Generalized linear differential equations in a Banach space: Continuous dependence on a parameter

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Abstract

The paper deals with integral equations in a Banach space X of the form

$$x(t) = \widetilde{x} + \int_a^t \mathrm{d}[A]\, x + f(t) - f(a), \quad t \in [a, b], \tag{0.1}$$

where $-\infty < a < b < \infty$, $\widetilde{x} \in X$, $f : [a, b] \to X$ is regulated on $[a, b]$, and $A(t)$ is for each $t \in [a, b]$ a linear bounded operator on X, while the mapping $A: [a, b] \to L(X)$ has a bounded variation on $[a, b]$. Such equations are called generalized linear differential equations. Our aim is to present new results on the continuous dependence of solutions of such equations on a parameter. In particular, in Sections 3 and 4 we give sufficient conditions ensuring that the sequence $\{x_n\}$ of the solutions of generalized linear differential equations

$$x_n(t) = \widetilde{x}_n + \int_a^t \mathrm{d}[A_n]\, x_n + f_n(t) - f_n(a), \quad t \in [a, b], \quad n \in \mathbb{N},$$

tends to the solution x of (0.1). Crucial assumptions of Section 3 are the uniform boundedness of the variations $\mathrm{var}_a^b A_n$ of A_n and uniform convergence of A_n to A. In Section 4, we present the extension of the classical result by Opial to the case $X \neq \mathbb{R}^n$, i.e. we do not require the uniform boundedness of $var_a^b A_n$ while the uniform convergence is replaced by a properly stronger concept. Finally in Section 5 we present a partial result for the case when the uniform convergence of A_n to A is violated.

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1 Introduction

The theory of generalized differential equations enables the investigation of continuous and discrete systems, including the equations on time scales, from the common standpoint. This fact can be observed in several papers related to special kinds of equations, such as e.g. those by Imaz and Vorel [11], Oliva and Vorel [22], Federson and Schwabik [4], Schwabik [24] or Slavík [30].

This paper is devoted to generalized linear differential equations of the form ((0.1)) in a Banach space X. A complete theory in case of $X = \mathbb{R}^n$ can be found, for instance, in the monographs by Schwabik [24] or Schwabik, Tvrdý and Vejvoda [29]. See also the pioneering paper by Hildebrandt [9]. As concerns integral equations in a general Banach space, it is worth to highlight the monograph by Hönig [10] having as a background the interior (Dushnik) integral. On the other hand, dealing with the Kurzweil-Stieltjes integral, the contributions by Schwabik in [26] and [27] represent the base of this paper.

In the case $X = \mathbb{R}^n$, for ordinary differential equations, fundamental results on the continuous dependence of solutions on a parameter based on the averaging principle have been delivered by Krasnoselskii and Krejn [13], Kurzweil and Vorel [15], Kurzweil[16], Opial [23] and Kiguradze [12]. In particular, the problem of continuous dependence gave an inspiration to Kurzweil to introduce the notion of generalized differential equation in the papers [16] and [17]. For linear ordinary differential equations, the most general result seems to be that given by Opial. An interesting observation is contained in the fundamental paper by Artstein [1]. A different approach can be found in the papers [18]–[20] by Meng Gang and Zhang Meirong dealing also with measure differential analogues of Sturm-Liouville equations and, in particular, describing the weak and weak*continuous dependence of related Dirichlet or Neumann eigenvalues on a potential.

After Kurzweil, problem of the continuous dependence for generalized differential equations has been treated by several authors, see e.g. Schwabik [24], Ashordia [2], Fraňková [5], Tvrdý [33], Halas [6], Halas and Tvrdý [7]. Up to now, to our knowledge, only Federson and Schwabik [4] dealt with the case of a general Banach space X. Our aim is to prove new results valid also for $X \neq \mathbb{R}^n$ and such that, on the contrary to all the above mentioned papers, they cover also the Opial's result.

2 Preliminaries

Throughout these notes X is a Banach space and $L(X)$ is the Banach space of bounded linear operators on X. By $\|\cdot\|_X$ we denote the norm in X. Similarly, $\|\cdot\|_{L(X)}$ denotes the usual operator norm in $L(X)$.

Assume that $-\infty < a < b < +\infty$ and $[a,b]$ denotes the corresponding closed interval.

A set $D=\{\alpha_0,\alpha_1,\dots,\alpha_m\}\subset[a,b]$ is said to be a division of $[a,b]$ if

$$a=\alpha_0<\alpha_1<\ldots<\alpha_m=b\,.$$

The set of all divisions of $[a,b]$ is denoted by $\mathcal{D}[a,b]$.

A function $f:[a,b]\to X$ is called a finite step function on $[a,b]$ if there exists a division $D=\{\alpha_0,\alpha_1,\dots,\alpha_m\}$ of $[a,b]$ such that f is constant on every open interval (α_{j-1},α_j), $j=1,2,\dots,m$.

For an arbitrary function $f:[a,b]\to X$ we set

$$\|f\|_\infty=\sup_{t\in[a,b]}\|f(t)\|_X$$

and

$$\operatorname{var}_a^b f=\sup_{D\in\mathcal{D}[a,b]}\sum_{j=1}^m\|f(\alpha_j)-f(\alpha_{j-1})\|_X$$

is the variation of f over $[a,b]$. If $\operatorname{var}_a^b f<\infty$ we say that f is a function of bounded variation on $[a,b]$. $BV([a,b],X)$ denotes the Banach space of functions $f:[a,b]\to X$ of bounded variation on $[a,b]$ equipped with the norm $\|f\|_{BV}=\|f(a)\|_X+\operatorname{var}_a^b f$.

Given $f:[a,b]\to X$, the function f is called regulated on $[a,b]$ if, for each $t\in[a,b)$ there is $f(t+)\in X$ such that

$$\lim_{s\to t+}\|f(s)-f(t+)\|_X=0\,,$$

and for each $t\in(a,b]$ there is $f(t-)\in X$ such that

$$\lim_{s\to t-}\|f(s)-f(t-)\|_X=0\,.$$

By $G([a,b],X)$ we denote the set of all regulated functions $f:[a,b]\to X$. For $t\in[a,b)$, $s\in(a,b]$ we put $\Delta^+f(t)=f(t+)-f(t)$ and $\Delta^-f(s)=f(s)-f(s-)$. Recall that

$$BV([a,b],X)\subset G([a,b],X)$$

cf. e.g. [26, 1.5]. Moreover, it is known that regulated function are uniform limits of finite step functions (see [10, Theorem I.3.1]).

In what follows, by an integral we mean the Kurzweil-Stieltjes integral. Let us recall its definition.

As usual, a *partition* of $[a,b]$ is a tagged system, i.e., a couple $P=(D,\xi)$ where $D\in\mathcal{D}[a,b]$, $D=\{\alpha_0,\,\alpha_1,\dots,\,\alpha_m\}$, and $\xi=(\xi_1,\dots,\,\xi_m)\in[a,b]^m$ with

$$\alpha_{j-1}\le\xi_j\le\alpha_j,\quad j=1,2,\dots,m\,.$$

The set of all partitions of $[a,b]$ is denoted by $\mathcal{P}[a,b]$. Furthermore, any function $\delta:[a,b]\to(0,\infty)$ is called a *gauge* on $[a,b]$. Given a gauge δ, the partition P is called *δ-fine*

$$[\alpha_{j-1},\alpha_j]\subset(\xi_j-\delta(\xi_j),\xi_j+\delta(\xi_j))$$

We remark that for an arbitrary gauge δ on $[a,b]$ there always exists a δ-fine partition of $[a,b]$. It is stated by the Cousin lemma (see [24, Lemma 1.4]).

For given functions $F:[a,b]\to L(X)$ and $g:[a,b]\to X$ and a partition $P=(D,\xi)$ of $[a,b]$, where $D=\{\alpha_0,\alpha_1,\dots,\alpha_m\}$, $\xi=(\xi_1,\dots,\xi_m)$, we define

$$S(\mathrm{d}F,g,P)=\sum_{j=1}^{m}[F(\alpha_j)-F(\alpha_{j-1})]\,g(\xi_j)\,.$$

We say that $I\in X$ is the Kurzweil-Stieltjes integral (or shortly KS-integral) of g with respect to F on $[a,b]$ and denote

$$I=\int_a^b \mathrm{d}[F]\,g$$

if for every $\varepsilon>0$ there exists a gauge δ on $[a,b]$ such that

$$\Big\|S(\mathrm{d}F,g,P)-I\Big\|_X<\varepsilon \quad \text{for all} \quad \delta-\text{fine partitions} \quad P \text{ of } \; [a,b]\,.$$

Analogously, we define the integral $\displaystyle\int_a^b F\,\mathrm{d}[g]$ using sums of the form

$$S(F,\mathrm{d}g,P)=\sum_{j=1}^{m}F(\xi_j)\,[g(\alpha_j)-g(\alpha_{j-1})]$$

For the reader's convenience some of the further results needed later are summarized in the following assertions:

2.1 . Proposition. *Let $F:[a,b]\to L(X)$ and $g:[a,b]\to X$.*

(i) [25, Proposition 10]
Let $F\in BV([a,b],L(X))$ and $g:[a,b]\to X$ be such that $\displaystyle\int_a^b \mathrm{d}[F]\,g$ exists. Then

$$\Big\|\int_a^b \mathrm{d}[F]\,g\Big\|_X\le(\mathrm{var}_a^b F)\,\|g\|_\infty\,.$$

(ii) [21, Lemma 2.2]
Let $F \in G([a,b],L(X))$ *and* $g \in BV([a,b],X)$ *be such that* $\int_a^b \mathrm{d}[F]\,g$ *exists. Then*

$$\Big\| \int_a^b \mathrm{d}[F]\,g \Big\|_X \le 2\,\|F\|_\infty\,\|g\|_{BV}.$$

(iii) [28, Corollary 14]
If $F \in BV([a,b],L(X))$ *and* $g \in BV([a,b],X)$ *then both the integrals* $\int_a^b F\,\mathrm{d}[g]$ *and* $\int_a^b \mathrm{d}[F]\,g$ *exist, the sum*

$$\sum_{a\le\tau<b} \Delta^+ F(\tau)\,\Delta^+ g(\tau) - \sum_{a<\tau\le b} \Delta^- F(\tau)\,\Delta^- g(\tau)$$

converges in X *and the equality*

$$\int_a^b F\,\mathrm{d}[g] + \int_a^b \mathrm{d}[F]\,g$$
$$= F(b)\,g(b) - F(a)\,g(a) - \sum_{a\le t<b} \Delta^+ F(t)\,\Delta^+ g(t) + \sum_{a<t\le b} \Delta^- F(t)\,\Delta^- g(t)$$

is true.

(iv) [21, Theorem 2.11]
Let $F \in BV([a,b],L(X))$ *and let* $g\colon [a,b] \to X$ *be bounded and such that the integral* $\int_a^b \mathrm{d}[F]\,g$ *exists. Then both the integrals*

$$\int_a^b H(s)\,\mathrm{d}_s\Big[\int_a^s \mathrm{d}[F]\,g\Big] \quad \text{and} \quad \int_a^b H\,\mathrm{d}[F]\,g$$

exist and the equality

$$\int_a^b H(s)\,\mathrm{d}\Big[\int_a^t \mathrm{d}[F]\,g\Big] = \int_a^b H\,\mathrm{d}[F]\,g$$

holds for each $H \in G([a,b],L(X))$.

In addition, we need the following convergence result.

2.2 . Theorem. *Let* $g,\ g_n \in G([a,b],X),\ F, F_n \in BV([a,b],L(X))$ *for* $n \in \mathbb{N}$. *Assume that*

$$\lim_{n\to\infty} \|g_n - g\|_\infty = 0, \qquad \lim_{n\to\infty} \|F_n - F\|_\infty = 0$$

and

$$\varphi^* := \sup\{\mathrm{var}_a^b F_n\,;n\in\mathbb{N}\} < \infty.$$

Then,

$$\lim_{n\to\infty}\left(\sup\left\{\left\|\int_a^t \mathrm{d}[F_n]\,g_n - \int_a^t \mathrm{d}[F]\,g\right\|_X ; t\in[a,b]\right\}\right) = 0\,. \tag{2.1}$$

Proof. Let $\varepsilon > 0$ be given. By [10, Theorem I.3.1], we can choose a finite step function $\widetilde{g}:[a,b]\to X$ such that

$$\|g - \widetilde{g}\|_\infty < \varepsilon\,.$$

Furthermore, let $n_0 \in \mathbb{N}$ be such that

$$\|g_n - g\|_\infty < \varepsilon \quad \text{and} \quad \|F_n - F\|_\infty < \varepsilon \quad \text{for } n \ge n_0.$$

For a fixed $t\in[a,b]$, by Proposition 2.1 (i) and (ii), we obtain for $n \ge n_0$

$$\begin{aligned}
&\left\|\int_a^t \mathrm{d}[F_n]\,g_n - \int_a^t \mathrm{d}[F]\,g\right\|_X \\
&\quad\le \left\|\int_a^t \mathrm{d}[F_n]\,(g_n - \widetilde{g})\right\|_X + \left\|\int_a^t \mathrm{d}[F_n - F]\,\widetilde{g}\right\|_X + \left\|\int_a^t \mathrm{d}[F]\,(\widetilde{g} - g)\right\|_X \\
&\quad\le (\mathrm{var}_a^t F_n)\,\|g_n - \widetilde{g}\|_\infty + 2\,\|F_n - F\|_\infty\,\|\widetilde{g}\|_{BV} + (\mathrm{var}_a^t F)\,\|\widetilde{g} - g\|_\infty \\
&\quad\le \varphi^*\,\big(\|g_n - g\|_\infty + \|g - \widetilde{g}\|_\infty\big) + 2\,\|\widetilde{g}\|_{BV}\,\varepsilon + (\mathrm{var}_a^b F)\,\varepsilon \\
&\quad\le \big(2\,\varphi^* + 2\,\|\widetilde{g}\|_{BV} + \mathrm{var}_a^b F\big)\,\varepsilon = K\,\varepsilon,
\end{aligned}$$

where $K = \big(2\,\varphi^* + 2\,\|\widetilde{g}\|_{BV} + \mathrm{var}_a^b F\big) \in (0,\infty)$ does not depend on n. This proves (2.1). □

2.3. Remark. In the case that X is a Hilbert space, Theorem 2.2 has been already given by Krejčí and Laurençot [14, Proposition 3.1] or Brokate and Krejčí [3, Proposition 1.10].

3 Continuous dependence on a parameter in the case of uniformly bounded variations

Given $A\in BV([a,b],L(X)),\ f\in G([a,b],X)$ and $\widetilde{x}\in X$, consider the integral equation

$$x(t) = \widetilde{x} + \int_a^t \mathrm{d}[A]\,x + f(t) - f(a)\,, \quad t\in[a,b]\,. \tag{3.1}$$

A function $x\colon[a,b]\to X$ is called a solution of (3.1) on $[a,b]$ if the integral $\displaystyle\int_a^b \mathrm{d}[A]\,x$ exists and x satisfies the equality (3.1) for each $t\in[a,b]$.

For our purposes the following property is crucial

$$\big[I-\Delta^- A(t)\big]^{-1}\in L(X)\quad\text{for all}\quad t\in(a,b]\,. \tag{3.2}$$

In particular, taking into account the closing remark in [26] we can see that the following result is a particular case of [26, Proposition 2.10].

3.1 . Proposition. *Let $A\in BV([a,b],L(X))$ satisfy (3.2) Then, for every $\widetilde{x}\in X$ and every $f\in G([a,b],X)$, the equation (3.1) possesses a unique solution x on $[a,b]$ and $x\in G([a,b],X)$.*

Moreover, if A and f are left-continuous on $(a,b]$, then x is also left-continuous on $(a,b]$.

In addition, the following two important auxiliary assertions are true:

3.2. Lemma. *Let $A\in BV([a,b],L(X))$ satisfy (3.2), $f\in G([a,b],X)$ and $\widetilde{x}\in X$ and let x be the corresponding solution of (3.1) on $[a,b]$. Then*

$$\operatorname{var}_a^b(x-f)\le(\operatorname{var}_a^b A)\,\|x\|_\infty<\infty \tag{3.3}$$

$$c_A{:=}\,\sup\{\big\|[I-\Delta^- A(t)]^{-1}\big\|_{L(X)}\,;t\in(a,b]\}\in(0,\infty), \tag{3.4}$$

and

$$\|x(t)\|_X\le c_A\,(\|\widetilde{x}\|_X+\|f(a)\|_X+\|f\|_\infty)\exp\big(c_A\operatorname{var}_a^t A\big)\quad\textit{for } t\in[a,b]\,. \tag{3.5}$$

Proof. i) Let $D=\{\alpha_0,\alpha_1,\dots,\alpha_m\}$ be an arbitrary division of $[a,b]$. Then

$$\sum_{j=1}^m\Big\|x(\alpha_j)-f(\alpha_j)-x(\alpha_{j-1})+f(\alpha_{j-1})\Big\|_X$$
$$=\sum_{j=1}^m\Big\|\int_{\alpha_{j-1}}^{\alpha_j}\mathrm{d}[A]\,x\Big\|_X\le\sum_{j=1}^m\big[(\operatorname{var}_{\alpha_{j-1}}^{\alpha_j}A)\,\|x\|_\infty\big]=(\operatorname{var}_a^b A)\,\|x\|_\infty<\infty,$$

i.e. (3.3) is true.

ii) For $t\in(a,b]$ such that $\|\Delta^- A(t)\|_{L(X)}<\frac{1}{2}$ we have

$$\big\|[I-\Delta^- A(t)]^{-1}\big\|_{L(X)}\le\frac{1}{1-\|\Delta^- A(t)\|_{L(X)}}<2$$

(cf. e.g. [31, Lemma 4.1-C]). Therefore, $0\le c_A<\infty$ due to the fact that the set

$$\{t\in[a,b];\,\|\Delta^- A(t)\|_{L(X)}\ge\frac{1}{2}\}$$

has at most finitely many elements. As the case $c_A=0$ is impossible, this proves (3.4).

iii) Now, let x be a solution of (3.1). Put $B(a)=A(a)$ and $B(t)=A(t-)$ for $t\in(a,b]$. Then, by [26, Corollary 2.6] and [26, Proposition 2.7], we get

$$A-B\in BV([a,b],L(X)),\quad \operatorname{var}_a^b B\le \operatorname{var}_a^b A$$

and

$$A(t)-B(t)=\Delta^- A(t),\quad \int_a^t \mathrm{d}[A-B]\,x=\Delta^- A(t)\,x(t)\quad \text{for } t\in(a,b].$$

Consequently

$$[I-\Delta^- A(t)]\,x(t)=\widetilde{x}+\int_a^t \mathrm{d}[B]\,x+f(t)-f(a)\quad \text{for } t\in(a,b]$$

and (cf. Proposition 2.1 (i))

$$\|x(t)\|_X\le K_1+K_2\int_a^t \mathrm{d}[h]\,\|x\|_X\quad \text{for}\ \ t\in[a,b],$$

where

$$K_1=c_A\left(\|\widetilde{x}\|_X+\|f(a)\|_X+\|f\|_\infty\right),\quad K_2=c_A\quad \text{and}\quad h(t)=\operatorname{var}_a^t B.$$

The function h is nondecreasing and, since B is left-continuous on $(a,b]$, h is also left-continuous on $(a,b]$. Therefore we can use the generalized Gronwall inequality (see e.g. [29, Lemma I.4.30] or [24, Corollary 1.43]) to get the estimate (3.5). □

3.3. Lemma. *Let* $A,\ A_n\in BV([a,b],L(X)), n\in\mathbb{N}$, *be such that* (3.2) *and*

$$\lim_{n\to\infty}\|A_n-A\|_\infty=0 \tag{3.6}$$

are satisfied. Then

$$\big[I-\Delta^- A_n(t)\big]^{-1}\in L(X) \tag{3.7}$$

for all $t\in(a,b]$ *and all* $n\in\mathbb{N}$ *sufficiently large. Moreover, there is* $\mu^*\in(0,\infty)$ *such that*

$$c_{A_n}:=\sup\{\big\|[I-\Delta^- A_n(t)]^{-1}\big\|_{L(X)}\,;t\in(a,b]\}\le\mu^* \tag{3.8}$$

for all $n\in\mathbb{N}$ *sufficiently large.*

Proof. First, notice that, since $A\in BV([a,b],L(X))$, the set

$$D:=\{t\in(a,b];\|\Delta^- A(t)\|_{L(X)}\ge\tfrac{1}{4}\}$$

has at most a finite number of elements.

Let c_A be defined as in (3.4). Then, as by (3.6) $\lim\limits_{n\to\infty}\|\Delta^- A_n-\Delta^- A\|_\infty=0$, there is $n_0\in\mathbb{N}$ such that

$$\|\Delta^- A_n(t)-\Delta^- A(t)\|_{L(X)}<\tfrac{1}{4}\,\min\{1,\tfrac{1}{c_A}\}\quad \text{for } t\in[a,b]\quad \text{and } n\ge n_0. \tag{3.9}$$

Thus,

$$\|\Delta^- A_n(t)\|_{L(X)}\le\|\Delta^- A(t)\|_{L(X)}+\|\Delta^- A_n(t)-\Delta^- A(t)\|_{L(X)}<\tfrac{1}{2}\quad \text{for}\ \ t\in[a,b]\setminus D,\ n\ge n_0\,.$$

By [31, Lemma 4.1-C], this implies that

$$[I-\Delta^- A_n(t)] \text{ is invertible and } \|[I-\Delta^- A_n(t)]^{-1}\|_{L(X)} < 2 \text{ for } t \in [a,b] \setminus D \text{ and } n \geq n_0.$$

Notice that, due to (3.2), the relation

$$I-\Delta^- A_n(t) = [I-\Delta^- A(t)]\,\big[I-[I-\Delta^- A(t)]^{-1}\,(\Delta^- A_n(t)-\Delta^- A(t))\big] \tag{3.10}$$

holds for all $t \in [a,b]$ and $n \in \mathbb{N}$. Denote

$$T_n(t) := [I-\Delta^- A(t)]^{-1}\,(\Delta^- A_n(t)-\Delta^- A(t)) \quad \text{for} \quad n \in \mathbb{N} \text{ and } \quad t \in [a,b].$$

Then (3.10) means that, $I-\Delta^- A_n(t)$ is invertible if and only if $I-T_n(t)$ is invertible.

Now, let $t \in D$ and $n \geq n_0$ be given. Then, due to (3.4) and (3.9), we have $\|T_n(t)\|_{L(X)} < \frac{1}{4}$. Consequently, by [31, Lemma 4.1-C], $I-T_n(t)$ and therefore also $[I-\Delta^- A_n(t)]$ are invertible. Moreover, taking into account (3.4) and (3.10), we can see that

$$\|[I-\Delta^- A_n(t)]^{-1}\|_{L(X)} \leq \tfrac{4}{3}\,c_A < 2\,c_A$$

is true.

To summarize, there exists $n_0 \in \mathbb{N}$ such that

$$[I-\Delta^- A_n(t)] \text{ is invertible and } \|[I-\Delta^- A_n(t)]^{-1}\|_{L(X)} \leq \mu^* = 2\,\max\{1, c_A\}$$

for all $t \in (a,b]$ and $n \geq n_0$. This completes the proof. □

The main result of this section is the following Theorem, which generalizes in a linear case the recent results by Federson and Schwabik [4]) and covers the results for generalized linear differential equations known for the case $X = \mathbb{R}^n$. Unlike [2], to prove it we do not utilize the variation-of-constants formula. Therefore it is not necessary to assume the additional condition

$$[I-\Delta^+ A(t)]^{-1} \in L(X), \quad t \in [a,b].$$

3.4. Theorem. *Let A, $A_n \in BV([a,b], L(X))$, f, $f_n \in G([a,b], X)$, $\widetilde{x}$, $\widetilde{x}_n \in X$ for $n \in \mathbb{N}$. Furthermore, let A satisfy (3.2), (3.6),*

$$\alpha^* := \sup\{\operatorname{var}_a^b A_n\,;\, n \in \mathbb{N}\} < \infty \tag{3.11}$$

and

$$\lim_{n\to\infty} \|\widetilde{x}_n - \widetilde{x}\|_X = 0 \quad \textit{and} \quad \lim_{n\to\infty} \|f_n - f\|_\infty = 0. \tag{3.12}$$

Then equation (3.1) has a unique solution x on $[a,b]$. Furthermore, for each $n \in \mathbb{N}$ large enough there is a unique solution x_n on $[a,b]$ to the equation

$$x_n(t) = \widetilde{x}_n + \int_a^t \mathrm{d}[A_n]\,x_n + f_n(t) - f_n(a)\,, \quad t \in [a,b] \tag{3.13}$$

and $\lim_{n\to\infty} \|x_n - x\|_\infty = 0$.

Proof. Due to (3.2) equation (3.1) has a unique solution x on $[a,b]$. Furthermore, by Lemma 3.2, there is $n_0 \in \mathbb{N}$ such that (3.7) is true for $n \geq n_0$. Hence, for each $n \geq n_0$, equation (3.13) possesses a unique solution x_n on $[a,b]$. Set

$$w_n = (x_n - f_n) - (x - f) \tag{3.14}$$

Then

$$w_n(t) = \widetilde{w}_n + \int_a^t \mathrm{d}[A_n]\, w_n + h_n(t) - h_n(a) \quad \text{for } n \in \mathbb{N} \text{ and } t \in [a,b],$$

where $\widetilde{w}_n = (\widetilde{x}_n - f_n(a)) - (\widetilde{x} - f(a))$ and

$$h_n(t) = \int_a^t \mathrm{d}[A_n - A]\,(x - f) + \Big(\int_a^t \mathrm{d}[A_n]\, f_n - \int_a^t \mathrm{d}[A]\, f \Big).$$

First, notice that according to (3.12) we have

$$\lim_{n\to\infty} \|\widetilde{w}_n\|_X = 0. \tag{3.15}$$

Furthermore, in view of Theorem 2.2, we have

$$\lim_{n\to\infty} \Big\| \int_a^t \mathrm{d}[A_n]\, f_n - \int_a^t \mathrm{d}[A]\, f \Big\|_X = 0.$$

Moreover, since $(x - f) \in BV([a,b],X)$ by (3.3), we get by Proposition 2.1 (ii)

$$\Big\| \int_a^t \mathrm{d}[A_n - A]\,(x - f) \Big\|_X \leq 2\,\|A_n - A\|_\infty\,\|x - f\|_{BV} \quad \text{for all } t \in [a,b].$$

Having in mind (3.6), we can see that the relation

$$\lim_{n\to\infty} \Big\| \int_a^t \mathrm{d}[A_n - A]\,(x - f) \Big\|_X = 0$$

holds. To summarize,

$$\lim_{n\to\infty} \|h_n\|_\infty = 0. \tag{3.16}$$

By (3.11) and by Lemmas 3.2 and 3.3 we have

$$\|w_n(t)\|_X \leq \mu^* \left(\|\widetilde{w}_n\|_X + \|h_n\|_\infty \right) \exp\left(\mu^* \operatorname{var}_a^b A_n \right) \quad \text{for all } t \in [a,b].$$

Consequently, using (3.15) and (3.16) we deduce that

$$\lim_{n\to\infty} \|w_n\|_X = 0.$$

Now, by (3.12) and (3.14) we conclude finally that $\lim\limits_{n\to\infty} \|x_n - x\|_\infty = 0.$ □

We will close this section by a comparison of Theorem 3.4 with two similar results presented for $\dim X < \infty$ by Schwabik in [24]. First, when restricted to the linear case, Theorem 8.2 from [24] modifies to

3.5 . Theorem. *Let $A,\ A_n \in BV([a,b],L(X))$ and $f_n(t) - f_n(a) = f(t) - f(a) = 0$ for $n \in \mathbb{N}$ and $t \in [a,b]$. Further, let a nondecreasing function $h : [a,b] \to \mathbb{R}$ be given such that*

$$\lim_{n\to\infty} A_n(t) = A(t) \quad \text{on} \quad [a,b], \tag{3.17}$$

$$\left\{ \begin{array}{c} \|A_n(t_2) - A_n(t_1)\|_{L(X)} \le |h(t_2) - h(t_1)|,\ \|A(t_2) - A(t_1)\|_{L(X)} \le |h(t_2) - h(t_1)| \\ \text{for } t_1, t_2 \in [a,b] \text{ and } n \in \mathbb{N}. \end{array} \right. \tag{3.18}$$

Let x_n, $n \in \mathbb{N}$, be solutions of (3.13) and let

$$\lim_{n\to\infty} \|x_n(t) - x(t)\|_X \quad \text{for } t \in [a,b].$$

Then $x \in BV([a,b],X)$ is a solution of (3.1) on $[a,b]$.

3.6 . Proposition. *Under the assumptions of Theorem 3.5 the relations (3.6) and (3.11) are satisfied.*

Proof. i) The relation (3.11) follows immediately from (3.18).

ii) Notice that (3.17) and (3.18) imply that

$$\left\{ \begin{array}{l} \|A_n(t-) - A_n(s)\|_{L(X)} \le |h(t-) - h(s)|,\ \|A(t-) - A(s)\|_{L(X)} \le |h(t-) - h(s)| \\ \qquad \text{for } t \in (a,b],\ s \in [a,b],\ n \in \mathbb{N}, \end{array} \right. \tag{3.19}$$

and

$$\left\{ \begin{array}{l} \|A_n(t+) - A_n(s)\|_{L(X)} \le |h(t+) - h(s)|,\ \|A(t+) - A(s)\|_{L(X)} \le |h(t+) - h(s)| \\ \qquad \text{for } t \in [a,b),\ s \in [a,b],\ n \in \mathbb{N}. \end{array} \right. \tag{3.20}$$

iii) Let $\varepsilon > 0$ and $t \in (a,b]$ be given and let us choose $s_0 \in (a,t)$ and $n_0 \in \mathbb{N}$ so that

$$|h(t-) - h(s_0)| < \frac{\varepsilon}{3} \quad \text{and} \quad \|A_n(s_0) - A(s_0)\|_{L(X)} < \frac{\varepsilon}{3} \quad \text{for } n \ge n_0. \tag{3.21}$$

Then, by (3.19) and (3.21),

$$\begin{aligned} \|A_n(t-) - A(t-)\|_{L(X)} &\le \|A_n(t-) - A_n(s_0)\|_{L(X)} + \|A_n(s_0) - A(s_0)\|_{L(X)} \\ &\quad + \|A(s_0) - A(t-)\|_{L(X)} \\ &< |h(t-) - h(s_0)| + \frac{\varepsilon}{3} + |h(t-) - h(s_0)| < \varepsilon. \end{aligned}$$

This means that

$$\lim_{n\to\infty} A_n(t-) = A(t-) \quad \text{holds for } t \in (a,b]. \tag{3.22}$$

Similarly, using (3.20) we get

$$\lim_{n\to\infty} A_n(t+) = A(t+) \quad \text{holds for } t \in [a,b). \tag{3.23}$$

iv) Now, suppose that (3.6) is not valid. Then there is $\widetilde{\varepsilon} > 0$ such that for any $\ell \in \mathbb{N}$ there exist $m_\ell \geq \ell$ and $t_\ell \in [a,b]$ such that

$$\|A_{m_\ell}(t_\ell) - A(t_\ell)\|_{L(X)} \geq \widetilde{\varepsilon}. \tag{3.24}$$

We may assume that $m_{\ell+1} > m_\ell$ for any $\ell \in \mathbb{N}$ and

$$\lim_{\ell\to\infty} t_\ell = t_0 \in [a,b]. \tag{3.25}$$

Let $t_0 \in (a,b]$ and assume that the set of those $\ell \in \mathbb{N}$ for which $t_\ell \in (a,t_0)$ has infinitely many elements, i.e. there is a sequence $\{\ell_k\} \subset \mathbb{N}$ such that $t_{\ell_k} \in (a,t_0)$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} t_{\ell_k} = t_0$. Denote $s_k = t_{\ell_k}$ and $B_k = A_{m_{\ell_k}}$ for $k \in \mathbb{N}$. Then, in view of (3.24), we have

$$s_k \in (a,t_0) \quad \text{for } k \in \mathbb{N}, \quad \lim_{k\to\infty} s_k = t_0 \tag{3.26}$$

and

$$\|B_k(s_k) - A(s_k)\|_{L(X)} \geq \widetilde{\varepsilon} \quad \text{for } k \in \mathbb{N}. \tag{3.27}$$

By (3.19), we have

$$\|A(t_0-) - A(s_k)\|_{L(X)} \leq h(t_0-) - h(k_n)$$

and

$$\|B_k(t_0-) - B_k(s_k)\|_{L(X)} \leq h(t_0-) - h(k_n)$$

for $k \in \mathbb{N}$. Therefore, by (3.22) and since $\lim\limits_{k\to\infty}(h(t_0-) - h(s_k)) = 0$ due to (3.26), we can choose $k_0 \in \mathbb{N}$ so that

$$\|B_{k_0}(t_0-) - A(t_0-)\|_{L(X)} < \frac{\widetilde{\varepsilon}}{3}$$

$$\|A(t_0-) - A(s_{k_0})\|_{L(X)} \leq h(t_0-) - h(s_{k_0}) < \frac{\widetilde{\varepsilon}}{3}$$

and

$$\|B_{k_0}(t_0-) - B_{k_0}(s_{k_0})\|_{L(X)} < \frac{\widetilde{\varepsilon}}{3}.$$

As a consequence, we get finally by (3.27)

$$\begin{aligned}\widetilde{\varepsilon} &\leq \|B_{k_0}(s_{k_0}) - A(s_{k_0})\|_{L(X)} \\ &\leq \|B_{k_0}(s_{k_0}) - B_{k_0}(t_0-)\|_{L(X)} + \|B_{k_0}(t_0-) - A(t_0-)\|_{L(X)} + \|A(t_0-) - A(s_{k_0})\|_{L(X)} < \widetilde{\varepsilon},\end{aligned}$$

a contradiction.

If $t_0 \in [a,b)$ and the set of those $\ell \in \mathbb{N}$ for which $t_\ell \in (a,t_0)$ has only finitely many elements, then there is a sequence $\{\ell_k\} \subset \mathbb{N}$ such that $t_{\ell_k} \in (t_0,b)$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} t_{\ell_k} = t_0$. As before, let $s_k = t_{\ell_k}$ and $B_k = A_{m_{\ell_k}}$ for $k \in \mathbb{N}$ and notice that

$$s_k \in (t_0,b) \quad \text{for } k \in \mathbb{N}, \quad \lim_{k\to\infty} s_k = t_0$$

and (3.27) are true. Arguing similarly as before we get that there is $k_0 \in \mathbb{N}$ such that

$$\begin{aligned}\widetilde{\varepsilon} &\le \|B_{k_0}(s_{k_0}) - A(s_{k_0})\|_{L(X)} \\ &\le \|B_{k_0}(s_{k_0}) - B_{k_0}(t_0+)\|_{L(X)} + \|B_{k_0}(t_0+) - A(t_0+)\|_{L(X)} + \|A(t_0+) - A(s_{k_0})\|_{L(X)} < \widetilde{\varepsilon},\end{aligned}$$

a contradiction. □

Similarly, when restricted to the linear case, Theorem 8.8 from [24] modifies to

3.7. Theorem. *Let $A, A_n \in BV([a,b], X)$, $f_n(t) - f_n(a) = f(t) - f(a) = 0$ for $n \in \mathbb{N}$ and $t \in [a,b]$. Furthermore, let (3.2) hold and let x be the corresponding solution of (3.1). Finally, let scalar nondecreasing and left-continuous on $(a,b]$ functions h_n, $n \in \mathbb{N}$, and h be given such that h is continuous on $[a,b]$ and*

$$\lim_{n\to\infty} A_n(t) = A(t) \quad \text{on} \quad [a,b], \tag{3.28}$$

$$\begin{cases} \|A_n(t_2) - A_n(t_1)\|_{L(X)} \le |h_n(t_2) - h_n(t_1)|, \ \|A(t_2) - A(t_1)\|_{L(X)} \le |h(t_2) - h(t_1)| \\ \qquad \text{for all } \ t_1, t_2 \in [a,b] \ \text{and} \ n \in \mathbb{N}, \end{cases} \tag{3.29}$$

$$\begin{cases} \limsup_{n\to\infty} \big[h_n(t_2) - h_n(t_1)\big] \le h(t_2) - h(t_1) \\ \qquad \text{whenever} \ a \le t_1 \le t_2 \le b. \end{cases} \tag{3.30}$$

Then, for any $n \in \mathbb{N}$ sufficiently large, equation (3.13) has a unique solution x_n on $[a,b]$ and

$$\lim_{n\to\infty} x_n(t) = x(t) \quad \text{uniformly on} \quad [a,b].$$

3.8. Proposition. *Under the assumptions of Theorem 3.7 the relations (3.6) and (3.11) are satisfied.*

Proof (taken from [33]). i) By (3.30) there is $n_0 \in \mathbb{N}$ such that

$$h_n(b) - h_n(a) \le h(b) - h(a) + 1 \quad \text{for all} \quad n \ge n_0.$$

Hence for any $n \in \mathbb{N}$ we have

$$\operatorname{var} A_n \le \alpha_0 = \max\Big(\big\{\operatorname{var} A_n\,; n \le n_0\big\} \cup \big\{h(b) - h(a) + 1\big\}\Big) < \infty.$$

Thus we conclude that (3.11) is true.

ii) Suppose that (3.6) does not hold. Then there is $\widetilde{\varepsilon} > 0$ such that for any $\ell \in \mathbb{N}$ there exist $m_\ell \ge \ell$ and $t_\ell \in [a,b]$ such that

$$\|A_{m_\ell}(t_\ell) - A(t_\ell)\|_{L(X)} \ge \widetilde{\varepsilon}. \tag{3.31}$$

We may assume that $m_{\ell+1} > m_\ell$ for any $\ell \in \mathbb{N}$ and

$$\lim_{\ell\to\infty} t_\ell = t_0 \in [a,b]. \tag{3.32}$$

Let $t_0 \in (a,b)$ and let an arbitrary $\varepsilon > 0$ be given. Since h is continuous, we may choose $\eta > 0$ in such a way that $t_0 - \eta, t_0 + \eta \in [a,b]$ and

$$h(t_0 + \eta) - h(t_0 - \eta) < \varepsilon. \tag{3.33}$$

Furthermore, by (3.28) there is $\ell_1 \in \mathbb{N}$ such that

$$\|A_{m_\ell}(t_0) - A(t_0)\|_{L(X)} < \varepsilon \quad \text{for all } \ \ell \geq \ell_1 \tag{3.34}$$

and by (3.29), (3.30) and (3.33) there is $\ell_2 \in \mathbb{N}$, $\ell_2 \geq \ell_1$, such that

$$\|A_{m_\ell}(\tau_2) - A_{m_\ell}(\tau_1)\|_{L(X)} \leq h(t_0 + \eta) - h(t_0 - \eta) + \varepsilon < 2\,\varepsilon \\ \text{whenever } \ \tau_1, \tau_2 \in (t_0 - \eta, t_0 + \eta) \text{ and } \ \ell \geq \ell_2. \tag{3.35}$$

The relations (3.28) and (3.35) imply immediately that

$$\left\{ \begin{array}{c} \|A(\tau_2) - A(\tau_1)\|_{L(X)} = \lim\limits_{\ell \to \infty} \|A_{m_\ell}(\tau_2) - A_{m_\ell}(\tau_1)\|_{L(X)} \leq 2\varepsilon \\ \text{whenever } \ \tau_1, \tau_2 \in (t_0 - \eta, t_0 + \eta). \end{array} \right. \tag{3.36}$$

Finally, let $\ell_3 \in \mathbb{N}$ be such that $\ell_3 \geq \ell_2$ and

$$|t_\ell - t_0| < \eta \quad \text{for all } \ \ell \geq \ell_3, \tag{3.37}$$

then in virtue of the relations (3.32)–(3.37) we have

$$\begin{aligned} &\|A_{m_\ell}(t_\ell) - A(t_\ell)\|_{L(X)} \\ &\qquad \leq \|A_{m_\ell}(t_\ell) - A_{m_\ell}(t_0)\|_{L(X)} + \|A_{m_\ell}(t_0) - A(t_0)\|_{L(X)} + \|A(t_0) - A(t_\ell)\|_{L(X)} \\ &\qquad \leq 5\,\varepsilon. \end{aligned}$$

Hence, choosing $\varepsilon < \frac{1}{5}\,\widetilde{\varepsilon}$, we obtain by (3.31) that

$$\widetilde{\varepsilon} > \|A_{m_\ell}(t_\ell) - A(t_\ell)\|_{L(X)} \geq \widetilde{\varepsilon}.$$

This being impossible, the relation (3.6) has to be true. The modification of the proof in the cases $t_0 = a$ or $t_0 = b$ is obvious. □

4 Continuous dependence on a parameter in the case of variations bounded with a weight

In this section we restrict ourselves to homogeneous generalized linear differential equations

$$x(t) = \widetilde{x} + \int_a^t \mathrm{d}[A]\,x, \quad t \in [a,b], \tag{4.1}$$

where, as before, $A \in BV([a,b], L(X))$ and $\widetilde{x} \in X$. As in the previous section we will assume that the fundamental existence assumption (3.2) is satisfied.

The main result of this section extends that obtained by Z. Opial for the case $\dim X < \infty$ in [23]. To this aim, we recall an estimate presented in [21].

4.1. Lemma. *If* $F\in G([a,b],L(X))$ *and* $G\in BV([a,b],L(X))$ *then*

$$\sum_{t\in[a,b)} \|\Delta^+F(t)\Delta^+G(t)\|_{L(X)} + \sum_{t\in(a,b]} \|\Delta^-F(t)\Delta^-G(t)\|_{L(X)} \le 2\,\|F\|_\infty \operatorname{var}_a^b G\,. \tag{4.2}$$

4.2. Theorem. *Let* $A,\ A_n \in BV([a,b],L(X))$ *and* $\widetilde{x},\ \widetilde{x}_n\in X$ *for* $n\in\mathbb{N}$*. Assume* (3.2) *and*

$$\lim_{n\to\infty} \|A_n - A\|_\infty \left(1+\operatorname{var}_a^b A_n\right) = 0 \tag{4.3}$$

and

$$\lim_{n\to\infty} \|\widetilde{x}_n - \widetilde{x}\|_X = 0. \tag{4.4}$$

Then (4.1) *has a unique solution* x *on* $[a,b]$*. Moreover, for each* $n\in\mathbb{N}$ *sufficiently large, the equation*

$$x_n(t) = \widetilde{x}_n + \int_a^t \mathrm{d}[A_n]\,x_n\,,\quad t\in[a,b] \tag{4.5}$$

has a unique solution x_n *on* $[a,b]$ *and* $\lim\limits_{n\to\infty}\|x_n-x\|_\infty = 0.$

Proof. First, notice that, since

$$\|A_n - A\|_\infty \le \|A_n - A\|_\infty \left(1+\operatorname{var}_a^b A_n\right) \quad \text{for all} \quad n\in\mathbb{N},$$

(4.3) implies (3.6). Therefore, by Lemma 3.3, there is $n_0\in\mathbb{N}$ such that (3.7) holds for each $t\in(a,b]$ and each $n\ge n_0$.

Assume $n\ge n_0$. Let x and x_n be the solutions on $[a,b]$ of (4.1) and (4.5), respectively. Then

$$(x_n(t)-x(t)) = (\widetilde{x}_n - \widetilde{x}) + \int_a^t \mathrm{d}[A]\,(x_n-x) + h_n(t) \quad \text{for} \quad t\in[a,b]\,, \tag{4.6}$$

where

$$h_n(t) = \int_a^t \mathrm{d}[A_n - A]\,x_n \quad \text{for} \quad t\in[a,b]\,. \tag{4.7}$$

By Lemma 3.2 we have

$$\|x_n - x\|_\infty \le c_A\,(\|\widetilde{x}_n-\widetilde{x}\|_X + \|h_n\|_\infty)\,\exp\,(c_A \operatorname{var}_a^b A). \tag{4.8}$$

(Notice that $h_n(a)=0$ for all k.) Thus, in view of the assumption (4.4), to prove the assertion of the theorem, we have to show that $\lim_{n\to\infty}\|h_n\|_\infty = 0$.

To this aim, we integrate by parts (cf. Proposition 2.1 (iii)) in the right-hand side of (4.7) and use Substitution Formula (cf. Proposition 2.1 (iv)). Then we get

$$h_n(t) = [A_n(t)- A(t)]\,x_n(t) - [A_n(a)-A(a)]\,\widetilde{x}_n - \int_a^t (A_n-A)\,\mathrm{d}[A_n]\,x_n - \Delta_a^t(A_n-A,x_n) \tag{4.9}$$

for $t\in[a,b]$, where

$$\Delta_a^t(A_n-A,x_n) = \sum_{a\le s<t} [\Delta^+(A_n(s)-A(s))\,\Delta^+x_n(s)] - \sum_{a<s\le t} [\Delta^-(A_n(s)-A(s))\,\Delta^-x_n(s)]\,. \tag{4.10}$$

Inserting the relations (cf. [26, Proposition 2.3])

$$\Delta^+ x_n(t) = \Delta^+ A_n(t)\, x_n(t) \quad \text{for } t\in[a,b) \quad \text{and} \quad \Delta^- x_n(t) = \Delta^- A_n(t)\, x_n(t) \quad \text{for } t\in(a,b]$$

into the right-hand side of (4.10) and using Lemma 4.1, we obtain the estimates

$$\|\Delta_a^t(A_n - A, x_n)\|_X \le 2\,\|A_n - A\|_\infty\,(\operatorname{var}_a^t A_n)\,\|x_n\|_\infty \quad \text{for } t\in[a,b]\,.$$

Hence

$$\|h_n(t)\|_X \le \|A_n - A\|_\infty\,\big(2 + 3\,(\operatorname{var}_a^t A_n)\big)\,\|x_n\|_\infty,$$

that is,

$$\|h_n\|_\infty \le \alpha_n\,\|x_n\|_\infty, \tag{4.11}$$

where $\alpha_n = \|A_n - A\|_\infty\Big(2+3\operatorname{var}_a^b A_n\Big)$. Note that, due to (4.3), we have

$$\lim_{n\to\infty} \alpha_n = 0. \tag{4.12}$$

We can see that to show that $\lim_{n\to\infty}\|h_n\|_\infty = 0$, it is sufficient to prove that the sequence $\{\|x_n\|_\infty\}$ is bounded. By (4.8) and (4.11) we have

$$\|x_n\|_\infty \le \|x_n - x\|_\infty + \|x\|_\infty \le c_A\,\Big(\|\widetilde{x}_n - \widetilde{x}\|_X + \alpha_n\|x_n\|_\infty\Big)\,\exp\,(c_A\operatorname{var}_a^b A) + \|x\|_\infty\,.$$

Hence

$$\big(1 - c_A\,\alpha_n\,\exp\,(c_A\operatorname{var}_a^b A)\big)\,\|x_n\|_\infty \le c_A\,\|\widetilde{x}_n - \widetilde{x}\|_X\,\exp\,(c_A\operatorname{var}_a^b A) + \|x\|_\infty \quad \text{for} \quad n \ge n_0.$$

By (4.4) and (4.12), there is $n_1 \ge n_0$ such that

$$\|\widetilde{x}_n - \widetilde{x}\|_X < 1 \quad \text{and} \quad c_A\,\alpha_n\,\exp\,(c_A\operatorname{var}_a^b A) < \tfrac{1}{2} \quad \text{for} \quad n \ge n_1.$$

In particular,

$$\|x_n\|_\infty < 2\,\big(c_A\,\exp\,(c_A\operatorname{var}_a^b A) + \|x\|_\infty\,\big) \quad \text{for} \quad n \ge n_1,$$

i.e. the sequence $\{\|x_n\|_\infty\}$ is bounded and this completes the proof. □

4.3. Remark. In comparison with Theorem 3.4, the uniform boundedness of variation (3.11) was not needed in Theorem 4.2. On the other hand, if (3.11) is assumed, Theorem 4.2 reduces to Theorem 3.4.

Let us note that, on the contrary to the finite dimensional case, in the case of a general Banach space X it is not possible to extend easily the convergence result Theorem 4.2 to the the nonhomogeneous equations.

5 Emphatic convergence

In this section we deal with the case that the uniform convergence is violated. The assumptions of Theorems 5.1 and 5.2 are related to the notion of emphatic convergence introduced by Kurzweil in [17]. More precisely, together with the locally uniform convergence we infer some control condition for points sufficiently close to the end points a, b of the interval $[a,b]$. These results extend the work of Halas and Tvrdý dealing with $X=\mathbb{R}^n$ (c.f. [6], [8] and [34]).

If $\{f_n\}$ is a sequence of X-valued functions defined on $[a,b]$, we say that it tends to f *locally uniformly on* $J\subset[a,b]$ if

$$\lim_{n\to\infty}\left(\sup\{\|f_n(t)-f(t)\|_X;t\in I\}\right)=0$$

for all closed subintervals $I\subset J$. In such a case we write $f_n\rightrightarrows f$ locally on J.

Of course, $f_n\rightrightarrows f$ locally on J implies

$$\lim_{n\to\infty}\|f_n(t)-f(t)\|_X=0 \quad \text{for all interior points } \ t \text{ of } \ J.$$

5.1. Theorem. *Let* A, $A_n\in BV([a,b],L(X))$, f, $f_n\in G([a,b],X)$, $\widetilde{x}$, $\widetilde{x}_n\in X$ *for* $n\in\mathbb{N}$. *Assume* (3.2), (4.4),

$$A_n\rightrightarrows A \text{ locally on } (a,b] \quad \text{and} \quad f_n\rightrightarrows f \text{ locally on } (a,b], \tag{5.1}$$

and that there is $N\in\mathbb{N}$ *such that* (3.7) *is true for all* $t\in(a,b]$ *and all* $n\in\mathbb{N}$ *such that* $n\ge N$.

Then, for $n\in\mathbb{N}$ *sufficiently large, there exist unique solutions* x *and* x_n *on* $[a,b]$ *to* (3.1) *and* (3.13), *respectively.*

In addition, let (3.11) *and*

$$\left.\begin{array}{l}\forall\,\varepsilon>0\ \exists\,\delta>0 \text{ such that } \forall\,t\in(a,a+\delta)\ \exists\,n_0\in\mathbb{N} \text{ such that}\\ \left\|\Big(\Delta^+A(a)\,\widetilde{x}+\Delta^+f(a)\Big)-\Big(x_n(t)-\widetilde{x}_n\Big)\right\|_X<\varepsilon \quad \text{for } n\ge n_0\end{array}\right\} \tag{5.2}$$

hold. Then

$$\lim_{n\to\infty}\|x_n(t)-x(t)\|_X=0 \quad \text{for any} \ \ t\in[a,b]$$

and $x_n\rightrightarrows x$ *locally on* $(a,b]$.

Proof. Without any loss of generality we may assume $A_n(a)=A(a)=0$ and $f_n(a)=f(a)=0$ for $n\in\mathbb{N}$. Due to assumptions (3.2) and (3.7), the existence and uniqueness of solutions to (3.1) and (3.13) are guaranteed by Proposition 3.1 . Denote by x and x_n the corresponding solutions.

Let $\varepsilon>0$ be given. Then, as x is regulated, there is $\delta_0>0$ such that

$$\|x(s)-x(a+)\|_X<\varepsilon \qquad \text{for all } s\in(a,a+\delta_0).$$

Furthermore, by (4.4) there is $n_1 \geq N$ such that

$$\|\widetilde{x}_n - \widetilde{x}\|_X < \varepsilon \quad \text{for } n \geq n_1 .$$

By (5.2) there is $\delta \in (0, \delta_0)$ such that for each $t \in (a, a+\delta)$ we can find $n_0 \geq n_1$ so that

$$\big\|\big(\Delta^+ A(a)\,\widetilde{x} + \Delta^+ f(a)\big) - \big(x_n(t) - \widetilde{x}_n\big)\big\|_X < \varepsilon \quad \text{for } \ n \geq n_0 .$$

To summarize, for any $t \in (a, a+\delta_0)$ and $n \geq n_0$, we have

$$\begin{aligned}
&\|x(t) - x_n(t)\|_X \\
&\qquad \leq \|x(t) - x(a+)\|_X + \|x(a+) - \widetilde{x} + \widetilde{x}_n - x_n(t)\|_X + \|\widetilde{x} - \widetilde{x}_n\|_X \\
&\qquad = \|x(t) - x(a+)\|_X + \|\Delta^+ A(a)\,\widetilde{x} + \Delta^+ f(a) + \widetilde{x}_n - x_n(t)\|_X + \|\widetilde{x} - \widetilde{x}_n\|_X < 3\,\varepsilon.
\end{aligned}$$

This implies also that $\lim_{n\to\infty} \|x_n(t) - x(t)\|_X = 0$ for all $t \in [a, a+\delta)$.

Now, let an arbitrary $c \in (a, a+\delta)$ be given. Then

$$\lim_{n\to\infty} x_n(c) = x(c).$$

Therefore, by Theorem 3.4 and due to the uniqueness of solutions to

$$x_n(t) = x_n(c) + \int_c^t \mathrm{d}[A_n]\,x_n + f_n(t) - f_n(c)\,, \quad t \in [c, b]$$

and

$$x(t) = x(c) + \int_c^t \mathrm{d}[A]\,x + f(t) - f(c)\,, \quad t \in [c, b],$$

x_n tend to x uniformly on $[c, b\,]$ as $n \to \infty$. More precisely,

$$\lim_{n\to\infty} \big(\sup\{\|x_n(t) - x(t)\|_X ; t \in [c, b]\}\big) = 0\,.$$

Since c was arbitrary, this means that $x_n(t) \to x(t)$ for each $t \in [a, b\,]$ and $x_n \rightrightarrows x$ locally on $(a, b\,]$. □

The result symmetrical to the previous theorem slightly differs. However, its proof is very similar.

5.2. Theorem. *Let $A,\ A_n \in BV([a,b\,], L(X)),\ f,\ f_n \in G([a,b\,], X),\ \widetilde{x},\ \widetilde{x}_n \in X$ for $n \in \mathbb{N}$. Assume (3.2), (4.4) and*

$$A_n \rightrightarrows A \ \textit{locally on } [a, b) \quad \textit{and} \quad f_n \rightrightarrows f \ \textit{locally on } [a, b) \tag{5.3}$$

and that there is $N \in \mathbb{N}$ such that (3.7) is true for all $t \in (a, b\,]$ and all $n \in \mathbb{N}$ such that $n \geq N$.

Then for $n \in \mathbb{N}$ sufficiently large there exist unique solutions x and x_n to (3.1) and (3.13), respectively.

Let, in addition, (3.11) *and*

$$\left.\begin{aligned}&\forall\,\varepsilon>0\ \exists\,\delta>0 \text{ such that } \forall\,t\in(b-\delta,b)\ \exists\,n_0\in\mathbb{N} \text{ such that}\\ &\Big\|\Big(\Delta^-A(b)[I-\Delta^-A(b)]^{-1}x(b-)+[I-\Delta^-A(b)]^{-1}\Delta^-f(b)\Big)-\Big(x_n(b)-x_n(t)\Big)\Big\|_X<\varepsilon\\ &\hfill \text{for } n\geq n_0\end{aligned}\right\}\tag{5.4}$$

hold. Then $x_n \rightrightarrows x$ *locally on* $[a,b)$.

Proof. Similarly to the previous theorem, the existence and uniqueness of solutions to (3.1) and (3.13) are guaranteed by Proposition 3.1. Denote by x and x_n the corresponding solutions. Then, due to Theorem 3.4 and due to the uniqueness of solutions to

$$x_n(t)=x_n(a)+\int_a^t \mathrm{d}[A_n]\,x_n+f_n(t)-f_n(a)\,,\quad t\in[a,c]$$

and

$$x(t)=x(a)+\int_a^t \mathrm{d}[A]\,x+f(t)-f(a)\,,\quad t\in[a,c],$$

the sequence $\{x_n\}$ tends for each $c\in[a,b)$ to x uniformly on $[a,c]$ as $n\to\infty$. In particular,

$$\lim_{n\to\infty} x_n(t)=x(t)\quad\text{for all}\ \ t\in[a,b).$$

It remains to show that $\lim_{n\to\infty}\|x_n(b)-x(b)\|_X=0$.

Let $\varepsilon>0$ be given. By (5.4), there is $\delta>0$ such that for each $t\in(b-\delta,b)$ we can find $n_1\in\mathbb{N}$ such that $n_1\geq N$ and

$$\Big\|\Big(\Delta^-A(b)[I-\Delta^-A(b)]^{-1}x(b-)+[I-\Delta^-A(b)]^{-1}\Delta^-f(b)\Big)-\Big(x_n(b)-x_n(t)\Big)\Big\|_X<\varepsilon$$

holds for all $n\geq n_0$. As x is regulated on $[a,b]$, we can also assume that

$$\|x(s)-x(b-)\|_X<\varepsilon\qquad\text{for all } s\in(b-\delta,b).$$

Now, choose and arbitrary $\tau\in(b-\delta,b)$. Then $x_n(\tau)\to x(\tau)$. Hence there is $n_0\geq n_1$ such that

$$\|x_n(\tau)-x(\tau)\|_X<\varepsilon\quad\text{for}\ \ n\geq n_0.$$

To summarize, for $n\geq n_0$ we have

$$\begin{aligned}&\|x(b)-x_n(b)\|_X\\ &\quad\leq\|x(b)-(\Delta^-A(b)\,x(b)+\Delta^-f(b))-x(\tau)\|_X+\|x(\tau)-x_n(\tau)\|_X\\ &\qquad+\|x_n(\tau)+\Delta^-A(b)\,x(b)+\Delta^-f(b)-x_n(b)\|_X\\ &\quad=\|x(b-)-x(\tau)\|_X+\|x(\tau)-x_n(\tau)\|_X\\ &\qquad+\|(x_n(\tau)-x_n(b))-\Delta^-A(b)\,[I-\Delta^-A(b)]^{-1}\,(x(b-)+\Delta^-f(b))\|_X\end{aligned}$$

$$= \|x(b-) - x(\tau)\|_X + +\|x(\tau) - x_n(\tau)\|_X$$
$$+ \|(x_n(\tau) - x_n(b)) - \Delta^- A(b)\,[I - \Delta^- A(b)]^{-1}\,x(b-) - [I - \Delta^- A(b)]^{-1}\,\Delta^- f(b))\|_X$$
$$< 3\,\varepsilon,$$

where we made use of the following well-known relations:

$$\Delta^- x(b) = \Delta^- A(b)\,x(b) + \Delta^- f(b), \quad x(b) = [I - \Delta^- A(b)]^{-1}\,(x(b-) + \Delta^- f(b))$$

and

$$I + \Delta^- A(b)[I - \Delta^- A(b)]^{-1} = [I - \Delta^- A(b)]^{-1}.$$

Therefore $\lim_{n\to\infty} \|x_n(b) - x(b)\|_X = 0$ and this completes the proof. □

5.3 . Remark. Let us notice that, due to Lemma 3.3, we can, instead of:

there is $N \in \mathbb{N}$ such that (3.7) is true for all $t \in (a, b\,]$ and all $n \in \mathbb{N}$ such that $n \geq N$

assume only

there are an $N \in \mathbb{N}$ and $\Delta < 0$ such that (3.7) is true for all $t \in (a, a + \Delta]$ and all $n \in \mathbb{N}$ such that $n \geq N$.

5.4 . Remark. It is easy to combine Theorems 5.1 and 5.2 to formulate a corresponding result for the case that the uniform convergence is violated at finitely many points in $[a, b\,]$. We leave it to the reader.

References

[1] Artstein Z.: Continuous dependence on parameters: On the best possible results. *J. Differential Equations* **19** (1975), 214–225.

[2] Ashordia, M.: On the correctness of linear boundary value problems for systems of generalized ordinary differential equations. *Proc.Georgian Acad.Sci.Math.* **1** (1993), No. 4, 385–394.

[3] Brokate, M. and Krejčí, P.: Duality in the space of regulated functions and the play operator. *Math. Z.* **245** (2003), 667–688.

[4] Federson, M and Schwabik, Š: Generalized ordinary differential equations approach to impulsive retarded functional differential equations. *Differential and Integral Equations* **19** (2006), 1201–1234.

[5] Fraňková, D.: Continuous dependence on a parameter of solutions of generalized differential equations. *Časopis pěst. mat.***114** (1989), 230–261.

[6] Halas, Z.: Continuous dependence of solutions of generalized linear ordinary differential equations on a parameter. *Mathematica Bohemica* **132** (2007), 205–218.

[7] Halas, Z. and Tvrdý, M.: Continuous dependence of solutions of generalized linear differential equations on a parameter. *Funct. Differ. Equ.* **16** (2009), 299–313.

[8] Halas, Z. and Tvrdý, M.: Approximated Solutions of Generalized Linear Differential Equations. (Preprint 141 (2008), Math. Inst., Acad. Sci. Czech Rep., updated version available at http://www.math.cas.cz/ tvrdy/HaTv.pdf .)

[9] Hildebrandt, T.H.: On systems of linear differentio-Stieltjes integral equations. *Illinois J. Math.* **3** (1959) 352–373.

[10] Hönig, Ch.S.: *Volterra Stieltjes-integral equations*. North Holland and American Elsevier, Mathematics Studies 16. Amsterdam and New York, 1975.

[11] Imaz, C. and Vorel, Z.: Generalized ordinary differential equations in Banach spaces and applications to functional equations. *Bol. Soc. Mat. Mexicana* **11** (1966) 47–59.

[12] Kiguradze, I.: Boundary value problems for systems of ordinary differential equations. *Boundary Value Problems for Systems of Ordinary Differential Equations* (in Russian). Itogi Nauki Tekh., Ser. Sovrem. Probl. Mat., Novejshie Dostizh. 30 (1987), 3-103; English transl.: J. Sov. Math. **43** (1988), 2259-2339.

[13] Krasnoselskij, M.A. and Krein, S.G.: On the averaging principle in nonlinear mechanics (in Russian). *Uspekhi mat. nauk* **10** (3) (1955), 147–152.

[14] Krejčí, P. and Laurençot, P.: Generalized variational inequalities. *J. Convex Anal.* **9** (2002), 159–183.

[15] Kurzweil, J. and Vorel, Z.: Continuous dependence of solutions of differential equations on a parameter. *Czechoslovak Math.J.* **23 (82)** (1957), 568–583.

[16] Kurzweil, J.: Generalized ordinary differential equation and continuous dependence on a parameter. *Czechoslovak Math.J.* **7(82)** (1957), 418–449.

[17] Kurzweil, J: Generalized ordinary differential equations. *Czechoslovak Math.J.* **24 (83)** (1958), 360–387.

[18] Meng, G. and Zhang, M.: Continuity in weak topology: First order linear system of ODE. *Acta Math. Sinica, Engl. Ser.* **26** (2010), 1287–1298.

[19] Meng, G. and Zhang, M.: Measure differential equations I. Continuity of solutions in measures with weak*topology. *Tsinghua University, preprint* (2009).

[20] Meng, G. and Zhang, M.: Measure differential equations I. Continuity of solutions in measures with weak*topology. *Tsinghua University, preprint* (2009).

[21] Monteiro G. and Tvrdý M.: On Kurzweil-Stieltjes integral in Banach space. *Preprint, Institute of Mathematics, AS CR, Prague* 2011-1-4 [http://www.math.cas.cz/preprint/pre-236.pdf].

[22] Oliva, F. and Vorel, Z.: Functional equations and generalized ordinary differential equations. *Bol. Soc. Mat. Mexicana* **11** (1966), 40–46.

[23] Opial, Z.: Continuous Parameter Dependence in Linear Systems of Differential Equations. *J. Differential Equations* **3** (1967), 571-579.

[24] Schwabik, Š.: *Generalized Ordinary Differential Equations.* World Scientific. Singapore, 1992.

[25] Schwabik, Š.: Abstract Perron-Stieltjes integral. *Math. Bohem.* **121**(1996), 425–447.

[26] Schwabik, Š.: Linear Stieltjes integral equations in Banach spaces. *Math. Bohem.* **124** (1999), 433–457.

[27] Schwabik, Š.: Linear Stieltjes integral equations in Banach spaces II; Operator valued solutions. *Math. Bohem.* **125** 2000, 431–454.

[28] Schwabik, Š.: A note on integration by parts for abstract Perron-Stieltjes integrals. *Math. Bohem.* **126** 2001, 613–629.

[29] Schwabik, Š., Tvrdý, M. and Vejvoda, O.: *Differential and Integral Equations: Boundary Value Problems and Adjoint.* Academia and Reidel. Praha and Dordrecht, 1979

[30] Slavík, A.: Dynamic equations on time scales and generalized ordinary differential equations.

[31] Taylor, A.: *Introduction to Functional Analysis.* Wiley, 1958.

[32] Tvrdý, M.: Regulated functions and the Perron-Stieltjes integral. *Čas.pěst.mat.* **114** (1989), 187–209.

[33] Tvrdý, M: On the continuous dependence on a parameter of solutions of initial value problems for linear generalized differential equations. *Funct.Differ.Equ.* **5** (1999), 483–498.

[34] Tvrdý, M.: Differential and Integral equations in the Space of Regulated Functions. *Mem. Differential Equations Math. Phys.* 25 (2002), 1-104.