Robust and guaranteed a posteriori error estimator for singularly perturbed diffusion-reaction problems

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Outline

- Review of AEE
- Diffusion reaction problem: $$-\Delta u + \kappa^2 u = f \text{ in } \Omega$$ $$u = 0 \text{ on } \partial\Omega$$
- Standard equilibrated residuals
  - upper bound with no constant
  - elementwise local
  - non-robust if $\kappa \to \infty$
  - not computable
- Improvements
  - robust fluxes
  - computable
- Conclusions

A Posterior Error Estimates (AEE)

Definition

- $\|e\| \approx \eta \quad (\text{or } \|e\| \leq \eta, \quad \text{or } \eta \leq \|e\|)$
- $\eta = \eta(u_h, f, \Omega, \mathcal{T}_h, \dots)$

Properties

- efficient and reliable $\quad C_1\eta \leq \|e\| \leq C_2\eta$
- guaranteed bounds $\quad \|e\| \leq \eta \quad \text{or} \quad \eta \leq \|e\|$
- robust (w.r.t. κ^2) $\quad C_1 \neq C_1(\kappa^2) \quad C_2 \neq C_2(\kappa^2)$
- asymptotically exact $\quad \lim_{h\to 0} \frac{\eta}{\|e\|} = 1, \quad I_{\text{eff}} = \frac{\eta}{\|e\|}$
- local $\quad \eta^2 = \sum_{K\in\mathcal{T}_h} \eta_K^2$

Types of AEE

- Explicit residual: efficient, reliable, local
- Implicit residual:
  - Hierarchical: lower bound, reliable, (global)
  - Local Dirichlet: lower bound, reliable, local
  - Local Neumann: efficient, upper bound, local
- Error Majorants: upper bound, global
- Postprocessing: asymptotically exact, local
- Quantity of interest: no energy norm

Model Problem

- Classical formulation: $\kappa = \text{const.} > 0$

$$-\Delta u + \kappa^2 u = f \quad \text{in } \Omega \subset \mathbb{R}^2$$
$$u = 0 \quad \text{on } \partial\Omega$$

- Weak formulation:
$V = H_0^1(\Omega), \quad B(u,v) = \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x + \int_\Omega \kappa^2 uv \,\mathrm{d}x$

$$u \in V : \quad B(u,v) = \int_\Omega fv \,\mathrm{d}x \quad \forall v \in V$$

- Linear triangular FEM:
$V_h = \{v_h \in V : v_h|_K \in P^1(K), K \in \mathcal{T}_h\}$

$$u_h \in V_h : \quad B(u_h, v_h) = \int_\Omega f v_h \,\mathrm{d}x \quad \forall v_h \in V_h$$

Notation

- $e = u - u_h$
- Residual equation

$$e \in V : \quad B(e,v) = \int_\Omega fv \,\mathrm{d}x - B(u_h, v) \quad \forall v \in V$$

- $B_K(u,v) = \int_K \nabla u \cdot \nabla v \,\mathrm{d}x + \int_K \kappa^2 uv \,\mathrm{d}x$
- $|||v|||^2 = B(v,v)$
  $|||v|||_K^2 = B_K(v,v)$
- $H_E^1(K) = \{v \in H^1(K) : v = 0 \text{ on } \partial K \cap \partial\Omega\} \quad K \in \mathcal{T}_h$

Local Neumann Problems

- $\varepsilon_K \in H^1_E(K)$:

$$B_K(\varepsilon_K, v) = \int_K fv \,\mathrm{d}x - B_K(u_h, v) + \int_{\partial K} g_K v \,\mathrm{d}s \quad \forall v \in H^1_E(K)$$

Local Neumann Problems

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$$B_K(\varepsilon_K, v) = \int_K fv \,\mathrm{d}x - B_K(u_h, v) + \int_{\partial K} g_K v \,\mathrm{d}s \quad \forall v \in H^1_E(K)$$

- Remark:

$$\begin{aligned} -\Delta(\varepsilon_K + u_h) + \kappa^2(\varepsilon_K + u_h) &= f && \text{in } K \\ \nabla(\varepsilon_K + u_h) \cdot \mathbf{n}_K &= g_K && \text{on } \partial K \setminus \partial\Omega \\ \varepsilon_K + u_h &= 0 && \text{on } \partial K \cap \partial\Omega \end{aligned}$$

Local Neumann Problems

- $\varepsilon_K \in H^1_E(K)$:

$$B_K(\varepsilon_K, v) = \int_K fv \,\mathrm{d}x - B_K(u_h, v) + \int_{\partial K} g_K v \,\mathrm{d}s \quad \forall v \in H^1_E(K)$$

- Theorem: If $g_K|_\gamma + g_{K^*}|_\gamma = 0$ for $\gamma = \partial K \cap \partial K^*$ then $\|e\|^2 \leq \sum_{K\in\mathcal{T}_h} \|\varepsilon_K\|^2_K$.

- Proof: $e = u - u_h$

$$B(e, v) = \sum_{K\in\mathcal{T}_h} \left(\int_K fv \,\mathrm{d}x - B_K(u_h, v) + \int_{\partial K} g_K v \,\mathrm{d}s \right)$$

$$= \sum_{K\in\mathcal{T}_h} B_K(\varepsilon_K, v) \leq \left(\sum_{K\in\mathcal{T}_h} \|\varepsilon_K\|^2_K \right)^{\frac{1}{2}} \|v\|$$

$\square$

Local Neumann Problems

- $\varepsilon_K \in H^1_E(K)$:

$$B_K(\varepsilon_K, v) = \int_K fv \,\mathrm{d}x - B_K(u_h, v) + \int_{\partial K} g_K v \,\mathrm{d}s \quad \forall v \in H^1_E(K)$$

- Theorem: If $g_K = \partial u / \partial \mathbf{n}_K$ then $\|e\|^2 = \sum_{K \in \mathcal{T}_h} \|\varepsilon_K\|^2_K$.
- Proof:
  - $\kappa^2 > 0 \quad \Rightarrow \quad u = \varepsilon_K + u_h$

$$\begin{aligned} -\Delta(\varepsilon_K + u_h) + \kappa^2(\varepsilon_K + u_h) &= f && \text{in } K \\ \nabla(\varepsilon_K + u_h) \cdot \mathbf{n}_K &= g_K && \text{on } \partial K \setminus \partial\Omega \\ \varepsilon_K + u_h &= 0 && \text{on } \partial K \cap \partial\Omega \end{aligned}$$

  - $\kappa^2 = 0 \quad \Rightarrow \quad u = \varepsilon_K + u_h + C_K$ and $\|u - u_h\|_K = \|\varepsilon_K\|_K$

$\square$

Local Neumann Problems

- $\varepsilon_K \in H^1_E(K)$:

$$B_K(\varepsilon_K, v) = \int_K fv \,\mathrm{d}x - B_K(u_h, v) + \int_{\partial K} g_K v \,\mathrm{d}s \quad \forall v \in H^1_E(K)$$

- Standard choice of g_K:
  - $g_K|_\gamma + g_{K^*}|_\gamma = 0$
  - $g_K|_\gamma \in P^1(\gamma)$, $\gamma \subset \partial K$, $K \in \mathcal{T}_h$, $\quad g_K \approx \dfrac{\partial u|_K}{\partial \mathbf{n}_K}$ on ∂K
  - equilibration condition

Non-robustness

$$|||e|||^2 \leq \sum_{K\in\mathcal{T}_h} |||\varepsilon_K|||_K^2$$

Example (M.A.+I.B. 1999)

$$-u'' + \kappa^2 u = \cos \pi x \quad \text{in } (-1/2, 1/2) \qquad u(x) = \frac{\cos \pi x}{\pi^2 + \kappa^2}$$

$$u(\pm 1/2) = 0$$

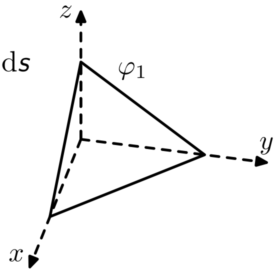
κ	$I_{\text{eff}}(g_K^{\text{equilib}})$	$I_{\text{eff}}(g_K^{\text{robust}})$
1	1.00	1.00
10^1	1.00	1.00
10^2	1.00	1.00
10^3	1.03	1.00
10^4	2.73	1.00
10^5	8.08	1.00
10^6	25.37	1.00

Construction of Fluxes

First order equilibration:

$$0 = \int_K f\varphi_i \,\mathrm{d}x - B_K(u_h, \varphi_i) + \int_{\partial K} g_K^{\text{equilib}} \varphi_i \,\mathrm{d}s$$

φ_i ... basis of $P^1(K) \cap H^1_E(K)$.

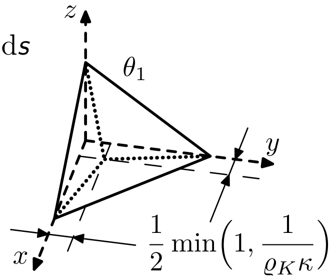


Robust fluxes:

$$0 \approx \int_K f\theta_i \,\mathrm{d}x - B_K(u_h, \theta_i) + \int_{\partial K} g_K^{\text{robust}} \theta_i \,\mathrm{d}s$$

$\theta_i \approx \mathcal{E}\varphi_i$ ... approximate minimum energy extension of φ_i

$\mathcal{E}v \in H^1(K)$:
$\mathcal{E}v = v$ on ∂K
$B_K(\mathcal{E}v, w) = 0 \quad \forall w \in H^1_0(K)$



Estimation of Local Errors

$$\varepsilon_K \in H^1_E(K), \quad v \in H^1_E(K),$$

$$B_K(\varepsilon_K, v) = \int_K fv \,\mathrm{d}x - B_K(u_h, v) + \int_{\partial K} g_K v \,\mathrm{d}s$$

Estimation of Local Errors

$$\varepsilon_K \in H^1_E(K), \quad v \in H^1_E(K), \quad \mathbf{y}_K \in \mathbf{H}(\mathrm{div}, K),$$

$$\begin{aligned} B_K(\varepsilon_K, v) = & \int_K f v \,\mathrm{d}x - B_K(u_h, v) \\ & + \int_{\partial K} g_K v \,\mathrm{d}s \\ & \underbrace{+ \int_K \mathrm{div}\,\mathbf{y}_K v \,\mathrm{d}x + \int_K \mathbf{y}_K \cdot \nabla v \,\mathrm{d}x - \int_{\partial K} \mathbf{y}_K \cdot \mathbf{n}_K v \,\mathrm{d}s}_{=0} \end{aligned}$$

Estimation of Local Errors

$$\varepsilon_K \in H^1_E(K), \quad v \in H^1_E(K), \quad \mathbf{y}_K \in \mathbf{H}(\mathrm{div}, K),$$

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Estimation of Local Errors

$$\varepsilon_K \in H^1_E(K), \quad v \in H^1_E(K), \quad \mathbf{y}_K \in \mathbf{H}(\mathrm{div}, K),$$

$$B_K(\varepsilon_K, v) = \int_K fv \,\mathrm{d}x - \int_K \nabla u_h \cdot \nabla v \,\mathrm{d}x - \int_K \kappa^2 u_h v \,\mathrm{d}x + \int_{\partial K} g_K v \,\mathrm{d}s + \int_K \mathrm{div}\, \mathbf{y}_K v \,\mathrm{d}x + \int_K \mathbf{y}_K \cdot \nabla v \,\mathrm{d}x - \int_{\partial K} \mathbf{y}_K \cdot \mathbf{n}_K v \,\mathrm{d}s$$

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$$r = f - \kappa^2 u_h$$

Estimation of Local Errors

$$\varepsilon_K \in H^1_E(K), \quad v \in H^1_E(K), \quad \mathbf{y}_K \in \mathbf{H}(\mathrm{div}, K),$$

$$\begin{aligned} B_K(\varepsilon_K, v) = & \int_K (r + \mathrm{div}\, \mathbf{y}_K) v \,\mathrm{d}x - \int_K \nabla u_h \cdot \nabla v \,\mathrm{d}x \\ & + \int_{\partial K} g_K v \,\mathrm{d}s \\ & + \int_K \mathbf{y}_K \cdot \nabla v \,\mathrm{d}x - \int_{\partial K} \mathbf{y}_K \cdot \mathbf{n}_K v \,\mathrm{d}s \end{aligned}$$

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$$r = f - \kappa^2 u_h$$

$$\mathbf{y}_K \cdot \mathbf{n}_K = g_K \quad \text{on } \partial K \setminus \partial\Omega$$

Estimation of Local Errors

$$\varepsilon_K \in H^1_E(K), \quad v \in H^1_E(K), \quad \mathbf{y}_K \in \mathbf{H}(\mathrm{div}, K),$$

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$$+ \int_K \mathbf{y}_K \cdot \nabla v \,\mathrm{d}x$$

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$$+ \int_K (\mathbf{y}_K - \nabla u_h) \cdot \nabla v \,\mathrm{d}x$$

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$$\varepsilon_K \in H^1_E(K), \quad v \in H^1_E(K), \quad \mathbf{y}_K \in \mathbf{H}(\mathrm{div}, K),$$

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$$+ \|\mathbf{y}_K - \nabla u_h\|_{0,K} \, \|\nabla v\|_{0,K}$$

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Estimation of Local Errors

$$\varepsilon_K \in H^1_E(K), \quad v \in H^1_E(K), \quad \mathbf{y}_K \in \mathbf{H}(\mathrm{div}, K),$$

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$$+ \frac{1}{2} \left\| \mathbf{y}_K - \nabla u_h \right\|^2_{0,K} + \frac{1}{2} \left\| \nabla v \right\|^2_{0,K}$$

$$r = f - \kappa^2 u_h$$
$$\mathbf{y}_K \cdot \mathbf{n}_K = g_K \quad \text{on } \partial K \setminus \partial \Omega$$

Estimation of Local Errors

$$\varepsilon_K \in H^1_E(K), \quad v \in H^1_E(K), \quad \mathbf{y}_K \in \mathbf{H}(\mathrm{div}, K),$$

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$$+ \frac{1}{2} \|\mathbf{y}_K - \nabla u_h\|^2_{0,K} + \frac{1}{2} \|\nabla v\|^2_{0,K}$$

$r = f - \kappa^2 u_h$

$\mathbf{y}_K \cdot \mathbf{n}_K = g_K \quad$ on $\partial K \setminus \partial\Omega$

$\frac{1}{2} \|\kappa v\|^2_{0,K} + \frac{1}{2} \|\nabla v\|^2_{0,K} = \frac{1}{2} |||v|||^2_K$

Estimation of Local Errors

$$\varepsilon_K \in H^1_E(K), \quad v \in H^1_E(K), \quad \mathbf{y}_K \in \mathbf{H}(\mathrm{div}, K),$$

$$B_K(\varepsilon_K, v) \leq \frac{1}{2} \left\| \frac{1}{\kappa}(r + \mathrm{div}\, \mathbf{y}_K) \right\|^2_{0,K}$$

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$$r = f - \kappa^2 u_h$$
$$\mathbf{y}_K \cdot \mathbf{n}_K = g_K \quad \text{on } \partial K \setminus \partial\Omega$$
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Estimation of Local Errors

$$\varepsilon_K \in H^1_E(K), \quad v \in H^1_E(K), \quad \mathbf{y}_K \in \mathbf{H}(\mathrm{div}, K),$$

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Estimation of Local Errors

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$$\frac{1}{2} |||\varepsilon_K|||_K^2 \le \frac{1}{2} \left\| \frac{1}{\kappa}(r + \mathrm{div}\,\mathbf{y}_K) \right\|_{0,K}^2$$

$$+ \frac{1}{2} \|\mathbf{y}_K - \nabla u_h\|_{0,K}^2$$

$$r = f - \kappa^2 u_h$$
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$$r = f - \kappa^2 u_h$$

$$\mathbf{y}_K \cdot \mathbf{n}_K = g_K \quad \text{on } \partial K \setminus \partial\Omega$$

Estimation of Local Errors

$$\varepsilon_K \in H^1_E(K), \quad v \in H^1_E(K), \quad \mathbf{y}_K \in \mathbf{H}(\mathrm{div}, K),$$

- Local estimate

$$|||\varepsilon_K|||^2_K \leq \left\| \frac{1}{\kappa}(r + \mathrm{div}\,\mathbf{y}_K) \right\|^2_{0,K} + \|\mathbf{y}_K - \nabla u_h\|^2_{0,K} \equiv \eta^2_K(\mathbf{y}_K)$$

$$r = f - \kappa^2 u_h$$
$$\mathbf{y}_K \cdot \mathbf{n}_K = g_K \quad \text{on } \partial K \setminus \partial\Omega$$

Estimation of Local Errors

$$\varepsilon_K \in H^1_E(K), \quad v \in H^1_E(K), \quad \mathbf{y}_K \in \mathbf{H}(\mathrm{div}, K),$$

- Local estimate

$$|||\varepsilon_K|||_K^2 \leq \left\| \frac{1}{\kappa}(r + \mathrm{div}\, \mathbf{y}_K) \right\|_{0,K}^2 + \|\mathbf{y}_K - \nabla u_h\|_{0,K}^2 \equiv \eta_K^2(\mathbf{y}_K)$$

- Global estimate

$$|||e|||^2 \leq \sum_{K \in \mathcal{T}_h} |||\varepsilon_K|||_K^2 \leq \sum_{K \in \mathcal{T}_h} \eta_K^2(\mathbf{y}_K)$$

$$r = f - \kappa^2 u_h$$

$$\mathbf{y}_K \cdot \mathbf{n}_K = g_K \quad \text{on } \partial K \setminus \partial\Omega$$

Estimation of Local Errors

$$\varepsilon_K \in H^1_E(K), \quad v \in H^1_E(K), \quad \mathbf{y}_K \in \mathbf{H}(\mathrm{div}, K),$$

- Local estimate

$$|||\varepsilon_K|||_K^2 \leq \left\| \frac{1}{\kappa}(r + \operatorname{div} \mathbf{y}_K) \right\|_{0,K}^2 + \|\mathbf{y}_K - \nabla u_h\|_{0,K}^2 \equiv \eta_K^2(\mathbf{y}_K)$$

- Theorem: If $\mathbf{y}_K = \nabla(u_h + \varepsilon_K)$ then $|||\varepsilon_K|||_K = \eta_K(\mathbf{y}_K)$.
- Proof:
  - $f - \kappa^2 u_h + \operatorname{div} \mathbf{y}_K = f - \kappa^2 u_h + \Delta(u_h + \varepsilon_K) = \kappa^2 \varepsilon_K$
  - $\mathbf{y}_K - \nabla u_h = \nabla \varepsilon_K$

□

$$r = f - \kappa^2 u_h$$

$$\mathbf{y}_K \cdot \mathbf{n}_K = g_K \quad \text{on } \partial K \setminus \partial\Omega$$

Choice of $\mathbf{y}_K$: (i) Minimization

Minimize $\eta_K^2(\mathbf{y}_K)$ over $\mathbf{W}^p(K) \subset \mathbf{H}(\mathrm{div}, K)$

$$\mathbf{W}^p(K) = \left\{\mathbf{y} \in [P^p(K)]^2 : \mathbf{y}\cdot\mathbf{n}_K = g_K\right\}$$
$$\mathbf{W}_0^p(K) = \left\{\mathbf{y} \in [P^p(K)]^2 : \mathbf{y}\cdot\mathbf{n}_K = 0\right\}$$

$$\mathbf{y}_K = \mathbf{y}_K^0 + \bar{\mathbf{y}}_K, \qquad \begin{array}{l} \mathbf{y}_K^0 \in \mathbf{W}_0^p(K) \\ \bar{\mathbf{y}}_K \in [P^1(K)]^2 \cap \mathbf{W}^p(K) \text{ uniquely given} \end{array}$$

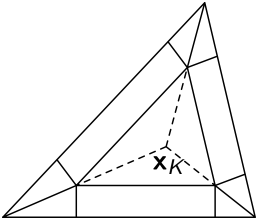
Find $\mathbf{y}_K^0 \in \mathbf{W}_0^p(K)$:

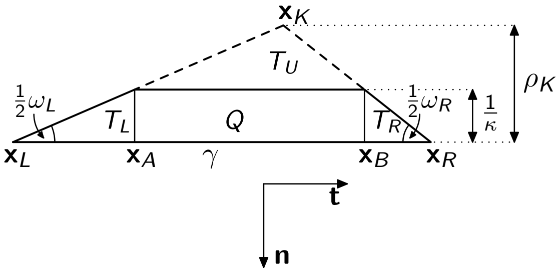
$$\int_K \mathrm{div}\,\mathbf{y}_K^0\,\mathrm{div}\,\mathbf{w}\,\mathrm{d}x + \int_K \kappa^2\mathbf{y}_K^0\mathbf{w}\,\mathrm{d}x$$
$$= -\int_K f\,\mathrm{div}\,\mathbf{w}\,\mathrm{d}x - \int_K \mathrm{div}\,\bar{\mathbf{y}}_K\,\mathrm{div}\,\mathbf{w}\,\mathrm{d}x - \int_K \kappa^2\bar{\mathbf{y}}_K\mathbf{w}\,\mathrm{d}x$$
$$\forall \mathbf{w} \in \mathbf{W}_0^p(K)$$

Choice of $\mathbf{y}_K$: (ii) Explicit

- $0 \le \kappa\rho_K \le 1$: $\quad \mathbf{y}_K^* = \bar{\mathbf{y}}_K, \;\; \bar{\mathbf{y}}_K \in [P^1(K)]^2, \;\; \bar{\mathbf{y}}_K \cdot \mathbf{n}_K = g_K$
- $\kappa\rho_K > 1$:

$$\mathbf{y}_K^*(\mathbf{x}) = \begin{cases} g_{K,\gamma}(\mathbf{x}_L)\lambda_L^{(L)}(\mathbf{x})\left(\mathbf{n} - \cot\dfrac{\omega_L}{2}\mathbf{t}\right) + g_{K,\gamma}(\mathbf{x}_A)\lambda_A^{(L)}(\mathbf{x})\mathbf{n}, & \mathbf{x} \in T_L \\ g_{K,\gamma}(\mathbf{x}_A + x\mathbf{t})(1-\kappa y)\mathbf{n}, & \mathbf{x} \in Q \\ g_{K,\gamma}(\mathbf{x}_R)\lambda_R^{(R)}(\mathbf{x})\left(\mathbf{n} + \cot\dfrac{\omega_R}{2}\mathbf{t}\right) + g_{K,\gamma}(\mathbf{x}_B)\lambda_B^{(R)}(\mathbf{x})\mathbf{n}, & \mathbf{x} \in T_R \\ 0, & \mathbf{x} \in T_U \end{cases}$$





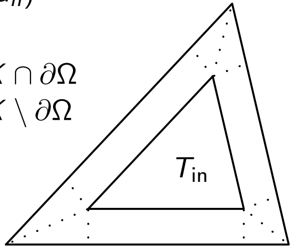
On Robustness ($\kappa \to \infty$)

- Robustness of ε_K (M.A.+I.B. 1999): $\|\varepsilon_K\|_K \le C_1 \|e\|_K$
- Need: $\eta_K(\mathbf{y}_K^*) \le C_2 \|\varepsilon_K\|_K$, i.e.,

$$\|\mathbf{y}_K^* - \nabla u_h\|_{0,K}^2 + \left\| \frac{1}{\kappa}(f - \kappa^2 u_h + \operatorname{div} \mathbf{y}_K^*) \right\|_{0,K}^2 \le C_2^2 \left(\|\nabla \varepsilon_K\|_{0,K}^2 + \|\kappa \varepsilon_K\|_{0,K}^2 \right)$$

- $$\mathbf{y}_K^* \approx \nabla(\varepsilon_K + u_h)$$
  $$-\operatorname{div} \mathbf{y}_K^* \approx -\Delta(\varepsilon_K + u_h) = f - \kappa^2(\varepsilon_K + u_h)$$

- $$\nabla(\varepsilon_K + u_h) = \begin{cases} f\kappa^{-1}\mathbf{n}_K & \text{in layer at } \partial K \cap \partial\Omega \\ g_K \mathbf{n}_K & \text{in layer at } \partial K \setminus \partial\Omega \\ 0 & \text{in } T_{\text{in}} \end{cases}$$

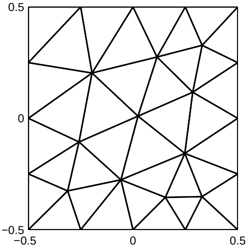


Numerical examples

$$-\Delta u + \kappa^2 u = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$

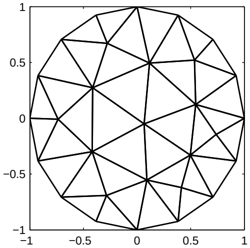
Example (A)

$$\Omega = (-1/2, 1/2)^2$$
$$f = \cos(\pi x)\cos(\pi y)$$
$$u = \frac{\cos(\pi x)\cos(\pi y)}{\pi^2 + \kappa^2}$$



Example (B)

$$\Omega = \{(x, y) : r < 1\}$$
$$f = 1 \qquad r = \sqrt{x^2 + y^2}$$
$$u = \frac{1}{\kappa^2}\left(1 - \frac{I_0(\kappa r)}{I_0(\kappa)}\right)$$



Results

$\eta_K(\mathbf{y}_K^*)$

Example (A)

κ	I_{eff}
0	2.76471
10^{-3}	2.76471
10^{-2}	2.76471
10^{-1}	2.76480
1	2.77374
10	3.14938
10^2	1.39897
10^3	1.00964
10^4	1.00060
10^5	1.00006
10^6	1.000006

Example (B)

κ	I_{eff}
0	—
10^{-3}	4.08953
10^{-2}	4.08952
10^{-1}	4.08838
1	3.95899
10	2.39917
10^2	1.07185
10^3	1.00210
10^4	1.00018
10^5	1.000018
10^6	1.0000018

Conclusions

$$-\Delta u + \kappa^2 u = f \quad \text{in } \Omega \subset \mathbb{R}^2$$
$$u = 0 \quad \text{on } \partial\Omega$$

$$|||u - u_h|||^2 \leq \sum_{K \in \mathcal{T}_h} \left(\left\| \frac{1}{\kappa}(r + \operatorname{div} \mathbf{y}_K^*) \right\|_{0,K}^2 + \|\mathbf{y}_K - \nabla u_h\|_{0,K}^2 \right)$$

- No constants
- Completely computable
- Guaranteed upper bound
- Elementwise local
- Robust for $\kappa^2 \in (0, \infty)$

Thank you for your attention

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