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Milan Tvrdý

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BOUNDARY VALUE PROBLEMS FOR LINEAR GENERALIZED DIFFERENTIAL EQUATIONS AND THEIR ADJOINTS

MILAN TVRDÝ, Praha

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0. INTRODUCTION

In [20] the boundary value problem

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)\,x + \int_a^b [\mathrm{d}_s G(t, s)]\,x(s) + f(t)\,, \tag{0,1}$$

$$\int_a^b [\mathrm{d}L(s)]\,x(s) = l \quad (-\infty < a < b < \infty) \tag{0,2}$$

was studied. The equation (0,1) was considered in the sense of Caratheodory. Solutions were sought as absolutely continuous functions on $[a, b]$, the $n \times n$-matrix function $A(t)$ and the n-vector function $f(t)$ were assumed to be L-integrable on $[a, b]$ and the $n \times n$-matrix function $G(t, s)$ was assumed to be of bounded variation on $[a, b]$ in s for any $t \in [a, b]$, while

$$\int_a^b (\|G(t, a)\| + \mathrm{var}_a^b\, G(t, .))^2\, \mathrm{d}t < \infty\,.$$

Further, the $m \times n$-matrix function $L(t)$ was assumed to be of bounded variation on $[a, b]$. Among other things, the adjoint to this problem was defined in such a way that the Fredholm alternative for the existence of a solution to (0, 1), (0, 2) held. Further a relation between the number of linearly independent solutions to the given and the adjoint homogeneous problems was established. Let for $t \in [a, b]$

$$C(t) = G(t, a+) - G(t, a)\,, \quad D(t) = G(t, b) - G(t, b-)\,,$$

$$G_0(t, s) = \begin{cases} G(t, a+) & \text{for} \quad s = a\,, \\ G(t, s) & \text{for} \quad a < s < b\,, \\ G(t, b-) & \text{for} \quad s = b \end{cases}$$

*) §4 was modified in accordance with the more recent results of Št. Schwabik in [16].

and

$$M = L(a+) - L(a), \quad N = L(b) - L(b-), \quad L_0(t) = \begin{cases} L(a+) & \text{for} \quad t = a, \\ L(t) & \text{for} \quad a < t < b, \\ L(b-) & \text{for} \quad t = b. \end{cases}$$

Then the adjoint boundary value problem to (0,1), (0,2) is to find a row n-vector function y of bounded variation on $[a, b]$ and a row m-vector λ such that

$$(0,1^*) \qquad y(t) = y(a) - \int_a^t y(s)\, A(s)\, \mathrm{d}s - \lambda(L_0(t) - L_0(a)) - \\ - \int_a^b y(s)\, (G_0(s, t) - G_0(s, a))\, \mathrm{d}s \quad \text{on} \quad [a, b],$$

$$(0,2^*) \qquad y(a) + \lambda M + \int_a^b y(s)\, C(s)\, \mathrm{d}s = 0, \quad y(b) - \lambda N - \int_a^b y(s)\, D(s)\, \mathrm{d}s = 0.$$

The given problem has a solution iff

$$\int_a^b y(s)\, f(s)\, \mathrm{d}s = \lambda l$$

for any solution (y, λ) of (0,1*), (0,2*). If the given problem is homogeneous and has exactly r linearly independent solutions on $[a, b]$, then (0,1*), (0,2*) has exactly $r + m - n$ linearly independent solutions on $[a, b]$. (The problem (0,1*), (0,2*) is the well-posed adjoint of the problem (0,1), (0,2).)

The equation (0,1*) is a generalized (ordinary) differential equation in the sense of J. Kurzweil ([8]) and it can be written in the form

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = D\left[-y\, B(t) - \lambda\, L_0(t) - \int_a^b y(s)\, G_0(s, t)\, \mathrm{d}s\right],$$

where $B(t)$ is the Lebesgue indefinite integral of $A(t)$. This suggests that generally we can expect the symmetry of the given boundary value problem and of its adjoint only if the original problem involves the generalized differential equation. Generalized differential equations were introduced by J. Kurzweil in [8]. Solutions to them are generally defined by means of the generalized Perron (Kurzweil) integral. In the special linear case

$$(0,3) \qquad \frac{\mathrm{d}x}{\mathrm{d}\tau} = D[A(t)\, x + C(t)\, x(a) + D(t)\, x(b) + \int_a^b [\mathrm{d}_s G(t, s)]\, x(s) + f(t)]$$

in which we are interested, the Perron-Stieltjes integral is fully sufficient. An n-vector function x is said to be a solution to (0,3) on $[a, b]$ if, given $t \in [a, b]$, there exist the Perron-Stieltjes integrals

$$\int_a^t [dA(s)]\, x(s) \quad \text{and} \quad \int_a^b [d_s G(t, s)]\, x(s)$$

and it holds

$$x(t) = x(a) + \int_a^t [dA(s)]\, x(s) + (C(t) - C(a))\, x(a) +$$

$$+ (D(t) - D(a))\, x(b) + \int_a^b [d_s(G(t, s) - G(a, s))]\, x(s) + f(t) - f(a)\,.$$

Generalized differential equations were studied by several authors (cf. e.g. [8]–[10], [12], [13] and [22]). In particular Št. Schwabik [13] treated the generalized linear differential equation

$$\frac{dx}{d\tau} = D[A(t)\, x + f(t)] \tag{0,4}$$

with A and f left-continuous on $[a, b]$ and of bounded variation on $[a, b]$. For the similar "differentio-Stieltjes-integral" equation

$$x(t) = x(a) + Y\int_a^t [dA(s)]\, x(s) + f(t) - f(a) \tag{0,5}$$

involving the Young integral T. H. Hildebrandt obtained in [6] the fundamental results (as e.g. existence and uniqueness of a solution in the class of bounded functions, fundamental matrix to the corresponding homogeneous equation, variation of-constants formula, relation to the adjoint equation) without supposing the continuity of A and f from the left or from the right. Under the assumptions guaranteeing the existence of a solution to (0,5) we may apply all these results to (0,4) (cf. [17]).

Another good reason for the investigation of boundary value problems for generalized differential equations is that by means of the transformation due to W. R. Jones [7] the integral boundary value problem (0,1), (0,2) is transferred to the simple two-point problem for the equation (0,3) with A, G, f appropriately defined. Even (cf. [19]) if $G(t, s) = G_1(t)\, G_2(s)$ we can convert (0,1), (0,2) into a two-point boundary value problem for an equation of the type (0,4). Finally, let us notice that also some practical problems (as e.g. those quoted in [18]) lead to boundary value problems for generalized differential equations.

Boundary value problems for generalized ordinary differential equations were for the first time mentioned in the book [1] by F. V. Atkinson (chapter XI). In [19] to the boundary value problems

$$\frac{dx}{d\tau} = D[A(t)\, x + f(t)]\,, \tag{0,4}$$

$$M\, x(a) + N\, x(b) = l \tag{0,6}$$

and

$$(0,7)\quad \frac{\mathrm{d}x}{\mathrm{d}\tau} = D\left[A(t)\,x + (H_1(t)\,Cx(a) + Dx(b)) + H_2(t)\int_a^b [\mathrm{d}G(s)]\,x(s) + f(t)\right],$$

$$(0,8)\qquad M\,x(a) + N\,x(b) + \int_a^b [\mathrm{d}L_1(s)]\,x(s) = l_1\,,\quad \int_a^b [\mathrm{d}L_2(s)]\,x(s) = l_2$$

the well-posed adjoints were found under the assumptions

(i) A, f are of bounded variation and regular on $[a, b]$ ($\Delta^+A(t) = A(t+) - A(t) = A(t) - A(t-) = \Delta^-A(t)$ for $a < t < b$, $\Delta^+A(a) = \Delta^-A(b) = 0$ and analogously for f), $(\Delta^+A(t))^2 = 0$ for $a < t < b$, M and N are constant matrices and l is a constant vector

and

(ii) A is absolutely continuous on $[a, b]$; H_1, H_2, G, L_1, L_2 and f are matrix functions of the appropriate types which are of bounded variation and regular on $[a, b]$; M, N, C, D are constant matrices such that the rank of the matrix $[M, N; C, D]$ is maximal; l_1 and l_2 are constant vectors,

respectively. (In [19] the Hildebrandt symbolics from [6] was used.) The latter problem covers that of R. N. BRYAN from [2].

In this paper we continue the investigations from [19]. We treat again the problems (0,4), (0,6) and (0,7), (0,8) and, moreover, the general problem (0,3), (0,2). For each of them we find the well-posed adjoint and prove the corresponding "alternative theorems". The first one is treated in §2. Instead of (i) we assume only A, f to be of bounded variation on $[a, b]$, $(\Delta^+A(a))^2 = (\Delta^-A(b))^2 = 0$, $(\Delta^+A(t))^2 = (\Delta^-A(t))^2$ and $\det[I - (\Delta^+A(t))^2] \neq 0$ for $a < t < b$. (I is the identity matrix.) This and the generalized Jones transformation allow us to obtain in §3 the desired theorems for the problem (0,7), (0,8) under less restrictive hypotheses than (ii). It suffices if A is of bounded variation and regular on $[a, b]$ and such that $\det[I - (\Delta^+A(t))^2] \neq 0$ for $a < t < b$. Furthermore, the matrices C, D need not be complementary to M, N and f need not be regular. Finally, §5 is devoted to the problem (0,3), (0,2), where A, C, D and f are of bounded variation and regular on $[a, b]$; $(\Delta^+A(t))^2 = 0$ for $a < t < b$; $G(t, s)$ is of bounded variation and regular on $[a, b]$ in each variable for the other variable fixed and of bounded two-dimensional variation on $[a, b] \times [a, b]$ (in the sense of Vitali); L is of bounded variation on $[a, b]$ and l is a constant vector. It is essential for our purpose to use the "alternative lemmas" for the "integro-algebraic" system

$$h(t) + P(t)\,c + \int_a^b K(t, s)\,\mathrm{d}h(s) = u(t) \quad \text{on} \quad [a, b],$$

$$Qc + \int_a^b R(s)\,\mathrm{d}h(s) = v\,.$$

These lemmas are stated in §4. They follows from the recent results of Št. Schwabik [16] concerning linear operator equations in Banach spaces. The theorems of §5 allow us to generalize the corresponding results from [20] for the classical boundary value problem (0,1), (0,2) to the case of $\|G(t, a)\| + \operatorname{var}_a^b G(t, .)$ L-integrable on $[a, b]$, but not necessarily square integrable.

The boundary value problems (0,4), (0,6) and (0,7), (0,2) are also studied in [21]. The assumptions are similar as here. Only instead of the regularity of the occurring functions we assume these functions to be left continuous.

1. PRELIMINARIES

1,1. Notation. Let $-\infty < a < b < \infty$. $J = [a, b]$ denotes the closed interval $a \leqq t \leqq b$ and $J^0 = (a, b)$ its interior. $\mathscr{R}_{p,q}$ denotes the space of all real $p \times q$-matrices equipped with the norm

$$A = (a_{i,j}) \in \mathscr{R}_{p,q} \to \|A\| = \max_{i=1,\ldots,p} \sum_{j=1}^{q} |a_{i,j}| .$$

($\mathscr{R}_{p,1} = \mathscr{R}_p$, $\mathscr{R}_{1,q} = \mathscr{R}_q^*$.) Given an arbitrary $A \in \mathscr{R}_{p,q}$, $\operatorname{rank}(A)$ denotes its rank and A' its transpose. If $p = q$ and $\det A \neq 0$, then A^{-1} denotes the inverse of A. The identity $p \times p$-matrix is denoted by I_p and the zero $p \times q$-matrix is denoted by $O_{p,q}$. (Usually we write briefly I or O.) Let us notice that for $x = (x_1, x_2, \ldots, x_p)' \in \mathscr{R}_p$

$$\|x'\| = \sum_{i=1}^{p} |x_i| \leqq p\|x\| = p \max_{i=1,\ldots,p} |x_i| .$$

$\mathscr{BV}_{p,q}$ denotes the Banach space (B-space) of all $p \times q$-matrix functions of bounded variation on J equipped with the norm

$$F \in \mathscr{BV}_{p,q} \to \|F\|_{\mathscr{BV}} = \|F(a)\| + \operatorname{var}_a^b F .$$

($\mathscr{BV}_{p,1} = \mathscr{BV}_p$, $\mathscr{BV}_{1,q} = \mathscr{BV}_q^*$.) (If no confusion may arise, indices are omitted.) Given $F \in \mathscr{BV}$, F_c denotes the continuous part of F, $F_b = F - F_c$ and, further, for $t \in J$

$$\Delta^+F(t) = F(t+) - F(t) , \quad \Delta^-F(t) = F(t) - F(t-) , \quad \Delta F(t) = F(t+) - F(t-) .$$

$$(\Delta^-F(a) = \Delta^+F(b) = 0 , \quad \Delta F(a) = \Delta^+F(a) , \quad \Delta F(b) = \Delta^-F(b).)$$

The function $F \in \mathscr{BV}$ is said to be regular on J if $\Delta^+F(t) = \Delta^-F(t)$ on J. (Thus $\Delta^+F(a) = \Delta^-F(b) = 0$.) The space of all $p \times q$-matrix functions of bounded variation and regular on J is denoted by $\mathscr{BV}_{p,q}^R$ (or briefly $\mathscr{BV}^R$). The space of all $p \times q$-matrix functions F of bounded variation on J such that $\Delta^+F(a) = \Delta^-F(b) = 0$ and $F(t+) = F(t-) = F(a)$ for all $t \in (a, b)$ is denoted by $\mathscr{N}_{p,q}$ (or briefly $\mathscr{N}$, $\mathscr{N}_{p,1} = \mathscr{N}_p$). (Evidently if $F \in \mathscr{N}$, then $F_c(t) = F(a)$ on J.)

Given an arbitrary matrix function $G(t, s)$ defined for $t, s \in J$ and given an arbitrary net subdivision $\sigma = \{a = t_0 < t_1 < \ldots < t_p = b,\ a = s_0 < s_1 < \ldots < s_q = b\}$ of $J \times J$, let us denote

$$\Delta\Delta_{j,k} G = \|G(t_{j+1}, s_{k+1}) - G(t_j, s_{k+1}) - G(t_{j+1}, s_k) + G(t_j, s_k)\| ,$$

$$w(G; \sigma) = \sum_{j=1}^{p} \sum_{k=1}^{q} \Delta\Delta_{j,k} G .$$

Then

$$w_{J \times J}(G) = \sup_\sigma w(G; \sigma)$$

denotes the Vitali two-dimensional variation of G on $J \times J$. (See [5], III §4.) The matrix function G is said to be of strongly bounded variation on $J \times J$ if

$$w_{J \times J}(G) + \operatorname{var}_a^b G(a, \cdot) + \operatorname{var}_a^b G(\cdot, a) < \infty .$$

The space of all $n \times n$-matrix functions of strongly bounded variation on $J \times J$ is denoted by $\mathscr{SBV}$. If $G \in \mathscr{SBV}$, then

$$\operatorname{var}_a^b G(\cdot, s) \leqq w_{J \times J}(G) + \operatorname{var}_a^b G(\cdot, a) \quad \text{for all} \quad s \in J ,$$
$$\operatorname{var}_a^b G(t, \cdot) \leqq w_{J \times J}(G) + \operatorname{var}_a^b G(a, \cdot) \quad \text{for all} \quad t \in J .$$

(See [5] or [14].) For $G \in \mathscr{SBV}$, we denote

$$\Delta_1^+ G(t, s) = G(t+, s) - G(t, s) , \qquad \Delta_2^+ G(t, s) = G(t, s+) - G(t, s) ,$$
$$\Delta_1^- G(t, s) = G(t, s) - G(t-, s) , \quad \Delta_2^- G(t, s) = G(t, s) - G(t, s-) ,$$
$$\Delta_1 G(t, s) = G(t+, s) - G(t-, s) , \quad \Delta_2 G(t, s) = G(t, s+) - G(t, s-) .$$

Let us notice that if $G(t, s) = G_1(t)\, G_2(s)$ with $G_1, G_2 \in \mathscr{BV}$, then $G \in \mathscr{SBV}$ (cf. [5]).

1,2. Integrals. Generally, $\int_c^d F\, \mathrm{d}G$ stands for the Perron-Stieltjes integral. Since we deal only with functions of bounded variation, the σ-Young integral (Y-integral) is fully sufficient for our purposes. For the definition of this integral see [5] II 19,3. Let us recall here only that if $F, G \in BV$, then the expression

$$(\mathrm{R, S}) \int_a^b F(t)\, \mathrm{d}G_c(t) + \sum_{t \in J} F(t)\, \Delta G(t)$$

(with the ordinary Riemann-Stieltjes integral) has a sense and its value defines the Y-integral $\int_a^b F(t)\, \mathrm{d}G(t)$. The relationship between various types of the Stieltjes integral was discussed in detail in [15]. The basic properties (substitution theorem, integration-by-parts formula, indefinite integrals etc.) of the Y-integral have been formulated and proved in [5]. We give here only some more complicated or less known formulas. (The list of properties of the Y-integral is given also in [19].)

Let F, G, $H(t, \cdot)$ and $H(\cdot, s) \in \mathscr{BV}$ for all $t, s \in J$, let $H(t, s)$ be bounded on $J \times J$ and let $K \in \mathscr{SBV}$. Then

$$(1,2,1) \quad \int_a^b F(t)\, \mathrm{d}G(t) + \int_a^b [\mathrm{d}F(t)]\, G(t) =$$
$$= F(b)\, G(b) - F(a)\, G(a) + \sum_{t \in J} [\Delta^- F(t)\, \Delta^- G(t) - \Delta^+ F(t)\, \Delta^+ G(t)]\,;$$

$$(1,2,2) \quad \int_a^b [\mathrm{d}F(t)] \left(\int_a^t H(t, s)\, \mathrm{d}G(s)\right) = \int_a^b \left(\int_s^b [\mathrm{d}F(t)]\, H(t, s)\right) \mathrm{d}G(s) +$$
$$+ \sum_{t \in J} [\Delta^- F(t)\, H(t, t)\, \Delta^- G(t) - \Delta^+ F(t)\, H(t, t)\, \Delta^+ G(t)]\,;$$

$$(1,2,3) \quad \Phi(t) = \int_a^t F(s)\, \mathrm{d}G(s) \in \mathscr{BV} \quad \text{and} \quad \Delta^+ \Phi(t) = F(t)\, \Delta^+ G(t)\,,$$
$$\Delta^- \Phi(t) = F(t)\, \Delta^- G(t) \quad \text{for all} \quad t \in J\,;$$

$$(1,2,4) \quad \int_a^b F(t) \left[\mathrm{d}_t \int_a^b K(t, s)\, \mathrm{d}G(s)\right] = \int_a^b \left(\int_a^b F(s)\, [\mathrm{d}_s K(s, t)]\right) \mathrm{d}G(t)\,;$$

$$(1,2,5) \quad \Psi(t) = \int_a^b G(s)\, \mathrm{d}_s K(s, t) \in \mathscr{BV}\,,$$

while

$$\Psi(t+) = \int_a^b G(s)\, \mathrm{d}_s K(s, t+) \quad \text{and} \quad \Psi(t-) = \int_a^b G(s)\, \mathrm{d}_s K(s, t-)\,;$$

$$(1,2,6) \quad \int_c^b F(t) \left[\mathrm{d}_t \int_d^b \mathrm{d}_s K(t, s)\, G(s)\right] = \int_d^b \left[\mathrm{d}_s \int_c^b F(t)\, \mathrm{d}_s K(t, s)\right] G(s)$$

for arbitrary $c, d \in J$.
(The statements (1,2,4)–(1,2,6) have been proved in [14]. The assertion (1,2,4) follows also from the integration-by-parts formula (1,2,1) and from the Dirichlet formula (1,2,2).)

1,3. Generalized linear ordinary differential equations. The fundamental results concerning generalized linear ordinary differential equations

$$(1,3,1) \qquad \frac{\mathrm{d}x}{\mathrm{d}\tau} = D[A(t)\, x + f(t)]$$

and

$$(1,3,2) \qquad \frac{\mathrm{d}x}{\mathrm{d}\tau} = D[A(t)\, x]$$

have been summarized in [17]. (Essentially they were proved by T. H. Hildebrandt in [6].) Let us recall here only the basic facts.

Let

$$(1,3,3) \qquad A \in \mathscr{BV}_{n,n}, \quad f \in \mathscr{BV}_{n,1} \quad \text{and}$$
$$\det\left[I - (\Delta^- A(t))^2\right] . \det\left[I - (\Delta^+ A(t))^2\right] \neq 0 \quad \text{for any} \quad t \in J .$$

Then there exist unique $n \times n$-matrix functions $U(t, s)$ and $V(t, s)$ defined on $J \times J$ and such that

$$(1,3,4) \qquad U(t, s) = I + \int_s^t [\mathrm{d}A(\tau)]\, U(\tau, s)$$

and

$$(1,3,5) \qquad V(t, s) = I + \int_s^t V(t, \sigma)\, \mathrm{d}A(\sigma)$$

for all $t, s \in J$. The functions U and V have the following properties.

(i) There exists $M < \infty$ such that

$$(1,3,6) \qquad \operatorname{var}_a^b U(\cdot, s) + \operatorname{var}_a^b V(\cdot, s) + \operatorname{var}_a^b U(t, \cdot) + \operatorname{var}_a^b V(t, \cdot) \leqq M$$
$$\text{for all} \quad t, s \in J .$$

(ii) Given arbitrary $t, s, r \in J$,

$$(1,3,7) \qquad U(t, r)\, U(r, s) = U(t, s), \quad U(t, t) = I,$$
$$V(t, r)\, V(r, s) = V(t, s), \quad V(t, t) = I .$$

(iii) If $A \in \mathscr{BV}^R$, then for an arbitrary $s \in J$ or $t \in J$, $U(\cdot, s)$ or $V(t, \cdot)$ is regular on J as well.

The notation U and V for the *evolution matrix functions* to (1,3,2) and to

$$(1,3,8) \qquad \frac{\mathrm{d}y'}{\mathrm{d}\tau} = D[-y'\, A(t)],$$

respectively, are kept throughout the paper. Let us notice that owing to (1,3,6) and (1,3,7) U and $V \in \mathscr{SBV}$.

Given an arbitrary $c \in \mathscr{R}_n$, the unique solution x to (1,3,1) on J such that $x(a) = c$ is given by

$$(1,3,9) \qquad x(t) = U(t, a)\,(c - f(a)) + f(t) - \int_a^t [d_s U(t, s)]\, f(s) \quad \text{on} \quad J .$$

If besides (1,3,3) also $(\Delta^+ A(t))^2 \equiv (\Delta^- A(t))^2$ on J, then the corresponding evolution functions U and V fulfil the relations

$$(1,3,10) \qquad V(t, s) = U(t, s) + V(t, s)\,(\Delta^+ A(s))^2 - (\Delta^+ A(t))^2\, U(t, s) \quad \text{for} \quad t, s \in J .$$

In particular,

$$(1,3,11) \qquad V(b, s) = U(b, s) + V(b, s)(\Delta^+ A(s))^2 \quad (V(b, a) = U(b, a))$$

and therefore $V(b, \cdot) - U(b, \cdot) \in \mathcal{N}$. Furthermore, if $(\Delta^+ A(t))^2 \equiv (\Delta^- A(t))^2 \equiv 0$ on J, then the evolution functions U and V coincide on J and if, moreover $A \in \mathscr{BV}^R$, then U is regular on J in both variables. For $f \in \mathscr{BV}^R$, the variation-of-constants formula (1,3,9) becomes

$$x(t) = U(t, a)\, c + \int_a^t U(t, s)\, \mathrm{d}f(s)\,.$$

1,4. Adjoint matrices. Let $M, N \in \mathscr{R}_{m,n}$ be such that rank $(M, N) = m$. Then there exist matrices $P, Q \in \mathscr{R}_{n,2n-m}$ and $\tilde{P}, \tilde{Q} \in \mathscr{R}_{n,m}$ such that

$$\text{(i)} \qquad \det \begin{pmatrix} \tilde{P}, & P \\ \tilde{Q}, & Q \end{pmatrix} \neq 0\,;$$

$$\text{(ii)} \qquad -MP + NQ = O_{n,2n-m} \quad \text{and} \quad -M\tilde{P} + N\tilde{Q} = I_m\,;$$

(iii) $P_1, Q_1 \in \mathscr{R}_{n,2n-m}$ and $\tilde{P}_1, \tilde{Q}_1 \in \mathscr{R}_{n,m}$ being arbitrary other matrices fulfilling (i) and (ii), there exist a regular $(2n - m) \times (2n - m)$-matrix R and a $(2n - m) \times$ $\times$ m-matrix S such that

$$P_1 = PR\,, \quad Q_1 = QR\,, \quad \tilde{P}_1 = \tilde{P} + PS\,, \quad \tilde{Q}_1 = \tilde{Q} + QS\,.$$

(For some more details see e.g. [3] XI or [19].)

The matrices P, Q and $\tilde{P}$, $\tilde{Q}$ fulfilling (i) and (ii) are called respectively *adjoint and complementary adjoint matrices to M, N.*

2. TWO-POINT BOUNDARY VALUE PROBLEM FOR GENERALIZED LINEAR ORDINARY DIFFERENTIAL EQUATION

Let $A \in \mathscr{BV}_{n,n}$, $f \in \mathscr{BV}_n$, M and $N \in \mathscr{R}_{m,n}$ and $l \in \mathscr{R}_m$. In this paragraph we consider the boundary value problem

$$(2,1) \qquad \frac{\mathrm{d}x}{\mathrm{d}\tau} = D[A(t)\, x + f(t)]\,,$$

$$(2,2) \qquad M\, x(a) + N\, x(b) = l\,.$$

Without any loss of generality we may assume $0 \leqq m \leqq 2n$ and rank $(M, N) = m$. Throughout this paragraph we assume

$$(2,3) \qquad \det\left[I - (\Delta^+ A(t))^2\right] \neq 0 \quad \text{and} \quad (\Delta^+ A(t))^2 = (\Delta^- A(t))^2 \quad \text{for any} \quad t \in J\,.$$

Lemma 2,1. *An n-vector function x is a solution to* (2,1), (2,2) *iff*

$$(2,4) \qquad x(t) = U(t, a)(c - f(a)) + f(t) - \int_a^t [d_s U(t, s)] f(s) \quad on \quad J,$$

where the n-vector c fulfils the linear system

$$(2,5) \quad [M + NU(b, a)]\, c = N[V(b, a) f(a) - f(b) + \int_a^b [d_s V(b, s)] f(s)] + l\,.$$

Proof. Putting (1,3,9) into (2,2) we get that x is a solution to our boundary value problem iff it is given by (2,4), where c satisfies the system

$$[M + NU(b, a)]\, c = N[U(b, a) f(a) - f(b) + \int_a^b [d_s V(b, s)] f(s)] + l\,.$$

Since $U(b, \cdot) - V(b, \cdot) \in \mathcal{N}$ (cf. (1,3,11)),

$$\int_a^b [d_s V(b, s)] f(s) = \int_a^b [d_s U(b, s)] f(s)$$

wherefrom the lemma immediately follows.

By the well-known theorem from linear algebra we obtain.

Corollary 2,1. *The homogeneous boundary value problem*

$$(2,6) \qquad \frac{dx}{d\tau} = D[A(t)\, x]\,,$$

$$(2,7) \qquad M\, x(a) + N\, x(b) = l$$

has exactly $r = n$*-rank* $([M + NU(b, a)]$ *linearly independent solutions on* J.

Definition 2,1. Let P, Q be adjoint matrices to M, N, then the boundary value problem

$$(2,8) \qquad \frac{dy'}{d\tau} = D[-y'\, A(t)]\,,$$

$$(2,9) \qquad y'(a) P + y'(b)\, Q = 0$$

is said to be adjoint to (2,1), (2,2).

Remark 2,1. The condition (2,9) does not depend on the choice of the adjoint matrices (cf. Sec. 1,4 (iii)).

In some cases the following equivalent formulation of the adjoint boundary value problem appears to be more appropriate. To find an n-vector function y and an m-vector λ such that y is a solution to (2,8) on J and

$$(2,10) \qquad y'(a) + \lambda' M = 0\,, \quad y'(b) - \lambda' N = 0\,.$$

If the couple (y, λ) is a solution to (2,8), (2,10), then according to the definition of the matrices P, Q we have

$$y'(a) P + y'(b) Q = \lambda'(-MP + NQ) = 0 .$$

On the other hand, if y is a solution to (2,8), (2,9), then since the rank of the matrix $(-M, N)$ is maximal, there exists by Sec. 1,4 (ii) $\lambda \in \mathcal{R}_m$ such that $y'(a) = -\lambda' M$ and $y'(b) = \lambda' N$. Thus, both the formulations are really equivalent.

Lemma 2,2. *An n-vector function y is a solution to* (2,8), (2,9) *iff*

$$y'(t) = \gamma' NV(b, t) \quad on \quad J , \tag{2,11}$$

where $\gamma \in \mathcal{R}_m$ *satisfies the linear system*

$$\gamma'[M + NU(b, a)] = 0 . \tag{2,12}$$

Proof. By the definition of the evolution matrix function V, an n-vector function y is a solution to (2,8) on J iff

$$y'(t) = \delta' V(b, t) \quad \text{on} \quad J , \tag{2,13}$$

where δ is some constant n-vector. Let (y, λ) be a solution to (2,8), (2,10). Since by (2,13) $y'(b) = \delta' = \lambda' N$ and $y'(a) = \delta' V(b, a) = \lambda' NV(b, a) = -\lambda' M$, we have $y'(t) = \lambda' NV(b, t)$ on J and $\lambda'[M + NU(b, a)] = \lambda'[M - M] = 0$.

Corollary 2,2. *The adjoint boundary value problem* (2,8), (2,9) *has exactly* $r^* = m$*-rank* $(M + NU(b, a))$ *linearly independent solutions on J.*

Proof. By Lemma 2,2 the adjoint problem has at most $r^* = m$-rank $(M + NU(b, a))$ linearly independent on J solutions

$$y_j'(t) = \gamma_j' NV(b, t) \quad (j = 1, 2, \ldots, r^*) ,$$

where γ_j are linearly independent solutions of (2,12). Let these y_j be linearly dependent on J. Denoting by Γ the $r^* \times m$-matrix formed by the rows γ_j', we get that for some nonzero $k \in \mathcal{R}_{r^*}$ $k'\Gamma NV(b, a) = k'\Gamma NU(b, a) = 0$, while $\Gamma[M + NU(b, a)] = 0$. Since rank $(M, N) = m$ and by (1,3,7) det $U(b, a) \neq 0$, it follows that $k'\Gamma(M, N) = 0$ and $k'\Gamma = 0$, which contradicts the definition of Γ.

Theorem 2,1. *Let P, Q and* $\tilde{P}$, $\tilde{Q}$ *be respectively the adjoint and the complementary adjoint matrices to M, N. Then the given boundary value problem* (2,1), (2,2) *fulfilling* (2,3) *has a solution iff*

$$y'(b) f(b) - y'(a) f(a) - \int_a^b [\mathrm{d}y'(s)] f(s) = (y'(a) \tilde{P} + y'(b) \tilde{Q}) l \tag{2,14}$$

for any solution y of the adjoint problem (2,8), (2,9).

Proof. The system (2,5) has a solution iff

$$\gamma' l = \gamma' N \left\{ f(b) - V(b, a) f(a) - \int_a^b [\mathrm{d}_s V(b, s)] \, f(s) \right\}$$

for any solution γ of (2,12). Given $\gamma \in \mathscr{R}_m$, let us put $y'_\gamma(t) = \gamma' N V(b, t)$ on J. Then according to Lemma 2,1, the problem (2,1), (2,2) has a solution iff

$$\gamma' l = y'_\gamma(b) f(b) - y'_\gamma(a) f(a) - \int_a^b [\mathrm{d} y'_\gamma(s)] \, f(s)$$

for any solution γ of (2,12). For any such γ we have

$$y'_\gamma(a) \tilde{P} + y'_\gamma(b) \tilde{Q} = \gamma'[N V(b, a) \tilde{P} + N \tilde{Q}] = \gamma'[-M \tilde{P} + N \tilde{Q}] = \gamma' .$$

This together with Lemma 2,2 completes the proof.

Corollary 2,3. *Let $A \in \mathscr{BV}^R_{n,n}$, $\det [I - (\Delta^+ A(t))^2] \neq 0$ on J and $f \in \mathscr{BV}^R_n$. Then the problem* (2,1), (2,2) *has a solution iff*

$$\int_a^b y'(s) \, \mathrm{d} f(s) = (y'(a) \tilde{P} + y'(b) \tilde{Q}) \, l$$

for any solution y of the adjoint problem (2,8), (2,9).

(Since any solution y of (2,8) is regular on J (cf. Sec. 1,3 (iii)), the proof follows from (2,14) by integration-by-parts formula (1,2,1).)

Remark 2,2. The condition (2,14) is independent of the choice of the matrices P, Q, $\tilde{P}$, $\tilde{Q}$ (cf. Sec. 1,4 (iii)).

From Lemmas 2,1 and 2,2 we can, analogously as in Theorem 2,1, derive the necessary and sufficient condition for the existence of a solution to (2,1), (2,2) in the following form:

Theorem 2,2. *The problem* (2,1), (2,2) *fulfilling* (2,3) *has a solution iff*

$$y'(b) f(b) - y'(a) f(a) - \int_a^b [\mathrm{d} y'(s)] f(s) = \lambda' l$$

for any solution (y, λ) of the adjoint problem (2,8), (2,10).

Corollaries 2,1 and 2,2 yield immediately the following

Theorem 2,3. *Let the homogeneous boundary value problem* (2,6), (2,7) *have exactly r linearly independent solutions on J, then the adjoint problem* (2,8), (2,9) *has exactly $r^* = r + m - n$ linearly independent solutions on J.*

Analogously as Theorem 2,2 we could prove the following

Theorem 2,4. *Let* (2,3) *hold and let* $g \in \mathscr{BV}_n$ *and* $p, q \in \mathscr{R}_n$. *Then the boundary value problem to find an n-vector function y and an m-vector* λ *in such a way that y is a solution on J to the equation*

$$\frac{\mathrm{d}y'}{\mathrm{d}\tau} = D[-y' A(t) + g'(t)]$$

and

$$y'(a) + \lambda' M = p', \quad y'(b) - \lambda' N = q'$$

has a solution iff

$$g'(b)\, x(b) - g'(a)\, x(a) - \int_a^b g'(s)\, \mathrm{d}x(s) = q'\, x(b) - p'\, x(a)$$

for any solution x of the homogeneous problem (2,6), (2,7).

3. INTEGRAL BOUNDARY VALUE PROBLEM FOR GENERALIZED LINEAR INTEGRODIFFERENTIAL EQUATION WITH A DEGENERATE KERNEL

Let

(3,1) $\quad A \in \mathscr{BV}^R_{n,n}, \quad \det(I - (\Delta^+ A(t))^2) \neq 0$ on J; $\quad C$ and $D \in \mathscr{BV}^R_{n,n}$;

$\quad H \in \mathscr{BV}^R_{n,p}, \quad K \in \mathscr{BV}_{p,n}; \quad f \in \mathscr{BV}_n; \quad L_1 \in \mathscr{BV}_{m_1,n}, \quad L_2 \in \mathscr{BV}_{m_2,n}$;

$\quad M$ and $N \in \mathscr{R}_{m_1,n}; \quad l_1 \in \mathscr{R}_{m_1}$ and $l_2 \in \mathscr{R}_{m_2}$.

We are looking for a solution x of the equation

$$\text{(3,2)} \quad \frac{\mathrm{d}x}{\mathrm{d}\tau} = D\left[A(t)\, x + C(t)\, x(a) + D(t)\, x(b) + H(t) \int_a^b [\mathrm{d}K(s)]\, x(s) + f(t)\right]$$

on J which is of bounded variation on J and such that

$$\text{(3,3)} \qquad M\, x(a) + N\, x(b) + \int_a^b [\mathrm{d}L_1(s)]\, x(s) = l_1,$$

$$\int_a^b [\mathrm{d}L_2(s)]\, x(s) = l_2.$$

Without any loss of generality we may assume

(3,4) $\quad K, L_1$ and $L_2 \in \mathscr{BV}^R; \quad 0 \leqq m_1 \leqq 2n$ and $\operatorname{rank}(M, N) = m_1$.

(Given $F \in \mathscr{BV}$ with $\Delta^+ F(a) = \Delta^- F(b) = 0$, there exists $F_1 \in \mathscr{BV}^R$ such that $F - F_1 \in \mathscr{N}$.)

Analogously as in [19] we put for $t \in J$

$$(3,5) \qquad q(t) = \int_a^t [\mathrm{d}L_1(s)]\, x(s)\,, \quad r(t) = \int_a^t [\mathrm{d}L_2(s)]\, x(s)\,, \quad u(t) = x(a)\,,$$

$$v(t) = x(b)\,, \qquad w(t) = \int_a^t [\mathrm{d}K(s)]\, x(s)\,, \quad z(t) = w(b)\,.$$

It follows that

$$M\, x(a) + N\, x(b) + q(b) = l_1\,, \quad r(b) = l_2\,, \quad q(a) = 0\,, \quad r(a) = 0\,, \quad u(a) = x(a)\,,$$
$$v(a) = x(b)\,, \quad w(a) = 0\,, \quad z(a) = w(b)\,.$$

Let us denote for $t \in J$

$$\mathscr{A}(t) = \begin{bmatrix} A(t) & 0 & 0 & C(t) & D(t) & 0 & H(t) \\ L_1(t) & 0 & 0 & 0 & 0 & 0 & 0 \\ L_2(t) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ K(t) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathscr{M} = \begin{bmatrix} M & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{m_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{m_2} & 0 & 0 & 0 & 0 \\ I_n & 0 & 0 & -I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -I_n & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_p & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_p \end{bmatrix},$$

$$\mathscr{N} = \begin{bmatrix} N & I_{m_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{m_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ I_n & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -I_p & 0 \end{bmatrix},$$

$$\varphi(t) = \begin{bmatrix} f(t) \\ O_{m+2n+2p,1} \end{bmatrix}, \qquad \varkappa = \begin{bmatrix} l_1 \\ l_2 \\ O_{m+2n+2p,1} \end{bmatrix},$$

$$m = m_1 + m_2\,, \quad \mu = 2m + 2n + 2p\,, \quad \nu = m + 3n + 2p\,.$$

Then $\mathscr{A} \in \mathscr{BV}^R_{\nu,\nu}$, $\det(I - (\Delta^+\mathscr{A}(t))^2) \neq 0$ on J, $\varphi \in \mathscr{BV}_\nu$, $\mathscr{M}$ and $\mathscr{N} \in \mathscr{R}_{\mu,\nu}$, $\operatorname{rank}(\mathscr{M}, \mathscr{N}) = \mu$, $\varkappa \in \mathscr{R}_\mu$ and the problem (3,2), (3,3) is equivalent to the two-point problem

$$\text{(3,6)} \qquad \frac{\mathrm{d}\xi}{\mathrm{d}\tau} = D[\mathscr{A}(t)\,\xi + \varphi(t)]\,,$$

$$\text{(3,7)} \qquad \mathscr{M}\,\xi(a) + \mathscr{N}\,\xi(b) = \varkappa$$

in the following sense. If x is a solution to (3,2), (3,3) and if the functions q, r, u, v, w, z are given by (3,5), then the ν – vector function $\xi = (x', q', r', u', v', w', z')'$ is a solution to (3,6), (3,7). On the other hand, if $\xi = (x', q', r', u', v', w', z')'$ is a solution to (3,6), (3,7), then the n-vector function x formed by its first n components is a solution to (3,2), (3,3) and the other components q, r, u, v, w and z fulfil (3.5).

By Theorem 2,2, the problem (3,6), (3,7) has a solution iff

$$\text{(3,8)} \qquad \eta'(b)\,\varphi(b) - \eta'(a)\,\varphi(a) - \int_a^b [\mathrm{d}\eta'(s)]\,\varphi(s) = \lambda'\varkappa$$

for any solution (η, λ) of the problem

$$\text{(3,9)} \qquad \frac{\mathrm{d}\eta'}{\mathrm{d}\tau} = D[-\eta'\,\mathscr{A}(t)]\,,$$

$$\text{(3,10)} \qquad \eta'(a) + \lambda'\mathscr{M} = 0\,, \quad \eta'(b) - \lambda'\mathscr{N} = 0\,.$$

According to the definition of $\mathscr{A}, \mathscr{M}, \mathscr{N}, \varphi$ and $\varkappa$, the problem (3,9), (3,10) is equivalent (in the same sense as (3,2), (3,3) is equivalent to (3,6), (3,7)) to the problem of finding $y \in \mathscr{BV}_n$, $\lambda_1 \in \mathscr{R}_{m_1}$ and $\lambda_2 \in \mathscr{R}_{m_2}$ such that y is a solution on J of the equation

$$\text{(3,11)} \qquad \frac{\mathrm{d}y'}{\mathrm{d}\tau} = D\left[-y'\,A(t) - \lambda_1'\,L_1(t) - \lambda_2'\,L_2(t) - \int_a^b y'(s)\,\mathrm{d}H(s)\,K(t)\right],$$

and

$$\text{(3,12)} \qquad y'(a) + \lambda_1'M + \int_a^b y'(s)\,\mathrm{d}C(s) = 0\,, \quad y'(b) - \lambda_1'N - \int_a^b y'(s)\,\mathrm{d}D(s) = 0\,.$$

Thus we have

Theorem 3,1. *The boundary value problem* (3,2), (3,3) *fulfilling* (3,1) *and* (3,4) *has a solution iff*

$$\text{(3,13)} \qquad y'(b)\,f(b) - y'(a)\,f(a) - \int_a^b [\mathrm{d}y'(s)]\,f(s) = \lambda_1'l_1 + \lambda_2'l_2$$

for any solution $(y, \lambda_1, \lambda_2)$ *of the adjoint problem* (3,11), (3,12).

Remark 3,1. Let us notice that the equation (3,2) admits in our case only solutions with bounded variation on J. This follows from the relationship between the equations (3,2) and (3,6). Symmetrically we can show that if $g \in \mathscr{BV}_n$ and $p, q \in \mathscr{R}_n$, then the problem of finding an n-vector function y, $\lambda_1 \in \mathscr{R}_{m_1}$ and $\lambda_2 \in \mathscr{R}_{m2}$ such that y is a solution on J of

$$(3,14) \qquad \frac{dy'}{d\tau} = D\left[-y' A(t) - \lambda_1' L_1(t) - \lambda_2' L_2(t) - \int_a^b y'(s)\, dH(s)\, K(t) + g'(t)\right]$$

and

$$(3,15) \qquad y'(a) + \lambda_1' M + \int_a^b y'(s)\, dC(s) = p', \quad y'(b) - \lambda_1' N - \int_a^b y'(s)\, dD(s) = q'$$

admits only such solutions $(y, \lambda_1, \lambda_2)$ that $y \in \mathscr{BV}_n$. Moreover, the following theorem is true.

Theorem 3,2. *Let* (3,1) *and* (3,4) *hold. Let* $g \in \mathscr{BV}_n$ *and* $p, q \in \mathscr{R}_n$. *Then the boundary value problem* (3,14), (3,15) *has a solution iff*

$$g'(b)\, x(b) - g'(a)\, x(a) - \int_a^b g'(s)\, dx(s) = q'\, x(b) - p'\, x(a)$$

for any solution x of the homogeneous problem.

$$(3,16) \qquad \frac{dx}{d\tau} = D\left[A(t)\, x + C(t)\, x(a) + D(t)\, x(b) + H(t) \int_a^b [dK(s)]\, x(s)\right],$$

$$(3,17) \qquad M\, x(a) + N\, x(b) + \int_a^b [dL_1(s)]\, x(s) = 0\,,$$

$$\int_a^b [dL_2(s)]\, x(s) = 0\,.$$

Remark 3,2. Let

$$C(t)\, x(a) + D(t)\, x(b) = F(t)\,(C\, x(a) + D\, x(b)) \quad \text{on} \quad J\,,$$

where $F \in \mathscr{BV}^R_{n,q}$ and $C, D \in \mathscr{R}_{q,n}$ are such that

$$\text{rank}\begin{bmatrix} M, & N \\ C, & D \end{bmatrix} = m_1 + q\,.$$

Let us choose $(2n - m_1 - q) \times n$-matrices $\tilde{C}, \tilde{D}$ arbitrarily in such a way that

$$\operatorname{rank} \begin{bmatrix} M, & N \\ C, & D \\ \tilde{C}, & \tilde{D} \end{bmatrix} = 2n$$

and let us denote

$$\tilde{M} = \begin{bmatrix} C \\ \tilde{C} \end{bmatrix}, \quad \tilde{N} = \begin{bmatrix} D \\ \tilde{D} \end{bmatrix}.$$

Then there exist adjoint and complementary adjoint matrices P, Q and $\tilde{P}$, Q such that

$$-\tilde{M}\tilde{P} + \tilde{N}\tilde{Q} = O_{2n-m_1,m_1}, \quad -\tilde{M}P + \tilde{N}Q = I_{2n-m_1}$$

(cf. [19] §4). Let us put $F_1(t) = (F(t), O_{n,2n-m_1-q})$ for $t \in J$. For any solution $(y, \lambda_1, \lambda_2)$ of (3,11), (3,12) we have

$$y'(a) P + y'(b) Q = \lambda_1'[-MP + NQ] + \left(\int_a^b y'(s)\,\mathrm{d}F_1(s)\right)[-\tilde{M}P + \tilde{N}Q] = $$
$$= \int_a^b y'(s)\mathrm{d}F_1(s)$$

and

$$y'(a)\tilde{P} + y'(b)\tilde{Q} = \lambda_1' .$$

Consequently, by Theorem 3,1, the problem (3,2), (3,3) has a solution iff

$$y'(b) f(b) - y'(a) f(a) - \int_a^b [\mathrm{d}y'(s)] f(s) = (y'(a)\tilde{P} + y'(b)\tilde{Q})\, l_1 + \lambda' l_2$$

for any solution (y, λ) *to*

$$\frac{\mathrm{d}y'}{\mathrm{d}\tau} = D\left[-y' A(t) - (y'(a)\tilde{P} + y'(b)\tilde{Q}) L_1(t) - \lambda' L_2(t) - \int_a^b y'(s)\,[\mathrm{d}H(s)]\,K(t)\right],$$
$$y'(a) P + y'(b) Q - \int_a^b y'(s)\,\mathrm{d}F_1 = 0$$

Analogously as in Theorem 6,2 of [19] we can deduce from Theorem 2,3 the following

Theorem 3,3. *Let* (3,1) *and* (3,4) *hold. Let the homogeneous problem* (3,16), (3,17) *have exactly r linearly independent solutions on J. Then its adjoint* (3,11), (3,12) *has exactly* $r^* = r + m - n$ *linearly independent solutions on J.*
(Couples $(y_j, \lambda_j) \in \mathscr{BV}_n \times \mathscr{R}_m$ $(j = 1, 2, \ldots, p)$ *are said to be linearly dependent on J if there exists a nonzero p-vector* $(k_1, k_2, \ldots, k_p)$ *such that* $k_1 y_1(t) + k_2 y_2(t) + \ldots + k_p y_p(t) \equiv 0$ *on J and* $k_1\lambda_1 + k_2\lambda_2 + \ldots + k_p\lambda_p = 0$.*)*

Thus we generalized Theorems 6,1 and 6,2 of [19].

Let

$$(4,1)\qquad K \in \mathscr{S}\mathscr{B}\mathscr{V}\,,\quad K(\cdot, s) \in \mathscr{B}\mathscr{V}^R_{n,n} \quad\text{and}\quad K(a, s) = 0 \quad\text{for any}\quad s \in J\,;$$
$$P \in \mathscr{B}\mathscr{V}^R_{n,n}\,,\quad P(a) = 0\,;\quad R \in \mathscr{B}\mathscr{V}_{m,n}\,;\quad u \in \mathscr{B}\mathscr{V}^R_n\,,\quad u(a) = 0\,;$$
$$Q \in \mathscr{R}_{m,n}\,,\quad v \in \mathscr{R}_m\,.$$

In §5 we shall reduce the boundary value problem (0,3), (0,2) to an "integro-algebriac" system of the form

$$(4,2)\qquad h(t) + P(t)\,c + \int_a^b K(t, s)\,\mathrm{d}h(s) = u(t) \quad\text{on}\quad J\,,$$
$$Qc + \int_a^b R(s)\,\mathrm{d}h(s) = v\,,$$

where $h \in \mathscr{B}\mathscr{V}_n$, regular on (a, b) and vanishing at a and $c \in \mathscr{R}_n$ are looked for. The system (4,2), the corresponding homogeneous system

$$(4,3)\qquad h(t) + P(t)\,c + \int_a^b K(t, s)\,\mathrm{d}h(s) = 0 \quad\text{on}\quad J\,,$$
$$Qc + \int_a^b R(s)\,\mathrm{d}h(s) = 0$$

and the adjoint system for $(\chi, \gamma) \in \mathscr{B}\mathscr{V}_n \times \mathscr{R}_m$

$$(4,4)\qquad \chi'(t) + \gamma'\,R(t) + \int_a^b \chi'(s)\,\mathrm{d}_s K(s, t) = 0 \quad\text{on}\quad J\,,$$
$$\gamma' Q \quad + \int_a^b \chi'(s)\,\mathrm{d}P(s) \quad = 0$$

shall be treated in this paragraph. In particular we shall prove two lemmas needed in §5.

Lemma 4,1. *The system* (4,2) *fulfilling* (4,1) *has a solution iff*

$$(4,5)\qquad \int_a^b \chi'(s)\,\mathrm{d}u(s) + \gamma' v = 0$$

for any solution (χ, γ) *to* (4,4).

Lemma 4,2. *Let* (4,1) *hold. The homogeneous systems* (4,3) *and* (4,4) *possess at most a finite number of linearly independent solutions on* J. *If* (4,3) *has exactly* r *linearly independent solutions on* J, *then* (4,4) *has exactly* $r^* = r + m - n$ *linearly independent solutions on* J.

The necessity of the condition (4,5) for the existence of a solution to (4,2) follows readily. In fact, let (h, c) and (χ, γ) be solutions to (4,2) and (4,4), respectively. Then according to (1,2,4) we have

$$(4,6) \qquad \int_a^b \chi'(t)\, \mathrm{d}u(t) + \gamma' v =$$

$$= \int_a^b \chi'(t) \left[\mathrm{d} \left\{ h(t) + P(t)\, c + \int_a^b K(t, s)\, \mathrm{d}h(s) \right\} \right] + \gamma' \left\{ Qc + \int_a^b R(t)\, \mathrm{d}h(t) \right\} =$$

$$= \int_a^b \left\{ \chi'(t) + \gamma' R(t) + \int_a^b \chi'(s)\, \mathrm{d}_s K(s, t) \right\} \mathrm{d}h(t) + \left\{ \gamma' Q + \int_a^b \chi'(t)\, \mathrm{d}P(t) \right\} c = 0 .$$

To prove the lemmas completely we have to make use of the following general theorem due to Št. Schwabik ([16]).

Proposition. *Let $\mathscr{X}$ be a Banach space and $\mathscr{X}^*$ its dual. Let $\mathscr{Y} \subset \mathscr{X}^*$ be a total space (i.e. $y(x) = 0$ for all $y \in \mathscr{Y}$ implies $x = 0$) which is also a Banach space. Let $T : \mathscr{X} \to \mathscr{X}$ be a completely continuous operator and let the operator $T' : \mathscr{Y} \to \to \mathscr{X}^*$ be such that $y(Tx) = T' y(x)$ for all $x \in \mathscr{X}$ and $y \in \mathscr{Y}$ and $T'(\mathscr{Y}) \subset \mathscr{Y}$. Then either (a)*
the equation

$$(4,7) \qquad x + Tx = \tilde{x}$$

has in $\mathscr{X}$ only one solution for any $\tilde{x} \in \mathscr{X}$
or (b)
the equation

$$(4,7') \qquad x + Tx = 0$$

has r, $0 < r < \infty$, linearly independent solutions in $\mathscr{X}$.

In the case (b), the equation

$$(4,8) \qquad y + T'y = 0$$

has also r linearly independent solutions in $\mathscr{Y}$.

Moreover, the equation (4,7) has a solution in $\mathscr{X}$ iff $y(\tilde{x}) = 0$ for any solution y of (4,8). Symmetrically, the equation

$$y + T'y = \tilde{y}$$

has a solution in $\mathscr{Y}$ for $\tilde{y} \in \mathscr{Y}$ iff $\tilde{y}(x) = 0$ for any solution x of (4.7').

(For the proof see [16].)

In the sequel we denote by $\mathscr{V}_n$ the space of all column n-vector functions of bounded variation on $[a, b]$, regular on (a, b) and vanishing at a. Furthermore, let us put $p = \max(m, n)$, $\mathscr{X} = \mathscr{V}_n \times \mathscr{R}_p$ and $\mathscr{Y} = \mathscr{BV}_n^* \times \mathscr{R}_p^*$. Then $\mathscr{X}$ and $\mathscr{Y}$ are Banach spaces equipped with the norms

$$x = (h, c) \in \mathscr{X} \to \|x\|_{\mathscr{X}} = \|h\|_{\mathscr{BV}} + \|c\| = \mathrm{var}_a^b\, h + \|c\|$$

and

$$y = (\chi', \gamma') \in \mathscr{Y} \to \|y\|_{\mathscr{Y}} = \|\chi'\|_{\mathscr{BV}} + \|\gamma'\| = \|\chi'(a)\| + \mathrm{var}_a^b\, \chi' + \|\gamma'\|,$$

respectively. The space $\mathscr{Y}$ can be identified with a subspace of $\mathscr{X}^*$. The value of the functional $y = (\chi', \gamma') \in \mathscr{Y}$ on $x = (h, c) \in \mathscr{X}$ is given by

$$\langle x, y\rangle = \int_a^b \chi'(t)\, \mathrm{d}h(t) + \gamma' c\,.$$

Furthermore, let us put

$$(4,9) \qquad \tilde{P}(t) = P(t)\,, \quad \tilde{Q} = \begin{bmatrix} Q \\ O_{n-m,n} \end{bmatrix} - I_n\,,$$

$$\tilde{R}(t) = \begin{bmatrix} R(t) \\ O_{n-m,n} \end{bmatrix}, \quad \tilde{v} = \begin{bmatrix} v \\ O_{n-m,1} \end{bmatrix} \quad \text{if} \quad m \leqq n \ (p = n)$$

and

$$\tilde{P}(t) = (P(t), O_{n,m-n})\,, \quad \tilde{Q} = (Q, O_{m,m-n}) - I_m$$

$$\tilde{R}(t) = R(t)\,, \quad \tilde{v} = v \quad \text{if} \quad m \geqq n \ (p = m)\,.$$

Then (4,2) becomes

$$(4,10) \qquad h(t) + \tilde{P}(t)\, c + \int_a^b K(t, s)\, \mathrm{d}h(s) = u(t) \quad \text{on } J\,,$$

$$c + \tilde{Q}c + \int_a^b \tilde{R}(s)\, \mathrm{d}h(s) = \tilde{v}\,,$$

where $(h, c) \in \mathscr{X}$ is looked for.

Let us notice that while the systems (4,10) and (4,2) are in the case $p = n$ identical, they differ a little from each other if $p = m$. In the latter case we have the following relationship between the systems (4,2) and (4,10). If (h, c) is a solution to (4,2) and if d is an arbitrary m-vector such that the n-vector formed by its first n components coincides with c, then (h, d) is a solution to (4,10). On the other hand, if (h, d) is a solution to (4,10), then (h, c) where c is an n-vector formed by the first n components of d is a solution to (4,2). This indicates that if $p = m$, then the homogeneous system

$$(4,11) \qquad h(t) + \tilde{P}(t)\, c + \int_a^b K(t, s)\, dh(s) = 0 \quad \text{on} \quad J\,,$$

$$c + \tilde{Q}c + \int_a^b \tilde{R}(s)\, \mathrm{d}h(s) = 0$$

has exactly $m - n$ linearly independent solutions on J more than the system (4,3).

Let us put

$$T : x = (h, c) \in \mathscr{X} \to \begin{pmatrix} \tilde{P}(t)\, c + \int_a^b K(t, s)\, \mathrm{d}h(s) \\ \tilde{Q}c \quad + \int_a^b \tilde{R}(s)\, \mathrm{d}h(s) \end{pmatrix}.$$

Then T is a linear continuous mapping of $\mathscr{X}$ into itself. (Clearly $T(\mathscr{X}) \subset \mathscr{BV}_n \times \mathscr{R}_p$. Moreover, since K is by (4,1) bounded on $J \times J$ and $K(t+, \cdot), K(t-, \cdot) \in \mathscr{BV}_{n,n}$ for any $t \in (a, b)$, we have by the convergence theorem for Y-integrals ([5] II 19,3,14) for any $h \in \mathscr{BV}_n$ and $t \in (a, b)$

$$\lim_{\tau \to 0+} \int_a^b K(t + \tau, s)\, \mathrm{d}h(s) = \int_a^b K(t+, s)\, \mathrm{d}h(s)$$

and

$$\lim_{\tau \to 0-} \int_a^b K(t + \tau, s)\, \mathrm{d}h(s) = \int_a^b K(t-, s)\, \mathrm{d}h(s).$$

Consequently

$$\varphi(t) = \int_a^b K(t, s)\, \mathrm{d}h(s) \in \mathscr{BV}_n^R$$

for any $h \in \mathscr{BV}_n$. It follows immediately that $T(\mathscr{X}) \subset \mathscr{X}$.) Furthermore, given $x \in \mathscr{X}$ and $y \in \mathscr{Y}$, we have by (4,6) $\langle Tx, y\rangle = \langle x, T'y\rangle$, where

$$T' : y = (\chi', \gamma') \in \mathscr{Y} \to \begin{pmatrix} \gamma' \tilde{R}(t) + \int_a^b \chi'(s)\, \mathrm{d}_s K(s, t) \\ \gamma' \tilde{Q} \quad + \int_a^b \chi'(s)\, \mathrm{d}\tilde{P}(s) \end{pmatrix} \in \mathscr{Y}.$$

Since by Theorem 3,2 of [14] T is completely continuous and by Lemma 5,1 of [14] $\mathscr{Y}$ is a total subspace of $\mathscr{X}^*$, all the assumptions of Proposition are fulfilled. Applying Proposition to the systems (4,10) and (4,11) we get Lemmas 4,1 and 4,2 immediately. (The adjoint systems (4,4) and

$$\text{(4,12)} \qquad \begin{aligned} &\chi'(t) + \gamma' \tilde{R}(t) + \int_a^b \chi'(s)\, \mathrm{d}_s K(s, t) = 0 \quad \text{on} \quad J, \\ &\gamma' \quad + \gamma' \tilde{Q} \quad + \int_a^b \chi'(s)\, \mathrm{d}\tilde{P}(s) \quad = 0 \end{aligned}$$

are in a similar relationship as the systems (4,3) and (4,11). If $p = n$, then the system (4,12) has exactly $n - m$ linearly independent solutions on J more than the system (4,4).)

Analogously as Lemma 4,1 we can prove the symmetrical assertion

Lemma 4,3. *Let* (4,1) *hold and let* $\alpha \in \mathscr{BV}_n$, $\beta \in \mathscr{R}_n$. *Then the system of equations for* $(\chi, \gamma) \in \mathscr{BV}_n \times \mathscr{R}_m$

$$\chi'(t) + \gamma' R(t) + \int_a^b \chi'(s)\, d_s K(s, t) = \alpha'(t) \quad \text{on} \quad J,$$

$$\gamma' Q + \int_a^b \chi'(s)\, dP(s) = \beta'$$

has a solution iff

$$\int_a^b \alpha'(s)\, dh(s) + \beta' c = 0$$

for any solution (h, c) *to* (4,3).

5. INTEGRAL BOUNDARY VALUE PROBLEM FOR GENERALIZED LINEAR INTEGRODIFFERENTIAL EQUATION WITH A GENERAL KERNEL

Let

(5,1) $$A \in \mathscr{BV}^R_{n,n}, \quad (\Delta^+ A(t))^2 \equiv 0 \text{ on } J, \quad G \in \mathscr{SBV},$$
$$G(\cdot, s) \in \mathscr{BV}^R_{n,n} \text{ for any } s \in J, \quad C \text{ and } D \in \mathscr{BV}^R_{n,n}, \quad f \in \mathscr{BV}^R_n,$$
$$M \text{ and } N \in \mathscr{R}_{m,n}, \quad L \in \mathscr{BV}_{m,n} \text{ and } l \in \mathscr{R}_m.$$

Let us consider the boundary value problem (P): to find an n-vector function x of bounded variation on J which is on J a solution to the generalized linear integro-differential equation

(5,2) $$\frac{dx}{d\tau} = D\left[A(t)\, x + C(t)\, x(a) + D(t)\, x(b) + \int_a^b [d_s G(t, s)]\, x(s) + f(t)\right]$$

and fulfils the additional condition

(5,3) $$M\, x(a) + N\, x(b) + \int_a^b [dL(s)]\, x(s) = l.$$

Without any loss of generality we may assume further

(5,4) $$C(a) = D(a) = 0, \quad G(a, s) \equiv 0 \text{ on } J, \quad L(a) = 0,$$
$$G(t, .) \in \mathscr{BV}^R_{n,n} \text{ for any } t \in J, \quad L \in \mathscr{BV}^R_{m,n}.$$

Let U denote the evolution matrix function for the equation

$$\frac{dx}{d\tau} = D[A(t)\, x]$$

According to (1,3,10) the function U coincides with the evolution matrix function V corresponding to the adjoint equation

$$\frac{dy'}{d\tau} = D[-y' A(t)] .$$

Denoting $U(t, a) = X(t)$ and $U(a, t) = Y(t)$ for $t \in J$, we get $Y(t) = X^{-1}(t)$ on J and $U(t, s) = X(t) X^{-1}(s)$ on $J \times J$, while X and $X^{-1} \in \mathscr{BV}^R_{n,n}$ and $X(a) = X^{-1}(a) = I$. The variation-of-constants formula (1,3,9) now becomes

$$(5,5) \qquad x(t) = X(t)\, c + X(t) \int_a^t X^{-1}(s)\, dh(s) - X(t)\, \Delta^- X^{-1}(t)\, \Delta^- h(t) +$$
$$+ X(t) \int_a^t X^{-1}(s)\, df(s) - X(t)\, \Delta^- X^{-1}(t)\, \Delta^- f(t) = X(t)\, c + X(t) \int_a^t X^{-1}(s)\, dh(s) +$$
$$+ \int_a^t H(t, s)\, dh(s) + X(t) \int_a^t X^{-1}(s)\, df(s) + \int_a^t H(t, s)\, df(s) ,$$

where

$$H(t, s) = \begin{cases} 0 & \text{for} \quad t, s \in J \quad \text{and} \quad t \neq s , \\ -X(t)\, \Delta^- X^{-1}(t) & \text{for} \quad t = s , \end{cases}$$

$c \in \mathscr{R}_n$ and

$$(5,6) \qquad h(t) = C(t)\, x(a) + D(t)\, x(b) + \int_a^b [d_s G(t, s)]\, x(s) \quad \text{on} \quad J .$$

Evidently, $h \in \mathscr{BV}^R_n$ and $h(a) = 0$. Inserting (5,5) into (5,3) and (5,6) and applying Dirichlet's formula (1,2,2) we find that an n-vector function x is a solution to the problem (P) iff x is given by (5,5), where $h \in \mathscr{V}_n$ and $c \in \mathscr{R}_n$ satisfy the "integro-algebraic" system

$$(5,7) \qquad h(t) - \left\{ C(t) + D(t)\, X(b) + \int_a^b [d_s G(t, s)]\, X(s) \right\} c -$$
$$- \int_a^b \left\{ D(t)\, X(b)\, X^{-1}(s) + \int_s^b [d_\sigma G(t, \sigma)]\, (X(\sigma)\, X^{-1}(s) + H(\sigma, s)) \right\} dh(s) =$$
$$= \int_a^b \left\{ D(t)\, X(b)\, X^{-1}(s) + \int_s^b [d_\sigma G(t, \sigma)]\, (X(\sigma)\, X^{-1}(s) + H(\sigma, s)) \right\} df(s) \quad \text{on} \quad J ,$$

$$\left\{ M + N\, X(b) + \int_a^b [dL(s)]\, X(s) \right\} c + \int_a^b \left\{ N\, X(b)\, X^{-1}(s) + \right.$$
$$\left. + \int_s^b [dL(\sigma)]\, (X(\sigma)\, X^{-1}(s) + H(\sigma, s)) \right\} dh(s) =$$
$$= l - \int_a^b \left\{ N\, X(b)\, X^{-1}(s) + \int_s^b [dL(\sigma)]\, (X(\sigma)\, X^{-1}(s) + H(\sigma, s)) \right\} df(s) .$$

Let us denote for $t, s \in J$

$$(5,8) \qquad K(t, s) = -D(t)\,X(b)\,X^{-1}(s) - \int_s^b [\mathrm{d}_\sigma G(t, \sigma)]\,(X(\sigma)\,X^{-1}(s) + H(\sigma, s)),$$

$$P(t) \quad = -C(t) - D(t)\,X(b) - \int_a^b [\mathrm{d}_\sigma G(t, \sigma)]\,X(\sigma),$$

$$u(t) \quad = -\int_a^b K(t, \sigma)\,\mathrm{d}f(\sigma),$$

$$Q \quad = M + N\,X(b) + \int_a^b [\mathrm{d}L(\sigma)]\,X(\sigma),$$

$$R(t) \quad = N\,X(b)\,X^{-1}(t) + \int_t^b [\mathrm{d}L(\sigma)]\,(X(\sigma)\,X^{-1}(t) + H(\sigma, t)),$$

$$v \quad = l - \int_a^b R(\sigma)\,\mathrm{d}f(\sigma).$$

Lemma 5,1. *Let Φ, $\Psi \in \mathscr{BV}_{n,n}$ be such that*

$$\sup_{t,s\in J} \|\Phi(t)\,\Psi(s)\| = M < \infty.$$

Then

$$\Xi(t, s) = \int_s^b [\mathrm{d}_\sigma G(t, \sigma)]\,\Phi(\sigma)\,\Psi(s) \in \mathscr{SBV}.$$

Proof. Let $\sigma = \{a = t_0 < t_1 < \ldots < t_p = b, \quad a = s_0 < s_1 < \ldots < s_q = b\}$ be an arbitrary net subdivision of $J \times J$. Then

$$w(\Xi; \sigma) = \sum_{j=1}^{p} \sum_{k=1}^{q} \left\| \int_{s_k}^b [\mathrm{d}_\sigma(G(t_j, \sigma) - G(t_{j-1}, \sigma))]\,\Phi(\sigma)\,\Psi(s_k) - \right.$$

$$\left. - \int_{s_{k-1}}^b [\mathrm{d}_\sigma(G(t_j, \sigma) - G(t_{j-1}, \sigma))]\,\Phi(\sigma)\,\Psi(s_{k-1}) \right\| \leqq$$

$$\leqq \sum_{j=1}^{p} \sum_{k=1}^{q} \left\| \int_{s_{k-1}}^{s_k} [\mathrm{d}_\sigma(G(t_j, \sigma) - G(t_{j-1}, \sigma))]\,\Phi(\sigma)\,\Psi(s_{k-1}) \right\| +$$

$$+ \left\| \int_{s_k}^b [\mathrm{d}_\sigma(G(t_j, \sigma) - G(t_{j-1}, \sigma))]\,\Phi(\sigma)\,(\Psi(s_k) - \Psi(s_{k-1})) \right\| \leqq$$

$$\leqq M \left\{ \sum_{j=1}^{p} \sum_{k=1}^{q} \Delta\Delta_{j,k} G + \left(\sum_{j=1}^{p} \operatorname{var}_a^b (G(t_j, \cdot) - G(t_{j-1}, \cdot)) \right) \cdot \right.$$

$$\left. \cdot \left(\sum_{k=1}^{q} \|\Psi(s_k) - \Psi(s_{k-1})\| \right) \right\} \leqq$$

$$\leqq M \left\{ w_{J\times J}(G) + (\operatorname{var}_a^b \Psi) \left(\sum_{j=1}^{p} \operatorname{var}_a^b (G(t_j, \cdot) - G(t_{j-1}, \cdot)) \right) \right\}.$$

Given an arbitrary subdivision $\{a = \tau_0 < \tau_1 < \ldots < \tau_r = b\}$ of J,

$$\sum_{j=1}^{p} \left(\sum_{i=1}^{r} \| G(t_j, \tau_i) - G(t_j, \tau_{i-1}) - G(t_{j-1}, \tau_i) + G(t_{j-1}, \tau_{i-1}) \| \leqq w_{J \times J}(G) \,.$$

Hence

$$\sum_{j=1}^{p} \operatorname{var}_a^b (G(t_j, .) - G(t_{j-1}, \cdot)) \leqq w_{J \times J}(G)$$

and

$$w(\Xi; \sigma) \leqq M w_{J \times J}(G) \; (1 + \operatorname{var}_a^b \Psi) \,.$$

Clearly $\Xi(a, s) \equiv 0$ on J and $\operatorname{var}_a^b \Xi(\cdot, a) < \infty$ (cf. (1,2,5)). Hence $\Xi \in \mathscr{SBV}$ follows.

Lemma 5,2. *Let us put*

$$\Omega(t, s) = \begin{cases} G(t, s+) & \text{for} \quad t, s \in J \quad \text{and} \quad s < b \,, \\ G(t, b) & \text{for} \quad t, s \in J \quad \text{and} \quad s = b \,. \end{cases}$$

Then $\Omega \in \mathscr{SBV}$.

Proof. Let $\sigma = \{a = t_0 < t_1 < \ldots < t_p = b, \; a = s_0 < s_1 < \ldots < s_q = b\}$ be an arbitrary net subdivision of $J \times J$. Given an arbitrary $\delta > 0$, we have

$$w_\delta(G; \sigma) = \sum_{j=1}^{p} \sum_{k=1}^{q} \| G(t_j, s_k + \delta) - G(t_j, s_{k-1} + \delta) - G(t_{j-1}, s_k + \delta) + \\ + G(t_{j-1}, s_{k-1} + \delta) \| \leqq w_{J \times J}(G) \,,$$

where $G(t, s) = G(t, b)$ if $s \geqq b$. Consequently also

$$w(\Omega; \sigma) = \lim_{\delta \to 0+} w_\delta(G; \sigma) \leqq w_{J \times J}(G) \,.$$

Lemma 5,3. *The functions* (5,8) *fulfil all the assumptions* (4,1).

Proof. By Lemmas 5,1 and 5,2

$$K(t, s) = -D(t)\, X(b)\, X^{-1}(s) - \int_s^b [\mathrm{d}_\sigma G(t, \sigma)]\, X(\sigma)\, X^{-1}(s) + \\ + \Delta_2^+ G(t, s)\, X(s)\, \Delta^- X^{-1}(s) \in \mathscr{SBV} \,.$$

The remainder is trivial.

Definition. The problem to find an n-vector function y of bounded variation on J and an m-vector λ such that y is a solution on J to the generalized linear integro-differential equation

$$\text{(5,9)} \qquad \frac{\mathrm{d}y'}{\mathrm{d}\tau} = D\left[-y'\, A(t) - \lambda'\, L(t) - \int_a^b y'(s)\, \mathrm{d}_s G(s, t) \right],$$

while

$$(5,10)\quad y'(a) + \lambda' M + \int_a^b y'(s)\,\mathrm{d}C(s) = 0\,, \quad y'(b) - \lambda' N - \int_a^b y'(s)\,\mathrm{d}D(s) = 0$$

is called *the adjoint boundary value problem* (P_0^*) *to the problem* (P).

Remark 5,1. It follows immediately from (1,2,3) and (1,2,5) that $y \in \mathscr{BV}^R$ for any solution (y, λ) of (P_0^*).

In the following we shall need the variation-of-constants formula for the equation

$$(5,11)\qquad \frac{\mathrm{d}y'}{\mathrm{d}\tau} = D[-y'\,A(t) + g'(t)]\,.$$

Lemma 5,4. *Let* $A \in \mathscr{BV}^R_{n,n}$, $(\Delta^+ A(t))^2 \equiv 0$ *on* J *and* $g \in \mathscr{BV}^R_n$. *Given an arbitrary* $\gamma \in \mathscr{R}_n$, *the unique solution* y *of* (5,11) *on* J *such that* $y'(b) = \gamma'$ *is given by*

$$(5,12)\qquad y'(t) = \gamma' U(b, t) - \int_t^b [\mathrm{d}g'(s)]\,(U(s, t) + H(s, t))\,,$$

where

$$(5,13)\qquad H(s, t) = \begin{cases} 0 & \text{for } \ t, s \in J\,, \quad t \neq s\,, \\ \Delta^+ X(t)\,X^{-1}(t) & \text{for } \ t = s\,. \end{cases}$$

Remark 5,2. It follows from (1,2,3), (1,3,4) and (1,3,5) that under our assumptions (5,1),

$$H(t, t) = \Delta^+ X(t)\,X^{-1}(t) = \Delta^+ A(t) = \Delta^- A(t) = -X(t)\,\Delta^- X^{-1}(t)$$

and

$$H(t, a) = H(a, t) = H(t, b) = H(b, t) = 0$$

for any $t \in J$. Consequently for any $t \in J$

$$\int_t^b [\mathrm{d}g'(s)]\,H(s, t) = \Delta^+ g'(t)\,\Delta^+ X(t)\,X^{-1}(t) = \Delta^+ g'(t)\,\Delta^+ A(t)\,.$$

Proof (of Lemma 5,4). We show here only that (5,12) is really a solution of (5,11) on J. The uniqueness can be proved similarly as it was done in [17] for the equation (1,3,1).

Let us denote $\varphi'(t) = \Delta^+ g'(t)\,\Delta^+ A(t)$ for $t \in J$. Then $\varphi'(t+) = \varphi'(t-) = \varphi'(a) = \varphi'(b) = 0$ for any $t \in J$. Hence for any $t \in J$ $\Delta^+ \varphi'(t) = -\varphi'(t)$ and by (1,2,1)

$$\int_t^b \varphi'(s)\,\mathrm{d}A(s) = -\varphi'(t)\,A(t) - \int_t^b [\mathrm{d}\varphi'(s)]\,A(s) - \Delta^+ \varphi'(t)\,\Delta^+ A(t) =$$

$$= -\varphi'(t)\,A(t) - \Delta^+ \varphi'(t)\,A(t) - \Delta^+ \varphi'(t)\,\Delta^+ A(t) = \Delta^+ g'(t)\,(\Delta^+ A(t))^2 = 0\,.$$

Furthermore by (1,2,2) for any $t \in J$

$$\int_t^b \left(\int_s^b [\mathrm{d}g'(\sigma)]\, U(\sigma, s) \right) \mathrm{d}A(s) = \int_t^b [\mathrm{d}g'(s)] \left(\int_t^s U(s, \sigma)\, \mathrm{d}A(\sigma) \right) + \Delta^+ g'(t)\, \Delta^+ A(t)\,.$$

Thus inserting (5,12) into

$$\int_t^b y'(s)\, \mathrm{d}A(s)\,,$$

we get by (1,3,5) for any $t \in J$

$$\int_t^b y'(s)\, \mathrm{d}A(s) = \gamma' \int_t^b U(b, s)\, \mathrm{d}A(s) - \int_t^b [\mathrm{d}g'(s)] \int_t^s U(s, \sigma)\, \mathrm{d}A(\sigma) - \Delta^+ g'(t)\, \Delta^+ A(t) =$$

$$= \gamma' U(b, t) - \int_t^b [\mathrm{d}g'(s)]\, U(s, t) - \Delta^+ g'(t)\, \Delta^+ A(t) - \gamma' + g'(b) - g'(t) =$$

$$= y'(t) - y'(b) + g'(b) - g'(t)\,.$$

It follows immediately that y is a solution to (5,11) on J.

Theorem 5,1. *The boundary value problem* (P) *fulfilling* (5,1) *and* (5,4) *has a solution iff*

$$\int_a^b y'(s)\, \mathrm{d}f(s) = \lambda' l$$

for any solution (y, λ) *of the adjoint problem* (P_0^*).

Proof. Sufficiency. By Lemmas 4,1 and 5,3 the given problem (P) has a solution iff given an arbitrary solution (χ, γ) of the system

(5,14)
$$\chi'(t) + \gamma' R(t) + \int_a^b \chi'(s)\, \mathrm{d}_s K(s, t) = 0 \quad \text{on} \quad J\,,$$

$$\gamma' Q + \int_a^b \chi'(s)\, \mathrm{d}P(s) = 0\,,$$

(where $K(t, s)$, $R(t)$, $P(t)$ and Q are defined by (5,8)) it holds

(5,15)
$$0 = \int_a^b \chi'(t)\, \mathrm{d}u(t) + \gamma' v =$$

$$= - \int_a^b \chi'(t) \left[\mathrm{d}_t \int_a^b K(t, s)\, \mathrm{d}f(s) \right] + \gamma' l - \gamma' \int_a^b R(t)\, \mathrm{d}f(t)\,.$$

Let (χ, γ) be an arbitrary solution of (5,14). Then according to (1,2,4)

$$\int_a^b \chi'(t)\, \mathrm{d}u(t) + \gamma' v = - \int_a^b \left(\int_a^b \chi'(s)\, [\mathrm{d}_s K(s, t)] + \gamma' R(t) \right) \mathrm{d}f(t) + \gamma' l =$$

$$= \int_a^b \chi'(t)\, \mathrm{d}f(t) + \gamma' l\,.$$

We shall show that $(\chi, -\gamma)$ is a solution to (P_0^*). By $(5{,}14)_1$ we have

(5,16)

$$\chi'(t) = -\gamma' R(t) - \int_a^b \chi'(s)\, \mathrm{d}_s K(s, t) = \left\{-\gamma' N + \int_a^b \chi'(s)\, \mathrm{d}D(s)\right\} X(b)\, X^{-1}(t) +$$

$$+ \int_t^b \left[\mathrm{d}_s \left\{-\gamma' L(s) + \int_a^b \chi'(\sigma)\, [\mathrm{d}_\sigma G(\sigma, s)]\right\}\right] (X(s)\, X^{-1}(t) + H(s, t)),$$

since by (1,2,6)

$$\int_a^b \chi'(s) \left[\mathrm{d}_s \int_t^b [\mathrm{d}_\sigma G(s, \sigma)]\, (X(\sigma)\, X^{-1}(t) + H(\sigma, t))\right] =$$

$$= \int_t^b \left[\mathrm{d}_s \int_a^b \mathrm{x}'(\sigma)\, [\mathrm{d}_\sigma G(\sigma, s)]\right] (X(s)\, X^{-1}(t) + H(s, t)).$$

($H(s, t)$ is defined by (5,13) and by Remark 5,2 has the same meaning as in (5,5).) It follows immediately from Lemma 5,4 that $\chi(t)$ is a solution of (5,9) on J such that

$$\chi'(b) = -\gamma' N + \int_a^b \chi'(s)\, \mathrm{d}D(s).$$

Finally, since by Remark 5,2 for any $s \in J$

$$\int_a^b [\mathrm{d}_\sigma G(s, \sigma)]\, H(\sigma, a) = 0 \quad \text{and} \quad \int_a^b [\mathrm{d}L(\sigma)]\, H(\sigma, a) = 0,$$

$(5{,}14)_2$ and (5,16) yield

$$\chi'(a) = -\gamma' N\, X(b) - \gamma' \int_a^b [\mathrm{d}L(\sigma)]\, X(\sigma) + \int_a^b \chi'(\sigma)\, \mathrm{d}D(\sigma)\, X(b) +$$

$$+ \int_a^b \chi'(s) \left[\mathrm{d}_s \int_a^b [\mathrm{d}_\sigma G(s, \sigma)]\, X(\sigma)\right] = \gamma' M - \int_a^b \chi'(s)\, \mathrm{d}C(s).$$

Consequently, the couple $(\chi, -\gamma)$ is a solution to the problem (P_0^*). This completes the proof of the first part of the theorem.

Necessity. Let $x, y \in \mathscr{BV}_n^R$ and $\lambda \in \mathscr{R}_m$. Then by (1,2,1), (1,2,6) and by the substitution theorem for Y-integrals ([5] II 19,3,7)

(5,17)

$$\int_a^b y'(t) \left[\mathrm{d}_t \left(x(t) - \int_a^t [\mathrm{d}A(s)]\, x(s) - C(t)\, x(a) - D(t)\, x(b) - \int_a^b [\mathrm{d}_s G(t, s)]\, x(s)\right)\right] +$$

$$+ \int_a^b \left[\mathrm{d}_t \left(y'(t) + \int_a^t y'(s)\, \mathrm{d}A(s) + \lambda' L(t) + \int_a^b y'(s)\, \mathrm{d}_s G(s, t)\right)\right] x(t) =$$

$$= \left[y'(b) - \int_a^b y'(t)\, \mathrm{d}D(t)\right] x(b) - \left[y'(a) + \int_a^b y'(t)\, \mathrm{d}C(t)\right] x(a) + \lambda' \int_a^b [\mathrm{d}L(t)]\, x(t).$$

In particular, if x and (y, λ) are solutions to (P) and (P_0^*), respectively, then (5,17) yields

$$\int_a^b y'(t)\,\mathrm{d}f(t) = \lambda' \left\{M\,x(a) + N\,x(b) + \int_a^b [\mathrm{d}L(t)]\,x(t)\right\} = \lambda' l\,.$$

Remark 5,3. Let us put

$$U : x \in \mathscr{BV}_n \to Ux = (h, c) \in \mathscr{V}_n \times \mathscr{R}_n\,,$$

where

$$c = x(a) \quad \text{and} \quad h(t) = C(t)\,x(a) + D(t)\,x(b) + \int_a^b [\mathrm{d}_s G(t, s)]\,x(s) \quad \text{on} \quad J$$

and

$$V : (h, c) \in \mathscr{V}_n \times \mathscr{R}_n \to V(h, c) = x \in \mathscr{BV}_n\,,$$

where

$$x(t) = U(t, a)\,c + h(t) - \int_a^t [\mathrm{d}_s U(t, s)]\,h(s) \quad \text{on} \quad J\,.$$

We know from the foregoing that if x is a solution to the homogeneous boundary value problem (P_0) on J

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = D\left[A(t)\,x + C(t)\,x(a) + D(t)\,x(b) + \int_a^b [\mathrm{d}_s G(t, s)]\,x(s)\right],$$

$$M\,x(a) + N\,x(b) + \int_a^b [\mathrm{d}L(s)]\,x(s)\,,$$

then Ux is a solution to the system

$$(5,18) \qquad h(t) + P(t)\,c + \int_a^b K(t, s)\,\mathrm{d}h(s) = 0 \quad \text{on} \quad J\,,$$

$$Qc + \int_a^b R(s)\,\mathrm{d}h(s) = 0\,.$$

Conversely, if (h, c) is a solution to (5,18), then $V(h, c)$ is a solution to (P_0). Obviously, x and (h, c) being solutions to (P_0) and (5,18), respectively, then $VUx = x$ and $UV(h, c) = (h, c)$. Thus we have a linear one-to-one correspondence between the solutions of (P_0) and of (5,18). In particular, it follows from Lemma 4,2 that the problem (P_0) has at most a finite number of linearly independent solutions on J. Moreover, if (P_0) has exactly r linearly independent solutions on J, then (5,18) has also exactly r linearly independent solutions on J and its adjoint (5,14) has exactly $r^* = r + m - n$ linearly independent solutions on J. Since by the proof of Theorem 5,1 a couple $(y, \lambda) \in \mathscr{BV}_n \times \mathscr{R}_m$ is a solution to (P_0^*) iff $(y, -\lambda)$ is a solution to (5,14), the following theorem is true.

Theorem 5,2. *Let the conditions* (5,1) *and* (5,4) *be fulfilled. Then both the homogeneous boundary value problem* (P_0) *and its adjoint* (P_0^*) *have at most a finite number of linearly independent solutions on J. Moreover, if* (P_0) *has exactly r linearly independent solutions on J, then* (P_0^*) *has exactly* $r^* = r + m - n$ *linearly independent solutions on J.*

To complete the Fredholm theory for the boundary value problem (P) we shall show that the problem (P_0) is a well-posed adjoint to the nonhomogeneous problem (P^*) corresponding to (P_0^*).

Theorem 5,3. *Let p and* $q \in \mathscr{R}_n$, $g \in \mathscr{BV}_n^R$ *and let* (5,1) *and* (5,4) *hold. Then the boundary value problem* (P^*) *to find* $(y, \lambda) \in \mathscr{BV}_n \times \mathscr{R}_m$ *such that y is a solution on J to*

$$\frac{\mathrm{d}y'}{\mathrm{d}\tau} = D\left[-y' A(t) - \lambda' L(t) - \int_a^b y'(s)\,\mathrm{d}_s G(s, t) + g'(t)\right]$$

and

$$y'(a) + \lambda' M + \int_a^b y'(s)\,\mathrm{d}C(s) = p', \quad y'(b) - \lambda' N - \int_a^b y'(s)\,\mathrm{d}D(s) = q',$$

has a solution iff

$$\int_a^b [\mathrm{d}g'(s)]\, x(s) = q'\, x(b) - p'\, x(a)$$

for any solution x to (P_0).

Proof. By Lemma 5,4 and by the formula (1,2,6), the problem (P^*) is equivalent to the system of equations for $(y, \lambda, \gamma) \in \mathscr{BV}_n \times \mathscr{R}_m \times \mathscr{R}_n$

$$y'(t) = \gamma' U(b, t) + \lambda' \int_t^b [\mathrm{d}L(s)]\,(U(s, t) + H(s, t)) + \tag{5,19}$$

$$+ \int_a^b y'(s)\left[\mathrm{d}_s \int_t^b [\mathrm{d}_\sigma G(s, \sigma)]\,(U(\sigma, t) + H(\sigma, t))\right] - \int_t^b [\mathrm{d}g'(s)]\,(U(s, t) + H(s, t))$$

$$\text{on} \quad J,$$

$$(y'(a) =)\, \gamma' U(b, a) + \lambda' \int_a^b [\mathrm{d}L(s)]\, U(s, a) + \int_a^b y'(s)\left[\mathrm{d}_s \int_a^b [\mathrm{d}_\sigma G(s, \sigma)]\, U(\sigma, a)\right] -$$

$$- \int_a^b [\mathrm{d}g'(s)]\, U(s, a) = -\lambda' M - \int_a^b y'(s)\,\mathrm{d}C(s) + p',$$

$$(y'(b) =)\, \gamma' = \lambda' N + \int_a^b y'(s)\,\mathrm{d}D(s) + q',$$

where $H(s, t)$ is given by (5,13). Inserting $(5,19)_3$ into $(5,19)_1$ and $(5,19)_2$, we reduce the system (5,19) to the system for $(y, \lambda) \in \mathscr{BV}_n \times \mathscr{R}_m$

$$(5,20) \qquad y'(t) - \lambda' \left\{NU(b, t) + \int_t^b [\mathrm{d}L(s)] (U(s, t) + H(s, t)\right\} -$$

$$- \int_a^b y'(s) \left[\mathrm{d}_s \left\{D(s) U(b, t) + \int_t^b [\mathrm{d}_\sigma G(s, \sigma)] (U(\sigma, t) + H(\sigma, t))\right\}\right] =$$

$$= - \int_t^b [\mathrm{d}g'(s)] (U(s, t) + H(s, t)) + q'U(b, t) \quad \text{on} \quad J,$$

$$\lambda' \left\{M + NU(b, a) + \int_a^b [\mathrm{d}L(s)] U(s, a)\right\} +$$

$$+ \int_a^b y'(s) \left[\mathrm{d}_s \left\{C(s) + D(s) U(b, a) + \int_a^b [\mathrm{d}_\sigma G(s, \sigma)] U(\sigma, a)\right\}\right] =$$

$$= p' - q' U(b, a) + \int_a^b [\mathrm{d}g'(s)] U(s, a).$$

The system (5,20) has a solution iff

$$0 = \int_a^b \left\{q' U(b, t) - \int_t^b [\mathrm{d}g'(s)] (U(s, t) + H(s, t))\right\} \mathrm{d}h(t) + \left\{p' - q' U(b, a) + \right.$$

$$\left. + \int_a^b [\mathrm{d}g'(s)] U(s, a)\right\} c = p'c - q' \left\{U(b, a) c - \int_a^b U(b, t) \,\mathrm{d}h(t)\right\} +$$

$$+ \int_a^b [\mathrm{d}g'(t)] \left\{U(t, a) c - \int_a^t (U(t, s) + H(t, s)) \,\mathrm{d}h(s)\right\}$$

for any solution $(h, c) \in \mathscr{BV}_n \times \mathscr{R}_n$ to the system (5,18) (cf. (5,8) and Lemma 4,3). Let us denote for $t \in J$

$$x(t) = U(t, a) c - \int_a^t (U(t, s) + H(t, s)) \,\mathrm{d}h(s).$$

Then $x(t)$ is a solution of the equation

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = D[A(t) x - h(t)]$$

on J. Furthermore, $x(a) = c$,

$$h(t) = - \left\{C(t) + D(t) U(b, a) + \int_a^b [\mathrm{d}_\sigma G(t, \sigma)] (U(\sigma, a) + H(\sigma, a))\right\} c +$$

$$+ \int_a^b \left\{D(t) U(b, s) + \int_s^b [\mathrm{d}_\sigma G(t, \sigma)] (U(\sigma, s) + H(\sigma, s))\right\} \mathrm{d}h(s) =$$

$$= - C(t)\,c - D(t)\left\{U(b,a)\,c - \int_a^b U(b,s)\,\mathrm{d}h(s)\right\} -$$

$$- \int_a^b [\mathrm{d}_s G(t,s)]\left\{U(s,a)\,c - \int_a^s (U(s,\sigma) + H(s,\sigma))\,\mathrm{d}h(\sigma)\right\} =$$

$$= -\left\{C(t)\,x(a) + D(t)\,x(b) + \int_a^b [\mathrm{d}_s G(t,s)]\,x(s)\right\}$$

and

$$0 = \left\{M + N\,U(b,a) + \int_a^b [\mathrm{d}L(\sigma)]\,U(\sigma,a)\right\} c -$$

$$- \int_a^b \left\{N\,U(b,s) + \int_s^b [\mathrm{d}L(\sigma)]\,(U(\sigma,s) + H(\sigma,s))\right\} \mathrm{d}h(s) =$$

$$= M\,x(a) + N\,x(b) + \int_a^b [\mathrm{d}L(s)]\,x(s)\,.$$

This completes the proof.

Remark 5,3. Let us consider the boundary value problem (π)

$$\text{(5,21)} \qquad \frac{\mathrm{d}x}{\mathrm{d}t} = A(t)\,x + C(t)\,x(a) + D(t)\,x(b) + \int_a^b [\mathrm{d}_s G(t,s)]\,x(s) + f(t)\,,$$

$$\text{(5,3)} \qquad M\,x(a) + Nx(b) + \int_a^b [\mathrm{d}L(s)]\,x(s) = l$$

where

(5,22) A, C and D are $n \times n$-matrix functions L-integrable on J; f is an n-vector function L-integrable on J; G is an $n \times n$-matrix function such that $\mathrm{var}_a^b\, G(t,\cdot) < \infty$ for any $t \in J$ and

$$\int_a^b [\mathrm{var}_a^b\, G(t,\cdot)]\,\mathrm{d}t < \infty\,;$$

$$M, N \in \mathscr{R}_{m \cdot n}\,; \qquad L \in \mathscr{BV}_{m \cdot n} \quad \text{and} \quad l \in \mathscr{R}_m\,.$$

Solutions to (5,21) are sought as absolutely continuous functions on J fulfilling (5,21) almost everywhere on J. (Let us notice that

$$\left\|\int_a^b [\mathrm{d}_s G(t,s)]\,x(s)\right\| \leqq [\mathrm{var}_a^b\, G(t,\cdot)]\,\|x\|_{\mathscr{BV}}$$

for any $x \in \mathscr{BV}_n$ and $t \in J$. Hence owing to (5,22)

$$\int_a^b \left\|\int_a^b [\mathrm{d}_s G(t,s)]\,x(s)\right\| \mathrm{d}t < \infty\,.)$$

Without any loss of generality we may assume L and $G(t, \cdot)$ to be regular on J for any $t \in J$.

Let us put

$$A_1(t) = \int_a^t A(s)\,\mathrm{d}s\,, \quad C_1(t) = \int_a^t C(s)\,\mathrm{d}s\,, \quad D_1(t) = \int_a^t D(s)\,\mathrm{d}s\,, \quad f_1(t) = \int_a^t f(s)\,\mathrm{d}s$$

and

$$G_1(t, s) = \int_a^t G(\tau, s)\,\mathrm{d}\tau\,.$$

Then the equation (5,21) is evidently equivalent with the generalized integrodifferential equation

$$(5{,}23) \qquad \frac{\mathrm{d}x}{\mathrm{d}\tau} = D\left[A_1(t)\,x + C_1(t)\,x(a) + D_1(t)\,x(b) + \int_a^b [\mathrm{d}_s G_1(t, s)]\,x(s) + f_1(t)\right]$$

(cf. (1,2,6)). The functions A_1, C_1, D_1, f_1 and L certainly fulfil the conditions (5,1), while $f_1(a) = 0$ and $C_1(a) = D_1(a) = G_1(a, s) = 0$ for any $s \in J$. Moreover, given an arbitrary net subdivision $\{a = t_0 < t_1 < \ldots < t_p = b, a = s_0 < s_1 < \ldots < s_q = b\}$ of $J \times J$, we have

$$\sum_{j=1}^{p} \sum_{k=1}^{q} \Delta\Delta_{j,k} G_1 = \sum_{j=1}^{p} \sum_{k=1}^{q} \left\| \int_{t_{j-1}}^{t_j} [G(\tau, s_k) - G(\tau, s_{k-1})] \right\| \mathrm{d}\tau \leqq$$

$$\leqq \int_a^b \sum_{k=1}^{q} \|G(\tau, s_k) - G(\tau, s_{k-1})\|\,\mathrm{d}\tau \leqq \int_a^b [\operatorname{var}_a^b G(\tau, \cdot)]\,\mathrm{d}\tau < \infty$$

and

$$\sum_{j=1}^{p} \|G_1(t, s_j) - G_1(t, s_{j-1})\| \leqq \int_a^b [\operatorname{var}_a^b G(\tau, \cdot)]\,\mathrm{d}\tau < \infty \quad \text{for any} \quad t \in J\,.$$

Thus $G_1 \in \mathcal{SBV}$. Applying Theorems 5,1 – 5,3 to the problem (5,23), (5,3) we get immediately the following statements generalizing Theorems 3,1 and 3,2 of [20].

Corollary 5,1. *The boundary value problem* (π) *fulfilling* (5,22) *has a solution iff*

$$\int_a^b y'(s)\,f(s)\,\mathrm{d}s = \lambda' l$$

for any solution $(y, \lambda) \in \mathcal{BV}_n \times \mathcal{R}_m$ *of its adjoint* (π_0^*)

$$\frac{\mathrm{d}y'}{\mathrm{d}\tau} = D\left[-y' A_1(t) - \lambda' L(t) - \int_a^b y'(s)\,G(s, t)\,\mathrm{d}s\right],$$

$$y'(a) + \lambda' M + \int_a^b y'(s)\,C(s)\,\mathrm{d}s = 0\,, \quad y'(b) - \lambda' N - \int_a^b y'(s)\,D(s)\,\mathrm{d}s = 0\,.$$

Corollary 5,2. *Let* (5,22) *hold. Then both the homogeneous problem* (π_0)

$$\frac{dx}{dt} = A(t)\,x + C(t)\,x(a) + D(t)\,x(b) + \int_a^b [d_s G(t, s)]\,x(s)\,,$$

$$M\,x(a) + N\,x(b) + \int_a^b [dL(s)]\,x(s) = 0$$

and its adjoint (π_0^*) *have at most a finite number of linearly independent solutions on* J. *Moreover, if* (π_0) *has exactly* r *linearly independent solutions on* J, *then* (π_0^*) *has exactly* $r^* = r + m - n$ *linearly independent solutions on* J.

Corollary 5,3. *Let* (5,22) *hold. Let* $A_1(t) = \int_a^t A(s)\,ds$ *on* J *and* $g \in \mathscr{BV}_n^R$; $p, q \in \mathscr{R}_n$. *Then the problem to find* $(y, \lambda) \in \mathscr{BV}_n \times \mathscr{R}_m$ *such that* y *is a solution on* J *to the generalized integrodifferential equation*

$$\frac{dy'}{d\tau} = D\left[-y'\,A_1(t) - \lambda'\,L(t) - \int_a^b y'(s)\,G(s, t)\,ds + g'(t)\right]$$

and

$$y'(a) + \lambda' M + \int_a^b y'(s)\,C(s)\,ds = p'\,, \quad y'(b) - \lambda' N - \int_a^b y'(s)\,D(s)\,ds = q'$$

has a solution iff

$$\int_a^b [dg'(s)]\,x(s) = q'\,x(b) - p'\,x(a)$$

for any solution x *to* (π_0).

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Author's address: 115 67 Praha 1, Žitná 25, ČSSR (Matematický ústav ČSAV v Praze).