y}^*] \mathbf{A} = \mathbf{0}, \quad \mathbf{y}^*(1) = \mathbf{0}.$$

Every solution $\mathbf{y}$ of (2,9) on $[0, 1]$ satisfies

(2,10) $$\mathbf{y}^*(s) + \mathbf{y}^*(s) \mathbf{A}(s) + \int_s^1 \mathrm{d}[\mathbf{y}^*(r)] \mathbf{A}(r) = \mathbf{0} \quad \text{on} \quad [0, 1],$$
$$\mathbf{y}^*(1) = \mathbf{0}.$$

Clearly, the function $\mathbf{y}(s) = \mathbf{0}$, $s \in [0, 1]$ is a solution of (2,9) on $[0, 1]$.

2.3. Lemma. *Under the assumptions of Lemma 2.2, every solution $\mathbf{y} : [0, 1] \to R_n$ of (2,9) on $[0, 1]$ possesses the onesided limits $\mathbf{y}(t+)$, $\mathbf{y}(t-)$ and the relations*

(2,11) $$\mathbf{y}^*(t+) = \mathbf{y}^*(t) [\mathbf{I} + \Delta^+ \mathbf{A}(t)]^{-1}, \quad \mathbf{y}^*(t-) = \mathbf{y}^*(t) [\mathbf{I} - \Delta^- \mathbf{A}(t)]^{-1}$$

hold for $t \in [0, 1)$ and $t \in (0, 1]$, respectively.

Proof. Given a solution $\mathbf{y} : [0, 1] \to R_n$ of (2,9) on $[0, 1]$, $t \in [0, 1)$ and $\delta > 0$, we have by (2,10)

$$\mathbf{y}^*(t + \delta) + \mathbf{y}^*(t + \delta) \mathbf{A}(t + \delta) - \mathbf{y}^*(t) - \mathbf{y}^*(t) \mathbf{A}(t) = \int_t^{t+\delta} \mathrm{d}[\mathbf{y}^*(r)] \mathbf{A}(r).$$

Since by [7] I.4.12

$$\lim_{\delta \to 0+} \left(\int_t^{t+\delta} \mathrm{d}[\mathbf{y}^*(r)] \mathbf{A}(r) - \mathbf{y}^*(t + \delta) \mathbf{A}(t) + \mathbf{y}^*(t) \mathbf{A}(t) \right) = \mathbf{0},$$

it follows that

$$\lim_{\delta \to 0+} (\mathbf{y}^*(t + \delta) + \mathbf{y}^*(t + \delta) \mathbf{A}(t + \delta) - \mathbf{y}^*(t) - \mathbf{y}^*(t + \delta) \mathbf{A}(t)) =$$
$$= \lim_{\delta \to 0+} \left(\int_t^{t+\delta} \mathrm{d}[\mathbf{y}^*(r)] \mathbf{A}(r) - \mathbf{y}^*(t + \delta) \mathbf{A}(t) + \mathbf{y}^*(t) \mathbf{A}(t) \right) = \mathbf{0},$$

i.e.

$$\lim_{\delta \to 0+} (\mathbf{y}^*(t + \delta) + \mathbf{y}^*(t + \delta)(\mathbf{A}(t + \delta) - \mathbf{A}(t))) =$$

$$= \lim_{\delta \to 0+} \mathbf{y}^*(t + \delta)(\mathbf{I} + \mathbf{A}(t + \delta) - \mathbf{A}(t)) = \mathbf{y}^*(t).$$

Since $\lim_{\delta \to 0+} \mathbf{I} + \mathbf{A}(t + \delta) - \mathbf{A}(t) = \mathbf{I} + \Delta^+ \mathbf{A}(t)$ is a nonsingular $n \times n$-matrix, we conclude that the limit $\mathbf{y}^*(t+)$ exists and, furthermore,

$$\mathbf{y}^*(t+) = \lim_{\delta \to 0+} \mathbf{y}^*(t + \delta) = \lim_{\delta \to 0+} (\mathbf{y}^*(t + \delta)[\mathbf{I} + \mathbf{A}(t + \delta) - \mathbf{A}(t)] \, .$$

$$. [\mathbf{I} + \mathbf{A}(t + \delta) - \mathbf{A}(t)]^{-1}) = \mathbf{y}^*(t)[\mathbf{I} + \Delta^+ \mathbf{A}(t)]^{-1} .$$

Analogously we can obtain the existence of $\mathbf{y}^*(t-)$ and the second relation in (2,11) for $t \in (0, 1]$.

2.4. Lemma. *Under the assumptions of Lemma* 2.2, *every solution* $\mathbf{y} : [0, 1] \to R_n$ *of* (2,9) *is bounded on* $[0, 1]$ *and satisfies*

$$\sum_{t \in (0,1]} |\Delta^+ \mathbf{y}(t)| + \sum_{t \in [0,1)} |\Delta^- \mathbf{y}(t)| < \infty .$$

Proof. Since $\mathbf{y}$ possesses onesided limits on $[0, 1]$ by Lemma 2.3, for every $t_0 \in [0, 1]$ there exist $\delta > 0$ and $M > 0$ such that $|\mathbf{y}(t)| \leqq M$ on $(t_0 - \delta, t_0 + \delta) \cap$ $\cap [0, 1]$. Using the Heine-Borel Covering Theorem we can easily show the boundedness of $\mathbf{y}$ on $[0, 1]$. Let $K = \sup_{t \in [0,1]} |\mathbf{y}^*(t)|$.

Furthermore, $\mathbf{y}$ has at most countable number of points of discontinuity in $[0, 1]$ and the series in question are well defined. By (2,11)

$$|\Delta^+ \mathbf{y}^*(t)| = |-\mathbf{y}^*(t+) \Delta^+ \mathbf{A}(t)| \leqq K |\Delta^+ \mathbf{A}(t)| \quad \text{for} \quad t \in [0, 1)$$

and consequently

$$\sum_{t \in [0,1)} |\Delta^+ \mathbf{y}^*(t)| \leqq \sum_{t \in [0,1)} K |\Delta^+ \mathbf{A}(t)| \leqq K \operatorname{var}_0^1 \mathbf{A} < \infty .$$

Similarly

$$\sum_{t \in (0,1]} |\Delta^- \mathbf{y}^*(t)| \leqq K \operatorname{var}_0^1 \mathbf{A} < \infty .$$

2.5. Lemma. *Let* $\mathbf{f} : [a, b] \to R_n$ *possess the onesided limits* $\mathbf{f}(t+)$ *on* $[a, b)$ *and* $\mathbf{f}(t-)$ *on* $(a, b]$ *and let*

$$\sum_{t \in [a,b)} |\Delta^+ \mathbf{f}(t)| + \sum_{t \in (a,b]} |\Delta^- \mathbf{f}(t)| < \infty . \tag{2,12}$$

Then for any $\mathbf{g} : [a, b] \to R_n$ *of bounded variation on* $[a, b]$ *both the integrals*

$$\int_a^b \mathrm{d}[\mathbf{f}^*(t)]\, \mathbf{g}(t) \quad \text{and} \quad \int_a^b \mathbf{f}^*(t)\, \mathrm{d}[\mathbf{g}(t)] \tag{2,13}$$

exist and the integration-by-parts formula holds in the form

$$\int_a^b \mathbf{f}^*(t)\, \mathrm{d}[\mathbf{g}(t)] + \int_a^b \mathrm{d}[\mathbf{f}^*(t)]\, \mathbf{g}(t) = \mathbf{f}^*(b)\, \mathbf{g}(b) - \mathbf{f}^*(a)\, \mathbf{g}(a) - \tag{2,14}$$
$$- \sum_{a \leqq \tau < b} \Delta^+ \mathbf{f}^*(\tau)\, \Delta^+ \mathbf{g}(\tau) + \sum_{a < \tau \leqq b} \Delta^- \mathbf{f}^*(\tau)\, \Delta^- \mathbf{g}(\tau)\,.$$

Proof. Let us put

$$\mathbf{f}_b(t) = \sum_{a \leqq \tau < t} \Delta^+ \mathbf{f}(\tau) - \sum_{a < \tau \leqq t} \Delta^- \mathbf{f}(\tau) \quad \text{for} \quad t \in [a, b]\,.$$

Obviously $\operatorname{var}_a^b \mathbf{f}_b < \infty$, $\Delta^+ \mathbf{f}_b(t) = \Delta^+ \mathbf{f}(t)$ on $[a, b)$, $\Delta^- \mathbf{f}_b(t) = \Delta^- \mathbf{f}(t)$ on $(a, b]$ and it is a matter of routine to show that the function $\mathbf{f}_c : [a, b] \to R_n$ given by

$$\mathbf{f}_c(t) = \mathbf{f}(t) - \mathbf{f}_b(t) \quad \text{on} \quad [a, b]$$

is continuous on $[a, b]$. Consequently, the integrals

$$\int_a^b \mathrm{d}[\mathbf{f}_c^*(t)]\, \mathbf{g}(t)\,, \quad \int_a^b \mathrm{d}[\mathbf{f}_b^*(t)]\, \mathbf{g}(t)\,, \quad \int_a^b \mathbf{f}_c^*(t)\, \mathrm{d}[\mathbf{g}(t)]\,, \quad \int_a^b \mathbf{f}_b^*(t)\, \mathrm{d}[\mathbf{g}(t)]$$

as well as (2,13) all exist. Applying Kurzweil's Integration-by-parts Theorem ([5]) we obtain readily (2,14).

2.6. Lemma. *Under the assumptions of Lemma* 2.2 *the homogeneous initial value problem* (2,9) *possesses only the trivial solution* $\mathbf{y}(t) \equiv \mathbf{0}$ on $[0, 1]$.

Proof. Let $\mathbf{y} : [0, 1] \to R_n$ satisfy (2,9) (or (2,10)). By Lemmas 2.4 and 2.5 both the integrals

$$\int_t^1 \mathrm{d}[\mathbf{y}^*(r)]\, \mathbf{A}(r) \quad \text{and} \quad \int_t^1 \mathbf{y}^*(r)\, \mathrm{d}[\mathbf{A}(r)]$$

exist and

$$\int_t^1 \mathrm{d}[\mathbf{y}^*(r)]\, \mathbf{A}(r) = - \int_t^1 \mathbf{y}^*(r)\, \mathrm{d}[\mathbf{A}(r)] + \mathbf{y}^*(1)\, \mathbf{A}(1) - \mathbf{y}^*(t)\, \mathbf{A}(t) -$$
$$- \sum_{t \leqq \tau < 1} \Delta^+ \mathbf{y}^*(\tau)\, \Delta^+ \mathbf{A}(\tau) + \sum_{t < \tau \leqq 1} \Delta^- \mathbf{y}^*(\tau)\, \Delta^- \mathbf{A}(\tau)$$

for any $t \in [0, 1]$. Inserting (2,10) we obtain

$$\mathbf{y}^*(t) = \int_t^1 \mathbf{y}^*(r)\, \mathrm{d}[\mathbf{A}(r)] + \sum_{t \leqq \tau < 1} \Delta^+ \mathbf{y}^*(\tau)\, \Delta^+ \mathbf{A}(\tau) - \Delta^- \mathbf{y}^*(\tau)\, \Delta^- \mathbf{A}(\tau) \quad \text{on} \quad [0, 1]\,.$$

Since $\mathbf{y}$ is bounded on $[0, 1]$ by Lemma 2.4, it follows that $\mathbf{y}^*$ is of bounded variation on $[0, 1]$

$$(\mathrm{var}_0^1\, \mathbf{y}^* \leqq (\sup_{t\in[0,1]} |\mathbf{y}^*(t)| + \sum_{0\leqq t<1} |\Delta^+\mathbf{y}^*(t)| + \sum_{0<t\leqq 1} |\Delta^-\mathbf{y}^*(t)|)\, \mathrm{var}_0^1\, \mathbf{A})\,.$$

Let us define

$$\xi(t) = \mathrm{var}_0^t\, \mathbf{A} \quad \text{for} \quad t \in [0, 1]\,.$$

For a given $s \in [0, 1)$ we have $(\mathbf{y}^*(1) = \mathbf{0})$

$$\mathrm{var}_s^1\, \mathbf{y}^* \leqq \int_s^1 |\mathbf{y}^*(r)|\, \mathrm{d}[\xi(r)] + \sum_{s\leqq\tau<1} |\Delta^+\mathbf{y}(\tau)|\, |\Delta^+\mathbf{A}(\tau)| +$$
$$+ \sum_{s<\tau\leqq 1} |\Delta^-\mathbf{y}^*(\tau)|\, |\Delta^-\mathbf{A}(\tau)| \leqq$$
$$\leqq (\mathrm{var}_s^1\, \mathbf{y}^*)\, (\xi(1-) - \xi(s) + \sum_{s\leqq\tau<1} |\Delta^+\mathbf{A}(\tau)| + \sum_{s<\tau\leqq 1} |\Delta^-\mathbf{A}(\tau)|)\,.$$

Obviously

$$\lim_{s\to 1-} [\xi(1-) - \xi(s) + \sum_{s\leqq\tau<1} |\Delta^+\mathbf{A}(\tau)| + \sum_{s<\tau\leqq 1} |\Delta^-\mathbf{A}(\tau)|] = 0$$

and hence such $s \in [0, 1)$ can be found that

$$\mathrm{var}_s^1\, \mathbf{y}^* \leqq \tfrac{1}{2}\, \mathrm{var}_s^1\, \mathbf{y}^*\,,$$

i.e. $\mathbf{y}^*(t) = \mathbf{0}$ on $[s, 1]$. Let $s^* \in [0, 1]$ be the infimum of such s. Then $\mathbf{y}^*(t) = \mathbf{0}$ on $(s^*, 1]$ and according to (2,11) also $\mathbf{y}^*(s^*) = \mathbf{0}$. Assume that $s^* > 0$. By the same argument as above we can deduce that there is $s' \in [0, s^*)$ such that $\mathbf{y}^*(t) = \mathbf{0}$ on $[s', s^*]$, i.e. $\mathbf{y}^*(t) = \mathbf{0}$ on $[s'. 1]$. This contradicts the definition of s^*. Thus $s^* = 0$ and $\mathbf{y}^*(t) \equiv \mathbf{0}$ on $[0, 1]$.

Lemmas 2.2 and 2.6 together yield the following theorem.

2.7. Theorem. *Let $\varphi \in BV_n$, $\mathbf{A} : [0, 1] \to L(R_n)$, $\mathbf{A} \in BV$ and let (1,4) hold. Then the initial value problem (2,8) has for every $\mathbf{c} \in R_n$ a unique solution $\mathbf{y} : [0, 1] \to R_n$ on $[0, 1]$. This solution has a bounded variation on $[0, 1]$ and is given on $[0, 1]$ by (2,7) (where $\mathbf{X} : [0, 1] \to L(R_n)$ is an arbitrary fundamental matrix solution of (1,8)).*

2.8. Remark. Assume that $\mathbf{B} : [0, 1] \to L(R_n)$ and $\mathbf{g} : [0, 1] \to R_n$ are Lebesgue integrable on $[0, 1]$ (all their components are Lebesgue integrable on $[0, 1]$). Let us consider the ordinary linear differential equation

$$\dot{\mathbf{x}} = \mathbf{B}(t)\, \mathbf{x} + \mathbf{g}(t) \tag{2,15}$$

in the sense of Carathéodory. A function $\mathbf{x} : [a, b] \to R_n$ is its solution on $[a, b] \subset$

$\subset [0, 1]$ if

$$\mathbf{x}(t) = \mathbf{x}(s) + \int_s^t \mathbf{B}(\tau)\, \mathbf{x}(\tau)\, \mathrm{d}\tau + \int_s^t \mathbf{g}(\tau)\, \mathrm{d}\tau \tag{2,16}$$

holds for every $t, s \in [a, b]$. Define

$$\mathbf{A}(t) = \int_0^t \mathbf{B}(\tau)\, \mathrm{d}\tau\,, \quad \mathbf{f}(t) = \int_0^t \mathbf{g}(\tau)\, \mathrm{d}\tau\,,$$

then $\mathbf{A}$ and $\mathbf{f}$ are absolutely continuous on $[0, 1]$ and the relation (2,16) becomes

$$\mathbf{x}(t) = \mathbf{x}(s) + \int_s^t \mathrm{d}[\mathbf{A}(\tau)]\, \mathbf{x}(\tau) + \mathbf{f}(t) - \mathbf{f}(s)$$

or equivalently

$$\mathrm{d}\mathbf{x} = \mathrm{d}[\mathbf{A}]\, \mathbf{x} + \mathrm{d}\mathbf{f}\,.$$

Consider now the usual (formal) adjoint equation to (2,15)

$$\dot{\mathbf{y}}^* = -\mathbf{y}^*\, \mathbf{B}(t) - \psi^*(t)\,. \tag{2,17}$$

This equation written in the integral form becomes

$$\mathbf{y}^*(t) = \mathbf{y}^*(s) - \int_s^t \mathbf{y}^*(\tau)\, \mathbf{B}(\tau)\, \mathrm{d}\tau - \int_s^t \psi^*(\tau)\, \mathrm{d}\tau$$

or

$$\mathbf{y}^*()) = \mathbf{y}^*(s) - \int_s^t \mathbf{y}^*(\tau)\, \mathrm{d}[\mathbf{A}(\tau)] - (\varphi^*(t) - \varphi^*(s)) \tag{2,18}$$

where

$$\varphi^*(t) = \int_0^t \psi^*(\tau)\, \mathrm{d}\tau\,.$$

The integral in (2,18) may be replaced by

$$\mathbf{y}^*(t)\, \mathbf{A}(t) - \mathbf{y}^*(s)\, \mathbf{A}(s) - \int_s^t \mathrm{d}[\mathbf{y}^*(\tau)]\, \mathbf{A}(\tau)\,.$$

Thus $\mathbf{y} : [0, 1] \to R_n$ is a (Carathéodory) solution to (2,17) on $[0, 1]$ if and only if

$$\mathbf{y}^*(t) = \mathbf{y}^*(s) - \mathbf{y}^*(t)\, \mathbf{A}(t) + \mathbf{y}^*(s)\, \mathbf{A}(s) + \int_s^t \mathrm{d}[\mathbf{y}^*(\tau)]\, \mathbf{A}(\tau) - \varphi^*(t) + \varphi^*(s)$$

for all $t, s \in [0, 1]$ or equivalently if and only if it is a solution to

$$\mathrm{d}\mathbf{y}^* + \mathrm{d}[\mathbf{y}^*\mathbf{A}] - \mathrm{d}[\mathbf{y}^*]\, \mathbf{A} = -\mathrm{d}\varphi^*$$

on $[0, 1]$.

Thus, if (2,15) is rewritten as $\mathrm{d}\mathbf{x} = \mathrm{d}[\mathbf{A}]\,\mathbf{x} + \mathrm{d}\mathbf{f}$, then both (2,18) and (2,2) are equivalent to its adjoint (2,17). This means that both (2,18) and (2,2) may be considered as generalized forms of (2,17).

3. BOUNDARY VALUE PROBLEM

3.1. Assumptions. In the sequel we assume that $\mathbf{A} : [0, 1] \to L(R_n)$, $\mathbf{A} \in BV$ and (1,4) holds, i.e.

$$\det(\mathbf{I} - \Delta^{-}\mathbf{A}(t)) \neq 0 \text{ for } t \in (0, 1]\,,$$

$$\det(\mathbf{I} + \Delta^{+}\mathbf{A}(t)) \neq 0 \text{ for } t \in [0, 1)\,.$$

Furthermore $\mathbf{K} : [0, 1] \to L(R_n, R_m)$ is of bounded variation on $[0, 1]$, $\mathbf{f} \in BV_n$ and $\mathbf{r} \in R_m$.

Let us consider the boundary value problem of finding a solution $\mathbf{x} : [0, 1] \to R_n$ of

(3,1) $$\mathrm{d}\mathbf{x} = \mathrm{d}[\mathbf{A}]\,\mathbf{x} + \mathrm{d}\mathbf{f}$$

on $[0, 1]$ which fulfils also the side condition

(3,2) $$\int_0^1 \mathrm{d}[\mathbf{K}(t)]\,\mathbf{x}(t) = \mathbf{r}\,.$$

3.2. Remark. Let us mention that the side condition

(3,3) $$\mathbf{M}\,\mathbf{x}(0) + \mathbf{N}\,\mathbf{x}(1) + \int_0^1 \mathrm{d}[\mathbf{H}(t)]\,\mathbf{x}(t) = \mathbf{r}$$

with $\mathbf{M}, \mathbf{N} \in L(R_n, R_m)$, $\mathbf{H} : [0, 1] \to L(R_n, R_m)$, $\mathbf{H} \in BV$ assumes the from (3.2) if we put

$$\mathbf{K}(t) = \begin{cases} -\mathbf{M} + \mathbf{H}(0) & \text{for } t = 0\,, \\ \mathbf{H}(t) & \text{for } 0 < t < 1\,, \\ \mathbf{N} + \mathbf{H}(1) & \text{for } t = 1\,. \end{cases}$$

Using the variation-of-constants formula for generalized linear differential equations we obtain the following algebraic solvability condition.

3.3. Lemma. *The boundary value problem* (3,1), (3,2) *has a solution of and only if*

(3,4) $$\gamma^* \left\{ \int_0^1 \mathrm{d}[\mathbf{K}(t)]\,\mathbf{f}(t) - \int_0^1 \mathrm{d}[\mathbf{K}(t)]\,\mathbf{X}(t) \int_0^t \mathrm{d}[\mathbf{X}^{-1}(s)]\,\mathbf{f}(s) \right\} = \gamma^* \mathbf{r}$$

holds for every $\gamma \in R_m$ *such that*

(3,5) $$\gamma^* \int_0^1 \mathrm{d}[\mathbf{K}(t)]\,\mathbf{X}(t) = \mathbf{0}\,.$$

Proof. By the variation-of-constants formula (1,10), $\mathbf{x} : [0, 1] \to R_n$ is a solution to (3,1) on $[0, 1]$ if and only if $(\mathbf{X}(0) = \mathbf{X}^{-1}(0) = \mathbf{I})$

$$(3,6) \quad \mathbf{x}(t) = \mathbf{X}(t)\,\mathbf{c} + \mathbf{f}(t) - \mathbf{f}(0) - \mathbf{X}(t)\int_0^t \mathrm{d}[\mathbf{X}^{-1}(s)]\,(\mathbf{f}(s) - \mathbf{f}(0)) \quad \text{on} \quad [0, 1]$$

for some $\mathbf{c} \in R_n$. Inserting (3,6) into the left-hand side of (3,2) we obtain

$$\int_0^1 \mathrm{d}[\mathbf{K}(t)]\,\mathbf{x}(t) = \left(\int_0^1 \mathrm{d}[\mathbf{K}(t)]\,\mathbf{X}(t)\right)\mathbf{c} + \int_0^1 \mathrm{d}[\mathbf{K}(t)]\,(\mathbf{f}(t) - \mathbf{f}(0)) -$$
$$- \int_0^1 \mathrm{d}[\mathbf{K}(t)]\,\mathbf{X}(t)\int_0^t \mathrm{d}[\mathbf{X}^{-1}(s)]\,(\mathbf{f}(s) - \mathbf{f}(0)) =$$
$$= \left(\int_0^1 \mathrm{d}[\mathbf{K}(t)]\,\mathbf{X}(t)\right)\mathbf{c} + \int_0^1 \mathrm{d}[\mathbf{K}(t)]\,(\mathbf{f}(t) - \mathbf{f}(0)) -$$
$$- \int_0^1 \mathrm{d}[\mathbf{K}(t)]\,\mathbf{X}(t)\int_0^t \mathrm{d}[\mathbf{X}^{-1}(s)]\,\mathbf{f}(s) +$$
$$+ \int_0^1 \mathrm{d}[\mathbf{K}(t)]\,\mathbf{X}(t)\,(\mathbf{X}^{-1}(t) - \mathbf{I})\,\mathbf{f}(0) = \left(\int_0^1 \mathrm{d}[\mathbf{K}(t)]\,\mathbf{X}(t)\right)\mathbf{c} +$$
$$+ \int_0^1 \mathrm{d}[\mathbf{K}(t)]\,\mathbf{f}(t) - \int_0^1 \mathrm{d}[\mathbf{K}(t)]\,\mathbf{X}(t)\int_0^t \mathrm{d}[\mathbf{X}^{-1}(s)]\,\mathbf{f}(s) - \int_0^1 \mathrm{d}[\mathbf{K}(t)]\,\mathbf{X}(t)\mathbf{f}(0)\,.$$

This implies that $\mathbf{x} : [0, 1] \to R_n$ is a solution to the b.v. problem (3,1), (3,2) if and only if it is given by (3,6), where $\mathbf{c} \in R_n$ is such that

$$(3,7) \quad \left(\int_0^1 \mathrm{d}[\mathbf{K}(t)]\,\mathbf{X}(t)\right)\mathbf{c} = \mathbf{r} + \int_0^1 \mathrm{d}[\mathbf{K}(t)]\,\mathbf{X}(t)\,\mathbf{f}(0) - \int_0^1 \mathrm{d}[\mathbf{K}(t)]\,\mathbf{f}(t) +$$
$$+ \int_0^1 \mathrm{d}[\mathbf{K}(t)]\,\mathbf{X}(t)\int_0^t \mathrm{d}[\mathbf{X}^{-1}(s)]\,\mathbf{f}(s)\,.$$

In particular, our b.v. problem (3,1), (3,2) possesses a solution if and only if the linear algebraic equation (3,7) has a solution $\mathbf{c} \in R_n$, i.e. if and only if

$$(3,8) \quad \gamma^*\mathbf{r} = \gamma^*\left\{- \int_0^1 \mathrm{d}[\mathbf{K}(t)]\,\mathbf{X}(t)\,\mathbf{f}(0) + \int_0^1 \mathrm{d}[\mathbf{K}(t)]\,\mathbf{f}(t) -\right.$$
$$\left.- \int_0^1 \mathrm{d}[\mathbf{K}(t)]\,\mathbf{X}(t)\int_0^t \mathrm{d}[\mathbf{X}^{-1}(s)]\,\mathbf{f}(s)\right\}$$

holds for every $\gamma \in R_m$ such that (3,5) holds. Since for every such γ the first term on the right-hand side of (3,8) vanishes, the assertion of the lemma follows readily.

3.4. Remark. A function $\boldsymbol{x} : [0, 1] \to R_n$ is a solution to the homogeneous b.v. problem

$$\mathrm{d}\boldsymbol{x} = \mathrm{d}[\boldsymbol{A}]\, \boldsymbol{x}\,, \tag{3,9}$$

$$\int_0^1 \mathrm{d}[\boldsymbol{K}(t)]\, \boldsymbol{x}(t) = \boldsymbol{0} \tag{3,10}$$

if and only if $\boldsymbol{x}(t) = \boldsymbol{X}(t)\, \boldsymbol{c}$ on $[0, 1]$, where $\boldsymbol{c} \in R_n$ satisfies

$$\left(\int_0^1 \mathrm{d}[\boldsymbol{K}(t)]\, \boldsymbol{X}(t)\right) \boldsymbol{c} = \boldsymbol{0}\,. \tag{3,11}$$

Consequently, if $n - k$ $(0 \leqq k \leqq n)$ is the rank of the $m \times n$-matrix

$$\boldsymbol{D} = \int_0^1 \mathrm{d}[\boldsymbol{K}(t)]\, \boldsymbol{X}(t)\,,$$

then the homogeneous b.v. problem (3,9), (3,10) possesses exactly k linearly independent (in the evident sense) solutions $\boldsymbol{x}(t) = \boldsymbol{X}(t)\, \boldsymbol{c}_j$, $j = 1, 2, \ldots, k$, where $\{\boldsymbol{c}_j\}_{j=1,2,\ldots,k}$ is a basis of the solutions to (3,11) in R_n. The problem (3,9), (3,10) is then said to be compatible of the order k. In particular, if the rank of $\boldsymbol{D}$ equals n, then $k = 0$ and the problem (3,9), (3,10) possesses only the trivial solution $\boldsymbol{x}(t) \equiv 0$ on $[0, 1]$. In this case it is called incompatible.

Now, let us turn our attention to the relation (3,4). If we put

$$\boldsymbol{Q}(t, s) = \begin{cases} \boldsymbol{X}(t)\, \boldsymbol{X}^{-1}(s) & \text{for } 0 \leqq s \leqq t \leqq 1\,, \\ \boldsymbol{X}(t)\, \boldsymbol{X}^{-1}(t) = \boldsymbol{I} & \text{for } 0 \leqq t \leqq s \leqq 1\,, \end{cases}$$

then $\boldsymbol{Q}$ is of bounded two-dimensional Vitali variation on $[0, 1] \times [0, 1]$, $\mathrm{var}_0^1\, \boldsymbol{Q}(0, \cdot) + \mathrm{var}_0^1\, \boldsymbol{Q}(\cdot, 0) < \infty$ and

$$\boldsymbol{X}(t) \int_0^t \mathrm{d}[\boldsymbol{X}^{-1}(s)]\, \boldsymbol{f}(s) = \int_0^1 \mathrm{d}_s[\boldsymbol{Q}(t, s)]\, \boldsymbol{f}(s) \quad \text{for} \quad t \in [0, 1]\,.$$

By [6] Lemma 2.2 or [7] Theorem 6.22 we have

$$\int_0^1 \mathrm{d}[\boldsymbol{K}(t)] \left(\int_0^1 \mathrm{d}_s[\boldsymbol{Q}(t, s)]\, \boldsymbol{f}(s)\right) = \int_0^1 \mathrm{d}_s \left[\int_0^1 \mathrm{d}[\boldsymbol{K}(t)]\, \boldsymbol{Q}(t, s)\right] \boldsymbol{f}(s)\,.$$

Hence

$$\int_0^1 \mathrm{d}[\boldsymbol{K}(t)]\, \boldsymbol{X}(t) \int_0^t \mathrm{d}[\boldsymbol{X}^{-1}(s)]\, \boldsymbol{f}(s) = \int_0^1 \mathrm{d}_s \left[\int_0^1 \mathrm{d}[\boldsymbol{K}(t)]\, \boldsymbol{Q}(t, s)\right] \boldsymbol{f}(s) =$$

$$= \int_0^1 \mathrm{d}_s \left[\int_0^s \mathrm{d}[\boldsymbol{K}(t)]\, \boldsymbol{Q}(t, s) + \int_s^1 \mathrm{d}[\boldsymbol{K}(t)]\, \boldsymbol{Q}(t, s)\right] \boldsymbol{f}(s) =$$

$$= \int_0^1 \mathrm{d}_s \left[\int_0^s \mathrm{d}[\boldsymbol{K}(t)] + \int_s^1 \mathrm{d}[\boldsymbol{K}(t)]\, \boldsymbol{X}(t)\, \boldsymbol{X}^{-1}(s)\right] \boldsymbol{f}(s)\,,$$

i.e.

$$(3,12)\qquad \int_0^1 d[\boldsymbol{K}(t)]\, \boldsymbol{X}(t) \int_0^t d[\boldsymbol{X}^{-1}(s)]\, \boldsymbol{f}(s) = \int_0^1 d[\boldsymbol{K}(s)]\, \boldsymbol{f}(s) + $$
$$+ \int_0^1 d_s\left[\int_s^1 d[\boldsymbol{K}(t)]\, \boldsymbol{X}(t)\, \boldsymbol{X}^{-1}(s)\right] \boldsymbol{f}(s)\,.$$

Inserting this into the left-hand side of (3,4) we get

$$\gamma^* \left\{\int_0^1 d[\boldsymbol{K}(t)]\, \boldsymbol{f}(t) - \int_0^1 d[\boldsymbol{K}(t)]\, \boldsymbol{X}(t) \int_0^t d[\boldsymbol{X}^{-1}(s)]\, \boldsymbol{f}(s)\right\} =$$
$$= \gamma^* \left\{- \int_0^1 d_s\left[\int_s^1 d[\boldsymbol{K}(t)]\, \boldsymbol{X}(t)\, \boldsymbol{X}^{-1}(s)\right] \boldsymbol{f}(s)\right\}.$$

To summarize:

3.5. Lemma. *The b.v. problem* (3,1), (3,2) *has a solution if and only if*

$$\int_0^1 d_s\left[\int_s^1 d[\gamma^*\, \boldsymbol{K}(t)\, \boldsymbol{X}(t)\, \boldsymbol{X}^{-1}(s)\right] \boldsymbol{f}(s) + \gamma^* \boldsymbol{r} = 0$$

for every solution $\gamma \in R_m$ *of* (3,5).

This reformulation of the solvability condition 3.3 enables us to prove the following statement.

3.6. Theorem. *Under the assumptions* 3.1 *the b.v. problem* (3,1), (3,2) *possesses a solution if and only if*

$$(3,13)\qquad \int_0^1 d[\boldsymbol{y}^*(s)]\, \boldsymbol{f}(s) + \gamma^* \boldsymbol{r} = 0$$

for any function $\boldsymbol{y} : [0, 1] \to R_n$ *and any constant* $\gamma \in R_m$ *such that* $\boldsymbol{y}$ *is a solution to*

$$(3,14)\qquad d\boldsymbol{y}^* + d[\boldsymbol{y}^* \boldsymbol{A}] - d[\boldsymbol{y}^*]\, \boldsymbol{A} = -d[\gamma^* \boldsymbol{K}]$$

on [0, 1] (cf. 2.1) and

$$(3,15)\qquad \boldsymbol{y}^*(0) = \boldsymbol{y}^*(1) = \boldsymbol{0}\,.$$

3.7. Definition. The problem of determining a function $\boldsymbol{y} : [0, 1] \to R_n$ and a constant $\gamma \in R_m$ such that $\boldsymbol{y}$ is a solution to (3,14) (in the sense of Definition 2.1) and (3,15) is called *the adjoint boundary value problem* to the b.v. problem (3,1), (3,2) (or (3,9), (3,10)). It will be denoted as the b.v. problem (3,14), (3,15).

Proof of Theorem 3.6. By Theorem 2.7, a function $\boldsymbol{y} : [0, 1] \to R_n$ is a solution to (3,14) on [0, 1] such that $\boldsymbol{y}(1) = \boldsymbol{0}$ (for $\gamma \in R_m$ fixed) if and only if

$$(3,16)\qquad \boldsymbol{y}^*(s) = \gamma^* \int_s^1 d[\boldsymbol{K}(t)]\, \boldsymbol{X}(t)\, \boldsymbol{X}^{-1}(s) \quad \text{on} \quad [0, 1]\,.$$

Hence a couple $(\mathbf{y}, \gamma)$ is a solution to the b.v. problem (3,14), (3,15) if and only if $\mathbf{y}$ is given by (3,16) and

$$\mathbf{y}^*(0) = \gamma^*\left(\int_0^1 \mathrm{d}[\mathbf{K}(t)]\, \mathbf{X}(t)\right) = \mathbf{0}\,,$$

i.e. γ satisfies (3.5). Thus the assertion of the theorem is equivalent with that of Lemma 3.5.

3.8. Remark. The set $\mathscr{Y} = \{(\mathbf{y}_j, \gamma_j),\ j = 1, 2, \ldots, q\}$ of couples $\mathbf{y}_j : [0, 1] \to R_n$, $\gamma_j \in R_m$ is linearly dependent on $[0, 1]$ if there are $\lambda_j \in R_1$, $j = 1, 2, \ldots, q$ such that $|\lambda_1| + |\lambda_2| + \ldots + |\lambda_q| > 0$,

$$\sum_{j=1}^{q} \lambda_j\, \mathbf{y}_j(t) \equiv \mathbf{0} \quad \text{on} \quad [0, 1] \quad \text{and} \quad \sum_{j=1}^{q} \lambda_j \gamma_j = \mathbf{0}\,.$$

The set $\mathscr{Y}$ is linearly independent on $[0, 1]$ if it is not linearly dependent on $[0, 1]$. As usual, we say that the b.v. problem (3,14), (3,15) has exactly q linearly independent solutions if there exists a set $\mathscr{Y}$ of its q solutions which is linearly independent on $[0, 1]$, while the set $\mathscr{Y} \cup \{(\mathbf{y}, \gamma)\}$ is linearly dependent for any solution $(\mathbf{y}, \gamma)$ of the b.v. problem (3,14), (3,15). Similarly for the b.v. problem (3,9), (3,10).

If the b.v. problem (3,9), (3,10) has exactly k linearly independent solutions (i.e. the rank of $\mathbf{D}$ equals $n - k$, cf. 3.4), then its adjoint (3,14), (3,15) has exactly $k^* = m - n + k$ linearly independent solutions $(\mathbf{y}_j, \gamma_j)$, $j = 1, 2, \ldots, k^*$, where $\{\gamma_j\}_{j=1,2,\ldots,k^*}$ is a basis of the space of solutions to (3,5) and $\mathbf{y}_j : [0, 1] \to R_n$, $j = 1, 2, \ldots, k^*$ are given by (3,16) with $\gamma = \gamma_j$. This means also that the adjoint b.v. problem (3,14), (3,15) is incompatible if and only if $m = n - k$, i.e. the rank of $\mathbf{D}$ equals m.

3.9. Remark. If the side condition (3,2) is written in the form (3,3), then the adjoint b.v. problem (3,14), (3,15) reduces to the system of equations for $\mathbf{y} : [0, 1] \to R_n$ and $\gamma \in R_m$

(3,17) $$\mathbf{y}^*(s) + \mathbf{y}^*(s)\, \mathbf{A}(s) + \int_0^1 \mathrm{d}[\mathbf{y}^*(t)]\, \mathbf{A}(t) = \gamma^*(\mathbf{H}(1) - \mathbf{H}(s)) \quad \text{for} \quad 0 < s < 1\,,$$

$$\int_0^1 \mathrm{d}[y^*(t)]\, \mathbf{A}(t) = \gamma^*(\mathbf{H}(1) + \mathbf{N} - \mathbf{H}(0) + \mathbf{M})\,, \quad \mathbf{y}^*(0) = \mathbf{y}^*(1) = \mathbf{0}\,.$$

On the other hand. inserting (1,10) into (3,3) and repeating the above procedure we obtain that the b.v. problem (3,1), (3,3) possesses a solution if and only if

(3,18) $$\mathbf{z}^*(1)\, \mathbf{f}(1) - \mathbf{z}^*(0)\, \mathbf{f}(0) - \int_0^1 \mathrm{d}[\mathbf{z}^*(t)]\, \mathbf{f}(t) = \lambda^* \mathbf{r}$$

for every $\mathbf{z} : [0, 1] \to R_n$ and $\lambda \in R_m$ such that

$$(3,19) \quad \mathbf{z}^*(s) + \mathbf{z}^*(s)\, \mathbf{A}(s) + \int_s^1 \mathrm{d}[\mathbf{z}^*(t)]\, \mathbf{A}(t) = \lambda^*(\mathbf{H}(1) - \mathbf{H}(s)) \quad \text{on} \quad [0, 1]$$

i.e.

$$\mathrm{d}\mathbf{z}^* + \mathrm{d}[\mathbf{z}^*\mathbf{A}] - \mathrm{d}[\mathbf{z}^*]\, \mathbf{A} = -\mathrm{d}[\lambda^*\mathbf{H}]$$

and

$$(3,20) \qquad \mathbf{z}^*(0) = -\lambda^*\mathbf{M}\,, \quad \mathbf{z}^*(1) = \lambda^*\mathbf{N}\,.$$

It is easy to see that the relations

$$(3,21) \qquad \mathbf{z}^*(s) = \begin{cases} -\lambda^*\mathbf{M}\,, & s = 0\,, \\ \mathbf{y}^*(s)\,, & 0 < s < 1\,, \\ \lambda^*\mathbf{N}\,, & s = 1 \end{cases} \qquad \lambda = \gamma$$

$$\mathbf{y}^*(s) = \begin{cases} \mathbf{0}\,, & s = 0\,, \\ \mathbf{z}^*(s)\,, & 0 < s < 1\,, \\ \mathbf{0}\,, & s = 1 \end{cases}$$

define a one-to-one correspondence between the solutions of the systems (3,17) and (3,19), (3,20). Furthermore, given $\mathbf{y}, \mathbf{f} \in BV_n$ and $\mathbf{z} \in BV_n$ such that $\mathbf{z}^*(s) = \mathbf{y}^*(s)$ on $(0, 1)$, we have

$$\int_0^1 \mathrm{d}[\mathbf{z}^*(s)]\, \mathbf{f}(s) = -\mathbf{z}^*(1)\, \mathbf{f}(1) + \mathbf{z}^*(0)\, \mathbf{f}(0) + \int_0^1 \mathrm{d}[\mathbf{y}^*(s)]\, \mathbf{f}(s)\,.$$

We conclude that the solvability conditions (3,13) and (3,18) using respectively the adjoints (3,17) and (3,19), (3,20) are equivalent.

3.10. Remark. It was derived in [10] that also the system

$$(3,22) \quad \mathbf{y}^*(s) + \int_s^1 \mathrm{d}[\mathbf{y}^*(r)]\, \mathbf{A}(r) + \mathbf{y}^*(s)\, \mathbf{A}(s+) + \lambda^*\, \mathbf{K}(s) = \mathbf{0} \quad \text{on} \quad [0, 1]\,,$$

$$\mathbf{y}^*(0) = \mathbf{y}^*(1) = \mathbf{0}$$

may serve as an adjoint problem to the b.v. problem (3,1), (3,2). In particular, Theorem 3.6 is true if the system (3,14), (3,15) is replaced by (3,22). Let the couples $(\mathbf{y}, \lambda)$ and $(\mathbf{z}, \lambda)$ satisfy respectively (3,14), (3,15) and (3,22) and let $\mathbf{u}(s) = \mathbf{y}(s) - \mathbf{z}(s)$. Then

$$\mathbf{u}^*(s) + \mathbf{u}^*(s)\, \mathbf{A}(s) + \int_s^1 \mathrm{d}[\mathbf{u}^*(r)]\, \mathbf{A}(r) = -\mathbf{y}^*(s)\, \Delta^+\mathbf{A}(s)$$

and according to Theorem 2.7 and [7], I.4.23

$$\mathbf{u}^*(s) = \int_s^1 \mathrm{d}[\mathbf{y}^*(\tau)\, \Delta^+\mathbf{A}(\tau)]\, \mathbf{X}(\tau)\, \mathbf{X}^{-1}(s) = -\, \mathbf{y}^*(s)\, \Delta^+\mathbf{A}(s)\,, \quad s \in [0, 1]\,.$$

Let us notice that $\boldsymbol{u}^*(s+) = \boldsymbol{u}^*(s-) = \boldsymbol{0}$ on $(0, 1)$, $\boldsymbol{u}^*(0+) = \boldsymbol{u}^*(0) = \boldsymbol{u}^*(1-) = \boldsymbol{u}^*(1) = \boldsymbol{0}$ and consequently (cf. [7], I. 5.5)

$$\int_0^1 \mathrm{d}[\boldsymbol{u}^*(s)]\, \boldsymbol{f}(s) = 0 \quad \text{for any} \quad \boldsymbol{f} \in BV_n .$$

4. THE GREEN MATRIX

In this section we shall consider the b.v. problem (3,1), (3,2)

$$\mathrm{d}\boldsymbol{x} = \mathrm{d}[\boldsymbol{A}]\, \boldsymbol{x} + \mathrm{d}\boldsymbol{f}, \quad \int_0^1 \mathrm{d}[\boldsymbol{K}(t)]\, \boldsymbol{x}(t) = \boldsymbol{r}$$

fulfilling the assumptions 3.1. i.e. $\boldsymbol{A} : [0, 1] \to L(R_n)$, $\boldsymbol{A} \in BV$, $\det(\boldsymbol{I} + \Delta^+ \boldsymbol{A}(t)) \neq 0$ on $[0, 1)$, $\det(\boldsymbol{I} - \Delta^- \boldsymbol{A}(t)) \neq 0$ on $(0, 1]$ ((1,4) holds), $\boldsymbol{K} : [0, 1] \to L(R_n, R_m)$, $\boldsymbol{K} \in BV$, $\boldsymbol{f} \in BV_n$ and $\boldsymbol{r} \in R_m$. As in the previous section, $\boldsymbol{X} : [0, 1] \to L(R_n)$ denotes the fundamental matrix solution to $\mathrm{d}\boldsymbol{x} = \mathrm{d}[\boldsymbol{A}]\, \boldsymbol{x}$ on $[0, 1]$ such that $\boldsymbol{X}(0) = \boldsymbol{I}$ and

$$\boldsymbol{D} = \int_0^1 \mathrm{d}[\boldsymbol{K}(t)]\, \boldsymbol{X}(t) . \tag{4,1}$$

It was already shown that $\boldsymbol{x} : [0, 1] \to R_n$ is a solution to the b.v. problem (3,1), (3,2) if and only if

$$\boldsymbol{x}(t) = \boldsymbol{X}(t)\, \boldsymbol{c} + \boldsymbol{f}(t) - \boldsymbol{f}(0) - \boldsymbol{X}(t) \int_0^t \mathrm{d}[\boldsymbol{X}^{-1}(s)]\, (\boldsymbol{f}(s) - \boldsymbol{f}(0)) \quad \text{on} \quad [0, 1] \tag{4,2}$$

and $\boldsymbol{c} \in R_n$ satisfies (3,7). In particular, the b.v. problem (3,1), (3,2) possesses a unique solution for every $\boldsymbol{f} \in BV_n$, $\boldsymbol{r} \in R_m$ if and only if

$$m = n \quad \text{and} \quad \det(\boldsymbol{D}) \neq 0 . \tag{4,3}$$

(i.e. both the homogeneous b.v. problem (3,9), (3,10) and its adjoint (3,14), (3,15) are incompatible.)

In the rest of the paper we shall assume (4,3). In this case, for any $\boldsymbol{f} \in BV_n$ and $\boldsymbol{r} \in R_n$, the function (4,2), where

(4,4)

$$\boldsymbol{c} = \boldsymbol{D}^{-1}\boldsymbol{r} - \boldsymbol{D}^{-1} \int_0^1 \mathrm{d}[\boldsymbol{K}(t)]\, \boldsymbol{f}(t) + \boldsymbol{f}(0) + \boldsymbol{D}^{-1} \int_0^1 \mathrm{d}[\boldsymbol{K}(t)]\, \boldsymbol{X}(t) \int_0^t \mathrm{d}[\boldsymbol{X}^{-1}(s)]\, \boldsymbol{f}(s) ,$$

is the unique solution of the given b.v. problem (3,1), (3,2). Inserting (4,4) into (4,2) and applying (3,12) we obtain

$$\boldsymbol{x}(t) = \boldsymbol{X}(t)\, \boldsymbol{D}^{-1}\boldsymbol{r} - \boldsymbol{X}(t)\, \boldsymbol{D}^{-1} \int_0^1 \mathrm{d}[\boldsymbol{K}(s)]\, \boldsymbol{f}(s) +$$

$$+ \mathbf{X}(t)\, \mathbf{D}^{-1} \int_0^1 d[\mathbf{K}(\tau)]\, \mathbf{X}(\tau) \int_0^\tau d[\mathbf{X}^{-1}(s)]\, \mathbf{f}(s) + \mathbf{f}(t) -$$

$$- \mathbf{X}(t) \int_0^t d[\mathbf{X}^{-1}(s)]\, \mathbf{f}(s) = \mathbf{X}(t)\, \mathbf{D}^{-1} \mathbf{r} + \mathbf{f}(t) +$$

$$+ \mathbf{X}(t)\, \mathbf{D}^{-1} \int_0^1 d_s \left[\int_s^1 d[\mathbf{K}(\tau)]\, \mathbf{X}(\tau)\, \mathbf{X}^{-1}(s) \right] \mathbf{f}(s) -$$

$$- \mathbf{X}(t)\, \mathbf{D}^{-1} \int_0^t d_s \left[\int_0^1 d[\mathbf{K}(\tau)]\, \mathbf{X}(\tau)\, \mathbf{X}^{-1}(s) \right] \mathbf{f}(s) \quad \text{for} \quad t \in [0, 1]$$

(cf. (4,1)). Hence

$$\mathbf{x}(t) = \mathbf{X}(t)\, \mathbf{D}^{-1} \mathbf{r} + \mathbf{f}(t) + \mathbf{X}(t)\, \mathbf{D}^{-1} \int_t^1 d_s \left[\int_s^1 d[\mathbf{K}(\tau)]\, \mathbf{X}(\tau)\, \mathbf{X}^{-1}(s) \right] \mathbf{f}(s) - \tag{4,5}$$

$$- \mathbf{X}(t)\, \mathbf{D}^{-1} \int_0^t d_s \left[\int_0^s d[\mathbf{K}(\tau)]\, \mathbf{X}(\tau)\, \mathbf{X}^{-1}(s) \right] \mathbf{f}(s) \quad \text{for} \quad t \in [0, 1].$$

Now, let us define $\mathbf{G} : [0, 1] \times [0, 1] \to L(R_n)$ by

$$\mathbf{G}(t, s) = \begin{cases} -\mathbf{X}(t)\, \mathbf{D}^{-1} \int_0^s d[\mathbf{K}(\tau)]\, \mathbf{X}(\tau)\, \mathbf{X}^{-1}(s) & \text{for} \quad 0 \leqq s < t \leqq 1, \\ \mathbf{X}(t)\, \mathbf{D}^{-1} \int_s^1 d[\mathbf{K}(\tau)]\, \mathbf{X}(\tau)\, \mathbf{X}^{-1}(s) & \text{for} \quad 0 \leqq t < s \leqq 1, \end{cases} \tag{4,6}$$

$$\mathbf{G}(t, t) \quad \text{arbitrary for} \quad t \in [0, 1]$$

and calculate (using the properties of the Perron-Stieltjes integral, cf. [7] or [4])

$$\int_0^1 d_s[\mathbf{G}(t, s)]\, \mathbf{f}(s) - \int_0^t d_s \left[-\mathbf{X}(t)\, \mathbf{D}^{-1} \int_0^s d[\mathbf{K}(\tau)]\, \mathbf{X}(\tau)\, \mathbf{X}^{-1}(s) \right] \mathbf{f}(s) -$$

$$- \int_t^1 d_s \left[\mathbf{X}(t)\, \mathbf{D}^{-1} \int_s^1 d[\mathbf{K}(\tau)]\, \mathbf{X}(\tau)\, \mathbf{X}^{-1}(s) \right] \mathbf{f}(s) =$$

$$= \left[\mathbf{G}(t, t) + \mathbf{X}(t)\, \mathbf{D}^{-1} \int_0^t d[\mathbf{K}(\tau)]\, \mathbf{X}(\tau)\, \mathbf{X}^{-1}(t) \right] \mathbf{f}(t) -$$

$$- \left[\mathbf{G}(t, t) - \mathbf{X}(t)\, \mathbf{D}^{-1} \int_t^1 d[\mathbf{K}(\tau)]\, \mathbf{X}(\tau)\, \mathbf{X}^{-1}(t) \right] \mathbf{f}(t) =$$

$$= \mathbf{X}(t)\, \mathbf{D}^{-1} \left(\int_0^1 d[\mathbf{K}(\tau)]\, \mathbf{X}(\tau) \right) \mathbf{X}^{-1}(t)\, \mathbf{f}(t) =$$

$$= \mathbf{X}(t)\, \mathbf{D}^{-1} \mathbf{D} \mathbf{X}^{-1}(t)\, \mathbf{f}(t) = \mathbf{f}(t)$$

for any $t \in [0, 1]$. This together with (4,5) yields.

4.1. Theorem. *Let* $\mathbf{A} : [0, 1] \to L(R_n)$, $\mathbf{A} \in BV$ *fulfil* (1,4) *and let* (4,3) *hold. Then for any* $\mathbf{f} \in BV_n$ *and* $\mathbf{r} \in R_n$ *the b.v. problem* (3,1), (3,2) *possesses a unique solution* $\mathbf{x} : [0, 1] \to R_n$ *and this solution is given by*

$$\mathbf{x}(t) = \mathbf{X}(t)\, \mathbf{D}^{-1}\mathbf{r} + \int_0^1 \mathrm{d}_s[\mathbf{G}(t, s)]\, \mathbf{f}(s) \quad \text{on} \quad [0, 1]\,, \tag{4,7}$$

where $\mathbf{G} : [0, 1] \times [0, 1] \to L(R_n)$ *and* $\mathbf{D} \in L(R_n)$ *have been defined in* (4,6) *and* (4,1), *respectively.*

4.2. Definition. Any function $\mathbf{G} : [0, 1] \times [0, 1] \to L(R_n)$ fulfilling (4,6) (with $\mathbf{D}$ given by (4,1)) is called *the Green matrix* of the b.v. problem (3,1), (3,2).

We shall show that Green's matrices $\mathbf{G}(t, s)$ not only offer a representation of solutions to the b.v. problem (3,1), (3,2) but possess also the other usual properties of Green matrices (cf. [1] or [2]). The following theorem describes their continuity properties.

4.3. Theorem. *Let the assumptions of Theorem* 4.1 *hold. Any Green's matrix satisfies*

(i) $\mathbf{G}(t, 0) = \mathbf{0}$ for $0 < t \leqq 1$, $\mathbf{G}(t, 1) = \mathbf{0}$ *for* $0 \leqq t < 1$;

(ii) $\mathbf{G}(t +, s) = [\mathbf{I} + \Delta^+\mathbf{A}(t)]\, \mathbf{G}(t, s)$ *for* $t \in [0, 1)$, $s \in [0, 1]$, $s \neq t$,

$\mathbf{G}(t -, s) = [\mathbf{I} - \Delta^-\mathbf{A}(t)]\, \mathbf{G}(t, s)$ *for* $t \in (0, 1]$, $s \in [0, 1]$, $s \neq t$;

(iii) $\mathbf{G}(t+, t) - \mathbf{G}(t-, t) = -\mathbf{I} - \Delta^+\mathbf{A}(t)\, \mathbf{X}(t)\, \mathbf{D}^{-1} \int_0^t \mathrm{d}[\mathbf{K}(\tau)]\, \mathbf{X}(\tau)\, \mathbf{X}^{-1}(t) +$
$+ \Delta^-\mathbf{A}(t)\, \mathbf{X}(t)\, \mathbf{D}^{-1} \int_t^1 \mathrm{d}[\mathbf{K}(\tau)]\, \mathbf{X}(\tau)\, \mathbf{X}^{-1}(t)$ *for* $t \in (0, 1)$;

(iv) $\mathbf{G}(t, s+) = [\mathbf{G}(t, s) - \mathbf{X}(t)\, \mathbf{D}^{-1}\, \Delta^+\mathbf{K}(s)]\, [\mathbf{I} + \Delta^+\mathbf{A}(s)]^{-1}$ *for* $t \in [0, 1]$, $s \in [0, 1)$, $s \neq t$;

$\mathbf{G}(t, s-) = [\mathbf{G}(t, s) + \mathbf{X}(t)\, \mathbf{D}^{-1}\, \Delta^-\mathbf{K}(s)]\, [\mathbf{I} - \Delta^-\mathbf{A}(s)]^{-1}$ *for* $t \in [0, 1]$, $s \in (0, 1]$, $s \neq t$;

(v) $\mathbf{G}(t, t+) - \mathbf{G}(t, t-) = \mathbf{X}(t)\, \mathbf{D}^{-1}\{\int_0^t \mathrm{d}[\mathbf{K}(\tau)]\, \mathbf{X}(\tau)\, \mathbf{X}^{-1}(t)\, [\mathbf{I} + \Delta^-\mathbf{A}(t)]^{-1} +$
$+ \int_t^1 \mathrm{d}[\mathbf{K}(\tau)]\, \mathbf{X}(\tau)\, \mathbf{X}^{-1}(t)\, [\mathbf{I} - \Delta^-\mathbf{A}(t)]^{-1} -$
$- \Delta^-\mathbf{K}(t)\, [\mathbf{I} - \Delta^-\mathbf{A}(t)]^{-1} - \Delta^+\mathbf{K}(t)\, [\mathbf{I} + \Delta^+\mathbf{A}(t)]^{-1}\}$ *for* $t \in (0, 1)$.

Proof. The relations (i) follow immediately from the definition. Since for any $t \in [0, 1)$

$$\Delta^+\mathbf{X}(t) = \Delta^+\mathbf{A}(t)\, \mathbf{X}(t)$$

(cf. (1,5)), we have for any $s \in [0, 1]$ and $t \in [0, s)$

$$\mathbf{G}(t+, s) - \mathbf{G}(t, s) = \Delta^+ \mathbf{X}(t)\, \mathbf{D}^{-1} \int_s^1 \mathrm{d}[\mathbf{K}(\tau)]\, \mathbf{X}(\tau)\, \mathbf{X}^{-1}(s) =$$

$$= \Delta^+ \mathbf{A}(t)\, \mathbf{X}(t)\, \mathbf{D}^{-1} \int_s^1 \mathrm{d}[\mathbf{K}(\tau)]\, \mathbf{X}(\tau)\, \mathbf{X}^{-1}(s) = \Delta^+ \mathbf{A}(t)\, \mathbf{G}(t, s)\,.$$

By the same argument we get that $\mathbf{G}(t+, s) - \mathbf{G}(t, s) = \Delta^+ \mathbf{A}(t)\, \mathbf{G}(t, s)$ holds also for $s \in [0, 1]$ and $t \in (s, 1]$. Analogously, using the equality

$$\Delta^- \mathbf{X}(t) = \Delta^- \mathbf{A}(t)\, \mathbf{X}(t) \quad \text{for} \quad t \in (0, 1]\,,$$

we can prove the second relation in (ii).

As concerns (iii), we have for any $t \in (0, 1)$ (cf. (1,5))

$$\mathbf{G}(t+, t) - \mathbf{G}(t-, t) =$$

$$= -\mathbf{X}(t+)\, \mathbf{D}^{-1} \int_0^t \mathrm{d}[\mathbf{K}(\tau)]\, \mathbf{X}(\tau)\, \mathbf{X}^{-1}(t) - \mathbf{X}(t-)\, \mathbf{D}^{-1} \int_t^1 \mathrm{d}[\mathbf{K}(\tau)]\, \mathbf{X}(\tau)\, \mathbf{X}^{-1}(t) =$$

$$= -[\mathbf{I} + \Delta^+ \mathbf{A}(t)]\, \mathbf{X}(t)\, \mathbf{D}^{-1} \int_0^t \mathrm{d}[\mathbf{K}(\tau)]\, \mathbf{X}(\tau)\, \mathbf{X}^{-1}(t) -$$

$$- [\mathbf{I} - \Delta^- \mathbf{A}(t)]\, \mathbf{X}(t)\, \mathbf{D}^{-1} \int_t^1 \mathrm{d}[\mathbf{K}(\tau)]\, \mathbf{X}(\tau)\, \mathbf{X}^{-1}(t) = -\mathbf{X}(t)\, \mathbf{D}^{-1} \mathbf{D}\, \mathbf{X}^{-1}(t) -$$

$$- \Delta^+ \mathbf{A}(t)\, \mathbf{X}(t)\, \mathbf{D}^{-1} \int_0^t \mathrm{d}[\mathbf{K}(\tau)]\, \mathbf{X}(\tau)\, \mathbf{X}^{-1}(t) +$$

$$+ \Delta^- \mathbf{A}(t)\, \mathbf{X}(t)\, \mathbf{D}^{-1} \int_t^1 \mathrm{d}[\mathbf{K}(\tau)]\, \mathbf{X}(\tau)\, \mathbf{X}^{-1}(t)\,.$$

It is known (cf. [7], I.4.12 or [4] Theorem 1.3.5) that

$$(4,8) \quad \lim_{\delta \to 0+} \int_{s+\delta}^1 \mathrm{d}[\mathbf{K}(\tau)]\, \mathbf{X}(\tau) = \int_s^1 \mathrm{d}[\mathbf{K}(\tau)]\, \mathbf{X}(\tau) - \Delta^+ \mathbf{K}(s)\, \mathbf{X}(s) \quad \text{for} \quad s \in [0, 1)\,,$$

$$\lim_{\delta \to 0+} \int_0^{s-\delta} \mathrm{d}[\mathbf{K}(\tau)]\, \mathbf{X}(\tau) = \int_0^s \mathrm{d}[\mathbf{K}(\tau)]\, \mathbf{X}(\tau) - \Delta^- \mathbf{K}(s)\, \mathbf{X}(s) \quad \text{for} \quad s \in (0, 1]\,,$$

$$\lim_{\delta \to 0+} \int_{s-\delta}^1 \mathrm{d}[\mathbf{K}(\tau)]\, \mathbf{X}(\tau) = \int_s^1 \mathrm{d}\mathbf{K}(\tau)]\, \mathbf{X}(\tau) + \Delta^- \mathbf{K}(s)\, \mathbf{X}(s) \quad \text{for} \quad s \in (0, 1]\,,$$

$$\lim_{\delta \to 0+} \int_0^{s+\delta} \mathrm{d}[\mathbf{K}(\tau)]\, \mathbf{X}(\tau) = \int_0^s \mathrm{d}[\mathbf{K}(\tau)]\, \mathbf{X}(\tau) + \Delta^+ \mathbf{K}(s)\, \mathbf{X}(s) \quad \text{for} \quad s \in [0, 1)\,.$$

Furthermore, by (1,9) and Lemma 2.3

$$(4,9) \qquad \mathbf{X}^{-1}(s+) = \mathbf{X}^{-1}(s)\, [\mathbf{I} + \Delta^+ \mathbf{A}(s)]^{-1} \quad \text{for} \quad s \in [0, 1)\,,$$

$$\mathbf{X}^{-1}(s-) = \mathbf{X}^{-1}(s)\, [\mathbf{I} - \Delta^- \mathbf{A}(s)]^{-1} \quad \text{for} \quad s \in (0, 1]\,.$$

Consequently,

$$\mathbf{G}(t, s+) = -\mathbf{X}(t)\,\mathbf{D}^{-1}\left\{\int_0^s \mathrm{d}[\mathbf{K}(\tau)]\,\mathbf{X}(\tau) + \Delta^+\mathbf{K}(s)\,\mathbf{X}(s)\right\}\mathbf{X}^{-1}(s)\,[\mathbf{I} + \Delta^+\mathbf{A}(s)]^{-1}$$

$$\text{for} \quad t \in (0, 1]\,, \quad s \in [0, t)\,,$$

$$\mathbf{G}(t, s+) = \mathbf{X}(t)\,\mathbf{D}^{-1}\left\{\int_s^1 \mathrm{d}[\mathbf{K}(\tau)]\,\mathbf{X}(\tau) - \Delta^+\mathbf{K}(s)\,\mathbf{X}(s)\right\}\mathbf{X}^{-1}(s)\,[\mathbf{I} + \Delta^+\mathbf{A}(s)]^{-1}$$

$$\text{for} \quad t \in [0, 1)\,, \quad s \in [t, 1)\,,$$

$$\mathbf{G}(t, s-) = -\mathbf{X}(t)\,\mathbf{D}^{-1}\left\{\int_0^s \mathrm{d}[\mathbf{K}(\tau)]\,\mathbf{X}(\tau) - \Delta^-\mathbf{K}(s)\,\mathbf{X}(s)\right\}\mathbf{X}^{-1}(s)\,[\mathbf{I} - \Delta^-\mathbf{A}(s)]^{-1}$$

$$\text{for} \quad t \in (0, 1]\,, \quad s \in (0, t]$$

and

$$\mathbf{G}(t, s-) = \mathbf{X}(t)\,\mathbf{D}^{-1}\left\{\int_s^1 \mathrm{d}[\mathbf{K}(\tau)]\,\mathbf{X}(\tau) + \Delta^-\mathbf{K}(s)\,\mathbf{X}(s)\right\}\mathbf{X}^{-1}(s)\,[\mathbf{I} - \Delta^-\mathbf{A}(s)]^{-1}$$

$$\text{for} \quad t \in [0, 1)\,, \quad s \in (t, 1]\,.$$

The relations (iv) and (v) follow immediately.

Up to now it has not been necessary to define the Green matrix $\mathbf{G}(t, s)$ for $t = s$. The following calculation shows that if the values $\mathbf{G}(s, s)$ are appropriately chosen, then the function $\mathbf{Z} = \mathbf{G}(\cdot, s)$ is for any $s \in [0, 1]$ a solution to the boundary value problem consisting of the generalized linear matrix differential equation

$$\mathrm{d}\mathbf{Z} = \mathrm{d}[\mathbf{A}]\,\mathbf{Z} + \mathrm{d}\mathbf{F}$$

with some $\mathbf{F} = \mathbf{F}(\cdot, s) : [0, 1] \rightarrow L(R_n)$ and of the side condition

$$\int_0^1 \mathrm{d}[\mathbf{K}(t)]\,\mathbf{Z}(t) = \mathbf{0}\,.$$

Let $s \in (0, 1)$ and $0 \leqq t_1 < s < t_2 \leqq 1$. Since (cf. [7] I.4.21)

$$\int_{t_1}^s \mathrm{d}[\mathbf{A}(\tau)]\left(\mathbf{G}(\tau, s) - \mathbf{X}(\tau)\,\mathbf{D}^{-1}\int_s^1 \mathrm{d}[\mathbf{K}(\varrho)]\,\mathbf{X}(\varrho)\,\mathbf{X}^{-1}(s)\right) =$$

$$= \Delta^-\mathbf{A}(s)\left(\mathbf{G}(s, s) - \mathbf{X}(s)\,\mathbf{D}^{-1}\int_s^1 \mathrm{d}[\mathbf{K}(\varrho)]\,\mathbf{X}(\varrho)\,\mathbf{X}^{-1}(s)\right),$$

we have by (1,6)

$$\int_{t_1}^s \mathrm{d}[\mathbf{A}(\tau)]\,\mathbf{G}(\tau, s) = (\mathbf{X}(s) - \mathbf{X}(t_1))\,\mathbf{D}^{-1}\int_s^1 \mathrm{d}[\mathbf{K}(\varrho)]\,\mathbf{X}(\varrho)\,\mathbf{X}^{-1}(s) +$$

$$+ \Delta^-\mathbf{A}(s)\left(\mathbf{G}(s, s) - \mathbf{X}(s)\,\mathbf{D}^{-1}\int_s^1 \mathrm{d}[\mathbf{K}(\varrho)]\,\mathbf{X}(\varrho)\,\mathbf{X}^{-1}(s)\right) =$$

$$= \mathbf{X}(s)\, \mathbf{D}^{-1} \int_s^1 \mathrm{d}[\mathbf{K}(\varrho)]\, \mathbf{X}(\varrho)\, \mathbf{X}^{-1}(s) - \mathbf{G}(t_1, s) +$$

$$+ \Delta^- \mathbf{A}(s) \left(\mathbf{G}(s, s) - \mathbf{X}(s)\, \mathbf{D}^{-1} \int_s^1 \mathrm{d}[\mathbf{K}(\varrho)]\, \mathbf{X}(\varrho)\, \mathbf{X}^{-1}(s) \right),$$

i.e.

(4,10)
$$\int_{t_1}^s \mathrm{d}[\mathbf{A}(\tau)]\, \mathbf{G}(\tau, s) = \mathbf{G}(s, s) - \mathbf{G}(t_1, s) -$$

$$- (\mathbf{I} - \Delta^- \mathbf{A}(s)) \left(\mathbf{G}(s, s) - \mathbf{X}(s)\, \mathbf{D}^{-1} \int_s^1 \mathrm{d}[\mathbf{K}(\varrho)]\, \mathbf{X}(\varrho)\, \mathbf{X}^{-1}(s) \right).$$

Similarly

(4,11)
$$\int_s^{t_2} \mathrm{d}[\mathbf{A}(\tau)]\, \mathbf{G}(\tau, s) = \mathbf{G}(t_2, s) - \mathbf{G}(s, s) +$$

$$+ [\mathbf{I} + \Delta^+ \mathbf{A}(s)] \left\{ \mathbf{G}(s, s) + \mathbf{X}(s)\, \mathbf{D}^{-1} \int_0^s \mathrm{d}[\mathbf{K}(\tau)]\, \mathbf{X}(\tau)\, \mathbf{X}^{-1}(s) \right\}.$$

This yields for $0 \leqq t_1 < s < t_2 \leqq 1$

(4,12)
$$\int_{t_1}^{t_2} \mathrm{d}[\mathbf{A}(\tau)]\, \mathbf{G}(\tau, s) = \mathbf{G}(t_2, s) - \mathbf{G}(t_1, s) + \mathbf{I} +$$

$$+ \Delta^- \mathbf{A}(s) \left\{ \mathbf{G}(s, s) - \mathbf{X}(s)\, \mathbf{D}^{-1} \int_s^1 \mathrm{d}[\mathbf{K}(\varrho)]\, \mathbf{X}(\varrho)\, \mathbf{X}^{-1}(s) \right\} +$$

$$+ \Delta^+ \mathbf{A}(s) \left\{ \mathbf{G}(s, s) + \mathbf{X}(s)\, \mathbf{D}^{-1} \int_0^s \mathrm{d}[\mathbf{K}(\varrho)]\, \mathbf{X}(\varrho)\, \mathbf{X}^{-1}(s) \right\}.$$

Furthermore,

(4,13)
$$\int_{t_1}^{t_2} \mathrm{d}[\mathbf{A}(\tau)]\, \mathbf{G}(\tau, s) = \mathbf{G}(t_2, s) - \mathbf{G}(t_1, s)$$

for $0 \leqq t_1 < t_2 \leqq 1$ and $s \notin [t_1, t_2]$. In fact, if e.g. $t_1 < t_2 < s$, then

$$\int_{t_1}^{t_2} \mathrm{d}[\mathbf{A}(\tau)]\, \mathbf{G}(\tau, s) = \int_{t_1}^{t_2} \mathrm{d}[\mathbf{A}(\tau)]\, \mathbf{X}(\tau)\, \mathbf{D}^{-1} \int_s^1 \mathrm{d}[\mathbf{K}(\varrho)]\, \mathbf{X}(\varrho)\, \mathbf{X}^{-1}(s) =$$

$$= [\mathbf{X}(t_2) - \mathbf{X}(t_1)]\, \mathbf{D}^{-1} \int_s^1 \mathrm{d}[\mathbf{K}(\varrho)]\, \mathbf{X}(\varrho)\, \mathbf{X}^{-1}(s).$$

Similarly for $s < t_1 < t_2$. In the case $s = 0$ or $s = 1$, (4,13) holds for $0 < t_1 < t_2 \leqq 1$ or $0 \leqq t_1 < t_2 < 1$, respectively (cf. 4.3 (i)). In particular, we have

4.4. Proposition. For any $s \in [0, 1]$ the matrix valued function

$$\mathbf{Z} : t \in [0, 1] \to \mathbf{G}(t, s) \in L(R_n)$$

is a solution to the generalized matrix linear differential equation

$$\mathrm{d}\mathbf{Z} = \mathrm{d}[\mathbf{A}]\,\mathbf{Z}$$

on the intervals $[0, s)$ and $(s, 1]$ (i.e. (4,13) holds for all $t_1, t_2 \in [0, s)$ or $t_1, t_2 \in (0, s]$).

Let us turn our attention to the side condition. Given $s \in [0, 1]$, it is

$$\int_0^1 \mathrm{d}[\mathbf{K}(\tau)]\,\mathbf{G}(\tau, s) = \left(\int_0^s \mathrm{d}[\mathbf{K}(\tau)]\,\mathbf{X}(\tau)\right)\mathbf{D}^{-1}\left(\int_s^1 \mathrm{d}[\mathbf{K}(\tau)]\,\mathbf{X}(\tau)\right)\mathbf{X}^{-1}(s) +$$

$$+ \Delta^-\mathbf{K}(s)\left\{\mathbf{G}(s, s) - \mathbf{X}(s)\,\mathbf{D}^{-1}\int_s^1 \mathrm{d}[\mathbf{K}(\tau)]\,\mathbf{X}(\tau)\,\mathbf{X}^{-1}(s)\right\} -$$

$$- \left(\int_s^1 \mathrm{d}[\mathbf{K}(\tau)]\,\mathbf{X}(\tau)\right)\mathbf{D}^{-1}\left(\int_0^s \mathrm{d}[\mathbf{K}(\tau)]\,\mathbf{X}(\tau)\right)\mathbf{X}^{-1}(s) +$$

$$+ \Delta^+\mathbf{K}(s)\left\{\mathbf{G}(s, s) + \mathbf{X}(s)\,\mathbf{D}^{-1}\int_0^s \mathrm{d}[\mathbf{K}(\tau)]\,\mathbf{X}(\tau)\,\mathbf{X}^{-1}(s)\right\},$$

wherefrom the relation

(4,14)
$$\int_0^1 \mathrm{d}[\mathbf{K}(\tau)]\,\mathbf{G}(\tau, s) =$$

$$= \Delta^-\mathbf{K}(s)\left\{\mathbf{G}(s, s) - \mathbf{X}(s)\,\mathbf{D}^{-1}\int_s^1 \mathrm{d}[\mathbf{K}(\tau)]\,\mathbf{X}(\tau)\,\mathbf{X}^{-1}(s)\right\} +$$

$$+ \Delta^+\mathbf{K}(s)\left\{\mathbf{G}(s, s) + \mathbf{X}(s)\,\mathbf{D}^{-1}\int_0^s \mathrm{d}[\mathbf{K}(\tau)]\,\mathbf{X}(\tau)\,\mathbf{X}^{-1}(s)\right\} \quad \text{for} \quad s \in [0, 1]$$

immediately follows by inserting (cf. (4,1))

$$\int_s^1 \mathrm{d}[\mathbf{K}(\tau)]\,\mathbf{X}(\tau) = \mathbf{D} - \int_0^s \mathrm{d}[\mathbf{K}(\tau)]\,\mathbf{X}(\tau).$$

It is apparent from (4,10), (4,11) or (4,14) that it would be convenient to define $\mathbf{G}(s, s)$ by either

(4,15)
$$\mathbf{G}(s, s) = \mathbf{X}(s)\,\mathbf{D}^{-1}\int_s^1 \mathrm{d}[\mathbf{K}(\tau)]\,\mathbf{X}(\tau)\,\mathbf{X}^{-1}(s), \quad s \in [0, 1]$$

or

(4,16)
$$\mathbf{G}(s, s) = -\mathbf{X}(s)\,\mathbf{D}^{-1}\int_0^s \mathrm{d}[\mathbf{K}(\varrho)]\,\mathbf{X}(\varrho)\,\mathbf{X}^{-1}(s), \quad s \in [0, 1],$$

i.e. to extend one of the relations defining $\mathbf{G}$ in (4,6) to the diagonal $t = s$ of $[0, 1] \times \times [0, 1]$. Let us assume (4,16). It means that the Green matrix $\mathbf{G}$ is now defined by

$$(4,17)\quad \mathbf{G}(t,s) = \begin{cases} -\mathbf{X}(t)\,\mathbf{D}^{-1}\displaystyle\int_0^s \mathrm{d}[\mathbf{K}(\varrho)]\,\mathbf{X}(\varrho)\,\mathbf{X}^{-1}(s) & \text{for} \quad 0 \leqq s \leqq t \leqq 1\,, \\ \mathbf{X}(t)\,\mathbf{D}^{-1}\displaystyle\int_s^1 \mathrm{d}[\mathbf{K}(\varrho)]\,\mathbf{X}(\varrho)\,\mathbf{X}^{-1}(s) & \text{for} \quad 0 \leqq t < s \leqq 1\,. \end{cases}$$

By (4,11) we have for $0 \leqq s \leqq t_2 \leqq 1$

$$(4,18)\qquad \int_s^{t_2} \mathrm{d}[\mathbf{A}(\tau)]\,\mathbf{G}(\tau,s) = \mathbf{G}(t_2,s) - \mathbf{G}(s,s)\,,$$

while (4,10) implies for $0 \leqq t_1 < s \leqq 1$

$$\int_{t_1}^s \mathrm{d}[\mathbf{A}(\tau)]\,\mathbf{G}(\tau,s) = \mathbf{G}(s,s) - \mathbf{G}(t_1,s) -$$

$$- (\mathbf{I} - \Delta^- \mathbf{A}(s))\left[-\mathbf{X}(s)\,\mathbf{D}^{-1}\left(\int_0^s \mathrm{d}[\mathbf{K}(\varrho)]\,\mathbf{X}(\varrho) + \int_s^1 \mathrm{d}[\mathbf{K}(\varrho)]\,\mathbf{X}(\varrho)\right)\mathbf{X}^{-1}(s)\right],$$

i.e.

$$(4,19)\qquad \int_{t_1}^s \mathrm{d}[\mathbf{A}(\tau)]\,\mathbf{G}(,\tau\, s) = \mathbf{G}(s,s) - \mathbf{G}(t_1,s) + (\mathbf{I} - \Delta^- \mathbf{A}(s))\,.$$

This leads to the following

4.5. Theorem. *Let* $\mathbf{A} : [0,1] \to L(R_n)$, $\mathbf{A} \in BV$ *fulfil* (1,4) *and let* (4,3) *hold. In addition, let us assume that* $\mathbf{K}$ *is left-continuous on* $(0, 1)$. *Let us put*

$$(4,20)\qquad \Delta(t,s) = \begin{cases} \mathbf{I} & \textit{for} \quad 0 < s \leqq t \leqq 1\,, \\ \mathbf{0} & \textit{for} \quad 0 \leqq t < s \leqq 1 \quad \textit{or} \quad t = s = 0\,. \end{cases}$$

If $\mathbf{G}(t,s)$ *is defined by* (4,17), *then the relations*

$$(4,21)\qquad \mathbf{G}(t_2,s) - \mathbf{G}(t_1,s) = \int_{t_1}^{t_2} \mathrm{d}[\mathbf{A}(\tau)]\,\mathbf{G}(\tau,s) -$$

$$- (\mathbf{I} - \Delta^- \mathbf{A}(s))\,(\Delta(t_2,s) - \Delta(t_1,s))$$

and

$$(4,22)\qquad \int_0^1 \mathrm{d}[\mathbf{K}(t)]\,\mathbf{G}(t,s) = \mathbf{0}$$

hold for every t_1, t_2 *and* $s \in [0,1]$.

Proof. The relation (4,21) follows from (4,18) and (4,19). The relation (4,22) follows from (4,14).

4.6. Remark. The equation (4,21) may be written in the form of a generalized linear matrix differential equation

$$\mathrm{d}\mathbf{G}(\cdot,s) + \mathrm{d}[\mathbf{A}]\,\mathbf{G}(\cdot,s) = \mathrm{d}[(\mathbf{I} - \Delta^- \mathbf{A}(s))\,\Delta(\cdot,s)]\,.$$

4.7. Remark. The assumption on the left-continuity of $\mathbf{K}$ on $(0, 1)$ does not mean any loss of generality. In fact, if $\mathbf{K}^{\sim}(0) = \mathbf{K}^{\wedge}(0) = \mathbf{K}(0)$, $\mathbf{K}^{\sim}(1) = \mathbf{K}^{\wedge}(1) = \mathbf{K}(1)$, $\mathbf{K}^{\sim}(t) = \mathbf{K}(t-)$ and $\mathbf{K}^{\wedge}(t) = \mathbf{K}(t+)$ on $(0, 1)$, then

$$\int_0^1 \mathrm{d}[\mathbf{K}^{\sim}(t)]\, \mathbf{x}(t) = \int_0^1 \mathrm{d}[\mathbf{K}^{\wedge}(t)]\, \mathbf{x}(t) = \int_0^1 \mathrm{d}[\mathbf{K}(t)]\, \mathbf{x}(t)$$

for every $\mathbf{x} \in BV_n$ (cf. [7] I.5.5). Obviously, an analogous assertion is true if $\mathbf{K}$ is supposed to be right-continuous on $(0, 1)$ and $\mathbf{G}(t, s)$ is defined by (4,6) and (4,15).

We close the paper by the investigation of the properties of the Green matrix $\mathbf{G}(t, s)$ with respect to the argument s.

4.8. Theorem. *Let* $\mathbf{A} : [0, 1] \to L(R_n)$, $\mathbf{A} \in BV$ *fulfil* (1,4) *and let* (4,3) *hold. Let the matrices* $\mathbf{G}(t, s)$, $\mathbf{H}(t)$ *and* $\Delta(t, s)$ *be given by* (4,17),

$$\mathbf{H}(t) = \mathbf{X}(t)\, \mathbf{D}^{-1}, \quad t \in [0, 1]$$

and (4,20), *respectively. Then for any* $t \in [0, 1]$ *the relations*

$$\text{(4,23)} \quad \mathbf{G}(t, s) + \mathbf{G}(t, s)\, \mathbf{A}(s) + \int_s^1 \mathrm{d}_\sigma[\mathbf{G}(t, \sigma)]\, \mathbf{A}(\sigma) - \mathbf{H}(t)\,(\mathbf{K}(1) - \mathbf{K}(s)) = -(\Delta(t, 1) - \Delta(t, s)), \quad s \in [0, 1]$$

$$\text{(4,24)} \quad \mathbf{G}(t, 0) = \mathbf{G}(t, 1) = \mathbf{0} \quad \text{if} \quad t \in [0, 1), \qquad \mathbf{G}(1, 0) = \mathbf{0}, \quad \mathbf{G}(1, 1) = -\mathbf{I}$$

hold.

Proof. The relations (4,24) follow immediately from (4,17). Obviously, we may write

$$\text{(4,25)} \quad \mathbf{G}(t, s) = -\mathbf{X}(t)\, \Delta(t, s)\, \mathbf{X}^{-1}(s) + \mathbf{X}(t)\, \mathbf{D}^{-1} \int_s^1 \mathrm{d}[\mathbf{K}(\varrho)]\, \mathbf{X}(\varrho)\, \mathbf{X}^{-1}(s) \quad \text{on} \quad [0, 1] \times [0, 1].$$

Using Theorem 2.7 we get

$$\text{(4,26)} \quad \int_s^1 \mathrm{d}_\sigma \left[\mathbf{X}(t)\, \mathbf{D}^{-1} \int_\sigma^1 \mathrm{d}[\mathbf{K}(\varrho)]\, \mathbf{X}(\varrho)\, \mathbf{X}^{-1}(\sigma)\right] \mathbf{A}(\sigma) = \mathbf{X}(t)\, \mathbf{D}^{-1}(\mathbf{K}(1) - \mathbf{K}(s)) - \mathbf{X}(t)\, \mathbf{D}^{-1} \int_s^1 \mathrm{d}[\mathbf{K}(\varrho)]\, \mathbf{X}(\varrho)\, \mathbf{X}^{-1}(s)\, (\mathbf{I} + \mathbf{A}(s)) \quad \text{for all} \quad t, s \in [0, 1].$$

For $0 \leqq t < s \leqq 1$ or $t = s = 0$, this is exactly (4,23). Now, if $0 < s \leqq t \leqq 1$, then in virtue of (4,20) and (1,9)

$$\int_s^1 \mathrm{d}_\sigma[-\mathbf{X}(t)\,\Delta(t,\sigma)\,\mathbf{X}^{-1}(\sigma)]\,\mathbf{A}(\sigma) =$$

$$= -\mathbf{X}(t)\int_s^t \mathrm{d}[\mathbf{X}^{-1}(\sigma)]\,\mathbf{A}(\sigma) + \mathbf{A}(t) =$$

$$= -\mathbf{X}(t)\,[\mathbf{X}^{-1}(t) + \mathbf{X}^{-1}(t)\,\mathbf{A}(t) - \mathbf{X}^{-1}(s) - \mathbf{X}^{-1}(s)\,\mathbf{A}(s)] + \mathbf{A}(t) =$$

$$= -\mathbf{I} + \mathbf{X}(t)\,\Delta(t,s)\,\mathbf{X}^{-1}(s) + \mathbf{X}(t)\,\Delta(t,s)\,\mathbf{X}^{-1}(s)\,\mathbf{A}(s)\,.$$

This together with (4,26) yields (4,23) also for $0 \leqq s \leqq t \leqq 1$.

4.9. Remark. In other words, the couple $\mathbf{G}(t, s)$, $\mathbf{H}(s)$ is for any $t \in (0, 1)$ a solution to the adjoint nonhomogeneous matrix boundary value problem

$$-\mathrm{d}\mathbf{G}(t,\cdot) - \mathrm{d}[\mathbf{G}(t,\cdot)\,\mathbf{A}] + \mathrm{d}[\mathbf{G}(t,\cdot)]\,\mathbf{A} - \mathrm{d}[\mathbf{H}(t)\,\mathbf{K}] = -\mathrm{d}\Delta(t,\cdot)\,,$$

$$\mathbf{G}(t,0) = \mathbf{G}(t,1) = \mathbf{0}\,.$$

References

[1] *Bryan R. H.:* A linear differential system with general linear boundary conditions, J. Differential Equations, *5* (1969), 38—48.

[2] *Cole R. H.:* General boundary conditions for an ordinary linear differential system, Trans. Amer. Math. Soc., *111* (1964), 521—550.

[3] *Krall A. M.:* The development of general differential and general differential-boundary systems, Rocky Mountain J. Math., *5* (1975), 493—542.

[4] *Kurzweil J.:* Generalized ordinary differential equations and continuous dependence on a parameter, Czechoslovak Math. J., *7* (*82*) (1957), 418—449.

[5] *Kurzweil J.:* On integration by parts, Czechoslovak Math. J. *8* (*83*) (1958), 356—359.

[6] *Schwabik Š.:* On an integral operator in the space of functions with bounded variation, Časopis pěst. mat., *97* (1972), 297—330.

[7] *Schwabik Š., Tvrdý M.* and *Vejvoda O.:* Differential and integral equations: Boundary value problems and adjoints, Academia, Praha & Reidel, Dordrecht, 1979.

[8] *Tvrdý M.:* Boundary value problems for linear generalized differential equations and their adjoints, Czechoslovak Math. J., *23* (*98*) (1973), 183—197.

[9] *Tvrdý M.:* Boundary value problems for generalized linear integrodifferential equations with left-continuous solutions, Časopis pěst. mat. *99* (1974), 147—157.

[10] *Tvrdý M.:* Fredholm-Stieltjes integral equations with linear constraints. Duality theory, and Green's Function, Časopis pěst. mat., *104* (1979), to appear.

[11] *Vejvoda O.* and *Tvrdý M.:* Existence of solutions to a linear integro-differential equation with additional conditions, Ann. Mat. Pura Appl. *89* (1971), 169—216.

Authors' address: 115 67 Praha 1, Žitná 25, ČSSR (Matematický ústav ČSAV).