Approximated Solutions of Generalized Linear Differential Equations

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Abstract. This contribution deals with systems of linear generalized linear differential equations of the form

$$x(t) = \widetilde{x} + \int_a^t \mathrm{d}[A(s)]\, x(s) + g(t) - g(a), \quad t \in [a, b],$$

where $-\infty < a < b < \infty$, A is an $n \times n$-complex matrix valued function, g is an n-complex vector valued function, A and g have bounded variation on $[a, b]$. The integrals are understood in the Kurzweil-Stieltjes sense.

Our aim is to present some new results on continuous dependence of solutions to linear generalized differential equations on parameters and initial data. In particular, we generalize in several aspects the known result by Ashordia. Our main goal consists in a more general notion of a solution to the given system. In particular, neither g nor x need not be of bounded variation on $[a, b]$ and, in general, they can be regulated functions.

The convergence effects studied in this contribution are, in some sense, very close to those described by Kurzweil and called by him the emphatic convergence.

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1 . Introduction

Starting with Kurzweil [10], generalized differential equations have been extensively studied by many authors, like e.g. Schwabik, Tvrdý and Vejvoda [19]–[21], [23]–[26], Ashordia [2], [3], Fraňková [5], [6]. Closely related and fundamental are also the contributions by Hildebrandt [8] and Hönig [9],

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In particular, see the monographs [21], [19], [25] and [9] and the references therein. Moreover, during several recent decades, the interest in their special cases like equations with impulses or discrete systems increased considerably, cf. e.g. monographs [15], [28], [4], [18] or [1].

The importance of generalized linear differential equations with regulated solutions consists in the fact that they enable us to treat in a unified way both continuous and discrete systems and, in addition, also systems with fast oscillating data.

In the paper we keep the following notation:

As usual, $\mathbb{N}$ is the set of natural numbers $(\mathbb{N} = \{1, 2, \dots\})$ and $\mathbb{C}$ stands for the set of complex numbers. $\mathbb{C}^{m\times n}$ is the space of complex matrices of the type $m \times n$, $\mathbb{C}^n = \mathbb{C}^{n\times 1}$ and $\mathbb{C}^1 = \mathbb{C}$. For a matrix

$$A = (a_{i,j})_{\substack{i=1,2,\dots,m \\ j=1,2,\dots,n}} \in \mathbb{C}^{m\times n},$$

its norm $|A|$ is defined by

$$|A| = \max_{j=1,2,\dots,n} \sum_{i=1}^{m} |a_{i,j}|.$$

In particular, we have $|x| = \sum_{i=1}^{n} |x_i|$ for $x \in \mathbb{C}^n$. The symbols I and 0 stand respectively for the identity and the zero matrix of the proper type. For an $n \times n$-matrix A, $\det[A]$ denotes its determinant.

If $-\infty < a < b < \infty$, then $[a, b]$ and (a, b) denote the corresponding closed and open intervals, respectively. Furthermore, $[a, b)$ and $(a, b]$ are the corresponding half-open intervals. When the intervals $[a, a)$ and $(b, b]$ occur, they are understood to be empty.

For an arbitrary function $F\colon [a, b] \to \mathbb{C}^{m\times n}$, we set

$$\|F\|_\infty = \sup\{|F(t)|\colon t \in [a, b]\}.$$

If $F_k\colon [a, b] \to \mathbb{C}^{m\times n}$, $k \in \mathbb{N}$, and $F\colon [a, b] \to \mathbb{C}^{m\times n}$ are such that

$$\lim_{k\to\infty} \|F_k - F\|_\infty = 0,$$

we say that F_k tends to F uniformly on $[a, b]$ and write $F_k \rightrightarrows F$ on $[a, b]$. If $I \subset \mathbb{R}$ and $F_k \rightrightarrows F$ on $[a, b]$ for each $[a, b] \subset I$, we say that F_k tends to F locally uniformly on I and write $F_k \rightrightarrows F$ locally on $[a, b]$.

The set $D=\{\alpha_0,\alpha_1,\dots,\alpha_m\}\subset[a,b]$, $m\in\mathbb{N}$, is called a division of the interval $[a,b]$, if

$$a=\alpha_0<\alpha_1<\,\dots\,<\alpha_m=b.$$

The set of all divisions of the interval $[a,b]$ is denoted by $\mathfrak{D}[a,b]$. For $D=\{\alpha_0,\alpha_1,\dots,\alpha_m\}\in\mathfrak{D}[a,b]$, we denote

$$|D|=\max_{\ell=1,2,\dots,m}(\alpha_\ell-\alpha_{\ell-1}).$$

If, for each $t\in[a,b)$ and $s\in(a,b]$, the function $F:[a,b]\to\mathbb{C}^{m\times n}$ possesses the limits

$$F(t+):=\lim_{\tau\to t+}F(\tau),\quad F(s-):=\lim_{\tau\to s-}F(\tau),$$

we say that the function F is regulated on the interval $[a,b]$. The set of all $m\times n$- matrix valued functions regulated on the interval $[a,b]$ is denoted by $G^{m\times n}[a,b]$. Furthermore, we denote

$$\begin{aligned}\Delta^+F(t)&=F(t+)-F(t)\quad\text{for }\ t\in[a,b),\quad \Delta^+F(b)=0,\\ \Delta^-F(s)&=F(s)-F(s-)\quad\text{for }\ s\in(a,b],\quad \Delta^-F(a)=0\end{aligned}$$

and

$$\Delta F(t)=F(t+)-F(t-)\quad\text{for }\ t\in(a,b).$$

It is known that, for each $F\in G^{m\times n}[a,b]$, the set of all points of its discontinuity on the interval $[a,b]$ is at most countable. Moreover, for each $\varepsilon>0$ there are at most finitely many points $t\in[a,b)$ such that $|\Delta^+F(t)|\geq\varepsilon$ and at most finitely many points $s\in(a,b]$ such that $|\Delta^-F(s)|\geq\varepsilon$. Clearly, each function regulated on $[a,b]$ is bounded on $[a,b]$, i.e. $\|F\|_\infty<\infty$ for all $F\in G^{m\times n}[a,b]$.

For a function $F:[a,b]\to\mathbb{C}^{m\times n}$ we denote by $\operatorname{var}_a^b F$ its variation over $[a,b]$. We say that F has a bounded variation on $[a,b]$ if $\operatorname{var}_a^b F<\infty$. The set of $m\times n$- complex matrix valued functions of bounded variation on $[a,b]$ is denoted by $BV^{m\times n}[a,b]$ and $\|F\|_{BV}=|F(a)|+\operatorname{var}_a^b F$. By $AC^{m\times n}[a,b]$ we denote the set of functions $F\colon[a,b]\to\mathbb{C}^{m\times n}$ such that each component f_{ij}, $i=1,\dots,m$, $j=1,\dots,n$, of F is absolutely continuous on the interval $[a,b]$. Similarly, $C^{m\times n}[a,b]$ stands for the set of functions $F:[a,b]\to\mathbb{C}^{m\times n}$ that are continuous on $[a,b]$. If $m=1$, we write $BV^n[a,b]$ instead of $BV^{n\times 1}[a,b]$. Analogously, $AC^n[a,b]=AC^{n\times 1}[a,b]$, $G^n[a,b]=G^{n\times 1}[a,b]$ and $C^n[a,b]=C^{n\times 1}[a,b]$. Obviously,

$$AC^{m\times n}[a,b]\subset BV^{m\times n}[a,b]\subset G^{m\times n}[a,b] \text{ and } C^{m\times n}[a,b]\subset G^{m\times n}[a,b].$$

Finally, a function $f:[a,b]\to\mathbb{C}$ is called a *finite step function on* $[a,b]$ if there is a division $\{\alpha_0,\alpha_1,\dots,\alpha_m\}\in\mathfrak{D}[a,b]$ of $[a,b]$ such that f is constant on every open interval (α_{j-1},α_j), $j=1,2,\dots,m$. The set of all finite step functions on $[a,b]$ is denoted by $S[a,b]$, $S^{m\times n}[a,b]$ is the set of all $m\times n$-matrix valued functions whose arguments are finite step functions and $S^{n\times 1}[a,b]=S^n[a,b]$. It is known that the set $S^{m\times n}[a,b]$ is dense in $G^{m\times n}[a,b]$ with respect to the supremum norm, i.e.

$$\begin{cases} \text{for each } \varepsilon>0 \text{ and each } F\in G^{m\times n}[a,b] \\ \text{there is an } \widetilde{F}\in S^{m\times n}[a,b] \text{ such that } \|F-\widetilde{F}\|_\infty<\varepsilon. \end{cases} \tag{1.1}$$

The integrals which occur in this paper are the Perron-Stieltjes ones. For the original definition, see A.J. Ward [27] or S. Saks [17]. We use the equivalent summation definition due to J. Kurzweil [10] (cf. also e.g. [12] or [21]). We call this integral the Kurzweil-Stieltjes integral, in short the KS-integral. For the reader's convenience, let us recall the definition of the KS-integral.

Let $-\infty<a<b<\infty$. For given $m\in\mathbb{N}$, $D=\{\alpha_0,\alpha_1,\dots,\alpha_m\}\in\mathfrak{D}[a,b]$ and $\xi=(\xi_1,\xi_2,\dots,\xi_m)\in[a,b]^m$, the couple $P=(D,\xi)$ is called a partition of $[a,b]$ if

$$\alpha_{j-1}\le\xi_j\le\alpha_j \quad \text{for } j=1,2,\dots,m.$$

The set of all partitions of the interval $[a,b]$ is denoted by $\mathfrak{P}[a,b]$.

An arbitrary positive valued function $\delta\colon[a,b]\to(0,\infty)$ is called a gauge on $[a,b]$. Given a gauge δ on $[a,b]$, the partition

$$P=(D,\xi)=\big(\{\alpha_0,\alpha_1,\dots,\alpha_m\},\,(\xi_1,\xi_2,\dots,\xi_m)\big)\in\mathfrak{P}[a,b]$$

is said to be δ-fine, if

$$[\alpha_{j-1},\alpha_j]\subset(\xi_j-\delta(\xi_j),\xi_j+\delta(\xi_j)) \quad \text{for } j=1,2,\dots,m.$$

The set of all δ-fine partitions of $[a,b]$ is denoted by $\mathfrak{A}(\delta;[a,b])$.

For functions $f,\, g:\,[a,b]\to\mathbb{C}$ and a partition $P\in\mathfrak{P}[a,b]$,

$$P=\big(\{\alpha_0,\alpha_1,\dots,\alpha_m\},\,(\xi_1,\xi_2,\dots,\xi_m)\big),$$

we define

$$\Sigma(f\,\Delta g;P)=\sum_{i=1}^m[g(\alpha_i)-g(\alpha_{i-1})]\,f(\xi_i).$$

We say that $I\in\mathbb{C}$ is the KS-integral of f with respect to g from a to b if

$$\begin{cases}\text{for each } \varepsilon>0 \text{ there is a gauge } \delta \text{ on } [a,b] \text{ such that}\\[1ex] |I-\Sigma(f\,\Delta g;P)|<\varepsilon \quad \text{for all} \quad P\in\mathfrak{A}\,(\delta;[a,b]).\end{cases}$$

In such a case we write

$$I=\int_a^b \mathrm{d}[g]\,f \quad\text{or}\quad I=\int_a^b \mathrm{d}[g(t)]\,f(t).$$

It is well known that the KS-integral $\int_a^b \mathrm{d}[g]\,f$ exists provided $f\in G[a,b]$ and $g\in BV[a,b]$. Taking into account [25, Theorem 2.3.8], we can formulate the following fundamental assertion.

1.1. Theorem. *If $f,\,g\in G[a,b]$ and at least one of the functions $f,\,g$ has a bounded variation on $[a,b]$, then the integral $\int_a^b f\,\mathrm{d}[g]$ exists. Furthermore,*

$$\Big|\int_a^b \mathrm{d}[g]\,f\Big|\le 2\,\|g\|_\infty\,\big(|f(a)|+\mathrm{var}_a^b f\big) \;\; \textit{if} \;\; f\in BV[a,b] \textit{ and } g\in G[a,b], \tag{1.2}$$

and

$$\Big|\int_a^b \mathrm{d}[g]\,f\Big|\le (\mathrm{var}_a^b\, g)\,\|f\|_\infty \qquad \textit{if} \;\; f\in G[a,b] \textit{ and } g\in BV[a,b]. \tag{1.3}$$

Furthermore, if $f\in BV[a,b]$ and $g,g_k\in G[a,b]$ for $k\in\mathbb{N}$, then

$$\lim_{k\to\infty}\|g_k-g\|_\infty=0 \quad\text{implies}\quad \lim_{k\to\infty}\Big\|\int_a^t \mathrm{d}[g_k-g]\,f\Big\|_\infty=0.$$

Further basic properties of the Perron-Stieltjes integral with respect to scalar regulated functions were described in [22] (see also [25]).

Given an $q \times n$-matrix valued function F and an $m \times q$-matrix valued function G defined on $[a, b]$ and such that all the integrals

$$\int_a^b \mathrm{d}[g_{i,k}(t)]\, f_{k,j}(t) \quad (i = 1, 2, \dots, m\,;\; k = 1, 2, \dots, q;\; j = 1, 2, \dots, n)$$

exist (i.e. they have finite values), the symbol

$$\int_a^b \mathrm{d}[G(t)]\, F(t) \quad \Big(\text{or more simply} \quad \int_a^b \mathrm{d}[G]\, F\,\Big)$$

stands for the $m \times n$-matrix with the entries

$$\sum_{k=1}^{q} \int_a^b \mathrm{d}[g_{i,k}]\, f_{k,j}, \quad i = 1, 2, \dots, m\,,\; j = 1, 2, \dots, n.$$

The extension of the results obtained in [22] or [25] for scalar real valued functions to complex vector valued or matrix valued functions is obvious and hence for the basic facts concerning integrals with respect to regulated functions we will refer to the corresponding assertions from [22] or [25].

Next assertion seems not to be available in the literature.

1.2 . Lemma. *Let* $g, g_k \in G^n[a, b]$, $A, A_k \in BV^{n\times n}[a, b]$ *for* $k \in \mathbb{N}$. *Furthermore, let*

$$g_k \rightrightarrows g \quad \textit{on} \;\; [a, b], \tag{1.4}$$

$$\alpha^* := \sup\{\mathrm{var}_a^b A_k :\; k \in \mathbb{N}\} < \infty, \tag{1.5}$$

and

$$A_k \rightrightarrows A \quad \textit{on} \;\; [a, b]. \tag{1.6}$$

Then

$$\int_a^t \mathrm{d}[A_k]\, g_k \rightrightarrows \int_a^t \mathrm{d}[A]\, g \quad \textit{on} \;\; [a, b].$$

Proof. Let $\varepsilon > 0$ be given. By (1.1) and (1.4), we can find $u \in S^n[a, b]$ and $k_0 \in \mathbb{N}$ such that

$$\|g - u\|_\infty < \varepsilon, \quad \|g_k - u\|_\infty < \varepsilon \quad \text{and} \quad \|A_k - A\|_\infty < \varepsilon \quad \text{for} \;\; k \geq k_0.$$

Furthermore, since $S^n[a,b] \subset BV^n[a,b]$, using (1.2) we obtain

$$\begin{aligned}
&\left|\int_a^t \mathrm{d}[A_k]\,g_k - \int_a^t \mathrm{d}[A]\,g\right| \\
&\quad = \left|\int_a^t \mathrm{d}[A_k]\,(g_k-u) + \int_a^t \mathrm{d}[A_k - A]\,u + \int_a^t \mathrm{d}[A]\,(u-g)\right| \\
&\quad \le \alpha^*\,\varepsilon + 2\,(|u(a)| + \operatorname{var}_a^b u)\,\varepsilon + \alpha^*\,\varepsilon = 2\,(\alpha^* + |u(a)| + \operatorname{var}_a^b u)\,\varepsilon
\end{aligned}$$

for $t \in [a,b]$ and $k \ge k_0$, wherefrom our assertion readily follows. $\square$

2. Generalized differential equations

Let $A \in BV^{n\times n}[a,b]$, $g \in BV^n[a,b]$ and $\widetilde{x} \in \mathbb{C}^n$. Consider an integral equation

$$x(t) = \widetilde{x} + \int_a^t \mathrm{d}[A(s)]\,x(s) + g(t) - g(a). \tag{2.1}$$

We say that a function $x\colon [a,b] \to \mathbb{C}^n$ is *a solution* of (2.1) on the interval $[a,b]$ if

$$\int_a^b \mathrm{d}[A(s)]\,x(s) \in \mathbb{C}^n$$

and equality (2.1) is satisfied for all $t \in [a,b]$. The equation (2.1) is usually called an initial value problem for the *generalized linear differential equation* (with initial condition $x(a) = \widetilde{x}$). Such problems with solutions having values in the space $\mathbb{R}^n$ of real n-vectors have been thoroughly investigated e.g. in the monographs [19] or [21]. The extension of the results presented therein to the complex case is mostly straightforward. In this section we will describe the basics needed later. Special attention is paid to the features whose extension to the complex case seems not to be so straightforward.

For our purposes the following property is crucial:

$$\det\big[I - \Delta^- A(t)\big] \ne 0 \quad \text{for all} \quad t \in [a,b]. \tag{2.2}$$

(Recall that we put $\Delta^- A(a) = 0$.) Its importance is well illustrated by the next assertion which is a fundamental existence result for the equation (2.1).

2.1. Theorem. *Let* $A \in BV^{n\times n}[a,b]$ *satisfy* (2.2). *Then, for each* $\widetilde{x} \in \mathbb{C}^n$ *and each* $g \in G^n[a,b]$, *equation* (2.1) *has a unique solution* x *on* $[a,b]$ *and* $x \in G^n[a,b]$. *Moreover,* $x - g \in BV^n[a,b]$.

Proof follows from [23, Proposition 2.5].

Furthermore, analogously to [21, Theorem III.1.7] where $g \in BV^n[a,b]$, we have

2.2. Theorem. *Let* $A \in BV^{n\times n}[a,b]$ *satisfy* (2.2). *Then*

$$c_A := \sup\{\big|[I - \Delta^- A(t)]^{-1}\big| \colon t \in [a,b]\} < \infty \tag{2.3}$$

and

$$|x(t)| \le c_A\,(|\widetilde{x}| + 2\,\|g\|_\infty)\,\exp(c_A \operatorname{var}_a^t A) \text{ for } t \in [a,b] \tag{2.4}$$

holds for each $\widetilde{x} \in \mathbb{C}^n$, $g \in G^n[a,b]$ *and each solution* x *of* (2.1) *on* $[a,b]$.

Proof. First, notice that for $t \in [a,b]$ such that $|\Delta^- A(t)| < \frac{1}{2}$ we have

$$\big|[I - \Delta^- A(t)]^{-1}\big| = \left|\sum_{k=1}^{\infty}(\Delta^- A(t))^k\right| \le \sum_{k=1}^{\infty}|\Delta^- A(t)|^k = \frac{1}{1 - |\Delta^- A(t)|} < 2.$$

Therefore, (2.3) follows from the fact that the set $\{t \in [a,b] \colon |\Delta^- A(t)| \ge \frac{1}{2}\}$ has at most finitely many elements.

Now, let x be a solution of (2.1). Put $B(a) = A(a)$ and $B(t) = A(t-)$ for $t \in (a,b]$. Then, as in the proof of [21, Theorem III.1.7], we get

$$A - B \in BV^{n\times n}[a,b], \quad \operatorname{var}_a^b B \le \operatorname{var}_a^b A$$

and

$$A(t) - B(t) = \Delta^- A(t), \quad \int_a^t \mathrm{d}[A-B]\,x = \Delta^- A(t)\,x(t) \quad \text{for } t \in [a,b].$$

Consequently

$$x(t) = [I - \Delta^- A(t)]^{-1}\left(\widetilde{x} + g(t) - g(a) + \int_a^t \mathrm{d}[B]\,x\right)$$

and

$$|x(t)| \le K_1 + K_2 \int_a^t \mathrm{d}[h]\,|x| \quad \text{for } t \in [a,b],$$

where

$$K_1 = c_A\,(|\widetilde{x}| + 2\,\|g\|_\infty)\,, \quad K_2 = c_A \quad \text{and} \quad h(t) = \operatorname{var}_a^t B \quad \text{for} \quad t \in [a,b].$$

The function h is nondecreasing and, since B is left-continuous on $(a,b]$, h is also left-continuous on $(a,b]$. Therefore we can use the generalized Gronwall inequality (see e.g. [21, Lemma I.4.30] or [19, Corollary 1.43]) to get the estimate (2.4). □

2.3. Corollary. *Let $A \in BV^{n\times n}[a,b]$ satisfy (2.2). Then for each $\widetilde{x} \in \mathbb{C}^n$, $g \in G^n[a,b]$ and each solution x of (2.1) on $[a,b]$, the estimate*

$$\operatorname{var}_a^b(x-g) \le c_A\,(\operatorname{var}_a^b A)\,(|\widetilde{x}| + 2\,\|g\|_\infty)\,\exp(c_A \operatorname{var}_a^b A).$$

is true, where c_A is defined by (2.3).

Proof. By (2.4), we have

$$\|x\|_\infty \le c_A\,(|\widetilde{x}| + 2\,\|g\|_\infty)\,\exp(c_A \operatorname{var}_a^b A).$$

Therefore

$$\begin{aligned} \operatorname{var}_a^b(x-g) &\le \big(\operatorname{var}_a^b A\big)\,\|x\|_\infty \\ &\le c_A\,\big(\operatorname{var}_a^b A\big)\,(|\widetilde{x}| + 2\,\|g\|_\infty)\,\exp(c_A \operatorname{var}_a^b A). \end{aligned}$$ □

2.4. Lemma. *Let $A \in BV^{n\times n}[a,b]$ satisfy (2.2) and let c_A be defined by (2.3). Then*

$$c_A = \Big(\inf\big\{\big|\big[I - \Delta^- A(t)\big]\;x\big|\colon\; t \in [a,b],\, x \in \mathbb{C}^n,\, |x| = 1\big\}\Big)^{-1}. \tag{2.5}$$

Proof. We have

$$\begin{aligned} c_A &= \sup\big\{|[I - \Delta^- A(t)]^{-1}|\colon\; t \in [a,b]\big\} \\ &= \sup\left\{\frac{|[I - \Delta^- A(t)]^{-1}|\,|[I - \Delta^- A(t)]\,x|}{|[I - \Delta^- A(t)]\,x|}\colon\; t \in [a,b],\, x \in \mathbb{C}^n,\, |x| = 1\right\} \\ &\ge \sup\left\{\frac{|x|}{|[I - \Delta^- A(t)]\,x|}\colon\; t \in [a,b],\, x \in \mathbb{C}^n,\, |x| = 1\right\} \\ &= \sup\left\{\frac{1}{|[I - \Delta^- A(t)]\,x|}\colon\; t \in [a,b],\, x \in \mathbb{C}^n,\, |x| = 1\right\} \\ &= \Big(\inf\big\{|[I - \Delta^- A(t)]\,x|\colon\; t \in [a,b],\, x \in \mathbb{C}^n,\, |x| = 1\big\}\Big)^{-1}. \end{aligned}$$

Thus, it remains to prove that the inequality

$$c_A \le \left(\inf\left\{\left|\left[I-\Delta^- A(t)\right]\, x\right| \colon\ t\in[a,b],\ x\in\mathbb{C}^n,\ |x|=1\right\}\right)^{-1} \tag{2.6}$$

is true, as well. To this aim, first let us notice that for each $t\in[a,b]$ there is a $z\in\mathbb{C}^n$ such that $|z|=1$ and

$$\left|[I-\Delta^- A(t)]^{-1}\right| = \left|[I-\Delta^- A(t)]^{-1}\, z\right|. \tag{2.7}$$

Indeed, let $t\in[a,b]$ and let $B=[I-\Delta^- A(t)]^{-1}$. Let $i_0\in\{1,2,\dots,n\}$ be such that $|B|=\sum_{j=1}^n |b_{i_0,j}|$ and let $z\in\mathbb{C}^n$ be such that $z_j=\operatorname{sgn}(b_{i_0,j})$ for $j=1,2,\dots,n$. Then $|z|=1$ and

$$\begin{aligned}|B\,z| &= \max_{i=1,2,\dots,n}\sum_{j=1}^n |b_{i,j}\,z_j| = \max_{i=1,2,\dots,n}\sum_{j=1}^n |b_{i,j}\ \operatorname{sgn}(b_{i_0,j})|\\ &\le \max_{i=1,2,\dots,n}\sum_{j=1}^n |b_{i,j}| = |B|.\end{aligned}$$

On the other hand, we have

$$|B| = \sum_{j=1}^n |b_{i_0,j}| = \left|\sum_{j=1}^n \operatorname{sgn}(b_{i_0,j})\, b_{i_0,j}\right| \le |B\,z|.$$

Therefore, we can conclude that (2.7) is true.

Now, due to (2.2), there is $w\in\mathbb{C}^n$ such that $z=[I-\Delta^- A(t)]\,w$. Inserting this instead of z into (2.7), we get

$$\begin{aligned}\left|[I-\Delta^- A(t)]^{-1}\right| &= \frac{|[I-\Delta^- A(t)]^{-1}\,[I-\Delta^- A(t)]\,w|}{|[I-\Delta^- A(t)]\,w|}\\ &= \frac{|w|}{|[I-\Delta^- A(t)]\,w|} = \frac{1}{\left|[I-\Delta^- A(t)]\,\left(\frac{w}{|w|}\right)\right|}\\ &\le \sup\left\{\frac{1}{|[I-\Delta^- A(t)]\,x|}\colon\ x\in\mathbb{C}^n,\ |x|=1\right\}.\end{aligned}$$

It follows that

$$\begin{aligned}c_A &\le \sup\left\{\frac{1}{|[I-\Delta^- A(t)]\,x|}\colon\ t\in[a,b],\ x\in\mathbb{C}^n,\ |x|=1\right\}\\ &= \left(\inf\{|[I-\Delta^- A(t)]\,x|\colon\ t\in[a,b],\ x\in\mathbb{C}^n,\ |x|=1\right)^{-1},\end{aligned}$$

i.e. (2.6) is true. This completes the proof. $\square$

The next result on the continuous dependence of solutions of generalized linear differential equations on a parameter generalizes the result due to M. Ashordia [2, Theorem 1]. Unlike [2] and [3], we do not utilize the variation-of-constants formula and therefore we need not assume that, in addition to (2.2), also the condition

$$\det[I+\Delta^+ A(t)] \neq 0 \quad \text{for all} \quad t\in[a,b]$$

is satisfied. Furthermore, both the nonhomogeneous part of the equation and the solutions may be only regulated functions (not necessarily of bounded variation).

2.5. Theorem. *Let* $A,\ A_k \in BV^{n\times n}[a,b]$, $g,\ g_k \in G^n[a,b]$, $\widetilde{x},\ \widetilde{x}_k \in \mathbb{C}^n$ *for* $k\in\mathbb{N}$. *Assume* (1.5), (1.6), (2.2),

$$\lim_{k\to\infty} \|g_k - g\|_\infty = 0 \tag{2.8}$$

and

$$\lim_{k\to\infty} \widetilde{x}_k = \widetilde{x}. \tag{2.9}$$

Then equation (2.1) *has a unique solution* x *on* $[a,b]$. *Furthermore, for each* $k\in\mathbb{N}$ *sufficiently large there exists a unique solution* x_k *on* $[a,b]$ *to the equation*

$$x(t) = \widetilde{x}_k + \int_a^t \mathrm{d}[A_k(s)]\,x(s) + g_k(t) - g_k(a) \tag{2.10}$$

and

$$\lim_{k\to\infty} \|x_k - x\|_\infty = 0. \tag{2.11}$$

Proof. *Step* 1. As in the first part of the proof of [2, Theorem 1], we can show that there is a $k_1\in\mathbb{N}$ such that

$$\det[I-\Delta^- A_k(t)] \neq 0 \quad \text{on} \quad (a,b]$$

holds for all $k\ge k_1$. In particular, (2.10) has a unique solution x_k for $k\ge k_1$.

Step 2. For $k\ge k_1$, put

$$c_{A_k} := \sup\{\big|[I-\Delta^- A_k(t)]^{-1}\big| \colon\ t\in(a,b]\}.$$

Then, by Lemma 2.4, we have

$$\begin{aligned}(c_{A_k})^{-1} &= \inf\left\{\left|\left[I-\Delta^- A_k(t)\right]\, x\right| \colon\, t\in[a,b],\, x\in\mathbb{C}^n,\, |x|=1\right\}\\ &\geq \inf\left\{\left|\left[I-\Delta^- A(t)\right]\, x\right| \colon\, t\in[a,b],\, x\in\mathbb{C}^n,\, |x|=1\right\}\\ &\quad - \sup\left\{\left|\left[\Delta^-(A_k(t)-A(t))\right]\, x\right| \colon\, t\in[a,b],\, x\in\mathbb{C}^n,\, |x|=1\right\}.\end{aligned}$$

By (1.6), we have

$$\Delta^- A_k \rightrightarrows \Delta^- A \quad \text{on} \quad [a,b].$$

Therefore, we can conclude that there is a $k_0\geq k_1$ such that

$$(c_{A_k})^{-1}\geq (c_A)^{-1}-(2\,c_A)^{-1}=(2\,c_A)^{-1} \quad \text{for} \quad k\geq k_0.$$

To summarize,

$$c_{A_k}\leq 2\,c_A<\infty \quad \text{for} \quad k\geq k_0. \tag{2.12}$$

Step 3. Set $w_k=(x_k-g_k)-(x-g)$. Then, for $k\geq k_0$,

$$w_k(t)=\widetilde{w}_k+\int_a^t \mathrm{d}[A_k]\,w_k+h_k(t)-h_k(a) \quad \text{for} \quad t\in[a,b],$$

where

$$h_k(t)=\int_a^t \mathrm{d}[A_k-A]\,(x-g)+\left(\int_a^t \mathrm{d}[A_k]\,g_k-\int_a^t \mathrm{d}[A]\,g\right) \quad \text{for} \quad t\in[a,b]$$

and

$$\widetilde{w}_k=(\widetilde{x}_k-g_k(a))-(\widetilde{x}-g(a))\,.$$

By (2.8) and (2.9) we can see that

$$\lim_{k\to\infty}\widetilde{w}_k=0. \tag{2.13}$$

Furthermore, since $x-g\in BV^n[a,b]$, using (1.6) and Theorem 1.1 we get

$$\int_a^t \mathrm{d}[A_k]\,(x-g)\rightrightarrows\int_a^t \mathrm{d}[A]\,(x-g) \quad \text{on} \quad [a,b].$$

Furthermore, due to (2.8) and Lemma 1.2, we have also

$$\int_a^t \mathrm{d}[A_k]\,g_k\rightrightarrows\int_a^t \mathrm{d}[A]\,g \quad \text{on} \quad [a,b].$$

To summarize,

$$\lim_{k\to\infty} \|h_k\|_\infty = 0. \tag{2.14}$$

On the other hand, applying Theorem 2.2 and taking into account the relation (2.12), we get

$$\|w_k\|_\infty \le 2\, c_A\, (|\widetilde{w}_k| + 2\, \|h_k\|_\infty)\, \exp(2\, c_A\, \alpha^*) \text{ for } k \ge k_0,$$

wherefrom, by virtue of (2.13) and (2.14), the relation

$$\lim_{k\to\infty} \|w_k\|_\infty = 0$$

follows. Finally, having in mind the assumptions (2.8) and (2.9), we conclude that the relation

$$\lim_{k\to\infty} \|x_k - x\|_\infty = 0$$

is true, as well. This completes the proof. □

2.6 . Remark. If (1.6) or (2.8) does not hold, the situation becomes much more difficult, see [6], [7] and [24]. The following two assertions may be deduced from the proof of [7, Theorem 2.2].

2.7 . Lemma. *Let* $A,\ A_k \in BV^{n\times n}[a,b],\quad g,\ g_k \in G^n[a,b],\quad \widetilde{x},\ \widetilde{x}_k \in \mathbb{C}^n$ *for* $k \in \mathbb{N}$. *Assume* (1.5), (2.2), (2.9) *and denote by* x *the solution of* (2.1) *on* $[a,b]$.

Furthermore, let

either

$$A_k \rightrightarrows A \quad \textit{and} \quad g_k \rightrightarrows g \quad \textit{locally on} \quad (a,b], \tag{2.15}$$

$$\begin{cases} \forall\, \varepsilon > 0\ \exists\, \delta > 0 \textit{ such that } \forall\, \tau \in (a, a+\delta)\ \exists\, k_0 \in \mathbb{N} \textit{ such that} \\ \qquad |x_k(\tau) - \widetilde{x}_k - \Delta^+ A(a)\, \widetilde{x} - \Delta^+ g(a)| < \varepsilon \\ \textit{for all } k \ge k_0 \end{cases} \tag{2.16}$$

or

$$A_k \rightrightarrows A \quad \text{and} \quad g_k \rightrightarrows g \quad \text{locally on} \;\; [a,b), \tag{2.17}$$

$$\begin{cases} \forall\, \varepsilon > 0 \; \exists\, \delta > 0 \text{ such that } \; \forall\, \tau \in (b-\delta, b) \; \exists\, k_0 \in \mathbb{N} \;\; \text{such that} \\ \quad |x_k(b) - x_k(\tau) - \Delta^- A(b)\, [I - \Delta^- A(b)]^{-1}\, x(b-) \\ \qquad\qquad\qquad\qquad\qquad -[I - \Delta^- A(b)]^{-1}\, \Delta^- g(b)| < \varepsilon \\ \text{for all } \; k \geq k_0 \end{cases} \tag{2.18}$$

hold.

Then, for each $k \in \mathbb{N}$ sufficiently large, there exists a unique solution x_k on $[a,b]$ to the equation (2.10) ,

$$\lim_{k\to\infty} x_k(t) = x(t) \quad \text{for} \;\; t \in [a,b] \tag{2.19}$$

and $x_k \rightrightarrows x$ locally on $(a,b]$ or $[a,b)$, respectively.

2.8 . Notation. Let $A \in BV^{n\times n}[a,b]$, $g \in G^n[a,b]$ and let

$$D = \{\alpha_0, \alpha_1, \ldots, \alpha_m\} \in \mathfrak{D}[a,b]$$

be a division of $[a,b]$. Then we define

$$A_D(t) = \begin{cases} A(t) \quad \text{if} \;\; t \in D, \\ A(\alpha_{i-1}) + \dfrac{A(\alpha_i) - A(\alpha_{i-1})}{\alpha_i - \alpha_{i-1}}\,(t - \alpha_{i-1}) \\ \qquad\qquad \text{if} \;\; t \in (\alpha_{i-1}, \alpha_i) \;\; \text{for some} \;\; i \in \{1, 2, \ldots, m\}, \end{cases} \tag{2.20}$$

and

$$g_D(t) = \begin{cases} g(t) \quad \text{if} \;\; t \in D, \\ g(\alpha_{i-1}) + \dfrac{g(\alpha_i) - g(\alpha_{i-1})}{\alpha_i - \alpha_{i-1}}\,(t - \alpha_{i-1}) \\ \qquad\qquad \text{if} \;\; t \in (\alpha_{i-1}, \alpha_i) \;\; \text{for some} \;\; i \in \{1, 2, \ldots, m\}. \end{cases} \tag{2.21}$$

We have

2.9. Lemma. *Assume that* $A \in BV^{n\times n}[a,b]$, $g \in G^n[a,b]$. *Let* $D \in \mathfrak{D}[a,b]$, $D = \{\alpha_0, \alpha_1, \dots, \alpha_m\}$, *and let* A_D *and* g_D *be defined by* (2.20) *and* (2.21), *respectively. Then* $A_D \in AC^{n\times n}[a,b]$, $g_D \in AC^n[a,b]$ *and*

$$\operatorname{var}_a^b A_D \le \operatorname{var}_a^b A \quad \textit{and} \quad \|g_D\|_\infty \le \|g\|_\infty.$$

Proof. Obviously, $A_D \in AC^{n\times n}[a,b]$, $g_D \in AC^n[a,b]$.

Furthermore, for each $\ell = 1, 2, \dots, m$ and each $t \in [\alpha_{\ell-1}, \alpha_\ell]$ we have

$$\operatorname{var}_{\alpha_{\ell-1}}^{\alpha_\ell} A_D = |A(\alpha_\ell) - A(\alpha_{\ell-1})| \le \operatorname{var}_{\alpha_{\ell-1}}^{\alpha_\ell} A$$

and

$$\begin{aligned}|g_D(t)| &= |g(\alpha_{\ell-1}) + \frac{g(\alpha_\ell) - g(\alpha_{\ell-1})}{\alpha_\ell - \alpha_{\ell-1}}\,(t - \alpha_{\ell-1})| \\ &= |g(\alpha_{\ell-1})\frac{\alpha_\ell - t}{\alpha_\ell - \alpha_{\ell-1}} + g(\alpha_\ell)\frac{t - \alpha_{\ell-1}}{\alpha_\ell - \alpha_{\ell-1}}| \le \|g\|_\infty.\end{aligned}$$

Therefore,

$$\operatorname{var}_a^b A_D = \sum_{\ell=1}^m \operatorname{var}_{\alpha_{\ell-1}}^{\alpha_\ell} A_D \le \sum_{\ell=1}^m \operatorname{var}_{\alpha_{\ell-1}}^{\alpha_\ell} A = \operatorname{var}_a^b A \ \text{ and } \ \|g_D\|_\infty \le \|g\|_\infty.$$

$\square$

2.10. Theorem. *Assume that* $A \in BV^{n\times n}[a,b] \cap C^{n\times n}[a,b]$ *and* $g \in C^n[a,b]$. *Let* $\widetilde{x}, \widetilde{x}_k \in \mathbb{C}^n$, $k \in \mathbb{N}$, *be such that* (2.9) *holds. Furthermore, let the sequence* $\{D_k\} \subset \mathfrak{D}[a,b]$ *of divisions of the interval* $[a,b]$ *be such that*

$$D_{k+1} \supset D_k \quad \textit{for} \quad k \in \mathbb{N} \quad \textit{and} \quad \lim_{k\to\infty} |D_k| = 0. \tag{2.22}$$

Finally, let the sequences $\{A_k\} \subset AC^{n\times n}[a,b]$, $\{g_k\} \subset AC^n[a,b]$ *be given by*

$$A_k = A_{D_k} \quad \textit{and} \quad g_k = g_{D_k} \quad \textit{for} \quad k \in \mathbb{N}, \tag{2.23}$$

where A_{D_k} *and* g_{D_k} *are defined as in* (2.20) *and* (2.21).

Then the equation (2.1) *has a unique solution* x *on* $[a,b]$. *Furthermore, for each* $k \in \mathbb{N}$, *the equation* (2.10) *has a solution* x_k *on* $[a,b]$ *and* (2.11) *holds.*

Proof. *Step* 1. Since A is uniformly continuous on $[a,b]$, we have:

$$\begin{cases} \text{for each } \ \varepsilon > 0 \text{ there is a } \delta > 0 \text{ such that} \\ |A(t) - A(s)| < \frac{\varepsilon}{2} \\ \text{holds for all } \ t, s \in [a,b] \ \text{ such that } \ |t-s| < \delta\,. \end{cases} \tag{2.24}$$

Let $|D_{k_0}| < \delta$ and let $t \in [a,b]$ be given. Furthermore, let

$$\alpha_{\ell-1}, \alpha_\ell \in \mathcal{D}_{k_0} = \{\alpha_0, \alpha_1, \dots, \alpha_{p_{k_0}}\} \text{ and } t \in [\alpha_{\ell-1}, \alpha_\ell].$$

Then

$$|\alpha_\ell - \alpha_{\ell-1}| < \delta$$

and, according to (2.20) and (2.22)–(2.24), we get for $k \geq k_0$

$$\begin{aligned} |A_k(t) - A(t)| &\leq |A(\alpha_\ell) - A(\alpha_{\ell-1})| \left[\frac{t - \alpha_{\ell-1}}{\alpha_\ell - \alpha_{\ell-1}}\right] + |A(\alpha_{\ell-1}) - A(t)| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

As k_0 was chosen independently of t, we can conclude that (1.6) is true.

Step 2. Analogously we can show that (2.8) is true, as well.

Step 3. By Lemma 2.9, (1.5) holds. Moreover, as A and A_k, $k \in \mathbb{N}$, are continuous, the equations (2.1) and (2.10) have unique solutions by Theorem 2.1 and we can complete the proof using Theorem 2.5. $\square$

3. Approximated solutions

In this section we will continue the consideration of the topics mentioned at the close of the previous section. Our aim is to disclose the relationship between solutions of generalized linear differential equation and limits of solutions of corresponding approximating sequences of linear ordinary differential equations.

We start by introducing the following notations.

3.1. Notation. For given $g \in G^n[a,b]$ and $k \in \mathbb{N}$, we denote

$$\mathfrak{U}_k(g) = \left\{ t \in [a,b] : |\Delta^+ g(t)| \geq \frac{1}{k} \text{ or } |\Delta^- g(t)| \geq \frac{1}{k} \right\} \quad \text{and} \quad \mathfrak{U}(g) = \bigcup_{k=1}^{\infty} \mathfrak{U}_k(g).$$

Analogous symbols are used also for matrix valued function.

Of course, $\mathfrak{U}(g)$ is the set of points of discontinuity of the function g in $[a,b]$.

3.2. Definition. Let $A \in BV^{n\times n}[a,b]$, $g \in G^n[a,b]$ and let $P_k \in \mathfrak{D}[a,b]$ be a sequence of divisions of $[a,b]$ such that

$$|P_k| = \frac{1}{2^k} \quad \text{for} \quad k \in \mathbb{N}. \tag{3.1}$$

We say that the sequence $\{A_k,\, g_k\} \subset AC^{n\times n}[a,b] \times AC^n[a,b]$ is a *piecewise linear approximation* ($\mathcal{PL}$-approximation) of $(A,\, g)$ if there exists a sequence $\{D_k\} \subset \mathfrak{D}[a,b]$ of divisions of the interval $[a,b]$ such that

$$D_k \supset P_k \cup \mathfrak{U}_k(A) \cup \mathfrak{U}_k(g) \quad \text{for} \quad k \in \mathbb{N} \tag{3.2}$$

and A_k, g_k are for $k \in \mathbb{N}$ defined by (2.20), (2.21) and (2.23).

3.3. Remark. Let $\{A_k,\, g_k\}$ be a $\mathcal{PL}$-approximation of $(A,\, g)$, $\widetilde{x}_k = \widetilde{x} \in \mathbb{C}^n$ for $k \in \mathbb{N}$ and let x_k, $k \in \mathbb{N}$, be the corresponding solutions of (2.10).

By Lemma 2.9, we have

$$\operatorname{var}_a^b A_k \le \operatorname{var}_a^b A \quad \text{and} \quad \|g_k\|_\infty \le \|g\|_\infty \quad \text{for all} \quad k \in \mathbb{N}.$$

Furthermore, as A_k are continuous, due to (2.3), we have also $c_{A_k} = 1$ for all $k \in \mathbb{N}$. Hence, Corollary 2.3 yields

$$\operatorname{var}_a^b(x_k - g_k) \le (\operatorname{var}_a^b A)\,(|\widetilde{x}| + 2\,\|g\|_\infty)\,\exp(\operatorname{var}_a^b A) < \infty \quad \text{for all} \quad k \in \mathbb{N}$$

and, by Helly's Theorem, there is a subsequence $\{k_n\}$ of $\mathbb{N}$ and $y \in G^n[a,b]$ and such that

$$\lim_{n\to\infty}(x_{k_n}(t) - g_{k_n}(t)) = y(t) - g(t) \quad \text{for each} \quad t \in [a,b].$$

In particular,

$$\lim_{m\to\infty} x_{k_m}(t) = y(t) \quad \text{for all} \quad t \in [a,b] \text{ such that } \lim_{n\to\infty} g_{k_n}(t) = g(t).$$

Notice that if the set $\mathfrak{U}(g;[a,b])$ has at most a finite number of elements, then

$$\lim_{k\to\infty} g_k(t) = g(t) \quad \text{for all} \quad t \in [a,b].$$

3.4 . Definition. Let $A \in BV^{n\times n}[a,b]$, $g \in G^n[a,b]$ and $\widetilde{x} \in \mathbb{C}^n$. We say that $x^*\colon [a,b] \to \mathbb{C}^n$ is an *approximated solution* to equation (2.1) on the interval $[a,b]$ if there is a $\mathcal{PL}$-approximation $\{A_k, g_k\}$ of (A, g) such that

$$\lim_{k\to\infty} x_k(t) = x^*(t) \quad \text{for} \quad t \in [a,b] \tag{3.3}$$

holds for solutions $x_k,\ k \in \mathbb{N},$ of the corresponding approximating initial value problems

$$x_k' = A_k'(t)\, x_k + g_k'(t), \quad x_k(a) = \widetilde{x}, \quad k \in \mathbb{N}. \tag{3.4}$$

3.5. Remark. Notice that, using the language of Definitions 3.2 and 3.4, we can translate Theorem 2.10 into the following form:

Assume that $A \in BV^{n\times n}[a,b] \cap C^{n\times n}[a,b]$ *and* $g \in C^n[a,b]$. *Then, the equation* (2.1) *has a unique approximated solution* x^* *on* $[a,b]$ *and* x^* *coincides on* $[a,b]$ *with the solution of* (2.1).

In the rest of this paper we consider the case when the set $\mathfrak{U}(A) \cup \mathfrak{U}(g)$ of discontinuities of the coefficients $A,\ g$ is non empty. We will start with the simplest case $\mathfrak{U}(A) \cup \mathfrak{U}(g) = \{b\}$.

The next natural assertion will be useful for our purposes and, in our opinion, it is not available in literature.

3.6. Lemma. *Let* $A \in BV^{n\times n}[a,b]$. *Then*

$$\left\{ \begin{aligned} &\lim_{s\to t-} \frac{1}{t-s} \left(\int_s^t \exp\left([A(t)-A(s)] \frac{t-r}{t-s} \right) \mathrm{d}r \right) \\ &\quad = \int_0^1 \exp\big(\Delta^- A(t)\,(1-\sigma)\big)\, \mathrm{d}\sigma \quad \textit{if} \ \ t \in (a,b] \end{aligned} \right. \tag{3.5}$$

and

$$\left\{ \begin{aligned} &\lim_{s\to t+} \frac{1}{s-t} \left(\int_t^s \exp\left([A(s)-A(t)] \frac{s-r}{s-t} \right) \mathrm{d}r \right) \\ &\quad = \int_0^1 \exp\big(\Delta^+ A(t)\,(1-\sigma)\big)\, \mathrm{d}\sigma \quad \textit{if} \ \ t \in (a,b] \end{aligned} \right. \tag{3.6}$$

Proof. (i) Let $t \in (a,b]$, $s \in [a,t)$ and let $\varepsilon > 0$ be given. Then there is a $\delta > 0$ such that

$$|A(t-) - A(s)| < \varepsilon \quad \text{whenever} \quad t-s < \delta.$$

Now, taking into account that

$$|\exp(C)-\exp(D)|\le|C-D|\,\exp(|C|+|D|)\quad\text{holds for all}\ \ C,D\in\mathbb{C}^{n\times n},$$

we get

$$\begin{aligned}
&\left|\frac{1}{t-s}\int_s^t\left[\exp\left([A(t)-A(s)]\frac{t-r}{t-s}\right)-\exp\left(\Delta^-A(t)\frac{t-r}{t-s}\right)\right]\,\mathrm{d}r\right|\\
&\quad\le\frac{1}{t-s}\,|A(t-)-A(s)|\int_s^t\exp\big(|\Delta^-A(t)|\big)\,\mathrm{d}r\\
&\quad=|A(t-)-A(s)|\,\exp\big(|\Delta^-A(t)|\big)\le\varepsilon\,\exp\big(|\Delta^-A(t)|\big)
\end{aligned}$$

for $t-s<\delta$. Therefore,

$$\begin{aligned}
&\lim_{s\to t-}\frac{1}{t-s}\left(\int_s^t\exp\left([A(t)-A(s)]\frac{t-r}{t-s}\right)\,\mathrm{d}r\right)\\
&\quad=\lim_{s\to t-}\frac{1}{t-s}\left(\int_s^t\exp\left(\Delta^-A(t)\frac{t-r}{t-s}\right)\,\mathrm{d}r\right)\quad\text{if}\ \ t\in(a,b].
\end{aligned}$$

Now, it is easy to see that the substitution $\sigma=1-\frac{t-r}{t-s}$ in the second integral yields (3.5).

(ii) Similarly we would justify the relation (3.6). □

3.7 . Lemma. *Assume that* $A\in BV^{n\times n}[a,b]$ *and* $g\in G^n[a,b]$ *are continuous on* $[a,b)$, $\widetilde{x},\ \widetilde{x}_k\in\mathbb{C}^n$ *and* (2.9) *is true. Denote by* x *the solution of* (2.1) *on* $[a,b)$. *Furthermore, let* $\{A_k,\,g_k\}$ *of* $(A,\,g)$ *be an arbitrary* $\mathcal{PL}$ *- approximation of* $\{A,\,g\}$ *and let* x_k *be the corresponding solutions of problems* (3.4).

Put

$$x^*(t)=\begin{cases}x(t) & \text{if}\ \ t\in[a,b),\\ v(1) & \text{if}\ \ t=b,\end{cases}\tag{3.7}$$

where v *is a solution on* $[0,1]$ *of the initial value problem*

$$v'=[\Delta^-A(b)]\,v+[\Delta^-g(b)],\quad v(0)=x(b-).\tag{3.8}$$

Then

$$\lim_{k\to\infty}x_k(t)=x^*(t)\ \ \textit{for all}\ \ t\in[a,b]\ \ \textit{and}\ \ x_k\rightrightarrows x^*\ \ \textit{locally on}\ [a,b).$$

Proof. *Step* 1. Let $\{A_k,\, g_k\}$ be a $\mathcal{PL}$-approximation of $\{A,\, g\}$ and let $\{D_k\}$ be the corresponding sequence of divisions of $[a,b]$ fulfilling (3.1) and (3.2). Notice that, under our assumptions, $D_k = P_k$ for $k \in \mathbb{N}$. For $k \in \mathbb{N}$, put

$$\tau_k = \max\{t \in P_k\colon\, t < b\}.$$

By (3.1) we have $b - \frac{b-a}{2^k} \le \tau_k < b$ for $k \in \mathbb{N}$, and hence

$$\lim_{k\to\infty} \tau_k = b. \tag{3.9}$$

Now, for $k \in \mathbb{N}$ and $t \in [a,b]$, let us define

$$\widetilde{A}_k(t) = \begin{cases} A_k(t) & \text{if } \ t \in [a,\tau_k], \\ A(\tau_k) + \dfrac{A(b-) - A(\tau_k)}{b-\tau_k}\,(t-\tau_k) & \text{if } \ t \in (\tau_k, b], \end{cases}$$

$$\widetilde{g}_k(t) = \begin{cases} g_k(t) & \text{if } \ t \in [a,\tau_k], \\ g(\tau_k) + \dfrac{g(b-) - g(\tau_k)}{b-\tau_k}\,(t-\tau_k) & \text{if } \ t \in (\tau_k, b]. \end{cases}$$

Furthermore, let

$$\widetilde{A}(t) = \begin{cases} A(t) & \text{if } \ t \in [a,b), \\ A(b-) & \text{if } \ t = b, \end{cases} \qquad \widetilde{g}(t) = \begin{cases} g(t) & \text{if } \ t \in [a,b), \\ g(b-) & \text{if } \ t = b. \end{cases}$$

We have $\widetilde{A}_k \in AC^{n\times n}[a,b]$, $\widetilde{g}_k \in AC^n[a,b]$ for $k \in \mathbb{N}$, $\widetilde{A} \in BV^{n\times n}[a,b] \cap C^{n\times n}[a,b]$ and $\widetilde{g} \in C^n[a,b]$.

Consider problems (2.1), (3.4),

$$y_k' = \widetilde{A}_k'(t)\, y_k + \widetilde{g}_k'(t), \quad y_k(a) = \widetilde{x}_k, \quad k \in \mathbb{N}, \tag{3.10}$$

and

$$y(t) = \widetilde{x} + \int_a^t \mathrm{d}[\widetilde{A}]\, y + \widetilde{g}(t) - \widetilde{g}(a). \tag{3.11}$$

Let $\{x_k\}$ and $\{y_k\}$ be the sequences of solutions on $[a,b]$ of problems (3.4) and (3.10), respectively. We can see that, for each $k \in \mathbb{N}$, y_k coincides with x_k on $[a,\tau_k]$. Furthermore, by Theorem 2.1, equation (3.11) possesses a unique solution y on $[a,b]$, y is continuous on $[a,b]$ and $y=x$ on $[a,b)$. It's easy to see that $\widetilde{A}_k \rightrightarrows \widetilde{A}$ and $\widetilde{g}_k \rightrightarrows \widetilde{g}$ on $[a,b]$. Moreover, by Lemma 2.9,

$$\operatorname{var}_a^b \widetilde{A}_k \leq \operatorname{var}_a^b \widetilde{A} \leq \operatorname{var}_a^b A < \infty \quad \text{for all} \quad k \in \mathbb{N}. \tag{3.12}$$

Therefore, by Theorem 2.5, we get

$$y_k \rightrightarrows y \quad \text{on} \quad [a,b]. \tag{3.13}$$

In particular, since $x_k = y_k$ on $[a,\tau_k]$ and $\lim_{k\to\infty} \tau_k = b$ (see (3.9)), we have

$$\lim_{k\to\infty} x_k(t) = \lim_{k\to\infty} y_k(t) = y(t) = x(t) = x^*(t) \quad \text{for} \quad t \in [a,b). \tag{3.14}$$

Step 2. Next, we will prove that

$$\lim_{k\to\infty} x_k(\tau_k) = y(b). \tag{3.15}$$

Indeed, let $\varepsilon > 0$ be given and let $\delta > 0$ be such that

$$|y(t) - y(b)| < \frac{\varepsilon}{2} \quad \text{for} \quad t \in [b-\delta, b]$$

Further, by (3.13), there is a $k_0 \in \mathbb{N}$ such that

$$\tau_k \in [b-\delta, b) \quad \text{and} \quad \|y_k - y\|_\infty < \frac{\varepsilon}{2} \quad \text{whenever} \quad k \geq k_0.$$

Consequently,

$$\begin{aligned} |x_k(\tau_k) - y(b)| &\leq |x_k(\tau_k) - y(\tau_k)| + |y(\tau_k) - y(b)| \\ &= |y_k(\tau_k) - y(\tau_k)| + |y(\tau_k) - y(b)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

holds for $k \geq k_0$. This completes the proof of (3.15).

Step 3. On the intervals $[\tau_k, b]$, the equations from (3.4) reduce to the equations with constant coefficients

$$x_k' = B_k\, x_k + e_k, \tag{3.16}$$

where

$$B_k = \frac{A(b) - A(\tau_k)}{b - \tau_k} \quad \text{and} \quad e_k = \frac{g(b) - g(\tau_k)}{b - \tau_k}.$$

Their solutions x_k are on $[\tau_k, b]$ given by

$$x_k(t) = \exp\left(B_k\,(t - \tau_k)\right) x_k(\tau_k) + \left(\int_{\tau_k}^{t} \exp\left(B_k\,(t - r)\right) \mathrm{d}r\right) e_k.$$

In particular,

$$\begin{aligned} x_k(b) &= \exp\left(A(b) - A(\tau_k)\right) x_k(\tau_k) \\ &\quad + \frac{1}{b - \tau_k} \left(\int_{\tau_k}^{b} \exp\left([A(b) - A(\tau_k)]\, \frac{b - r}{b - \tau_k}\right) \mathrm{d}r\right) [g(b) - g(\tau_k)]. \end{aligned}$$

By Lemma 3.6, we have

$$\begin{aligned} &\lim_{k\to\infty} \frac{1}{b - \tau_k} \left(\int_{\tau_k}^{b} \exp\left([A(b) - A(\tau_k)]\frac{b - r}{b - \tau_k}\right) \mathrm{d}r\right) [g(b) - g(\tau_k)] \\ &\quad = \left(\int_0^1 \exp\left(\Delta^- A(b)\,(1 - s)\right) \mathrm{d}s\right) \Delta^- g(b). \end{aligned}$$

To summarize, due to (3.15), we have

$$\lim_{k\to\infty} x_k(b) = \exp\left(\Delta^- A(b)\right) y(b) + \left(\int_0^1 \exp\left(\Delta^- A(b)\,(1 - s)\right) \mathrm{d}s\right) \Delta^- g(b),$$

i.e.

$$\lim_{k\to\infty} x_k(b) = v(1) = x^*(b). \tag{3.17}$$

where v is a solution to (3.8) on $[0, 1]$. With respect to (3.7) and (3.14), this completes the proof. □

3.8. Remark. In particular, x^* defined in the previous lemma is an approximated solution of (2.1). Moreover, as it does not depend upon the choice of the approximating sequence $\{A_k,\, g_k\}$, we can see that it is the unique approximated solution of (2.1).

The following assertion concerns a situation only seemingly symmetric to that treated by Lemma 3.7. Similarly, to the proof of Lemma 3.7, we will deal with the modified equation

$$y(t) = \widetilde{x} + \int_a^t \mathrm{d}[\widetilde{A}]\, y + \widetilde{g}(t) - \widetilde{g}(a), \tag{3.18}$$

where $\widetilde{y} \in \mathbb{C}^n$ and

$$\widetilde{A}(t) = \begin{cases} A(a+) & \text{if } t = a, \\ A(t) & \text{if } t \in (a,b], \end{cases} \qquad \widetilde{g}(t) = \begin{cases} g(a+) + \widetilde{x} - \widetilde{y} & \text{if } t = a, \\ g(t) & \text{if } t \in (a,b] \end{cases} \tag{3.19}$$

with $\widetilde{y} \in \mathbb{C}^n$ to be determined later.

3.9 . Lemma. *Assume that* $A \in BV^{n\times n}[a,b]$ *and* $g \in G^n[a,b]$ *are continuous on* $(a,b]$, $\widetilde{x}, \widetilde{x}_k \in \mathbb{C}^n$ *and* (2.9) *holds. Furthermore, let* $\{A_k, g_k\}$ *of* (A, g) *be an arbitrary* $\mathcal{PL}$ *- approximation of* $\{A, g\}$ *and let* x_k *be the corresponding solutions of* (3.4) *Finally, let* w *be a solution of the initial value problem*

$$w' = [\Delta^+ A(a)]\, w + [\Delta^+ g(a)], \quad w(0) = \widetilde{x} \tag{3.20}$$

and let y *be a solution on* $[a,b]$ *of the equation* (3.18), *where* $\widetilde{y} = w(1)$.

Then

$$\lim_{k\to\infty} x_k(t) = y(t) \ \textit{ for all } t \in [a,b] \ \textit{ and } \ x_k \rightrightarrows y \ \textit{ locally on } (a,b]. \tag{3.21}$$

Proof. Let $\{D_k\}$ be the sequence of divisions of $[a,b]$ fulfilling (2.20) and (2.21). As in the previous proof, $D_k = P_k$ for $k \in \mathbb{N}$. Furthermore, let $\tau_k = \min\{t \in P_k\colon\ t > a\}$. By (2.22), we have $a + \frac{b-a}{2^k} \ge \tau_k > a$ for $k \in \mathbb{N}$, and hence $\lim_{k\to\infty} \tau_k = a$. We have

$$A_k \rightrightarrows \widetilde{A} \quad \text{and} \quad g_k \rightrightarrows \widetilde{g} \quad \text{locally on } (a,b].$$

Furthermore, similarly as in *Step* 1 of the proof of Lemma 3.7, inequalities (3.12) are true. Therefore, by Lemma 2.7, to prove (3.21) it suffices to show that

$$\begin{cases} \forall\, \varepsilon > 0\ \exists\, \delta > 0 \text{ such that } \ \forall\, \tau \in (a, a+\delta)\ \exists\, k_0 \in \mathbb{N} \ \text{ such that} \\ \qquad |x_k(\tau) - \widetilde{y}| < \varepsilon \\ \text{for all } \ k \ge k_0. \end{cases} \tag{3.22}$$

holds. We shall do it in 3 steps.

Step 1. We will show that

$$\lim_{k\to\infty} x_k(\tau_k) = \widetilde{y}. \tag{3.23}$$

Indeed, on the intervals $[a,\tau_k]$ the equations from (3.4) reduce to equations (3.16) with the coefficients

$$B_k = \frac{A(\tau_k) - A(a)}{\tau_k - a}, \quad e_k = \frac{g(\tau_k) - g(a)}{\tau_k - a}.$$

Their solutions x_k are on $[a,\tau_k]$ given by

$$x_k(t) = \exp(B_k\,(t-a))\,\widetilde{x} + \left(\int_a^t \exp\left(B_k\,(t-r)\right) \mathrm{d}r\right) e_k.$$

In particular,

$$\begin{aligned} x_k(\tau_k) &= \exp\left(A(\tau_k) - A(a)\right)\widetilde{x} && (3.24)\\ &\quad + \frac{1}{\tau_k - a}\left(\int_a^{\tau_k} \exp\left([A(\tau_k) - A(a)]\,\frac{\tau_k - r}{\tau_k - a}\right) \mathrm{d}r\right)[g(\tau_k) - g(\tau_k)]. && (3.25)\end{aligned}$$

By Lemma 3.6, we have

$$\begin{aligned}\lim_{k\to\infty} &\frac{1}{\tau_k - a}\left(\int_a^{\tau_k} \exp\left([A(\tau_k) - A(a)]\,\frac{\tau_k - r}{\tau_k - a}\right) \mathrm{d}r\right)[g(\tau_k) - g(a)]\\ &= \left(\int_0^1 \exp(\Delta^+ A(a)\,(1-s))\,\mathrm{d}s\right)\Delta^+ g(a)\end{aligned}$$

wherefrom (3.23) follows easily.

Step 2. We will show that

$$\begin{cases} \text{for all } \ \varepsilon > 0 \quad \text{there exist } \ \delta > 0 \text{ such that} \\ \qquad |x_k(t) - x_k(s)| < \varepsilon \\ \text{for all } \ k \in \mathbb{N} \ \text{ and } \ t,s \in (a, a+\delta). \end{cases} \tag{3.26}$$

holds. It what follows we denote by X_k the fundamental matrix solution of (3.4) on $[a,b]$ and $\Phi_k(t,s) = X_k(t)\, X_k^{-1}(s)$ for $t,s \in [a,b]$. Then

$$\Phi_k(t,s) = I + \int_s^t \mathrm{d}[A_k(r)]\, \Phi_k(r,s)$$

and

$$x_k(t) = \Phi_k(t,s)\, x_k(s) + \int_s^t \Phi_k(t,r)\, \mathrm{d}[g_k(r)]$$

for $t,s \in [a,b]$.

Let $t,s \in [a,b]$ be given such that $a < s \le t \le b$. With respect to Theorem 2.2, Corollary 2.3 and Lemma 2.9 (with $[s,t]$ on the place of $[a,b]$), we have

$$\begin{aligned} |\Phi_k(t,s)| &\le \exp(\mathrm{var}_s^t\, A_k) \le \exp(\mathrm{var}_s^t\, A) = \exp(\mathrm{var}_s^t\, \widetilde{A}) \\ &\le \exp\left(\mathrm{var}_a^t\, \widetilde{A}\right), \end{aligned}$$

$$\mathrm{var}_s^t\, \Phi_k(.,s) \le (\mathrm{var}_s^t\, A_k)\ \exp(\mathrm{var}_s^t\, A_k) \le (\mathrm{var}_a^t\, \widetilde{A})\ \exp(\mathrm{var}_a^t\, \widetilde{A})$$

and

$$|x_k(t)| \le (|\widetilde{x}_k| + 2\, \|g\|_\infty)\ \exp(\mathrm{var}_a^b\, \widetilde{A}).$$

Consequently,

$$\begin{aligned} |x_k(t) - x_k(s)| &\le |\Phi_k(t,s) - \Phi_k(s,s)|\, |x_k(s)| + 2\, (\mathrm{var}_s^t\, \Phi_k(.,s))\, \|g\|_\infty \\ &\le (\mathrm{var}_s^t\, \Phi_k(.,s))\, (|x_k(s)| + 2\, \|g\|_\infty) \le (\mathrm{var}_a^t\, \widetilde{A})\, \varkappa, \end{aligned}$$

where, due to (2.9),

$$\varkappa = \exp(\mathrm{var}_a^t\, \widetilde{A})\, (\sup_{k \in \mathbb{N}} |\widetilde{x}_k| + 2\, \|g\|_\infty)\ \exp(\mathrm{var}_a^b\, \widetilde{A}) < \infty.$$

Having in mind that $\widetilde{A}$ is right-continuous at a and hence

$$\lim_{t \to a+} \mathrm{var}_a^t\, \widetilde{A} = 0,$$

we can conclude that (3.26) is true.

Step 3. Now,we will complete the proof of (3.22). Indeed, let $\varepsilon > 0$ be given. Then, by *Step1* 1, we can choose $k_1 \in \mathbb{N}$ so that

$$|x_k(\tau_k) - \widetilde{y}| < \frac{\varepsilon}{2} \quad \text{whenever} \quad k \ge k_1. \tag{3.27}$$

Furthermore, by *Step* 2, there is $\delta > 0$ such that

$$\begin{cases} |x_k(t) - x_k(s)| < \frac{\varepsilon}{2} \\ \quad \text{holds for all } \ k \in \mathbb{N} \ \text{ and } \ t, s \in (a, a+\delta). \end{cases} \tag{3.28}$$

Then, if $\tau \in (a, a+\delta)$ and $k_0 \geq k_1$ is such that $\tau_k \leq \tau$. Then, by virtue of (3.27) and (3.28), we get

$$|x_k(\tau) - \widetilde{y}| \leq |x_k(\tau) - x_k(\tau_k)| + |x_k(\tau_k) - \widetilde{y}| < \varepsilon$$

and this completes the proof. □

3.10 . Remark. Let us notice that if $a < c < b$ and the functions x_1^* and x_2^* are respectively approximated solutions to

$$x(t) = \widetilde{x}_1 + \int_a^t \mathrm{d}[A]\, x + g(t) - g(a), t \in [a, c]$$

and

$$x(t) = \widetilde{x}_2 + \int_c^t \mathrm{d}[A]\, x + g(t) - g(c), t \in [c, b],$$

where $\widetilde{x}_2 = x_1^*(c)$, then the function

$$x^*(t) = \begin{cases} x_1^*(t) & \text{if } \ t \in [a, c], \\ x_2^*(t) & \text{if } \ t \in (c, b] \end{cases}$$

is a $\mathcal{PL}$ - approximated solution to (2.1).

3.11 . Theorem. *Assume that* $A \in BV^{n \times n}[a, b]$, $g \in G^n[a, b]$ *and*

$$\mathfrak{U}(A) \cup \mathfrak{U}(g) = \{s_1, s_2, \ldots, s_m\} \subset [a, b].$$

Then, for each $\widetilde{x} \in \mathbb{C}^n$, *there is exactly one approximated solution* x^* *of equation* (2.1) *on* $[a, b]$.

Moreover,

$$x^*(t) = w_\ell(1) + \int_{s_\ell}^t \mathrm{d}[\widetilde{A}_\ell]\, x^* + \widetilde{g}_\ell(t) - \widetilde{g}_\ell(s_\ell) \ \textit{for} \ \ t \in [s_\ell, s_{\ell+1}), \ \ell \in \mathbb{N} \cap [0, m],$$

$$x^*(t) = v_\ell(1) \qquad \textit{for} \ \ t = s_\ell, \ \ell \in \mathbb{N} \cap [1, m+1],$$

where $s_0 = a,\ s_{m+1} = b,\ w_0(1) = \widetilde{x}$ *and, for* $\ell \in \mathbb{N} \cap [0, m]$,

$$\widetilde{A}_\ell(t) = \begin{cases} A(s_\ell+) & \text{if } t = s_\ell, \\ A(t) & \text{if } t \in (s_\ell, s_{\ell+1}], \end{cases} \qquad \widetilde{g}_\ell(t) = \begin{cases} g(s_\ell+) & \text{if } t = s_\ell, \\ g(t) & \text{if } t \in (s_\ell, s_{\ell+1}] \end{cases}$$

and v_ℓ *and* w_ℓ *respectively denote the solutions on* $[0, 1]$ *of initial value problems*

$$v_\ell' = [\Delta^- A(s_\ell)]\, v_\ell + [\Delta^- g(s_\ell)], \quad v_\ell(0) = x^*(s_\ell-)$$

and

$$w_\ell' = [\Delta^+ A(s_\ell)]\, w_\ell + [\Delta^+ g(s_\ell)], \quad w_\ell(0) = x^*(s_\ell).$$

Proof. Having in mind Remark 3.10, we deduce the assertion of Theorem 3.11 by a successive use of Lemmas 3.7 and 3.9. To this aim it is sufficient to choose a division $D = \{\alpha_0, \alpha_1, \dots, \alpha_r\}$ of $[a, b]$ such that for each subinterval $[\alpha_{k-1}, \alpha_k]$, $k = 1, 2, \dots, r$, either the assumptions of Lemma 3.7 or the assumptions of Lemma 3.9 are satisfied with α_{k-1} in place of a and α_k in place of b. □

3.12 . Remark. The convergence effects appearing in Theorem 3.11 are related to the notion of *R-emphatic convergence* introduced by Kurzweil in [11]. Further references to this notion are e.g. Fraňková [5], [6], Halas [7], Schwabik [19], Tvrdý [25] and the unpublished thesis by Pelant [16]. A different point of view can be found in the papers [13] and [14] by Meng Gang and Zhang Meirong dealing with measure differential analogues of Sturm-Liouville equations and, in particular, describing the weak* continuous dependence of related Dirichlet or Neumann eigenvalues on a potential.

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