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GENERALIZED BOUNDARY VALUE PROBLEMS WITH ABSTRACT SIDE CONDITIONS AND THEIR ADJOINTS II

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0. PRELIMINARIES

Let $-\infty < a < b < \infty$. Let A be an $m \times m$-matrix valued function essentially bounded on $[a, b]$. Let F be a locally convex topological vector space and let H be a linear continuous mapping of the Sobolev space $W_m^{1,\infty}$ into F.

For $u \in W_m^{1,\infty}$, ℓu denotes the value of the differential expression

$$\ell u := u' + A(t)\, u\,.$$

This expression is defined a.e. on $[a, b]$ and $\ell u \in L_m^\infty$ for any $u \in W_m^{1,\infty}$. The symbol ℓ will be also used for the "maximal" operator

$$\ell : u \in W_m^{1,\infty} \to \ell u \in L_m^\infty\,.$$

Under our assumptions the graph

$$G = G(\ell) = \{(u, \ell u) \in L_m^\infty \times L_m^\infty : u \in W_m^{1,\infty}\} \tag{0,1}$$

of ℓ is certainly closed in $L_m^\infty \times L_m^\infty$. Hence when endowed with the usual operations and with the norm of $L_m^\infty \times L_m^\infty$

$$(u, \ell u) \in G \to \|u\|_\infty + \|\ell u\|_\infty\,,$$

G becomes a Banach space.

We shall consider the linear differential operator L acting on L_m^∞ defined on

$$D(L) = \{u \in L_m^\infty : u \in W_m^{1,\infty} \text{ and } Hu = 0\}$$

by

$$Lu := \ell u\,.$$

We shall use the notation introduced in the first part [1] of the paper. Given locally convex topological vector spaces X, Y and a linear operator T with the definition domain $D(T) \subset X$ and the range $R(T) \subset Y$, $N(T)$ denotes its null space and $G(T)$

its graph. X^* is the dual space to X and $[., u]_X$ denotes the linear continuous functional on X corresponding to $u \in X^*$. For $M \subset X$ and $N \subset X^*$ the symbols $M^\perp$ and ${}^\perp N$ are defined by

$$M^\perp = \{u \in X^* : [x, u]_X = 0 \text{ for all } x \in M\}$$

and

$${}^\perp N = \{x \in X : [x, u]_X = 0 \text{ for all } u \in N\},$$

respectively. Furthermore, $\mathrm{cl}^*(N)$ denotes the weak*-closure of N in X^* (with respect to the duality $[., .]_X$). If X is normed, then the norm on X is denoted by $\|.\|_X$ and $\overline{M}$ is the corresponding norm closure of $M \subset X$. In such a case it is possible also to equip X^* with the norm $\|u\|_{X^*} = \sup_{\|x\|_X \leqq 1} |[x, u]|$. The corresponding norm closure of $N \subset X^*$ is denoted by $\overline{N}$.

Let S be a linear operator acting from Y^* into X^* ($D(S) \subset Y^*$, $R(S) \subset X^*$). We say that the set $G(^*S)$ is the graph of the pre-adjoint relation $^*S$ to S if

$$G(^*S) = \{(x, y) \in X \times Y : [x, Su]_X = [y, u]_Y \text{ for all } u \in D(S)\},$$

i.e. $G(^*S) = {}^\perp G(-S)$, where the orthogonal complement of the graph $G(-S) = \{(-Su, u) : u \in D(S) \subset Y^*\}$ of $-S$ is considered with respect to the duality $[., .]_{X \times Y}$ on $(X \times Y) \times (X^* \times Y^*)$,

$$[(x, y), (u, v)]_{X \times Y} = [x, u]_X + [y, v]_Y .$$

$D(^*S) = \{x \in X : (x, y) \in G(^*S) \text{ for some } y \in Y\}$ is the definition domain of $^*S$, $R(^*S) = \{y \in Y : (x, y) \in G(^*S) \text{ for some } x \in X\}$ its range, $N(^*S) = \{x \in X : (x, 0) \in G(^*S)\}$ its null space and

$$^*Sx = \{y \in Y : (x, y) \in G(^*S)\} \quad \text{for} \quad x \in D(^*S).$$

$^*S$ is an operator if $^*Sx = 0$ for $x = 0$.

0.1. Lemma (cf. [2], Theorem 2.3). *Let X, Y be Banach spaces. If $S : D(S) \subset Y^* \to X^*$ is weakly*-closed in $X^* \times Y^*$ and $\overline{R(S)} = R(S)$, then $R(S)$ is weakly*-closed in X^*, $(^*S)^* = S$ and*

$$R(S) = N(^*S)^\perp, \quad {}^\perp R(S) = N^*(S), \tag{0,2}$$
$$R(^*S) = {}^\perp N(S), \quad R(^*S)^\perp = N(S).$$

C^m denotes the space of complex row m-vectors, $|.|$ is the norm on C^m, x^* denotes the conjugate transposition of $x \in C^m$; L^p_m $(1 \leqq p \leqq \infty)$ is the space of functions $x : [a, b] \to C^m$ for which

$$\|x\|_p = \left(\int_a^b |x(t)|^p \, dt\right)^{1/p} < \infty \quad \text{if} \quad 1 \leqq p < \infty$$

or

$$\|x\|_\infty = \sup_{t\in[a,b]} \text{ess}\, |x(t)| < \infty \quad \text{if} \quad p = \infty\,;$$

$W_m^{1,p}$ is the Sobolev space of functions $x : [a, b] \to C^m$ absolutely continuous on $[a, b]$ and such that their derivatives x' belong to L_m^p,

$$\|x\|_{1,p} = |x(a)| + \|x'\|_p\,.$$

Let $(1/p) + (1/q) = 1$ if $1 < p < \infty$, $q = \infty$ if $p = 1$, then L_m^q is the dual space to L_m^p with respect to the duality

$$[x, u]_L = \int_a^b u^* x \,\mathrm{d}t \quad \text{for} \quad x \in L_m^1 \quad \text{and} \quad u \in L_m^\infty$$

and $W_m^{1,q}$ is the dual space to $W_m^{1,p}$ with respect to the duality

$$[x, v]_W = v^*(a)\, x(a) + [x', v']_L \quad \text{for} \quad x \in W_m^{1,p} \quad \text{and} \quad v \in W_m^{1,q}\,.$$

1. NORMAL SOLVABILITY OF L

In the first part of the paper we proved that under our assumptions L has a closed range in L_m^∞, i.e. it is normally solvable in the usual sense. However, since we have no proper analytic representation of the dual space to L_m^∞ we cannot obtain an analytic form of the adjoint L^* to the operator L. This means that the relations (Fredholm Alternatives)

$$R(L) = {}^\perp N(L^*)\,, \quad R(L)^\perp = N(L^*)$$

which follow from the normal solvability give us no useful information. Nevertheless, we have a chance to obtain similar but more useful Fredholm type relations using the pre-adjoint $^*L$ of L. Since L_m^∞ is the dual space to L_m^1, the pre-adjoint $^*L$ to L is a linear relation in $L_m^1 \times L_m^1$ with the graph

(1,1) $$G(^*L) = \{(x, y) \in L_m^1 \times L_m^1 : [x, \ell u]_L = [y, u]_L \text{ for all } u \in D(L)\}\,,$$

definition domain

(1,2) $$D(^*L) = \{x \in L_m^1 : (x, y) \in G(^*L) \text{ for some } y \in L_m^1\}\,,$$

null space

(1,3) $$N(^*L) = \{x \in L_m^1 : [x, \ell u]_L = 0 \text{ for all } u \in D(L)\}$$

and values

(1,4) $$^*Lx = \{y \in L_m^1 : (x, y) \in G(^*L)\} \quad \text{for} \quad x \in D(^*L)\,.$$

If we show that L is weakly*-closed in $L_m^\infty \times L_m^\infty$ (with respect to the duality

$$[(x, y), (u, v)] = [x, u]_L + [y, v]_L \quad \text{for} \quad x, y \in L_m^1 \quad \text{and} \quad u, v \in L_m^\infty)\,,$$

then by Lemma 0.1 we obtain the formulas

$$(1,5) \qquad R(L) = N(*L)^{\perp}, \quad {}^{\perp}R(L) = N(*L),$$
$$R(*L) = {}^{\perp}N(L), \quad R(*L)^{\perp} = N(L).$$

After proving this we shall in the following section derive the analytic form of the pre-adjoint relation $*L$ to L. The following assumptions will be kept.

1.1. Assumptions. *A* is an $m \times m$-matrix valued function essentially bounded on $[a, b]$, $-\infty < a < b < \infty$; F is a locally convex topological vector space such that $F = (*F)^*$ for some locally convex topological vector space $*F$; H is a linear continuous mapping of the space $W_m^{1,\infty}$ into F such that $H = (*H)^*$ for some linear continuous mapping $*H$ of $*F$ into $W_m^{1,1}$.

1.2. Notation. We denote by J the linear operator (cf. (0,1))

$$J : (u, \ell u) \in G \subset L_m^{\infty} \times L_m^{\infty} \to u \in W_m^{1,\infty}.$$

Obviously,

$$(1,6) \qquad J_{-1}(N(H)) := \{(u, \ell u) \in G : Hu = 0\} = G(L)$$

is the graph of L.

1.3. Lemma. cl* $(N(H)) = N(H)$ (*the weak*-closure in* $W_m^{1,\infty}$ *with respect to the duality* $[.,.]_W$).

Proof. Let $u \in \mathrm{cl}^*(N(H))$. Then for each finite set $Z = \{z_1, z_2, \ldots, z_k\} \subset W_m^{1,1}$ there exists a sequence $\{u_j^{(Z)}\}_{j=1}^{\infty} \subset N(H)$ such that

$$[z, u_j^{(Z)}]_W \to [z, u]_W \quad \text{as} \quad j \to \infty$$

holds for any $z \in Z$. Let us choose an arbitrary $\varphi \in *F$. Then there exists a sequence $\{u_j^{(\varphi)}\}_{j=1}^{\infty} \subset N(H)$ such that

$$[*H\varphi, u_j^{(\varphi)}]_W \to [*H\varphi, u]_W \quad \text{as} \quad j \to \infty.$$

This means that

$$[\varphi, Hu]_{*F} = [\varphi, H(u - u_j^{(\varphi)})]_{*F} = [*H\varphi, u - u_j^{(\varphi)}]_W \to 0.$$

Since $\varphi \in *F$ was arbitrary, this implies that $Hu = 0$, i.e. $u \in N(H)$. This completes the proof.

1.4. Lemma. *The mapping J defined in* 1.2 *is continuous with respect to the corresponding weak*-topologies.*

Proof. Let $\varepsilon > 0$ be given and let Z be an arbitrary finite subset of $W_m^{1,1}$. To prove the lemma we have to show that there exist $\delta > 0$ and a finite subset W of $L_m^1 \times L_m^1$

such that for every $u \in W_m^{1,\infty}$ satisfying

$$\left|[x, u]_L + [y, \ell u]_L\right| < \delta \quad \text{for all} \quad (x, y) \in W$$

we have

$$\left|[z, u]_W\right| < \varepsilon \quad \text{for all} \quad z \in Z\,.$$

Recall that

$$[z, u]_W = u^*(a)\, z(a) + \int_a^b u'^* z' \,\mathrm{d}t$$

and

$$(1,7) \qquad [x, u]_L + [y, \ell u]_L = \int_a^b u^* x \,\mathrm{d}t + \int_a^b (u' + Au)^*\, y \,\mathrm{d}t =$$

$$= \int_a^b u^*(x + A^* y)\,\mathrm{d}t + \int_a^b u'^* y \,\mathrm{d}t =$$

$$= u^*(a) \int_a^b (x + A^* y)\,\mathrm{d}t + \int_a^b u'^* \left[\int_t^b (x + A^* y)\,\mathrm{d}\tau + y\right] \mathrm{d}t\,.$$

Now we shall prove

Auxiliary Assertion. *For any $z \in W_m^{1,1}$ there exist $x, y \in L_m^1$ such that*

(1,8)

$$\int_a^b (x + A^* y)\,\mathrm{d}t = z(a) \quad \text{and} \quad y(t) + \int_t^b (x + A^* y)\,\mathrm{d}\tau = z'(t) \quad \text{a.e. on} \quad [a, b]\,.$$

Proof (of Auxiliary Assertion). We have to show that for any $d \in C^m$ and $w \in L_m^1$ there exist $x, y \in L_m^1$ such that

$$(1,9) \qquad \int_a^b (x + A^* y)\,\mathrm{d}t = d\,,$$

$$y(t) + \int_t^b (x + A^* y)\,\mathrm{d}\tau = w(t) \quad \text{a.e. on} \quad [a, b]\,.$$

If x, y satisfy (1,9), then there certainly exists $\xi \in W_m^{1,1}$ such that $\xi = w - y$ a.e. and

(1,10)

$$\xi(t) = \int_t^b (x + A^*(w - \xi))\,\mathrm{d}\tau \quad \text{on} \quad [a, b]\,, \qquad d = \int_a^b (x + A^*(w - \xi))\,\mathrm{d}\tau\,.$$

Notice that then $\xi(a) = d$ and $\xi(b) = 0$.

On the other hand, if $\xi \in W_m^{1,1}$ and $x \in L_m^1$ fulfil (1,10), then the couple (x, y), $y = w - \xi$, fulfils (1,9).

Differentiating (1,10) we further obtain that our assertion holds if for any $g \in L_m^1$ and $d \in C^m$ there exists $x \in L_m^1$ such that the two-point boundary value problem

$$(1,11) \qquad -\xi' + A^*(t)\,\xi = g(t) + x(t) \quad \text{a.e. on} \quad [a, b]\,,$$
$$\xi(a) = d \quad \text{and} \quad \xi(b) = 0$$

has a solution $\xi \in W_m^{1,1}$.

Given $g \in L_m^1$ and $d \in C^m$, let us put

$$\xi(t) = \frac{b-t}{b-a}\,d \quad \text{for} \quad t \in [a, b]$$

and

$$x(t) = -\xi'(t) + A^*(t)\,\xi(t) - g(t) \quad \text{for a.e.} \quad t \in [a, b]\,.$$

Then evidently $\xi \in W_m^{1,1}$, $\xi(a) = d$, $\xi(b) = 0$ and ξ is a solution to the system (1,11). This completes the proof of Auxiliary Assertion.

Proof of Lemma 1.4 (continuation). Let Z be an arbitrary finite subset of $W_m^{1,1}$. Then by Auxiliary Assertion for any $z \in Z$ there exist x_z, $y_z \in L_m^1$ such that (1,8) holds when the symbols x, y are replaced by x_z and y_z, respectively. Let us denote

$$W := \{(x_z, y_z) : z \in Z\}\,.$$

Let $u \in W_m^{1,\infty}$ be such that

$$|[x, u]_L + [y, \ell u]_L| < \varepsilon \quad \text{for all} \quad (x, y) \in W\,.$$

Then for any $z \in Z$ we have in virtue of (1,7)

$$|[z, u]_W| = |[x_z, u]_L + [y_z, \ell u]_L| < \varepsilon\,.$$

This completes the proof of Lemma 1.4.

Now we can prove the following assertion.

1.5. Theorem. *Under Assumptions* 1.1 *the graph* $G(L)$ *of* L *is weakly*-closed in* $L_m^\infty \times L_m^\infty$.

Proof. By (1,6), $G(L) = J_{-1}(N(H))$. Since $N(H)$ is weakly*-closed in $W_m^{1,\infty}$ by Lemma 1.3 and $J : G \subset L_m^\infty \times L_m^\infty \to W_m^{1,\infty}$ is continuous with respect to the corresponding weak*-topologies by Lemma 1.4, it follows immediately that $G(L)$ is weakly*-closed in $L_m^\infty \times L_m^\infty$.

Since $R(L)$ is closed in L_m^∞ (cf. Theorem 4.3 of the first part [1] of this paper) and L is weakly*-closed in $L_m^\infty \times L_m^\infty$, it follows from Lemma 0.1 that $R(L)$ is weakly*-closed in L_m^∞.

1.6. Theorem. *Under Assumptions* 1.1, $R(L)$ *is weakly*-closed in* L_m^∞, $(^*L)^* = L$ *and the relations* (1,5) *hold.*

1.7. Remark. The results of this section also hold if we only assume the operator $H : W_m^{1,\infty} \to F$ to be continuous and such that its pre-adjoint relation $^*H$ is densely defined in $^*F$, i.e. $\overline{D(^*H)} = {}^*F$. (The last condition is fulfilled e.g. if H is weakly*-closed in $W_m^{1,\infty} \times F$. In fact, in this case we have $\overline{D(^*H)} = {}^{\perp}\{0\}$, cf. [2], Theorem 2.3.) The proof of Lemma 1.3 should be modified as follows:

Let $u \in \mathrm{cl}^*(N(H))$. Then for each $\varphi \in D(^*H) \subset {}^*F$ and each value $z \in {}^*H\varphi \subset W_m^{1,1}$ there exists a sequence $\{u_j^{(z)}\}_{j=1}^{\infty} \subset N(H)$ such that

$$[z, u_j^{(z)}]_W \to [z, u]_W \quad \text{as} \quad j \to \infty .$$

Consequently

$$[\varphi, Hu]_{*F} = [\varphi, H(u - u_j^{(z)})]_{*F} = [z, u - u_j^{(z)}]_W \to 0 ,$$

i.e. $[\varphi, Hu]_{*F} = 0$ for any $\varphi \in D(^*H)$. Since $\overline{D(^*H)} = {}^*F$, this implies that $Hu = 0$ and $u \in N(H)$.

2. PRE-ADJOINT RELATION

We want to find an analytic description of the pre-adjoint relation $^*L$ to L.

Let us assume 1.1.

2.1. Theorem. *The graph $G(^*L)$ of the pre-adjoint relation $^*L$ to L is the set of all couples $(y, v) \in L_m^1 \times L_m^1$ for which there exists $\psi \in L_m^1$ such that*

$$y + \psi \in W_m^{1,1} \;{}^{*)}, \tag{2,1}$$

$$v = \ell^+(y, \psi) := -(y + \psi)' + A^*y , \tag{2,2}$$

$$[y + \psi](b) = 0 \tag{2,3}$$

and

$$u^*(a)\,[y + \psi]\,(a) + \int_a^b u'^*\psi \,\mathrm{d}t = 0 \quad \textit{for all} \quad u \in D = D(L) . \tag{2,4}$$

Proof. a) Let $(y, v) \in L_m^1 \times L_m^1$ belong to $G(^*L)$. Then

$$0 = [y, \ell u]_L - [v, u]_L = \int_a^b [(u' + Au)^*\, y - u^*v]\,\mathrm{d}t = \tag{2,5}$$

$$= u^*(a) \int_a^b (A^*y - v)\,\mathrm{d}t + \int_a^b u'^* \left[y + \int_t^b (A^*y - v)\,\mathrm{d}\tau \right] \mathrm{d}t$$

*) The functions y, ψ are supposed to be defined everywhere on $[a, b]$.

for all $u \in D(L)$. Let $\psi \in L^1_m$ be such that

$$[y + \psi](t) + \int_t^b (A^*y - v)\,d\tau = 0 \quad \text{for any} \quad t \in [a, b].$$

Then $y + \psi \in W^{1,1}_m$, $[y + \psi](b) = 0$, $v = -(y + \psi)' + A^*y$ a.e. on $[a, b]$. Consequently, the couple (u, v) fulfils (2,1)–(2,3). Furthermore, since

$$\int_a^b (A^*y - v)\,dt = [y + \psi](a),$$

it follows from (2,5) that it fulfils also (2,4).

b) Let $(y, v) \in L^1_m \times L^1_m$ and let $\psi \in L^1_m$ be such that (2,1)–(2,4) hold. Then for any $u \in D(L)$ we have

$$\int_a^b u^*v\,dt = -\int_a^b u^*(y + \psi)'\,dt + \int_a^b u^*Ay\,dt =$$

$$= -u^*[y + \psi]|_a^b + \int_a^b u^{*\prime}[y + \psi]\,dt + \int_a^b u^*Ay\,dt =$$

$$= \int_a^b (u' + Au)^*\,y\,dt.$$

Hence $(y, v) \in G(^*L)$.

Let D_0' again denote the set of all derivatives $u' \in L^\infty_m$ of functions u from $D_0 = \{u \in D : u(a) = u(b) = 0\}$. Analogously as we obtained in the first part of this paper ([1]) the analytic description 4.6 of the adjoint relation L_0^* to the restriction L_0 of L on D_0 for the case $1 \leqq p < \infty$ from Theorem 4.5, we also can obtain in our present situation from Theorem 2.1 an analytic description of the pre-adjoint $^*L_0$ to L_0,

$$L_0 : u \in D_0 \to \ell u \in L^\infty_m \quad (D(L_0) = D_0).$$

2.2. Corollary. *$G(^*L_0)$ is the set of all $(y, v) \in L^1_m \times L^1_m$ for which there exists $\psi \in {}^\perp D_0'$ (the set of all $\chi \in L^1_m$ such that $[\chi, u']_L = 0$ for all $u \in D_0$) such that* (2,1) *and* (2,2) *hold.*

The following assertion is analogous to Theorem 4.8 of the first part [1] of this paper.

2.3. Theorem. *Let us assume* 1.1. *$G(^*L)$ is the set of all $(y, v) \in L^1_m \times L^1_m$ for which there exist $\zeta \in W^{1,1}_m$ and its derivative $\zeta' \in L^1_m$ such that*

(2,6) $$y + \zeta' \in W^{1,1}_m,$$

(2,7) $$v = \ell^+(y, \zeta') \quad \text{a.e. on} \quad [a, b],$$

(2,8) $$[y + \zeta'](a) = \zeta(a), \quad [y + \zeta'](b) = 0$$

and

$$(2,9) \qquad \zeta \in \overline{R(*H)} \quad \text{(the closure in } W_m^{1,1}) .$$

Proof. a) Let $y, v \in L_m^1$, $\zeta \in W_m^{1,1}$ and $\zeta' \in L_m^1$ be such that (2,6)–(2,9) hold. Obviously y, v and $\psi := \zeta'$ fulfil (2,1)–(2,3). Since H is weakly*-closed in $W_m^{1,\infty} \times F$, $\overline{R(*H)} = {}^{\perp}N(H) = {}^{\perp}D$ (with respect to the pairing $[.,.]_W$). Thus (2,9) implies that

$$u^*(a) [y + \psi] (a) + \int_a^b u^{*\prime}\psi \, dt = 0 \quad \text{for all} \quad u \in D ,$$

i.e. (2,4) holds and $(y, v) \in G(*L)$ according to Theorem 2.1.

b) On the other hand, if $(y, v) \in G(*L)$, then by Theorem 2.1 there exists $\psi \in L_m^1$ such that (2,1)–(2,4) hold. Let us put

$$(2,10) \qquad \zeta(a) = [y + \psi] (a) , \quad \zeta(t) = \zeta(a) + \int_a^t \psi \, d\tau \quad \text{on} \quad [a, b] .$$

Then the relations (2,6)–(2,8) follow directly from (2,1)–(2,3). Furthermore, we have by (2,4) and (2,10)

$$u^*(a) \zeta(a) + \int_a^b u'^*\zeta' \, dt = 0 \quad \text{for all} \quad u \in D .$$

It means that $\zeta \in {}^{\perp}D \subset W_m^{1,1}$ (with respect to the pairing $[.,.]_W$). Since ${}^{\perp}D = {}^{\perp}N(H) = \overline{R(*H)}$, the relation (2,9) follows immediately.

2.4. Remark. Notice that from the assumptions in 1.1 concerning H we have exploited in this section only the weak*-closedness of H in $W_m^{1,\infty} \times F$.

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