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ERGODICITY AND PARAMETER ESTIMATES FOR INFINITE-DIMENSIONAL FRACTIONAL ORNSTEIN-UHLENBECK PROCESS

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172
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ERGODICITY AND PARAMETER ESTIMATES FOR INFINITE-DIMENSIONAL FRACTIONAL ORNSTEIN-UHLENBECK PROCESS

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Abstract. Existence and ergodicity of a strictly stationary solution for linear stochastic evolution equations driven by cylindrical fractional Brownian motion are proved. Ergodic behaviour of non-stationary infinite-dimensional fractional Ornstein-Uhlenbeck processes is also studied. Based on these results, strong consistency of suitably defined families of parameter estimators is shown. The general results are applied to linear parabolic and hyperbolic equations perturbed by a fractional noise.

1. Introduction

The theory of stochastic equations in infinite dimensional spaces (or stochastic partial differential equations) driven by fractional Brownian motion is still at a very early stage. Linear stochastic equations in a Hilbert space with a cylindrical fractional Brownian motion are considered by Duncan, Maslowski and Pasik-Duncan in [4] and [6] where some results on the continuity and space regularity of sample paths are given and large time behaviour of solutions is investigated. In [5], similar analysis is carried out for bilinear equations. The same type of equations is investigated by Tindel, Tudor and Viens in [27], where a fractional Feynman-Kac formula is obtained. Grecksch and Anh [8] consider a semi-linear stochastic parabolic equation with additive noise term and prove existence and uniqueness of the solution. Semilinear evolution equations with a covariance type fractional Brownian motion with a non-additive noise term are studied by Maslowski and Nualart in [14] and Nualart and Vuillermot in [16], where existence, uniqueness and pathwise regularity are established. Hu, Oksendal and Zhang [11] treat the elliptic equation (the Poisson problem) and the heat equation with multiparameter fractional Gaussian noise (cf. also [9], where the stochastic heat equations with a multiplicative fractional noise is dealt with). Maslowski and Schmalfuss [15] show the existence of pathwise exponentially stable random fixed point for semilinear dissipative systems. The linear wave equation perturbed by fractional noise is studied by Caithamer [2].

The problem of parameter estimation for infinite dimensional stochastic equations has been studied also only recently. There exist several methods mainly of statistical origin. One of them is the maximum likelihood estimates method used by Huebner and Rozovskii in [12] to estimate parameters from a continuous observation of a solution to stochastic parabolic equations driven by Wiener process. Khasminskii and Milstein studied in [13] the estimation of the linearised drift for nonlinear SDEs. Estimation for some stochastic equations based on discrete observations was presented by Prakasa Rao in [22]. He also investigated the asymptotic properties of the maximum likelihood estimator and the Bayes estimator of the drift parameter in stochastic equations driven by fractional Brownian motion, see [23] and [24]. Goldys and Maslowski considered in [7] a controlled stochastic semilinear equation with the drift depending on the unknown parameter and with the Wiener process as the driving process. They showed that the maximum likelihood estimator is consistent for a class of bounded predictable controls. A problem of identification of

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some unknown parameters for a stochastic parabolic equation with a scalar fractional Brownian motion was also partially mentioned in [5].

In this paper we study linear stochastic evolution equations in Hilbert spaces. The driving process is fractional Brownian motion with Hurst parameter $H \in (0,1)$. Given separable Hilbert spaces U and V, we consider the equation

$$\begin{aligned} dX(t) &= AX(t)\,dt + \Phi\,dB^H(t), \\ X(0) &= x_0, \end{aligned} \tag{1}$$

where $(B^H(t), t \geq 0)$ is a standard cylindrical fractional Brownian motion on U, $A : \mathrm{Dom}(A) \to V$, $\mathrm{Dom}(A) \subset V$, A is the infinitesimal generator of a strongly continuous semigroup $(S(t), t \geq 0)$ on V, $\Phi \in \mathcal{L}(U,V)$ and $x_0 \in V$ is a random variable. The solution $(X^{x_0}(t), t \geq 0)$ to (1) is defined by the mild form,

$$X^{x_0}(t) = S(t)x_0 + Z(t), \quad t \geq 0, \tag{2}$$

where $(Z(t), t \geq 0)$ is the convolution integral

$$Z(t) = \int_0^t S(t-u)\Phi\,dB^H(u). \tag{3}$$

It is known ([4] and [5]) that the solution exists as a V-valued process under suitable conditions. If the semigroup is exponentially stable there exists a centered Gaussian limiting measure μ_∞ for the solution. We show that if the semigroup is exponentially stable there exists a strictly stationary solution $(X^{\tilde{x}}(t), t \geq 0)$ with probability distribution μ_∞ (see Theorem 3.1 for a precise statement that extends an earlier result in [15]).

We then prove that this stationary solution is ergodic, that is,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \varrho(X^{\tilde{x}}(t))\,dt \; = \int_V \varrho(y)\,\mu_\infty(dy),$$

almost surely as $T \to \infty$ for any measurable functional $\varrho : V \to \mathbf{R}$ such that $\mathbb{E}|\varrho(\tilde{x})| < \infty$ (cf. Theorem 4.6).

As a corollary we obtain a similar result for arbitrary solution, under some restrictions on the functional ϱ. For instance, in Theorem 4.9 we assume that ϱ is locally Lipschitz with the Lipschitz constant growing at most polynomially and the noise is of covariance type.

These results are employed in Section 5 where parameter-dependent equations are considered with a multiplicative parameter in the drift. Two families of estimators are defined and, based on the above ergodic theorems, their strong consistency is proved (Theorems 5.1 and 5.2, respectively).

Technically, some of these results are probably new even in the particular case, when the state space is finite-dimensional. However, in this case the problem is considerably simplified. We consider two basic examples where our general results are applied and conditions of particular theorems are verified: Linear stochastic heat (or parabolic) equation with Dirichlet-type boundary conditions and distributed fractional noise (Example 4.1) and an analogous problem for stochastic wave (or hyperbolic) equation (Example 4.2).

2. Preliminaries

Let $K_H(t,s)$ for $0 \leq s \leq t \leq T$ be the kernel function

$$K_H(t,s) = c_H(t-s)^{H-\frac{1}{2}} + c_H\left(\frac{1}{2} - H\right)\int_s^t (u-s)^{H-\frac{3}{2}}\left(1 - \left(\frac{s}{u}\right)^{\frac{1}{2}-H}\right)du$$

where

$$c_H = \left[\frac{2H\Gamma\left(H+\frac{1}{2}\right)\Gamma\left(\frac{3}{2}-H\right)}{\Gamma\left(2-2H\right)}\right]^{\frac{1}{2}},$$

where $\Gamma(\cdot)$ is the gamma function and $H \in (0,1)$.

Following [1] and [3], we will define a stochastic integral of a deterministic V-valued function with respect to a scalar fractional Brownian motion $(\beta^H(t), t \in \mathbf{R})$.

Let $\mathcal{K}_H^*\colon \mathcal{E} \to L^2([0,T],V)$ be the linear operator given by

$$\mathcal{K}_H^*\varphi(t) := \varphi(t)K_H(T,t) + \int_t^T (\varphi(s) - \varphi(t)) \frac{\partial K_H}{\partial s}(s,t)\, ds$$

for $\varphi \in \mathcal{E}$ where $\mathcal{E}$ is the linear space of V-valued step functions on $[0,T]$ and

$$\varphi(t) = \sum_{i=1}^{n-1} x_i \mathbb{1}_{[t_i,t_{i+1})}(t)$$

where $x_i \in V$, $i \in \{1,\dots,n-1\}$ and $0 = t_1 < \cdots < t_n = T$.

Setting

$$\int_0^T \varphi\, d\beta^H := \sum_{i=1}^{n} x_i \left(\beta^H(t_{i+1}) - \beta^H(t_i)\right)$$

it follows directly that

$$\mathbb{E}\left|\int_0^T \varphi\, d\beta^H\right|_V^2 = |\mathcal{K}_H^*\varphi|^2_{L^2([0,T],V)}.$$

Let $(\mathcal{H}, |\cdot|_{\mathcal{H}}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ be the Hilbert space obtained by the completion of the pre-Hilbert space $\mathcal{E}$ with respect to the inner product

$$\langle\varphi,\psi\rangle_{\mathcal{H}} := \langle\mathcal{K}_H^*\varphi, \mathcal{K}_H^*\psi\rangle_{L^2([0,T],V)}$$

for $\varphi,\psi \in \mathcal{E}$. The stochastic integral (2) is extended to $\mathcal{H}$ by the isometry (2). Thus $\mathcal{H}$ is the space of integrable functions and it is useful to obtain some more specific information, cf. [18] for details on integration theory.

If $H \in \left(\frac{1}{2},1\right)$, then it is easily verified that $\mathcal{H} \supset \tilde{\mathcal{H}}$ where $\tilde{\mathcal{H}}$ is the Banach space of Borel measurable functions with the norm $|\cdot|_{\tilde{\mathcal{H}}}$ given by

$$|\varphi|^2_{\tilde{\mathcal{H}}} := \int_0^T\int_0^T |\varphi(u)|_V|\varphi(v)|_V\phi(u-v)\, du\, dv$$

where

$$\phi(u) = H(2H-1)|u|^{2H-2}. \tag{4}$$

It may be verified that $\tilde{\mathcal{H}} \supset L^{1/H}([0,T],V)$ and consequently $\tilde{\mathcal{H}} \supset L^2([0,T],V)$. If $\varphi \in \tilde{\mathcal{H}}$ and $H > \frac{1}{2}$, then

$$\mathbb{E}\left|\int_0^T \varphi\, d\beta^H\right|_V^2 = \int_0^T\int_0^T \langle\varphi(u),\varphi(v)\rangle_V\, \phi(u-v)\, du\, dv.$$

If $H \in \left(0,\frac{1}{2}\right)$, then the space of integrable functions is smaller than $L^2([0,T],V)$. It is known that $\mathcal{H} \supset C^\beta([0,T],V)$ for each $\beta > \frac{1}{2} - H$ (see e.g. [10] Lemma 5.20) and in particular $\mathcal{H} \supset H^1([0,T],V)$. If $H \in \left(0,\frac{1}{2}\right)$, then the linear operator $\mathcal{K}_H^*$ may be described the composition

$$\mathcal{K}_H^*\varphi(t) = c_H t^{\frac{1}{2}-H} D_{T-}^{\frac{1}{2}-H}\left(u_{H-\frac{1}{2}}\varphi\right), \tag{5}$$

where

$$\left(D^\alpha_{T-}\psi\right)(t) = \frac{1}{\Gamma(1-\alpha)}\left(\frac{\psi(t)}{(T-t)^\alpha} + \alpha\int_t^T \frac{\psi(s)-\psi(t)}{(s-t)^{\alpha+1}}\, ds\right).$$

is a fractional derivative and $(u_{H-1/2}\varphi)(s) = s^{H-1/2}\varphi(s)$.

Definition 2.1. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a complete probability space. Let $(U, |\cdot|_U, \langle\cdot,\cdot\rangle_U)$ be a separable Hilbert space. A cylindrical process $\langle B^H,\cdot\rangle : \Omega\times\mathbf{R}\times U \to \mathbf{R}$ on $(\Omega,\mathcal{F},\mathbb{P})$ is called a standard cylindrical fractional Brownian motion with Hurst parameter $H \in (0,1)$ if

(1) For each $x \in U\setminus\{0\}$, $\frac{1}{|x|_U}\langle B^H(\cdot),x\rangle$ is a standard scalar fractional Brownian motion with Hurst parameter H.

(2) For $\alpha, \beta \in \mathbf{R}$ and $x, y \in U$,

$$\left\langle B^H(t), \alpha x + \beta y \right\rangle = \alpha \left\langle B^H(t), x \right\rangle + \beta \left\langle B^H(t), y \right\rangle \quad \text{a.s. } \mathbb{P}.$$

Note that $\langle B^H(t), x \rangle$ has the interpretation of the evaluation of the functional $B^H(t)$ at x although the process $B^H(\cdot)$ does not take values in U.

For $H = \frac{1}{2}$, this is the usual definition of a standard cylindrical Wiener process in U.

We will now define the stochastic integral $\int_0^T G\, d\beta^H$ for an operator-valued function $G : [0, T] \to \mathcal{L}(U, V)$.

Definition 2.2. Let $G : [0, T] \to \mathcal{L}(U, V)$, $(e_n, n \in \mathbf{N})$ be a complete orthonormal basis in U, $G(\cdot)e_n \in \mathcal{H}$ for $n \in \mathbf{N}$, and B^H be a standard cylindrical fractional Brownian motion in U. Let $\beta_n^H(t) := \left\langle B^H(t), e_n \right\rangle$ for $n \in \mathbf{N}$. Define

$$\int_0^T G\, dB^H := \sum_{n=1}^{\infty} \int_0^T G e_n\, d\beta_n^H$$

provided the infinite series converges in $L^2(\Omega, V)$.

Note that by condition 2 in Definition 2.1, the scalar processes $(\beta_n^H(t), t \in \mathbf{R}, n \in \mathbf{N})$ are independent.

Next we formulate an existence and regularity results for the solution of equation (1). In what follows, it is assumed that $(S(t), t \geq 0)$ is an analytic semigroup. In this case, there is a $\hat{\beta} \in \mathbf{R}$ such that the operator $\hat{\beta} I - A$ is uniformly positive on V. For each $\delta \geq 0$, let us define $(V_\delta, |\cdot|_\delta)$ a Hilbert space, where $V_\delta = \operatorname{Dom}\left((\hat{\beta} I - A)^\delta\right)$ with the graph norm topology such that

$$|x|_\delta = \left|(\hat{\beta} I - A)^\delta x\right|_V .$$

The shift $\hat{\beta}$ is fixed. The space V_δ does not depend on $\hat{\beta}$ because the norms are equivalent for different values of $\hat{\beta}$ satisfying the positivity condition.

Proposition 2.1. *Let $H \in (0, 1)$. Let $(S(t), t \geq 0)$ be a strong continuous analytic semigroup such that*

$$\text{(A1)} \qquad |S(t)\Phi|_{\mathcal{L}_2(U,V)} \leq c t^{-\gamma}, \quad t \in (0, T],$$

for some $T > 0$, $c > 0$ and $\gamma \in [0, H)$. Then $(Z(t), t \in [0, T])$ is a well-defined V_δ-valued process in $\mathcal{C}^\beta([0, T], V_\delta)$, a.s.-$\mathbb{P}$ for $\beta + \delta + \gamma < H, \beta \geq 0, \delta \geq 0$.

Proof. See [4] for $H > 1/2$ and [6] for $H < 1/2$. □

Remark 2.1. In the case $H > 1/2$ and $\delta = 0$ (i.e. $V_\delta = V$) we may drop the assumption of analyticity of the semigroup $(S(t), t \geq 0)$, cf. [4]. It may be also dropped (for $H > 1/2$) in the Theorem 3.1 and Section 4.1 below.

Proposition 2.2. *If* (A1) *is satisfied and the semigroup $(S(t), t \geq 0)$ is exponentially stable, i.e. there exist constants $M > 0$ and $\rho > 0$ such that for all $t \geq 0$*

$$\text{(A2)} \qquad |S(t)|_{\mathcal{L}(V)} \leq M e^{-\rho t},$$

then there is a Gaussian centered limiting measure $\mu_\infty = \mathcal{N}(0, Q_\infty)$ for $(X(t), t \geq 0)$ such that

$$w^* - \lim_{t \to \infty} \mu_t^{x_0} = \mu_\infty$$

for each initial condition $x_0 \in V$ where $\mu_t^{x_0} = \operatorname{Law}(X^{x_0}(t))$ and $\operatorname{Law}(\cdot)$ denotes the probability distribution.

Proof. See [4] for $H > 1/2$ and [6] for $H < 1/2$. □

Remark 2.2. It should be noted that for $H \neq 1/2$ the limiting measure is not "invariant" in the following sense: If the initial distribution is the limiting measure and the initial value is stochastically independent of B^H, the law for the solution does not remain the same.

The covariance Q_∞ has for $H > 1/2$ the following form:

$$Q_\infty = \int_0^\infty \int_0^\infty S(u) Q S^*(v) \phi(u-v)\, du\, dv,$$

where ϕ is given by (4). The form for $H < 1/2$ can be specified in terms of $\mathcal{K}_H^*$ and precise statement can be found in [6].

3. Strictly stationary solutions

Recall that a measurable V-valued process $(X(t), t \geq 0)$ is said to be strictly stationary, if for all $k \in \mathbf{N}$ and for all arbitrary positive numbers $t_1, t_2, \ldots, t_k$, the probability distribution of the V^k-valued random variable $(X(t_1+r), X(t_2+r), \ldots, X(t_k+r))$ does not depend on $r \geq 0$, i.e.

$$\operatorname{Law}(X(t_1+r), X(t_2+r), \ldots, X(t_k+r)) = \operatorname{Law}(X(t_1), X(t_2), \ldots, X(t_k)) \tag{6}$$

for all $t_1, t_2, \ldots, t_k, r \geq 0$.

Theorem 3.1. *If* (A1) *and* (A2) *are satisfied, then there exists a strictly stationary solution to* (1)*, i.e. there exists $\tilde{x}$, a random variable on* $(\Omega, \mathcal{F}, \mathbb{P})$*, such that $(X^{\tilde{x}}(t), t \geq 0)$ is a strictly stationary process with* $\operatorname{Law}(X^{\tilde{x}}(t)) = \mu_\infty$, $t \geq 0$. *In particular* $\operatorname{Law}(\tilde{x}) = \mu_\infty$.

Proof. For $t \geq 0$ let

$$\tilde{Z}_t := \int_{-t}^0 S(-u)\Phi\, dB^H(u).$$

It is clear that $\tilde{Z}_t$ is a V-valued random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ with probability law $\mu_t^0 = \mathcal{N}(0, Q_t)$. Let $n \in \mathbf{R}$. We will show that the limit

$$\tilde{x} = \lim_{n \to \infty} \tilde{Z}_n$$

exists in $L^2(\Omega, V)$ and that

$$X^{\tilde{x}}(t) = S(t)\tilde{x} + Z(t),$$

where $Z(t)$ is given by (3), is a stationary solution of (1).

First we show that $\tilde{Z}_n$ is a Cauchy sequence in $L^2(\Omega, V)$.

For all $n \geq m$ we have

$$\begin{aligned}
\mathbb{E}|\tilde{Z}_n - \tilde{Z}_m|_V^2 &= \mathbb{E}\left| \int_{-n}^0 S(-u)\Phi\, dB^H(u) - \int_{-m}^0 S(-u)\Phi\, dB^H(u) \right|_V^2 \\
&= \mathbb{E}\left| \int_{-n}^{-m} S(-u)\Phi\, dB^H(u) \right|_V^2.
\end{aligned}$$

Denoting by $\tilde{B}^H(u) = B^H(-u)$ an inverse process that is also a standard cylindrical fractional Brownian motion with stationary increments we have

$$\mathbb{E}|\tilde{Z}_n - \tilde{Z}_m|_V^2 = \mathbb{E}\left| \int_m^n S(u)\Phi\, d\tilde{B}^H(u) \right|_V^2.$$

First consider the case $H > 1/2$. We can use for example estimate (1.1.17) from [19] to show that

$$\begin{aligned}
\mathbb{E}\left| \int_m^n S(u)\Phi\, d\tilde{B}^H(u) \right|_V^2 &\leq \int_m^n \int_m^n |S(u)\Phi|_{\mathcal{L}_2(U,V)} |S(v)\Phi|_{\mathcal{L}_2(U,V)} \phi(u-v)\, du\, dv \\
&\leq \int_m^n \int_m^n |S(u-1)|_{\mathcal{L}(V)} |S(v-1)|_{\mathcal{L}(V)} |S(1)\Phi|_{\mathcal{L}_2(U,V)}^2 \phi(u-v)\, du\, dv
\end{aligned}$$

and by the exponential stability (A2)

$$\begin{aligned}
&\leq \int_m^n \int_m^n M^2 e^{-\rho(u-1)} e^{-\rho(v-1)} |S(1)\Phi|^2_{\mathcal{L}_2(U,V)} \phi(u-v)\, du\, dv \\
&\leq c_1 \int_m^\infty \int_m^\infty e^{-\rho u} e^{-\rho v} \phi(u-v)\, du\, dv \\
&\leq c_2 \int_m^\infty v^{2H-1} e^{-\rho v}\, dv.
\end{aligned}$$

The last integral on the right hand side tends to 0 as $m \to \infty$ and thus $(\tilde{Z}_t, t \geq 1)$ is Cauchy in $L^2(\Omega, V)$.

Next consider the case $H < 1/2$. Using (2) and (5) we have

$$\begin{aligned}
&\mathbb{E} \left| \int_m^n S(u)\Phi \, d\tilde{B}^H(u) \right|_V^2 = \sum_i \int_m^n |\mathcal{K}_H^*(S(\cdot)\Phi e_i)(r)|_V^2 \, dr \\
&\leq c \sum_i \int_m^n \frac{|S(r)\Phi e_i|_V^2 \, r^{2(H-1/2)} \, r^{2(1/2-H)}}{(n-r)^{1-2H}} \\
&\qquad + r^{1-2H} \left(\int_r^n \frac{|S(s)\Phi e_i s^{H-1/2} - S(r)\Phi e_i r^{H-1/2}|_V}{(s-r)^{3/2-H}} \, ds \right)^2 dr \\
&\leq \underbrace{c_1 \int_m^n \frac{|S(r)\Phi|^2_{\mathcal{L}_2(V)}}{(n-r)^{1-2H}} \, dr}_{I_1} + \underbrace{c_2 \sum_i \int_m^n r^{1-2H} \left[\int_r^n \frac{|S(r)\Phi e_i|_V |s^{H-1/2} - r^{H-1/2}|}{(s-r)^{3/2-H}} \, ds \right]^2 dr}_{I_2} \\
&\quad \underbrace{+ c_3 \sum_i \int_m^n r^{1-2H} \left[\int_r^n \frac{s^{H-1/2} |S(s)\Phi e_i - S(r)\Phi e_i|_V}{(s-r)^{3/2-H}} \, ds \right]^2 dr}_{I_3}.
\end{aligned}$$

From (A1) and exponential stability (A2) it follows that there exist constants $c > 0, \rho > 0, \gamma > 1$ such that

$$|S(r)\Phi|_{\mathcal{L}_2(V)} \leq c e^{-\rho r} r^{-\gamma}$$

for $r > 0$. Hence

$$I_1 \leq c_1 \int_m^n \frac{e^{-2\rho r}}{(n-r)^{1-2H} r^{2\gamma}} \, dr.$$

By substituting $\lambda = r - m$ and $T = n - m$ we get

$$\begin{aligned}
I_1 &\leq c_{11} e^{-2\rho m} \int_0^T \frac{e^{-2\rho\lambda}}{(T-\lambda)^{1-2H} (m+\lambda)^{2\gamma}} \, d\lambda \\
&\leq c_{12} e^{-2\rho m} \left(\int_0^{T-1} e^{-2\rho\lambda} \, d\lambda + \int_{T-1}^T \frac{1}{(T-\lambda)^{1-2H}} \, d\lambda \right) \\
&\leq c_{13} e^{-2\rho m} \left(\int_0^\infty e^{-2\rho\lambda} \, d\lambda + \int_0^1 \frac{1}{s^{1-2H} \, ds} \right) \\
&\leq c_{14} e^{-2\rho m},
\end{aligned}$$

which goes to zero as m goes to infinity. Next

$$
\begin{aligned}
I_2 &\leq c_{21} \int_m^n |S(r)\Phi|^2_{\mathcal{L}_2(V)} r^{1-2H} \left[\int_r^n \frac{s^{H-1/2} - r^{H-1/2}}{(s-r)^{3/2-H}}\, ds \right]^2 dr \\
&\leq c_{22} \int_m^n |S(r)\Phi|^2_{\mathcal{L}_2(V)} r^{1-2H} \left[r^{-1+2H} \right]^2 dr \\
&\leq c_{23} \int_m^n \frac{e^{-2\rho r}}{r^{1-2H+2\gamma}}\, dr \\
&\leq c_{24} \int_m^\infty e^{-2\rho r}\, dr,
\end{aligned}
$$

which also goes to zero as m goes to infinity. Finally, by setting $k_i(r,s) = \frac{|S(s)\Phi e_i - S(r)\Phi e_i|_V}{(s-r)^{3/2-H}}$, we get

$$
\begin{aligned}
I_3 &\leq c_{31} \int_m^n \sum_i \left(\int_r^{\min(n,r+1)} k_i(r,s)\, ds \right)^2 dr + c_{31} \int_m^n \sum_i \left(\int_{\min(n,r+1)}^n k_i(r,s)\, ds \right)^2 dr \\
&= \underbrace{c_{31} \int_m^n \sum_i \left(\int_r^{\min(n,r+1)} k_i(r,s)\, ds \right)^2 dr}_{J1} + \underbrace{c_{41} \int_m^{n-1} \sum_i \left(\int_{r+1}^n k_i(r,s)\, ds \right)^2 dr}_{J2},
\end{aligned}
$$

which converges to zero as m goes to infinity, because

$$
\begin{aligned}
J_1 &\leq c_{32} \int_m^n \left(\int_r^{\min(n,r+1)} \frac{|S(s-\frac{r}{2}) - S(\frac{r}{2})|_{\mathcal{L}(V)}}{(s-r)^{3/2-H}}\, ds \right)^2 \sum_i |S(\tfrac{r}{2})\Phi e_i|^2_V dr \\
&= c_{32} \int_m^n \left(\int_r^{\min(n,r+1)} \frac{|S(s-\frac{r}{2}) - S(\frac{r}{2})|_{\mathcal{L}(V)}}{(s-r)^{3/2-H}}\, ds \right)^2 |S(\tfrac{r}{2})\Phi|^2_{\mathcal{L}_2(V)} dr,
\end{aligned}
$$

which is for all $\beta > 0$

$$
\begin{aligned}
&\leq c_{33} \int_m^n \frac{e^{-2\rho\frac{r}{2}}}{(\frac{r}{2})^{2\gamma}} \left(\int_r^{\min(n,r+1)} \frac{|S(s-\frac{3r}{4}) - S(\frac{r}{4})|_{\mathcal{L}(V_\beta,V)}}{(s-r)^{3/2-H}}\, ds \right)^2 |S(\tfrac{r}{4})\Phi|^2_{\mathcal{L}(V,V_\beta)} \\
&\leq c_{34} \int_m^n \frac{e^{-\rho r}}{r^{2\gamma+2\beta}} \left(\int_r^{\min(n,r+1)} \frac{ds}{(s-r)^{3/2-H-\beta}} \right)^2 dr
\end{aligned}
$$

in particular for $1/2 > \beta > 1/2 - H$

$$
\begin{aligned}
&\leq c_{35} \int_m^n \frac{e^{-\rho r}}{r^{2\gamma+2\beta}} \left(\int_0^1 \frac{d\lambda}{\lambda^{3/2-H-\beta}} \right)^2 dr \\
&\leq c_{36} \int_m^\infty e^{-\rho r}\, dr \to 0 \quad \text{as} \quad m \to \infty.
\end{aligned}
$$

Also,

$$
\begin{aligned}
J_2 &\le c_{42}\int_m^{n-1}\sum_i\left(\int_{r+1}^n \frac{|S(s)\Phi e_i|_V}{(s-r)^{3/2-H}}ds\right)^2 dr + c_{42}\int_m^{n-1}\sum_i\left(\int_{r+1}^n \frac{|S(r)\Phi e_i|_V}{(s-r)^{3/2-H}}ds\right)^2 dr\\
&\le c_{43}\int_m^{n-1}\sum_i\left(\int_{r+1}^n \frac{|S(s-r)|_{\mathcal{L}(V)}|S(r)\Phi e_i|_V}{(s-r)^{3/2-H}}ds\right)^2 dr\\
&\quad + c_{42}\int_m^{n-1}\sum_i\left(\int_{r+1}^n \frac{|S(r)\Phi e_i|_V}{(s-r)^{3/2-H}}ds\right)^2 dr\\
&\le c_{44}\int_m^{n-1} |S(r)\Phi|^2_{\mathcal{L}_2(V)}\left(\int_{r+1}^n \frac{ds}{(s-r)^{3/2-H}}\right)^2 dr\\
&\le c_{45}\int_m^{n-1} \frac{e^{-2\rho r}}{r^{2\gamma}}\left(\int_{r+1}^\infty \frac{ds}{(s-r)^{3/2-H}}\right)^2 dr\\
&\le c_{46}\int_m^{n-1} e^{-2\rho r}\left(\int_1^\infty \frac{d\lambda}{\lambda^{3/2-H}}\right)^2 dr\\
&\le c_{47}\int_m^\infty e^{-2\rho r}\,dr \to 0 \quad \text{as} \quad m\to\infty.
\end{aligned}
$$

We have shown that $(\tilde{Z}_t, t\ge 1)$ is Cauchy in $L^2(\Omega,V)$ for $H\in(0,1)$. There is therefore a V-valued random variable $\tilde{x}$ such that $\tilde{Z}_t\to\tilde{x}$ in $L^2(\Omega,V)$. Clearly the probability distribution for $\tilde{x}$ is $\mu_\infty=\mathcal{N}(0,Q_\infty)$ where

$$Q_\infty=\lim_{t\to\infty} Q_t.$$

Now we are ready to show for all $k\in\mathbf{N}$ and all arbitrary times $t_1,t_2,\dots,t_k\ge 0$

$$\text{Law}\left(X^{\tilde{x}}(t_1+r),X^{\tilde{x}}(t_2+r),\dots,X^{\tilde{x}}(t_k+r)\right)=\text{Law}\left(X^{\tilde{x}}(t_1),X^{\tilde{x}}(t_2),\dots,X^{\tilde{x}}(t_k)\right)$$

for all $r\ge 0$.

Let $t\ge 0$. Then

$$
\begin{aligned}
X^{\tilde{x}}(t) &= S(t)\tilde{x}+Z(t)\\
&= S(t)\left(\lim_{n\to\infty}\int_{-n}^0 S(-u)\Phi dB^H(u)\right)+\int_0^t S(t-u)\Phi\, dB^H(u).
\end{aligned}
$$

Since $S(t)$ is a bounded operator on V and using the semigroup property

$$
\begin{aligned}
&= \lim_{n\to\infty}\int_{-n}^0 S(t-u)\Phi\, dB^H(u)+\int_0^t S(t-u)\Phi\, dB^H(u)\\
&= \lim_{n\to\infty}\int_{-n}^t S(t-u)\Phi\, dB^H(u).
\end{aligned}
$$

For $t,r\ge 0$ we have

$$X^{\tilde{x}}(t+r)=\lim_{n\to\infty}\int_{-n}^{t+r} S(t+r-u)\Phi\, dB^H(u).$$

Denoting by $B^H_r(u)=B^H(u-r)$ a process shifted in time by r,

$$
\begin{aligned}
&= \lim_{n\to\infty}\int_{-n-r}^{t} S(t-u)\Phi\, dB^H_r(u)\\
&= \lim_{n\to\infty}\int_{-n}^{t} S(t-u)\Phi\, dB^H_r(u),
\end{aligned}
$$

because $n+r\to\infty$ for arbitrary $r\ge 0$.

Since $B^H(u)$ and $B_r^H(u)$ have stationary increments, we have for all $k \in \mathbf{N}$ and arbitrary times $t_1, t_2, \ldots, t_k, \geq 0$

$$\begin{aligned}\mathrm{Law}&\left(\int_{-n}^{t_1} S(t_1-u)\Phi\, dB_r^H(u), \int_{-n}^{t_2} S(t_2-u)\Phi\, dB_r^H(u), \ldots, \int_{-n}^{t_k} S(t_k-u)\Phi\, dB_r^H(u)\right)\\ &= \mathrm{Law}\left(\int_{-n}^{t_1} S(t_1-u)\Phi\, dB^H(u), \int_{-n}^{t_2} S(t_2-u)\Phi\, dB^H(u), \ldots, \int_{-n}^{t_k} S(t_k-u)\Phi\, dB^H(u)\right)\end{aligned}$$

for all $r \geq 0$. Since

$$\begin{aligned}&\left(\int_{-n}^{t_1} S(t_1-u)\Phi\, dB_r^H(u), \int_{-n}^{t_2} S(t_2-u)\Phi\, dB_r^H(u), \ldots, \int_{-n}^{t_k} S(t_k-u)\Phi\, dB_r^H(u)\right)\\ &\qquad\xrightarrow[n\to\infty]{} \left(X^{\tilde{x}}(t_1+r), X^{\tilde{x}}(t_2+r), \ldots, X^{\tilde{x}}(t_k+r)\right)\end{aligned}$$

in $L^2(\Omega, V^k)$ and

$$\begin{aligned}&\left(\int_{-n}^{t_1} S(t_1-u)\Phi\, dB^H(u), \int_{-n}^{t_2} S(t_2-u)\Phi\, dB^H(u), \ldots, \int_{-n}^{t_k} S(t_k-u)\Phi\, dB^H(u)\right)\\ &\qquad\xrightarrow[n\to\infty]{} \left(X^{\tilde{x}}(t_1), X^{\tilde{x}}(t_2), \ldots, X^{\tilde{x}}(t_k)\right)\end{aligned}$$

in $L^2(\Omega, V^k)$, we deduce that

$$\mathrm{Law}\left(X^{\tilde{x}}(t_1+r), X^{\tilde{x}}(t_2+r), \ldots, X^{\tilde{x}}(t_k+r)\right) = \mathrm{Law}\left(X^{\tilde{x}}(t_1), X^{\tilde{x}}(t_2), \ldots, X^{\tilde{x}}(t_k)\right).$$

□

4. Ergodic theorems

4.1. **Ergodic theorem for a strictly stationary solution.** At first we recall the famous Birkhoff's theorem for strictly stationary processes.

Theorem 4.1 (Birkhoff's theorem)**.** *Let $(X^{\tilde{x}}(t), t \geq 0)$ be a V-valued strictly stationary process on $(\Omega, \mathcal{F}, \mathbb{P})$. Then for all measurable functionals $\varrho : V \to \mathbf{R}$ such that $\mathbb{E}|\varrho(\tilde{x})| < \infty$ there exists a measurable functional $\xi : \Omega \to \mathbf{R}$ such that*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \varrho(X_t^{\tilde{x}})\, dt = \xi, \quad a.s.\text{-}\mathbb{P}. \tag{7}$$

Proof. See e.g [25]. □

Recall that a V-valued strictly stationary process $(X(t), t \geq 0)$ is said to be *ergodic*, if ξ in (7) does not depend on $\omega \in \Omega$, i.e. ξ is deterministic, and $\xi = \mathbb{E}[\varrho(\tilde{x})]$.

Lemma 4.2. *Let $(Y(t), t \geq 0)$ be a $\mathbf{R}$-valued strictly stationary centered Gaussian process and let $R(t) := \mathbb{E}[Y(0)Y(t)]$ be the correlation function of the process $Y(t)$. Then the process $Y(t)$ is ergodic if $\lim_{t\to\infty} R(t) = 0$.*

Proof. See e.g [25]. □

If the semigroup $(S(t), t \geq 0)$ is analytic, it is well known (cf. [17]), that for all $\gamma > 0$ there exist constants $\hat{c}_\gamma, \tilde{c}_\gamma > 0$ such that $|S(t)z - z| \leq \hat{c}_\gamma t^\gamma |z|_\gamma$ and $|S(t)z|_\gamma \leq \tilde{c}_\gamma t^{-\gamma}|z|_V$, where $|z|_\gamma = |(\beta I - A)^\gamma z|_V$.

Lemma 4.3. *Let $S(t)$ be analytic and let $z \in V$ and $T > 0$ be arbitrary. Then for all $0 < s < r \leq T$,*

$$|(S^*(r) - S^*(s))z|_V \leq c_\gamma |r-s|^\gamma |s|^{-\gamma} |z|_V, \tag{8}$$

for a constant $c_\gamma < \infty$.

Proof.

$$\begin{aligned} |(S^*(r) - S^*(s))z)|_V &= \left|\left(S^*(r - \tfrac{s}{2}) - S^*(\tfrac{s}{2})\right) S^*(\tfrac{s}{2}) z\right|_V \\ &\leq \hat{c}_\gamma |r-s|^\gamma \left|S^*(\tfrac{s}{2})z\right|_\gamma \\ &\leq c_\gamma |r-s|^\gamma |s|^{-\gamma} |z|_V. \end{aligned}$$

□

The following two Lemmas 4.4 and 4.5 will be needed in the proof of the ergodic Theorem 4.6 below.

Lemma 4.4. *Let* (A1) *and* (A2) *be satisfied. Let $0 < H < 1/2$ and let $z \in V$ be arbitrary. Then*

$$\lim_{t\to\infty} \mathbb{E} \langle \tilde{x}, z \rangle_V \left\langle \int_0^t S(t-r)\Phi \, dB^H(r), z \right\rangle_V = 0$$

Proof. Let $\tilde{x}_n := \int_{-n}^0 S(-r)\Phi \, dB^H_r$. We have that $\tilde{x}_n \to \tilde{x}$ in $L^2(\Omega, V)$ and by the Cauchy-Schwarz inequality it follows that

$$\begin{aligned} \mathbb{E}|\tilde{x}_n - \tilde{x}|_V \cdot \left| \int_0^t S(t-r)\Phi \, dB^H(r) \right|_V &\leq \left\{\mathbb{E}|\tilde{x}_n - \tilde{x}|_V^2\right\}^{1/2} \left\{ \mathbb{E} \left| \int_0^t S(t-r)\Phi \, dB^H(r) \right|^2 \right\}^{1/2} \\ &\leq c \left\{\mathbb{E}|\tilde{x}_n - \tilde{x}|_V^2\right\}^{1/2} \sup_{t\geq 0} (\operatorname{Tr} Q_t)^{1/2} \end{aligned} \tag{9}$$

which goes to zero as $n \to \infty$ uniformly in t. It is therefore sufficient to show that for each $n \in \mathbf{N}$

$$\lim_{t\to\infty} \mathbb{E} \langle \tilde{x}_n, z \rangle \left\langle \int_0^t S(t-r)\Phi \, dB^H(r), z \right\rangle_V = 0. \tag{10}$$

Let $n \in \mathbf{N}$ be fixed. Then

$$\begin{aligned} &\mathbb{E} \langle \tilde{x}_n, z \rangle \left\langle \int_0^t S(t-r)\Phi \, dB^H(r), z \right\rangle_V \\ &= \mathbb{E} \left\{ \sum_i \int_{-n}^0 \langle S(-r)\Phi e_i, z \rangle_V \, d\beta_i^H \right\} \left\{ \sum_i \int_0^t \langle S(t-r)\Phi e_i, z \rangle_V \, d\beta_i^H \right\}. \end{aligned}$$

Since β_i are stochastically independent and have stationary increments, we get

$$\begin{aligned} &\mathbb{E} \langle \tilde{x}_n, z \rangle \left\langle \int_0^t S(t-r)\Phi \, dB^H(r), z \right\rangle_V \\ &= \sum_i \mathbb{E} \int_0^n \varphi_i(n-r) \, d\beta_i^H \int_n^{n+t} \varphi_i(t+n-r) \, d\beta_i^H, \end{aligned} \tag{11}$$

where $\varphi_i(r) = \langle S(r)\Phi e_i, z \rangle_V$. For a fixed $i \in \mathbf{N}$ (that is suppressed in the notation) set

$$\psi(r) = \begin{cases} \varphi(n-r) & \text{for } r \in [0,n] \\ 0 & \text{for } r \in [n, n+t], \end{cases}$$

$$\chi(r) = \begin{cases} 0 & \text{for } r \in [0,n] \\ \varphi(t+n-r) & \text{for } r \in [n, n+t]. \end{cases}$$

Then

$$\begin{aligned}&\mathbb{E}\int_0^n \varphi(n-r)\,d\beta_i^H \int_n^{n+t} \varphi(t+n-r)\,d\beta_i^H\\&= \int_0^{n+t} \left(\mathcal{K}_H^*\psi\right)(s) \left(\mathcal{K}_H^*\chi\right)(s)\,ds,\end{aligned}$$

where $\mathcal{K}_H^*$ is the operator defined in Section 2 (with $T = n+t$). Using the form (5) of the operator $\mathcal{K}_H^*$ we get

$$= \int_0^n \left(\mathcal{K}_H^*\psi\right)(s) \left(c_H s^{1/2-H} \int_n^{n+t} \frac{r^{H-1/2}\varphi(t+n-r)}{(r-s)^{3/2-H}}\,dr\right) ds. \tag{12}$$

Note that

$$\begin{aligned}s^{1/2-H}\int_n^{n+t} \frac{r^{H-1/2}\varphi(t+n-r)}{(r-s)^{3/2-H}}\,dr &\le \int_n^{n+t} \frac{\varphi(t+n-r)}{(r-s)^{3/2-H}}\,dr\\&\le \int_0^t \frac{|\varphi(\lambda)|}{(t+n-s-\lambda)^{3/2-H}}\,d\lambda.\end{aligned} \tag{13}$$

Let us now turn back to (11). By (12) and (13) we may estimate

$$\begin{aligned}&\sum_i \int_0^n \Bigg\{ \Big(\frac{|\varphi_i(n-s)|}{(n-s)^{1/2-H}} + s^{1/2-H} \int_s^n \frac{|r^{H-1/2}\varphi_i(n-r) - s^{H-1/2}\varphi_i(n-s)|}{(r-s)^{3/2-H}}\,dr \Big) \cdot\\&\quad\cdot \int_0^t \frac{|\varphi_i(\lambda)|}{(t+n-s-\lambda)^{3/2-H}}\,d\lambda \Bigg\}\, ds\\&= \int_0^n \Bigg\{ \underbrace{\sum_i \frac{|\varphi_i(n-s)|}{(n-s)^{1/2-H}} \int_0^{t-1} \frac{|\varphi_i(\lambda)|}{(t+n-s-\lambda)^{3/2-H}}\,d\lambda}_{I}\\&\quad + \underbrace{\sum_i \frac{|\varphi_i(n-s)|}{(n-s)^{1/2-H}} \int_0^1 \frac{|\varphi_i(r+t-1)|}{(1+n-s-r)^{3/2-H}}\,dr}_{II}\\&\quad + \underbrace{\sum_i s^{1/2-H} \int_s^n \frac{|r^{H-1/2}\varphi_i(n-r) - s^{H-1/2}\varphi_i(n-s)|}{(r-s)^{3/2-H}}\,dr \int_0^{t-1} \frac{|\varphi_i(\lambda)|}{(t+n-s-\lambda)^{3/2-H}}\,d\lambda}_{III}\\&\quad + \underbrace{\sum_i s^{1/2-H} \int_s^n \frac{|r^{H-1/2}\varphi_i(n-r) - s^{H-1/2}\varphi_i(n-s)|}{(r-s)^{3/2-H}}\,dr \int_0^1 \frac{|\varphi_i(\lambda+t-1)|}{(1+n-s-\lambda)^{3/2-H}}\,d\lambda}_{IV} \Bigg\}\, ds.\end{aligned}$$

We will now prove the convergence to zero as $t \to \infty$ for particular terms I - IV.
We have

$$I = \frac{1}{(n-s)^{1/2-H}} \int_0^{t-1} \frac{1}{(t+n-s-\lambda)^{3/2-H}} \sum_i |\varphi_i(n-s)| \cdot |\varphi_i(\lambda)|\,d\lambda$$

by Cauchy-Schwarz inequality

$$\begin{aligned}&\le \frac{1}{(n-s)^{1/2-H}} \int_0^{t-1} \frac{1}{(t+n-s-\lambda)^{3/2-H}} \left\{\sum_i \varphi_i^2(n-s)\right\}^{1/2} \left\{\sum_i |\varphi_i^2(\lambda)|\right\}^{1/2} d\lambda\\&= \frac{1}{(n-s)^{1/2-H}} \int_0^{t-1} \frac{1}{(t+n-s-\lambda)^{3/2-H}} |\Phi^* S^*(n-s)z| \cdot |\Phi^* S^*(\lambda)z|\,d\lambda\end{aligned}$$

using the estimate $|\Phi^* S^*(\cdot)z| \le |\Phi^*| \cdot |S^*(\cdot)z| \le c|S^*(\cdot)z|$

$$\le \frac{c_{11}}{(n-s)^{1/2-H}}|S^*(n-s)z| \int_0^\infty \frac{1}{(t+n-s-\lambda)^{3/2-H}}|S^*(\lambda)z|\mathbb{1}_{[\lambda\le t-1]}\,d\lambda,$$

which converges to zero as $t\to\infty$ for each $s\in(0,n)$ by dominated convergence theorem (DCT in what follows), because $|S^*(\lambda)z|\le ce^{-\omega\lambda}$. We also have

$$\begin{aligned} I &\le \frac{c_{12}}{(n-s)^{1/2-H}}|S^*(n-s)z|\int_0^\infty e^{-\omega\lambda}\,d\lambda \\ &\le \frac{c_{13}}{(n-s)^{1/2-H}}, \end{aligned}$$

which is an integrable majorant, and therefore (again by DCT) $\int_0^n I\,ds$ goes to zero. In a similar manner we have

$$\begin{aligned} II &\le \frac{c_{21}}{(n-s)^{1/2-H}}|S^*(n-s)z|\int_0^1 \frac{1}{(1+n-s-r)^{3/2-H}}|S^*(r+t-1)z|\,dr \\ &\le \frac{c_{22}}{(n-s)^{1/2-H}}|S^*(n-s)z|\int_0^1 \frac{e^{-\omega(r+t-1)}}{(1+n-s-r)^{3/2-H}}\,dr \end{aligned}$$

which converges pointwise to zero by Levi's theorem (for $t=1$ the integrand is less than $\frac{c}{(n-s)^{3/2-H}}$). We have

$$\begin{aligned} II &\le \frac{c_{23}}{(n-s)^{1/2-H}}|S^*(n-s)z|\int_0^1 \frac{dr}{(1+n-s-r)^{3/2-H}} \\ &\le \frac{c_{24}}{(n-s)^{1/2-H}}\frac{1}{(n-s)^{1/2-H}} \\ &= \frac{c_{24}}{(n-s)^{1-2H}}, \end{aligned}$$

which is integrable, and therefore (by DCT) $\int_0^n II\,ds$ goes to zero. Next we have (by adding and subtracting $r^{H-1/2}\varphi_i(n-s)$ in the numerator of the first term)

$$\begin{aligned} III &\le \sum_i s^{1/2-H}\left(|\varphi_i(n-s)|\int_s^n \frac{|r^{H-1/2}-s^{H-1/2}|}{(r-s)^{3/2-H}} + \int_s^n \frac{r^{H-1/2}|\varphi_i(n-r)-\varphi_i(n-s)|}{(r-s)^{3/2-H}}\,dr\right)\cdot \\ &\quad \cdot\int_0^{t-1}\frac{|\varphi_i(\lambda)|}{(t+n-s-\lambda)^{3/2-H}}\,d\lambda \\ &\le c_{31}\sum_i\left(s^{1/2-H}|\varphi_i(n-s)|s^{-1+2H} + \int_0^n \frac{|\varphi_i(n-r)-\varphi_i(n-s)|}{(r-s)^{3/2-H}}\,dr\right)\cdot \\ &\quad \cdot\int_0^{t-1}\frac{|\varphi_i(\lambda)|}{(t+n-s-\lambda)^{3/2-H}}\,d\lambda \\ &\le c_{32}s^{H-1/2}\int_0^{t-1}\frac{\sum_i|\varphi_i(n-s)|\cdot|\varphi_i(\lambda)|}{(t+n-s-\lambda)^{3/2-H}}\,d\lambda + c_{33}\int_s^n\int_0^{t-1}\frac{\sum_i|\varphi_i(\lambda)|\cdot|\varphi_i(n-r)-\varphi_i(n-s)|}{(t+n-s-\lambda)^{3/2-H}(r-s)^{3/2-H}}\,d\lambda\,dr \\ &\le c_{34}s^{H-1/2}|S^*(n-s)z|\int_0^\infty\frac{|S^*(\lambda)z|\mathbb{1}_{[\lambda\le t-1]}\,d\lambda}{(t+n-s-\lambda)^{3/2-H}} \\ &\quad + c_{35}\int_0^\infty\frac{|S^*(\lambda)z|\mathbb{1}_{[\lambda\le t-1]}\,d\lambda}{(t+n-s-\lambda)^{3/2-H}}\int_s^n\frac{|(S^*(n-r)-S^*(n-s))z|}{(r-s)^{3/2-H}}\,dr. \end{aligned}$$

The first two integrals on the right hand side are identical and converge to zero as $t \to \infty$ as shown in the above estimate of I, hence the whole right hand side tends to zero. Using the estimate (8) we have

$$\begin{aligned} III &\le c_{36} s^{H-1/2} |S^*(n-s)z| \frac{1}{(n-s)^{1/2-H}} + \frac{c_{37}}{(n-s)^{1/2-H}} \int_s^n \frac{c_\gamma |r-s|^\gamma |n-s|^{-\gamma}}{(r-s)^{3/2-H}}\, dr \\ &\le \frac{c_{36}|S^*(n-s)z|}{s^{1/2-H}(n-s)^{1/2-H}} + \frac{c_{38}}{(n-s)^{1-2H}} \end{aligned}$$

which is integrable and by DCT $\int_0^n III\, ds$ goes to zero. In a similar manner we have

$$\begin{aligned} IV &\le c_{41} s^{H-1/2} \int_0^1 \frac{\sum_i |\varphi_i(n-s)| \cdot |\varphi_i(\lambda+t-1)|}{(1+n-s-\lambda)^{3/2-H}}\, d\lambda \\ &\quad + c_{42} \int_s^n \int_0^1 \frac{\sum_i |\varphi_i(\lambda+t-1)| \cdot |\varphi_i(n-r) - \varphi_i(n-s)|}{(1+n-s-\lambda)^{3/2-H}(r-s)^{3/2-H}}\, d\lambda\, dr \\ &\le c_{43} s^{H-1/2} |S^*(n-s)z| \int_0^1 \frac{|S^*(\lambda+t-1)z|}{(1+n-s-\lambda)^{3/2-H}}\, d\lambda \\ &\quad + c_{44} \int_0^1 \frac{|S^*(\lambda+t-1)z|}{(1+n-s-\lambda)^{3/2-H}}\, d\lambda \int_s^n \frac{|(S^*(n-r) - S^*(n-s))z|}{(r-s)^{3/2-H}}\, dr \\ &\le c_{45} s^{H-1/2} e^{-\omega(t-1)} \int_0^1 \frac{d\lambda}{(1+n-s-\lambda)^{3/2-H}} \\ &\quad + c_{46} e^{-\omega t} \int_0^1 \frac{d\lambda}{(1+n-s-\lambda)^{3/2-H}} \int_s^n \frac{|(S^*(n-r) - S^*(n-s))z|}{(r-s)^{3/2-H}}\, dr \end{aligned}$$

which converges pointwise to zero as $t \to \infty$. Using the estimate (8) we have as in the previous case

$$IV \le \frac{c_{47}}{s^{1/2-H}(n-s)^{1/2-H}} + \frac{c_{48}}{(n-s)^{1/2-H}} \frac{c_\gamma}{(n-s)^{1/2-H}}$$

which is integrable, and therefore by DCT $\int_0^n IV\, ds$ tends to zero.

To recapitulate, we have proven that integral

$$\int_0^n (I + II + III + IV)\, ds$$

goes to zero, i.e. we have shown (10) which completes the proof. □

Lemma 4.5. *Let* (A1) *and* (A2) *be satisfied. Let* $1/2 < H < 1$ *and let* $z \in V$ *be arbitrary. Then*

$$\lim_{t\to\infty} \mathbb{E} \langle \tilde{x}, z\rangle_V \left\langle \int_0^t S(t-r)\Phi\, dB^H(r), z \right\rangle_V = 0$$

Proof. Set

$$\begin{aligned} \tilde{R}(t) &= \mathbb{E}\left[\langle \tilde{x}, z\rangle_V \left\langle \int_0^t S(t-r)\Phi\, dB^H(r), z \right\rangle_V \right] \\ &= \mathbb{E}\left[\left\langle \lim_{n\to\infty} \int_{-n}^0 S(-r)\Phi\, dB^H(r), z \right\rangle_V \left\langle \int_0^t S(t-r)\Phi\, dB^H(r), z \right\rangle_V \right] \\ &= \lim_{n\to\infty} \mathbb{E}\left[\left\langle \int_{-n}^0 S(-r)\Phi\, dB^H(r), z \right\rangle_V \left\langle \int_0^t S(t-r)\Phi\, dB^H(r), z \right\rangle_V \right] \\ &= \lim_{n\to\infty} \mathbb{E}\left[\left\langle \int_{-n}^t \psi(r)\, dB^H(r), z \right\rangle_V \left\langle \int_{-n}^t \chi(r)\, dB^H(r), z \right\rangle_V \right] = \lim_{n\to\infty} \left\langle \hat{Q}_{n,t} z, z \right\rangle_V, \end{aligned}$$

where

$$\psi(r) = \begin{cases} S(-r)\Phi & \text{for } r \in [-n, 0) \\ 0 & \text{for } r \in [0, t], \end{cases}$$

$$\chi(r) = \begin{cases} S(t-r)\Phi & \text{for } r \in [0, t] \\ 0 & \text{for } r \in [-n, 0). \end{cases}$$

and $\hat{Q}_{n,t}$ is the covariance operator of $\int_{-n}^{t} \psi(r)\, dB^H(r)$ and $\int_{-n}^{t} \chi(r)\, dB^H(r)$.

Since $H > 1/2$, we know the form of the covariance operator

$$\hat{Q}_{n,t} = \int_{-n}^{t} \int_{-n}^{t} \psi(r)\chi^*(s)\phi(r-s)\, dr\, ds, \tag{14}$$

where ϕ is given by (4). Therefore

$$\tilde{R}(t) \le \lim_{n\to\infty} \int_0^t \int_{-n}^0 |\Phi^* S^*(t-s)z|_V\, |\Phi^* S^*(-r)z|_V\, \phi(r-s)\, dr\, ds$$

and by the exponential stability (A2)

$$\begin{aligned} \tilde{R}(t) &\le M^2 |z|_V^2 \lim_{n\to\infty} \int_0^t \int_{-n}^0 e^{-\rho(t-s)} e^{-\rho(-r)} \phi(r-s)\, dr\, ds \\ &\le M^2 |z|_V^2 \int_0^\infty \int_0^\infty e^{-\rho s} e^{-\rho r} \hat{\phi}(t+r-s)\, dr\, ds, \end{aligned}$$

where

$$\hat{\phi}(u) = \begin{cases} \phi(u) & \text{for } u \ge 0 \\ 0 & \text{for } u < 0 \end{cases}$$

and using the fact that $\phi(-u) = \phi(u)$ for all $u \in \mathbf{R}$. Since the function $\hat{\phi}(t+r-s)$ is decreasing as $t \to \infty$ for all $s, r \in \mathbf{R}$, we deduce that $\tilde{R}(t)$ goes to 0 as $t \to \infty$ completing the proof by the Lebesgue monotone convergence theorem. □

Theorem 4.6 (Ergodic theorem for a strictly stationary solution)**.** *Let* $(X^{\tilde{x}}(t), t \ge 0)$ *be a* V*-valued strictly stationary solution to* (1)*. Let* $\varrho : V \to \mathbf{R}$ *be a measurable functional such that* $\mathbb{E}|\varrho(\tilde{x})| < \infty$*. Then*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \varrho(X^{\tilde{x}}(t))\, dt = \int_V \varrho(y)\, \mu_\infty(dy), \quad a.s.\text{-}\mathbb{P}.$$

Proof. Let $z \in V$ be arbitrary and let $(Y(t), t \ge 0)$ be a $\mathbf{R}$-valued process defined by

$$Y(t) := \left\langle X^{\tilde{x}}(t), z \right\rangle_V .$$

Then $Y(0) = \langle \tilde{x}, z \rangle_V$. We prove that the process $Y(t)$ is ergodic. By Lemma 4.2 we have to show that $\lim_{t\to\infty} R(t) = 0$.

$$\begin{aligned} R(t) &= \mathbb{E}[Y(0)Y(t)] \\ &= \mathbb{E}[\langle \tilde{x}, z \rangle_V \left\langle X^{\tilde{x}}(t), z \right\rangle_V] \\ &= \mathbb{E}\left[\langle \tilde{x}, z \rangle_V \left\langle S(t)(\tilde{x}) + \int_0^t S(t-r)\Phi\, dB^H(r), z \right\rangle_V \right] \\ &= \underbrace{\mathbb{E}\left[\langle \tilde{x}, z \rangle_V \langle S(t)(\tilde{x}), z \rangle_V \right]}_{I} + \underbrace{\mathbb{E}\left[\langle \tilde{x}, z \rangle_V \left\langle \int_0^t S(t-r)\Phi\, dB^H(r), z \right\rangle_V \right]}_{II} . \end{aligned}$$

We will estimate both terms separately.

The first term

$$\begin{aligned}\mathbb{E}|I| &= \mathbb{E}\left|\langle \tilde{x}, z\rangle_V \langle S(t)(\tilde{x}), z\rangle_V\right| \\ &\leq \mathbb{E}|S(t)\tilde{x}|_V\, |z|_V^2\, |\tilde{x}|_V\end{aligned}$$

and using the exponential stability bound (A2)

$$\leq Me^{-\rho t}|z|_V^2\, \mathbb{E}|\tilde{x}|_V^2,$$

which goes to 0 as $t \to \infty$, because $\mathbb{E}|\tilde{x}|_V^2 < \infty$. The second term is for $H = 1/2$ equal to zero, because $\tilde{x}$ and the convolution integral $\int_0^t S(t-r)\Phi\, dB^H(r)$ are stochastically independent and therefore

$$\begin{aligned}\mathbb{E}[II] &= \mathbb{E}\left[\langle \tilde{x}, z\rangle_V \left\langle \int_0^t S(t-r)\Phi\, dB^H(r), z\right\rangle_V\right] \\ &= \mathbb{E}\left[\langle \tilde{x}, z\rangle_V\right]\, \mathbb{E}\left[\left\langle \int_0^t S(t-r)\Phi\, dB^H(r), z\right\rangle_V\right] \\ &= 0.\end{aligned}$$

For $H \neq 1/2$ we obtain that $\mathbb{E}[II]$ goes to zero as $t \to \infty$ by Lemma 4.4 and 4.5 respectively. Thus the process $Y(t) = \left\langle X^{\tilde{x}}(t), z\right\rangle_V$ is ergodic for each $z \in V$.

Take $(h_n, n \in \mathbf{N})$ any orthonormal basis in V. Then

$$\mathbb{E}\left\langle \tilde{x}, h_n\right\rangle_V = \lim_{T\to\infty} \frac{1}{T}\int_0^T \left\langle X^{\tilde{x}}(t), h_n\right\rangle_V\, dt = 0$$

on $\Omega_n \subset \Omega$, $\mathbb{P}(\Omega_n) = 1$. On the other hand, by Theorem 4.1

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T X^{\tilde{x}}(t)\, dt = \xi$$

on $\Omega_0 \subset \Omega$, $\mathbb{P}(\Omega_0) = 1$. Taking $\Omega' = \bigcap_{n=0}^{\infty} \Omega_n$, we have $\mathbb{P}(\Omega') = 1$ and

$$\begin{aligned}0 &= \lim_{T\to\infty} \frac{1}{T}\int_0^T \left\langle X^{\tilde{x}}(t), h_n\right\rangle_V\, dt \\ &= \lim_{T\to\infty} \left\langle \frac{1}{T}\int_0^T X^{\tilde{x}}(t)\, dt, h_n\right\rangle_V \\ &= \left\langle \lim_{T\to\infty} \frac{1}{T}\int_0^T X^{\tilde{x}}(t)\, dt, h_n\right\rangle_V \\ &= \langle \xi, e_n\rangle_V\end{aligned}$$

on Ω'. Hence $\langle \xi, h_n\rangle_V = 0$ for each n on Ω', i.e. $\xi = 0$, a.s.-$\mathbb{P}$, and it follows that $X^{\tilde{x}}(t)$ is ergodic. □

4.2. **Ergodic theorems for an arbitrary solution.** In this section we will apply the previous results to the solution of (1) with arbitrary initial condition.

Theorem 4.7. *Let* (A1) *and* (A2) *be satisfied and let* $(X^{x_0}(t), t \geq 0)$ *be a solution to* (1). *Let* $\varrho : V \to \mathbf{R}$ *be a functional satisfying the global Lipschitz condition, i.e. there exists a constant* $L > 0$ *such that*

$$|\varrho(x) - \varrho(y)| \leq L|x-y|_V \tag{15}$$

for all $x, y \in V$. *Then*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \varrho(X^{x_0}(t))\, dt = \int_V \varrho(y)\, \mu_\infty(dy), \quad a.s.\text{-}\mathbb{P}.$$

for all $x_0 \in V$.

Proof. The desired convergence can be rewritten as

$$\lim_{T\to\infty}\left|\frac{1}{T}\int_0^T \varrho\big(X^{x_0}(t)\big)\,dt - \int_V \varrho(y)\,\mu_\infty(dy)\right| = 0, \quad \text{a.s.-}\mathbb{P},$$

for all $x_0 \in V$. Let $(X^{\tilde{x}}(t), t \geq 0)$ be a strictly stationary solution to (1). Then obviously

$$\left|\frac{1}{T}\int_0^T \varrho\big(X^{x_0}(t)\big)\,dt - \frac{1}{T}\int_0^T \varrho\big(X^{\tilde{x}}(t)\big)\,dt\right| \leq \frac{1}{T}\int_0^T \left|\varrho\big(X^{x_0}(t)\big) - \varrho\big(X^{\tilde{x}}(t)\big)\right| dt.$$

We will show that the right hand side goes to zero as $T \to \infty$. Using the Lipschitz assumption (15) and the exponential stability bound (A2) we get

$$\begin{aligned}
&\leq \frac{L}{T}\int_0^T \left|X^{x_0}(t) - X^{\tilde{x}}(t)\right|_V \,dt \\
&= \frac{L}{T}\int_0^T |S(t)(x_0-\tilde{x})|_V\,dt \\
&\leq \frac{L}{T}|x_0-\tilde{x}|_V \int_0^T Me^{-\rho t}\,dt,
\end{aligned}$$

which goes to zero as $T \to \infty$ completing the proof. □

The global Lipschitz continuity of ϱ is a rather restrictive condition. We may relax it (cf. Theorem 4.9 below) for $\Phi \in \mathcal{L}_2(U,V)$ which corresponds to the important case when the driving process is, in fact, a genuine V-valued fractional Brownian motion.

Definition 4.1. Let Q be a nonnegative, self-adjoint, trace class operator on V. A V-valued Gaussian process $(B_Q^H(t), t \in \mathbf{R})$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is called a fractional Brownian motion of covariance type with Hurst parameter $H \in (0,1)$ and covariance Q (or simply a fractional Q-Brownian motion with Hurst parameter H) if

(1) $\mathbb{E}B_Q^H(t) = 0$ for all $t \in \mathbf{R}$,
(2) $\mathrm{Cov}(B_Q^H(t), B_Q^H(s)) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H})Q$, for all $s, t \in \mathbf{R}$,
(3) $(B_Q^H(t), t \in \mathbf{R})$ has V-valued, continuous sample paths a.s.-$\mathbb{P}$.

Let $(B^H(t), t \in \mathbf{R})$ be a standard cylindrical fractional Brownian motion in U. Let $\Phi \in \mathcal{L}_2(U,V)$ and set $Q = \Phi\Phi^*$. Then there exists (see e.g. [19], Proposition 1.1.1) a Q-covariance fractional Brownian motion $(B_Q^H(t), t \in \mathbf{R})$ such that for all $z \in V$

$$\left\langle B_Q^H(t), z\right\rangle_V = \left\langle B^H(t), \Phi^* z\right\rangle, \quad \text{a.s.-}\mathbb{P}.$$

Moreover, the solution $(X^{\tilde{x}_0}(t), t \geq 0)$ of (1) is a.s.-$\mathbb{P}$ identical with the solution to

$$\begin{aligned} dX(t) &= AX(t)\,dt + dB_Q^H(t), \\ X(0) &= x_0. \end{aligned} \tag{16}$$

We recall some a.s.-$\mathbb{P}$ growth estimates and a representation of the solution to equation (16) that have been proved in [15].

Lemma 4.8. *Let $\Phi \in \mathcal{L}_2(U,V)$. Then*

$$\int_0^t S(t-r)\,dB_Q^H(r) = A\int_0^t S(t-r)B_Q^H(r)\,dr + B_Q^H(t), \quad a.s.\text{-}\mathbb{P}, \tag{17}$$

for $t \geq 0$. Let moreover $\delta \in (0,H)$. Then for any $\omega \in \Omega$, $\varepsilon > 0$ there exists a constant $k(\omega,\varepsilon,\delta)$ such that

$$|B_Q^H(t+\cdot) - B_Q^H(t)|_{\mathcal{C}^\delta([0,1],V)} + |B_Q^H(t)|_V \leq \varepsilon t^2 + k(\omega,\varepsilon) \tag{18}$$

for $t \in \mathbf{R}$.

Proof. Both statements are proved in [15] (cf. Lemmas 2.4 and 2.6 for (18) and Proposition 3.1 for (17)). In [15], only the case $H > 1/2$ is considered; however, for $H < 1/2$ the proof of (18) remains unchanged and the proof of (17) works after a small modificaton taking into account the different form of covariance of the solution. □

Theorem 4.9. *Let* (A1) *and* (A2) *be satisfied and let* $(X^{x_0}(t), t \geq 0)$ *be a solution to* (1) *with* $\Phi \in \mathcal{L}_2(U, V)$. *Let* $\varrho : V \to \mathbf{R}$ *be a functional satisfying the following local Lipschitz condition: let there exist real constants* $K > 0$ *and* $m \geq 0$ *such that*

$$|\varrho(x) - \varrho(y)| \leq K|x - y|_V (1 + |x|_V^m + |y|_V^m) \tag{19}$$

for all $x, y \in V$. *Then*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \varrho(X^{x_0}(t))\, dt = \int_V \varrho(y)\, \mu_\infty(dy), \qquad a.s.\text{-}\mathbb{P}. \tag{20}$$

Proof. Let $(X^{\tilde{x}}(t), t \geq 0)$ be a strictly stationary solution to (1). Then obviously

$$\begin{aligned}
&\left| \frac{1}{T} \int_0^T \varrho\big(X^{x_0}(t)\big)\, dt - \frac{1}{T} \int_0^T \varrho\big(X^{\tilde{x}}(t)\big)\, dt \right| \\
&\qquad\qquad \leq \frac{1}{T} \int_0^T \left| \varrho\big(X^{x_0}(t)\big) - \varrho\big(X^{\tilde{x}}(t)\big) \right| dt.
\end{aligned}$$

We will show that the right hand side goes to zero as $T \to \infty$. Using the local Lipschitz assumption (19) we get

$$\begin{aligned}
&\frac{1}{T} \int_0^T \left| \varrho\big(X^{x_0}(t)\big) - \varrho\big(X^{\tilde{x}}(t)\big) \right| dt \\
&\leq \frac{K}{T} \int_0^T \left| X^{x_0}(t) - X^{\tilde{x}}(t) \right|_V \left(1 + |X^{x_0}(t)|_V^m + |X^{\tilde{x}}(t)|_V^m \right) dt \\
&\leq \frac{K}{T} \int_0^T |S(t)(x_0 - \tilde{x})|_V \left(1 + |S(t)x_0 + Z(t)|_V^m + |S(t)\tilde{x} + Z(t)|_V^m \right) dt \\
&\leq \frac{K}{T} \int_0^T |S(t)|_{\mathcal{L}(V)} |x_0 - \tilde{x}|_V |z|_V \left(1 + c_1 |S(t)|_{\mathcal{L}(V)}^m (|x_0|_V^m + |\tilde{x}|_V^m) + c_2 |Z(t)|_V^m \right) dt
\end{aligned}$$

and in virtue of the exponential stability bound (A2)

$$\begin{aligned}
&\leq \frac{KM}{T} |x_0 - \tilde{x}|_V \, |z|_V \int_0^T e^{-\rho t} \left(1 + c_1 M e^{-\rho m t} (|x_0|_V^m + |\tilde{x}|_V^m) + c_2 |Z(t)|_V^m \right) dt \\
&\leq \frac{c_3}{T} \int_0^T e^{-\rho t} \left(1 + c_4 e^{-\rho m t} + c_2 |Z(t)|_V^m \right) dt. \qquad (21)
\end{aligned}$$

We need the last term on the right hand side to go to zero as $T \to \infty$. By Lemma 4.8 we have

$$\begin{aligned}
|Z(t)|_V &\leq \left| A \int_0^t S(t - r) B_Q^H(r)\, dr \right|_V + |B_Q^H(t)|_V \\
&\leq \left| A \int_0^{t-1} S(t - r) B_Q^H(r)\, dr \right|_V + \left| A \int_0^1 S(1 - s) B_Q^H(s + t - 1)\, ds \right|_V + |B_Q^H(t)|_V. \qquad (22)
\end{aligned}$$

The condition (A2) and analyticity of the semigroup $(S(t), t \geq 0)$ yield

$$\begin{aligned}
\left| A \int_0^{t-1} S(t - r) B_Q^H(r)\, dr \right|_V &\leq c_5 \int_0^{t-1} \frac{e^{-\rho(t-r)}}{(t - r)} |B_Q^H(r)|_V\, dr \\
&\leq c_5 \int_0^{t-1} e^{-\rho(t-r)} |B_Q^H(r)|_V\, dr \\
&\leq \varepsilon t^2 + k_1, \qquad (23)
\end{aligned}$$

for each $\varepsilon > 0$ and random constant $k_1 = k_1(\omega, \varepsilon), \omega \in \Omega$, by Lemma 4.8. Also, for each $\delta \in (0, H)$ there is a constant $k_2 = k_2(\delta)$ such that

$$\begin{aligned}\left| A \int_0^1 S(1-s) B_Q^H(s+t-1)\, ds \right|_V &\le \left| A \int_0^1 S(1-s)(B_Q^H(s+t-1) - B_Q^H(t-1))\, ds \right|_V \\ &\quad + \left| A \int_0^1 S(1-s) B_Q^H(t-1)\, ds \right|_V \\ &\le k_2 \left(|B_Q^H(\cdot + t - 1) - B_Q^H(t-1)|_{\mathcal{C}^\delta([0,1],V)} + |B_Q^H(t-1)|_V \right) \end{aligned} \tag{24}$$

by [26], p. 44. By (22), (23), (24) and Lemma 4.8 it follows that

$$|Z_t| \le k_3(1+t^2), \quad t \ge 0,$$

for a constant $k_3 = k_3(\omega), \omega \in \Omega$, which by (21) yields

$$\frac{1}{T} \int_0^T |\varrho(X^{x_0}(t) - \varrho(X^{\tilde{x}}(t)|\, dt \le \frac{k_4}{T} \int_0^T e^{-\rho m t}(1+t^2)^m\, dt$$

for a random constant $k_4 = k_4(\omega), \omega \in \Omega$, which tends to zero as $T \to \infty$. Now (20) follows by Theorem 4.6. □

Example 4.1. Consider the initial boundary value problem for stochastic parabolic equation

$$\begin{aligned} &\frac{\partial u}{\partial t}(t,x) = [Lu](t,x) + \xi(t,x), \quad (t,x) \in \mathbf{R}_+ \times D, \\ &u(0,x) = u_0(x), \quad x \in D, \\ &u(t,x) = 0, \quad t \in \mathbf{R}_+,\, x \in \partial D, \end{aligned} \tag{25}$$

where $D \subset \mathbf{R}^d$ is a bounded domain with a smooth boundary, L is a second order uniformly elliptic operator on D and η is a noise process that is the formal time derivative of a space dependent fractional Brownian motion.

To provide a rigorous meaning to (25), we rewrite the parabolic system as an infinite dimensional stochastic differential equation

$$dX(t) = AX(t)\, dt + \Phi\, dB^H(t) \tag{26}$$

for $t \ge 0$ where the space V is $L^2(D)$, $A = L|_{\mathrm{Dom}(A)}$ generates an exponentially stable strong continuous analytic semigroup $(S(t), t \ge 0)$ on V with $\mathrm{Dom}(A) = H^2(D) \cap H_0^1(D)$, $U = V$ and the noise ξ is modelled as the formal derivative $\Phi(dB^H/dt)$, $(B^H(t), t \ge 0)$ is a standard cylindrical fractional Brownian motion in U and $\Phi \in \mathcal{L}(V)$.

If $\Phi \in \mathcal{L}_2(V)$, which corresponds to the case where the fractional Brownian motion in (26) is of Q-covariance type, then it follows from Proposition 2.1 that there is a V_δ continuous solution to (26) for $\delta < H$.

An interesting case occurs if it is only assumed that $\Phi \in \mathcal{L}(V)$ so that $(B^H(t), t \ge 0)$ is only a standard cylindrical fractional Brownian motion. By standard estimates on the Green function for $dx/dt = Ax$ it follows that

$$|S(t)|_{\mathcal{L}_2(V)} \le ct^{-d/4}$$

for $t \in (0, T]$, $c > 0$ and d is the dimension of the underlying space. It follows from Proposition 2.1 with $\gamma = d/4$ that if

$$d/4 < H$$

then the condition (A1) is satisfied and hence there is a β-Hölder continuous solution in V_δ if $\beta + \delta \in [0, H - d/4)$. It is well known that the semigroup $(S(t), t \ge 0)$ is exponentially stable on V, so (A2) is satisfied and there is a limiting measure. If the standard cylindrical fractional Brownian motion in (26) is replaced by a standard cylindrical Wiener process, that is $H = 1/2$, then a V-valued solution exists only if $d = 1$.

The above analysis shows that the conditions (A1) and (A2) are satisfied and hence by Theorems 3.1 and 4.6 there exists a strictly stationary solution to (26) that is ergodic. If $\Phi \in \mathcal{L}_2(V)$, Theorem 4.9 may be applied as well, which shows the ergodic behaviour of solutions with arbitrary initial condition.

Example 4.2. Consider the initial boundary value problem for stochastic hyperbolic equation

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2}(t,x) + a\frac{\partial u}{\partial t}(t,x) = Lu(t,x) + \xi(t,x), \quad (t,x) \in \mathbf{R}_+ \times D,\\
&\frac{\partial u}{\partial t}(0,x) = u_1(x), \quad x \in D,\\
&u(0,x) = u_2(x), \quad x \in D,\\
&u(t,x) = 0, \quad (t,x) \in \mathbf{R}_+ \times \partial D,
\end{aligned} \tag{27}$$

where D, L and ξ satisfy the conditions in Example 4.1 and $a \geq 0$ is a real constant parameter.

We rewrite the hyperbolic system (27) as an infinite dimensional stochastic differential equation

$$\begin{aligned}
dX(t) &= AX(t)\,dt + \Phi\,dB^H(t)\\
X(0) &= x_0 = (u_1, u_2).
\end{aligned} \tag{28}$$

Let $\Lambda = L|_{\mathrm{Dom}(\Lambda)}$, $\mathrm{Dom}(\Lambda) = H_0^1(D) \cap H^1(D)$, $\mathrm{Dom}(A) = \mathrm{Dom}(\Lambda) \times \mathrm{Dom}(-\Lambda)^{1/2}$ and

$$A = \begin{pmatrix} 0 & I \\ \Lambda & -aI \end{pmatrix}$$

It is well known that A generates a strongly continuous semigroup in the space $V = \mathrm{Dom}(-\Lambda)^{1/2} \times L^2(D)$. Let $(B^H(t), t \geq 0)$ be a standard cylindrical fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$ and

$$\Phi = \begin{pmatrix} 0 & 0 \\ 0 & Q_2^{1/2} \end{pmatrix},$$

where $Q_2^{1/2}$ is a Hilbert-Schmidt operator on $L^2(D)$. It follows from Proposition 2.1 that the solution of (28) exists and has V-continuous sample paths.

If $a > 0$ then the semigroup $(S(t), t \geq 0)$ is exponentially stable and by Theorems 3.1 and 4.6 there exists a strictly stationary solution to (28) that is ergodic (cf. also Remark 2.1).

5. Parameter estimates based on ergodicity

In this section we present the results on parameter estimation in infinite dimensional equations that are based on the ergodic theorems proved in the previous section.

Consider the linear equation

$$\begin{aligned}
dX(t) &= \alpha\,AX(t)\,dt + \Phi\,dB^H(t),\\
X(0) &= x_0,
\end{aligned} \tag{29}$$

where $\alpha > 0$ is a real constant parameter, $(B^H(t), t \geq 0)$ is a standard cylindrical fractional Brownian motion in U and U is a separable Hilbert space, $A : \mathrm{Dom}(A) \to V$, $\mathrm{Dom}(A) \subset V$, A is the infinitesimal generator of an exponentially stable (strongly continuous) analytic semigroup $(S(t), t \geq 0)$ on the separable Hilbert space V, $\Phi \in \mathcal{L}_2(U, V)$ and $x_0 \in V$.

The operator αA is the infinitesimal generator of the semigroup $(S_\alpha(t), t \geq 0)$. Obviously $S_\alpha(t) = S(\alpha t)$ for all $t \geq 0$. The semigroup $(S_\alpha(t), t \geq 0)$ is also exponentially stable and there is a limiting measure $\mu_\infty^\alpha = \mathcal{N}(0, Q_\infty^\alpha)$.

For $H > 1/2$ we have

$$\begin{aligned}
Q_\infty^\alpha &= \int_0^\infty \int_0^\infty S_\alpha(u) Q S_\alpha^*(v) \phi(u-v)\,du\,dv\\
&= \frac{1}{\alpha^2} \int_0^\infty \int_0^\infty S(u) Q S^*(v) \phi\left(\frac{u}{\alpha} - \frac{v}{\alpha}\right) du\,dv\\
&= \frac{1}{\alpha^{2H}} Q_\infty,
\end{aligned}$$

where Q_∞ corresponds to the case $\alpha = 1$ (cf. Remark 2.2).

For $H < 1/2$ and $x, y \in V$ we have

$$\begin{aligned}
\langle Q^\alpha_\infty x, y\rangle_V &= \lim_{T\to\infty} \langle Q^\alpha_T x, y\rangle_V \\
&= \lim_{T\to\infty} \sum_{n=1}^\infty \mathbb{E} \left\langle \int_0^T S_\alpha(r)\Phi h_n \, d\beta_n^H(r), x \right\rangle_V \left\langle \int_0^T S_\alpha(r)\Phi h_n \, d\beta_n^H(r), y \right\rangle_V \\
&= \lim_{T\to\infty} \int_0^T \sum_{n=1}^\infty \langle \mathcal{K}_H^*(S_\alpha(\cdot)\Phi h_n)(r), x\rangle_V \langle \mathcal{K}_H^*(S_\alpha(\cdot)\Phi h_n)(r), x\rangle_V \, dr.
\end{aligned}$$

Using the representation (5) and a simple substitution theorem we also arrive at

$$\langle Q^\alpha_\infty x, y\rangle_V = \frac{1}{\alpha^{2H}} \langle Q_\infty x, y\rangle_V$$

for all $x, y \in V$ and therefore

$$Q^\alpha_\infty = \frac{1}{\alpha^{2H}} Q_\infty.$$

For $H = 1/2$ this equality is obvious.

Based on the above results, some strongly consistent families of estimators of the parameter α may be proposed.

Theorem 5.1. *Let* (A1) *and* (A2) *be satisfied and let* $(X^{x_0}(t), t \geq 0)$ *be a* V*-valued solution to* (29) *with* $\Phi \in \mathcal{L}_2(U, V)$*. Let* $z \in V$ *be arbitrary and let the limiting measure* μ_∞ *exists with covariance* Q_∞ *such that*

$$\langle Q_\infty z, z\rangle_V > 0.$$

Define

$$\hat\alpha_T := \left(\frac{\langle Q_\infty z, z\rangle_V}{\frac{1}{T}\int_0^T |\langle X^{x_0}(t), z\rangle_V|^2 \, dt} \right)^{\frac{1}{2H}}.$$

Then

$$\lim_{T\to\infty} \hat\alpha_T = \alpha, \quad a.s.\text{-}\mathbb{P}.$$

Proof. Let $z \in V$ be arbitrary. Let $\varrho : V \to \mathbf{R}$, $\varrho(y) = \langle y, z\rangle_V^2$, $y \in V$. Then all the conditions of Theorem 4.9 are satisfied with $m = 1$ and

$$\begin{aligned}
\lim_{T\to\infty} \frac{1}{T} \int_0^T \varrho\big(X^{x_0}(t)\big) \, dt &= \lim_{T\to\infty} \frac{1}{T} \int_0^T |\langle X^{x_0}(t), z\rangle_V|^2 \, dt \\
&= \int_V |\langle y, z\rangle_V|^2 \, \mu_\infty(dy) \\
&= \langle Q^\alpha_\infty z, z\rangle_V \\
&= \frac{1}{\alpha^{2H}} \langle Q_\infty z, z\rangle_V, \quad \text{a.s.-}\mathbb{P},
\end{aligned}$$

which completes the proof. $\square$

Theorem 5.2. *Let* (A1) *and* (A2) *be satisfied and let* $(X^{x_0}(t), t \geq 0)$ *be a* V*-valued solution to* (29) *with* $\Phi \in \mathcal{L}_2(U, V)$*,* $\Phi \neq 0$*. Define*

$$\hat\alpha_T := \left(\frac{\operatorname{Tr} Q_\infty}{\frac{1}{T}\int_0^T |X^{x_0}(t)|_V^2 \, dt} \right)^{\frac{1}{2H}}.$$

Then

$$\lim_{T\to\infty} \hat\alpha_T = \alpha.$$

Proof. Let $\varrho : V \to \mathbf{R}$, $\varrho(y) = |y|_V^2$, $y \in V$. Then all the conditions of Theorem 4.9 are satisfied with $m = 1$ and

$$\begin{aligned}\lim_{T\to\infty} \frac{1}{T}\int_0^T \varrho(X^{x_0}(t))\,dt &= \lim_{T\to\infty}\frac{1}{T}\int_0^T |X^{x_0}(t)|_V^2\,dt \\ &= \int_V |X^{x_0}(t)|_V^2\,d\mu_\infty \\ &= \operatorname{Tr} Q_\infty^\alpha \\ &= \frac{1}{\alpha^{2H}}\operatorname{Tr} Q_\infty.\end{aligned}$$

It remains to check that $\operatorname{Tr} Q_\infty \neq 0$. If we admit that $\operatorname{Tr} Q_\infty = 0$ then obviously the strictly stationary solution $X^{\tilde{x}} = 0, \text{a.s.-}\mathbb{P}$, i.e. $\tilde{x} = 0, \text{a.s.-}\mathbb{P}$, and $\int_0^t S(t-r)\Phi\,dB^H(r) = 0, \text{a.s.-}\mathbb{P}$, for each $t \geq 0$. It is easy to see that it contradicts the condition $\Phi \neq 0$. □

Remark 5.1. Theorems 5.1 and 5.2 may be applied to the equation from Example 4.1 where the operator L depends on a parameter $\alpha > 0$, $L(\alpha) = \alpha L$.

One observation path is sufficient to get a consistent almost sure estimate for $T \to \infty$. If we have more observations, we can propose similar estimates using mean values (see e.g. [19]). Some examples in parameter estimation including numerical simulations can be found in [20] and [21].

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