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Approximated Solutions of Generalized Linear Differential Equations

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Abstract. This contribution deals with systems of linear generalized linear differential equations of the form

$$x(t) = \widetilde{x} + \int_a^t \mathrm{d}[A(s)]\, x(s) + g(t) - g(a), \quad t \in [a,b],$$

where $-\infty < a < b < \infty$, A is an $n \times n$ -complex matrix valued function, g is an n -complex vector valued function, A and g have bounded variation on $[a,b]$. The integrals are understood in the Kurzweil-Stieltjes sense.

Our aim is to present some new results on continuous dependence of solutions to linear generalized differential equations on parameters and initial data. In particular, we generalize in several aspects the known result by Ashordia. Our main goal consists in a more general notion of a solution to the given system. In particular, neither g nor x need not be of bounded variation on $[a,b]$ and, in general, they can be regulated functions.

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1 . Introduction

Starting with Kurzweil [7], generalized differential equations have been extensively studied by many authors, like e.g. Hildebrandt [5], Schwabik, Tvrdý and Vejvoda [13]–[15], [17]–[19], Hönig [6], Ashordia [2], [3]. In particular, see the monographs [15], [13], [18] and [6] and the references therein. Moreover, during several recent decades, the interest in their special cases like equations with impulses or discrete systems increased considerably, cf. e.g. monographs [9], [21], [4], [12] or [1].

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The importance of generalized linear differential equations with regulated solutions consists in the fact that they enable us to treat in a unified way both continuous and discrete systems and, in addition, also systems with fast oscillating data.

In the paper we keep the following notation:

As usual, $\mathbb{N}$ is the set of natural numbers ($\mathbb{N} = \{1, 2, \dots\}$) and $\mathbb{C}$ stands for the set of complex numbers. $\mathbb{C}^{m \times n}$ is the space of complex matrices of the type $m \times n$, $\mathbb{C}^n = \mathbb{C}^{n \times 1}$ and $\mathbb{C}^1 = \mathbb{C}$. For a matrix

$$A = (a_{i,j})_{\substack{i=1,2,\dots,m \\ j=1,2,\dots,n}} \in \mathbb{C}^{m \times n},$$

its norm $|A|$ is defined by

$$|A| = \max_{j=1,2,\dots,n} \sum_{i=1}^{m} |a_{i,j}|.$$

In particular, we have $|x| = \sum_{i=1}^{n} |x_i|$ for $x \in \mathbb{C}^n$. The symbols I and 0 stand respectively for the identity and the zero matrix of the proper type. For an $n \times n$-matrix A, $\det[A]$ denotes its determinant.

If $-\infty < a < b < \infty$, then $[a, b]$ and (a, b) denote the corresponding closed and open intervals, respectively. Furthermore, $[a, b)$ and $(a, b]$ are the corresponding half-open intervals. When the intervals $[a, a)$ and $(b, b]$ occur, they are understood to be empty.

For an arbitrary function $F\colon [a, b] \to \mathbb{C}^{m \times n}$ we set

$$\|F\|_\infty = \sup\{|F(t)|\colon t \in [a, b]\}.$$

The set $\mathcal{D} = \{t_0, t_1, \dots, t_m\} \subset [a, b]$, $m \in \mathbb{N}$, is called a subdivision of the interval $[a, b]$, if $a = t_0 < t_1 < \dots < t_m = b$. The set of all subdivisions of the interval $[a, b]$ is denoted by $\mathfrak{D}[a, b]$. If, for each $t \in [a, b)$ and $s \in (a, b]$, the function $F : [a, b] \to \mathbb{C}^{m \times n}$ possesses limits

$$F(t+) := \lim_{\tau \to t} F(\tau), \quad F(s-) := \lim_{\tau \to s} F(\tau),$$

we say that the function F is regulated on the interval $[a, b]$. The set of all $m \times n$-matrix valued functions regulated on the interval $[a, b]$ is denoted by

$G^{m\times n}[a,b]$. Furthermore, we denote

$$\Delta^+F(t)=F(t+)-F(t) \quad \text{for} \quad t\in[a,b), \quad \Delta^+F(b)=0,$$
$$\Delta^-F(s)=F(s)-F(s-) \quad \text{for} \quad s\in(a,b], \quad \Delta^-F(a)=0$$

and

$$\Delta F(t)=F(t+)-F(t-) \quad \text{for} \quad t\in(a,b).$$

It is known that, for each $F\in G^{m\times n}[a,b]$, the set of all points of its discontinuity on the interval $[a,b]$ is at most countable. Moreover, for each $\varepsilon>0$ there are at most finitely many points $t\in[a,b)$ such that $|\Delta^+F(t)|\geq\varepsilon$ and at most finitely many points $s\in[a,b]$ such that $|\Delta^-F(s)|\geq\varepsilon$. Clearly, each function regulated on $[a,b]$ is bounded on $[a,b]$, i.e. $\|F\|_\infty<\infty$ for all $F\in G^{m\times n}[a,b]$.

For a function $F:[a,b]\to\mathbb{C}^{m\times n}$ we denote by $\operatorname{var}_a^b F$ its variation over $[a,b]$. We say that F has a bounded variation on $[a,b]$ if $\operatorname{var}_a^b F<\infty$. The set of $m\times n$-complex matrix valued functions of bounded variation on $[a,b]$ is denoted by $BV^{m\times n}[a,b]$ and $\|F\|_{BV}=|F(a)|+\operatorname{var}_a^b F$. By $AC^{m\times n}[a,b]$ we denote the set of functions $F:[a,b]\to\mathbb{C}^{m\times n}$ such that each component f_{ij}, $i=1,\dots,m$, $j=1,\dots,n$, of F is absolutely continuous on the interval $[a,b]$. Similarly, $C^{m\times n}[a,b]$ stands for the set of functions $F:[a,b]\to\mathbb{C}^{m\times n}$ that are continuous on $[a,b]$. Analogously to the spaces of functions of bounded variation, $AC^n[a,b]=AC^{n\times 1}[a,b]$, $G^n[a,b]=G^{n\times 1}[a,b]$ and $C^n[a,b]=C^{n\times 1}[a,b]$. Obviously, $AC^{m\times n}[a,b]\subset BV^{m\times n}[a,b]\subset G^{m\times n}[a,b]$ and $C^{m\times n}[a,b]\subset G^{m\times n}[a,b]$. Finally, a function $f:[a,b]\to\mathbb{C}$ is called a *finite step function on* $[a,b]$ if there is a division $\{\alpha_0,\alpha_1,\dots,\alpha_m\}\in\mathfrak{D}[a,b]$ of $[a,b]$ such that f is constant on every open interval (α_{j-1},α_j), $j=1,2,\dots,m$. The set of all finite step functions on $[a,b]$ is denoted by $S[a,b]$, $S^{m\times n}[a,b]$ is the set of all $m\times n$-matrix valued functions whose arguments are finite step functions and $S^{n\times 1}[a,b]=S^n[a,b]$. It is known that the set $S^{m\times n}[a,b]$ is dense in $G^{m\times n}[a,b]$ with respect to the supremum norm, i.e.

$$\begin{cases} \text{for each } \varepsilon>0 \text{ and each } F\in G^{m\times n}[a,b] \\ \text{there is an } \widetilde{F}\in S^{m\times n}[a,b] \text{ such that } \|F-\widetilde{F}\|_\infty<\varepsilon. \end{cases} \tag{1.1}$$

The integrals which occur in this paper are the Perron-Stieltjes ones. For the original definition, see A.J. Ward [20] or S. Saks [11]. We use the equivalent summation definition due to J. Kurzweil [7] (cf. also e.g. [8] or

[15]). We call this integral the Kurzweil-Stieltjes integral, in short the KS-integral. For the reader's convenience, let us recall the definition of the KS-integral.

Let $-\infty < a < b < \infty$. For a given $m \in \mathbb{N}$, a division $\mathcal{D}$ of the integral $[a,b]$, $\mathcal{D} = \{\alpha_0, \alpha_1, \dots, \alpha_m\} \in \mathfrak{D}[a,b]$ and $\xi = (\xi_1, \xi_2, \dots, \xi_m) \in [a,b]^m$, the couple $P = (D, \xi)$ is called a partition of $[a,b]$ if

$$\alpha_{j-1} \le \xi_j \le \alpha_j \quad \text{for} \quad j = 1, 2, \dots, m.$$

The set of all partitions of the interval $[a,b]$ is denoted by $\mathfrak{P}[a,b]$.

An arbitrary positive valued function $\delta\colon [a,b] \to (0,\infty)$ is called a gauge on $[a,b]$. Given a gauge δ on $[a,b]$, the partition

$$P = (D, \xi) = \big(\{\alpha_0, \alpha_1, \dots, \alpha_m\},\, (\xi_1, \xi_2, \dots, \xi_m)\big) \in \mathfrak{P}[a,b]$$

is said to be δ -fine, if

$$[\alpha_{j-1}, \alpha_j] \subset (\xi_j - \delta(\xi_j), \xi_j + \delta(\xi_j)) \quad \text{for} \quad j = 1, 2, \dots, m.$$

The set of all δ-fine partitions of $[a,b]$ is denoted by $\mathfrak{A}(\delta; [a,b])$.

For functions $f,\ g : [a,b] \to \mathbb{C}$ and a partition $P \in \mathfrak{P}[a,b]$,

$$P = (\{\alpha_0, \alpha_1, \dots, \alpha_m\},\, (\xi_1, \xi_2, \dots, \xi_m))\,,$$

we define

$$\Sigma(f\,\Delta g; P) = \sum_{i=1}^{m} f(\xi_i)\,[g(\alpha_i) - g(\alpha_{i-1})].$$

We say that $I \in \mathbb{C}$ is the KS-integral of f with respect to g from a to b if

$$\begin{cases} \text{for each } \varepsilon > 0 \text{ there is a gauge } \delta \text{ on } [a,b] \text{ such that} \\ |I - \Sigma(f\,\Delta g; P)| < \varepsilon \quad \text{for all} \quad P \in \mathfrak{A}(\delta; [a,b]). \end{cases}$$

In such a case we write

$$I = \int_a^b f\,\mathrm{d}[g] \quad \text{or} \quad I = \int_a^b f(t)\,\mathrm{d}[g(t)].$$

It is well known that the KS-integral $\int_a^b f\,\mathrm{d}g$ exists provided $f \in G[a,b]$ and $g \in BV[a,b]$. Taking into account [18, Theorem 2.3.8], we can formulate the following fundamental assertion.

1.1. Theorem. *If* $f, g \in G[a,b]$ *and at least one of the functions* f, g *has a bounded variation on* $[a,b]$, *then the integral* $\int_a^b f\,\mathrm{d}[g]$ *exists. Furthermore,*

$$\Big|\int_a^b f\,\mathrm{d}[g]\Big| \le 2\left(|f(a)| + \mathrm{var}_a^b f\right)\|g\|_\infty \text{ if } f \in BV[a,b] \text{ and } g \in G[a,b], \quad (1.2)$$

and

$$\Big|\int_a^b f\,\mathrm{d}[g]\Big| \le \|f\|_\infty\, var_a^b g \qquad \text{if } f \in G[a,b] \text{ and } g \in BV[a,b]. \quad (1.3)$$

Furthermore, if $f \in BV[a,b]$ and $g, g_k \in G[a,b]$ for $k \in \mathbb{N}$, then

$$\lim_{k\to\infty} \|g_k - g\|_\infty = 0 \quad \text{implies} \quad \lim_{k\to\infty} \Big\|\int_a^t f\,\mathrm{d}[g_k - g]\Big\|_\infty = 0.$$

Further basic properties of the Perron-Stieltjes integral with respect to scalar regulated functions were described in [16] (see also [18]).

Given an $m\times q$-matrix valued function F and an $q\times n$-matrix valued function G defined on $[a,b]$ and such that all the integrals

$$\int_a^b f_{i,k}(t)\,\mathrm{d}[g_{k,j}(t)] \quad (i = 1,2,\dots,m\,;\ k = 1,2,\dots,q;\ j = 1,2,\dots,n)$$

exist (i.e. they have finite values), the symbol

$$\int_a^b F(t)\,\mathrm{d}[G(t)] \quad \text{(or more simply} \quad \int_a^b F\,\mathrm{d}G)$$

stands for the $p\times n$ -matrix with the entries

$$\sum_{k=1}^q \int_a^b f_{i,k}\,\mathrm{d}[g_{k,j}], \quad i = 1,2,\dots,m\,,\ j = 1,2,\dots,n.$$

The extension of the results obtained in [16] or [18] for scalar real valued functions to complex vector valued or matrix valued functions is obvious and hence for the basic facts concerning integrals with respect to regulated functions we will refer to the corresponding assertions from [16] or [18].

The next natural assertion will be very useful for our purposes and, in our opinion, it is not available in literature.

1.2 . Lemma. *Let* $A \in BV^{n\times n}[a,b]$. *Then*

$$\left\{\begin{aligned}&\lim_{s\to t-}\frac{1}{t-s}\left(\int_s^t \exp\left([A(t)-A(s)]\frac{t-r}{t-s}\right)\,\mathrm{d}r\right)\\ &\quad=\lim_{s\to t-}\frac{1}{t-s}\left(\int_s^t \exp\left(\Delta^- A(t)\frac{t-r}{t-s}\right)\,\mathrm{d}r\right)\quad \textit{if}\ \ t\in(a,b]\end{aligned}\right. \tag{1.4}$$

and

$$\left\{\begin{aligned}&\lim_{s\to t+}\frac{1}{s-t}\left(\int_t^s \exp\left([A(s)-A(t)]\frac{s-r}{s-t}\right)\,\mathrm{d}r\right)\\ &\quad=\lim_{s\to t+}\frac{1}{s-t}\left(\int_t^s \exp\left(\Delta^+ A(t)\frac{s-r}{s-t}\right)\,\mathrm{d}r\right)\quad \textit{if}\ \ t\in[a,b).\end{aligned}\right. \tag{1.5}$$

Proof. (i) Let $t \in (a,b]$, $s \in [a,t)$ and let $\varepsilon > 0$ be given. Then there is a $\delta > 0$ such that

$$|A(t-)-A(s)| < \eta \quad \text{whenever} \ \ t-s<\varepsilon.$$

Now, taking into account that

$$|\exp(C)-\exp(D)| \le |C-D|\,\exp(|C|+|D|) \quad \text{holds for all} \ \ C,D\in\mathbb{R}^{n\times n},$$

we get

$$\begin{aligned}&\left|\frac{1}{t-s}\int_s^t\left[\exp\left([A(t)-A(s)]\frac{t-r}{t-s}\right)-\exp\left(\Delta^-A(t)\frac{t-r}{t-s}\right)\right]\,\mathrm{d}r\right|\\ &\quad\le\frac{1}{t-s}\,|A(t-)-A(s)|\int_s^t\exp\big(|\Delta^-A(t)|\big)\,\mathrm{d}r\\ &\quad=\varepsilon\,|A(t-)-A(s)|\,\exp\big(|\Delta^-A(t)|\big)\le\varepsilon\,\exp\big(|\Delta^-A(t)|\big)\end{aligned}$$

for $t-s<\delta$, wherefrom the validity of (1.4) immediately follows.

(ii) Similarly we would justify the relation (1.5). □

1.3 . Lemma. *Let* $x,\,x_k \in G^n[a,b]$, $A,\,A_k \in BV^{n\times n}[a,b]$ *for* $k\in\mathbb{N}$. *Furthermore, let*

$$\lim_{k\to\infty}\|x_k-x\|_\infty=0, \tag{1.6}$$

$$\alpha^* := \sup\{\operatorname{var}_a^b A_k\colon\ k\in\mathbb{N}\}<\infty, \tag{1.7}$$

and

$$\lim_{k\to\infty} \|A_k - A\|_\infty = 0. \tag{1.8}$$

Then

$$\lim_{k\to\infty} \left\| \int_a^t \mathrm{d}[A_k]\, x_k - \int_a^t \mathrm{d}[A]\, x \right\|_\infty = 0.$$

Proof. Let $\varepsilon > 0$ be given. By (1.1) and (1.6), we can find $u \in S^n[a,b]$ and $k_0 \in \mathbb{N}$ such that

$$\|x-u\|_\infty < \varepsilon, \quad \|x_k - u\|_\infty < \varepsilon \quad \text{and} \quad \|A_k - A\|_\infty < \varepsilon \quad \text{for} \quad k \ge k_0.$$

Furthermore, since $\mathrm{var}_a^b\, u < \infty$, using (1.2) we can see that for $t \in [a,b]$ and $k \ge k_0$ the relations

$$\left| \int_a^t \mathrm{d}[A_k]\, x_k - \int_a^t \mathrm{d}[A]\, x \right| = \left| \int_a^t \mathrm{d}[A_k]\,(x_k - u) + \int_a^t \mathrm{d}[A_k - A]\, u + \int_a^t \mathrm{d}[A]\,(u-x) \right|$$
$$\le \alpha^* \varepsilon + 2\,(\mathrm{var}_a^b\, u)\,\varepsilon + \alpha^* \varepsilon = 2\,(\alpha^* + \mathrm{var}_a^b\, u)\,\varepsilon$$

hold, wherefrom our assertion immediately follows. □

2. Generalized differential equations

Let $A \in BV^{n\times n}[a,b]$, $g \in BV^n[a,b]$ and $\widetilde{x} \in \mathbb{C}^n$. Consider an integral equation

$$x(t) = \widetilde{x} + \int_a^t \mathrm{d}[A(s)]\, x(s) + g(t) - g(a). \tag{2.1}$$

We say that a function $x\colon [a,b] \to \mathbb{C}^n$ is *a solution* of (2.1) on the interval $[a,b]$ if the integral

$$\int_a^b \mathrm{d}[A(s)]\, x(s)$$

has sense and equality (2.1) is satisfied for all $t \in [a,b]$. The equation (2.1) is usually called a *generalized linear differential equation.* Such equations with solutions having values in the space $\mathbb{R}^n$ of real n-vectors have been thoroughly investigated e.g. in the monographs [13] or [15]. The extension of the results presented therein to the complex case is mostly straightforward. In this section we will describe the basics needed later. Special attention is paid to the features whose extension to the complex case seems not to be so straightforward.

For our purposes the following property is crucial:

$$\det\big[I-\Delta^- A(t)\big]\neq 0 \quad \text{hold for each } t\in[a,b]. \tag{2.2}$$

(Recall that we put $\Delta^- A(b)=0$.) Its importance is well illustrated by the next assertion which is a fundamental existence result for the equation (2.1).

2.1 . Theorem. *Let $A\in BV^{n,n}[a,b]$ satisfy (2.2). Then, for each $\widetilde{x}\in\mathbb{C}^n$ and each $g\in G^n[a,b]$, equation (2.1) has a unique solution x on $[a,b]$ and $x\in G^n[a,b]$. Moreover, $x-g\in BV^n[a,b]$.*

Proof follows from [17, Proposition 2.5].

Furthermore, analogously to [15, Theorem III.1.7] where $g\in BV^n[a,b]$, we have

2.2 . Theorem. *Let $A\in BV^{n,n}[a,b]$ satisfy (2.2). Then*

$$c_A:=\sup\{\big|[I-\Delta^- A(t)]^{-1}\big|:\ t\in[a,b]\}<\infty \tag{2.3}$$

and

$$|x(t)|\leq c_A\,(|\widetilde{x}|+2\,\|g\|_\infty)\,\exp(c_A\,\mathrm{var}_a^t A)\ \textit{for}\ t\in[a,b], \tag{2.4}$$

holds for each $\widetilde{x}\in\mathbb{C}^n$, $g\in G^n[a,b]$ and each solution x of (2.1) on $[a,b]$.

Proof. First, notice that for $t\in[a,b]$ such that $|\Delta^- A(t)|<\frac{1}{2}$ we have

$$\big|[I-\Delta^- A(t)]^{-1}\big|=\left|\sum_{k=1}^{\infty}(\Delta^- A(t))^k\right|\leq\sum_{k=1}^{\infty}|\Delta^- A(t)|^k=\frac{1}{1-|\Delta^- A(t)|}<2.$$

Therefore, (2.3) follows from the fact that the set $\{t\in[a,b]:\ |\Delta^- A(t)|\geq\frac{1}{2}\}$ has at most finitely many elements.

Now, let x be a solution of (2.1). Put $B(a)=A(a)$ and $B(t)=A(t-)$ for $t\in(a,b]$. Then, as in the proof of [15, Theorem III.1.7], we get

$$A-B\in BV^{n\times n}[a,b],\quad \mathrm{var}_a^b B\leq\mathrm{var}_a^b A$$

and

$$A(t) - B(t) = \Delta^- A(t), \quad \int_a^t \mathrm{d}[A-B]\, x = \Delta^- A(t) \quad \text{for} \quad t \in [a,b].$$

Consequently

$$x(t) = [I - \Delta^- A(t)]^{-1} \left(\widetilde{x} + g(t) - g(a) + \int_a^t \mathrm{d}[B]\, x \right)$$

and

$$|x(t)| \le K_1 + K_2 \int_a^t \mathrm{d}[h]\, |x| \quad \text{for} \quad t \in [a,b],$$

where

$$K_1 = c_A \left(|\widetilde{x}| + 2\, \|g\|_\infty \right), \quad K_2 = c_A \quad \text{and} \quad h(t) = \operatorname{var}_a^t B \quad \text{for} \quad t \in [a,b].$$

The function h is nondecreasing and, since B is left-continuous on $(a,b]$, h is also left-continuous on $(a,b]$. Therefore we can use the generalized Gronwall inequality (see e.g. [15, Lemma I.4.30] or [13, Corollary 1.43]) to get the estimate (2.4). □

2.3. Corollary. *Let* $A \in BV^{n \times n}[a,b]$ *satisfy* (2.2). *Then for each* $\widetilde{x} \in \mathbb{C}^n$, $g \in G^n[a,b]$ *and each solution* x *of* (2.1) *on* $[a,b]$, *the estimate*

$$\operatorname{var}_a^b (x-g) \le c_A \,(\operatorname{var}_a^b A)\, (|\widetilde{x}| + 2\, \|g\|_\infty)\, \exp(c_A \operatorname{var}_a^b A).$$

is true, where c_A *is defined by* (2.3).

Proof. By (2.4), we have

$$\|x\|_\infty \le c_A \,(|\widetilde{x}| + 2\, \|g\|_\infty)\, \exp(c_A \operatorname{var}_a^b A).$$

Therefore

$$\begin{aligned} \operatorname{var}_a^b (x-g) &\le \left(\operatorname{var}_a^b A\right) \|x\|_\infty \\ &\le c_A \left(\operatorname{var}_a^b A\right) (|\widetilde{x}| + 2\, \|g\|_\infty)\, \exp(c_A \operatorname{var}_a^b A). \end{aligned}$$

□

2.4. Lemma. *Let* $A \in BV^{n \times n}[a,b]$ *satisfy* (2.2) *and let* c_A *be defined by* (2.3). *Then*

$$c_A = \left(\inf \left\{ \left| \left[I - \Delta^- A(t) \right] x \right| : t \in [a,b],\, x \in \mathbb{C}^n,\, |x| = 1 \right\} \right)^{-1}. \tag{2.5}$$

Proof. We have

$$\begin{aligned}
c_A &= \sup\left\{|[I-\Delta^- A(t)]^{-1}|\colon\, t\in[a,b]\right\}\\
&= \sup\left\{\frac{|[I-\Delta^- A(t)]^{-1}|\,|[I-\Delta^- A(t)]\,x|}{|[I-\Delta^- A(t)]\,x|}\colon\, t\in[a,b],\ x\in\mathbb{C}^n,\ |x|=1\right\}\\
&\geq \sup\left\{\frac{|x|}{|[I-\Delta^- A(t)]\,x|}\colon\, t\in[a,b],\ x\in\mathbb{C}^n,\ |x|=1\right\}\\
&= \sup\left\{\frac{1}{|[I-\Delta^- A(t)]\,x|}\colon\, t\in[a,b],\ x\in\mathbb{C}^n,\ |x|=1\right\}\\
&= \left(\inf\left\{|[I-\Delta^- A(t)]\,x|\colon\, t\in[a,b],\ x\in\mathbb{C}^n,\ |x|=1\right\}\right)^{-1}.
\end{aligned}$$

Thus, it remains to prove that the inequality

$$c_A \leq \left(\inf\left\{\left|\left[I-\Delta^- A(t)\right]\,x\right|\colon\, t\in[a,b],\ x\in\mathbb{C}^n,\ |x|=1\right\}\right)^{-1} \tag{2.6}$$

is true, as well. To this aim, first let us notice that for each $t\in[a,b]$ there is a $z\in\mathbb{C}^n$ such that $|z|=1$ and

$$\left|[I-\Delta^- A(t)]^{-1}\right| = \left|[I-\Delta^- A(t)]^{-1}\,z\right|. \tag{2.7}$$

Indeed, let $t\in[a,b]$ and let $B=[I-\Delta^- A(t)]^{-1}$. Let $i_0\in\{1,2,\dots,n\}$ be such that $|B|=\sum_{j=1}^n |b_{i_0,j}|$ and let $z\in\mathbb{C}^n$ be such that $z_i=\operatorname{sgn}(b_{i_0,2})$ for $i=1,2,\dots,n$. Then $|z|=1$. Furthermore,

$$\begin{aligned}
|B\,z| &= \max_{i=1,2,\dots,n}\sum_{j=1}^n |b_{i,j}\,z_j| = \max_{i=1,2,\dots,n}\sum_{j=1}^n |b_{i,j}\ \operatorname{sgn}(b_{i_0,j})|\\
&\leq \max_{i=1,2,\dots,n}\sum_{j=1}^n |b_{i,j}| = |B|.
\end{aligned}$$

On the other hand, we have

$$|B| = \sum_{j=1}^n |b_{i_0,j}| = \left|\sum_{j=1}^n \operatorname{sgn}(b_{i_0,j})\,b_{i_0,j}\right| \leq |B\,z|.$$

Therefore, we can conclude that (2.7) is true.

Now, due to (2.2), there is $w \in \mathbb{C}^n$ such that $z = [I - \Delta^- A(t)]\, w$. Inserting this instead of z into (2.7), we get

$$\begin{aligned}\left|[I - \Delta^- A(t)]^{-1}\right\| &= \frac{|[I - \Delta^- A(t)]^{-1}\, [I - \Delta^- A(t)]\, w|}{|[I - \Delta^- A(t)]\, w|} \\ &= \frac{|w|}{|[I - \Delta^- A(t)]\, w|} = \frac{1}{\left|[I - \Delta^- A(t)]\, \left(\frac{w}{|w|}\right)\right|} \\ &\le \sup\left\{\frac{1}{|[I - \Delta^- A(t)]\, x|} \colon x \in \mathbb{C}^n,\ |x| = 1\right\}.\end{aligned}$$

It follows that

$$\begin{aligned}c_A &\le \sup\left\{\frac{1}{|[I - \Delta^- A(t)]\, x|} \colon t \in [a,b],\ x \in \mathbb{C}^n,\ |x| = 1\right\} \\ &= \left(\inf\{|[I - \Delta^- A(t)]\, x| \colon t \in [a,b],\ x \in \mathbb{C}^n,\ |x| = 1\right)^{-1},\end{aligned}$$

i.e. (2.6) is true. This completes the proof □

The next fundamental result on the continuous dependence of solutions of generalized linear differential equations on a parameter generalizes the result due to M. Ashordia [2, Theorem 1]. Unlike [2] and [3], we do not utilize the variation-of-constants formula and therefore we need not assume that, in addition to (2.2), also the condition

$$\det[I + \Delta^+ A(t)] \neq 0 \quad \text{for all} \quad t \in [a,b]$$

is satisfied. Furthermore, both the nonhomogeneous part of the equation and the solutions may be only regulated functions (not necessarily of bounded variation).

2.5. Theorem. *Let* $A,\ A_k \in BV^{n\times n}[a,b]$, $g,\ g_k \in G^n[a,b]$, $\widetilde{x},\ \widetilde{x}_k \in \mathbb{C}^n$ *for* $k \in \mathbb{N}$. *Assume* (1.7), (1.8), (2.2),

$$\lim_{k\to\infty} \|g_k - g\|_\infty = 0 \tag{2.8}$$

and

$$\lim_{k\to\infty} \widetilde{x}_k = \widetilde{x}. \tag{2.9}$$

Then equation (2.1) *has a unique solution* x *on* $[a,b]$. *Furthermore, for each* $k\in\mathbb{N}$ *sufficiently large there exists a unique solution* x_k *on* $[a,b]$ *to the equation*

$$x(t)=\widetilde{x}_k+\int_a^t \mathrm{d}[A_k(s)]\,x(s)+g_k(t)-g_k(a) \tag{2.10}$$

and

$$\lim\|x_k-x\|_\infty=0. \tag{2.11}$$

Proof. *Step* 1. As in the first part of the proof of [2, Theorem 1], we can show that there is a $k_1\in\mathbb{N}$ such that

$$\det[I-\Delta^- A_k(t)]\neq 0 \quad \text{on} \quad [a,b]$$

holds for all $k\geq k_1$. In particular, (2.10) has a unique solution x_k for $k\geq k_1$.

Step 2. For $k\geq k_1$, put

$$c_{A_k}:=\sup\{\big|[I-\Delta^- A_k(t)]^{-1}\big|:\, t\in(a,b]\}<\infty$$

Then, by Lemma 2.4, we have

$$\begin{aligned}(c_{A_k})^{-1}&=\inf\big\{\big|\big[I-\Delta^- A_k(t)\big]\ x\big|:\, t\in[a,b],\, x\in\mathbb{C}^n,\, |x|=1\big\}\\ &\geq \inf\big\{\big|\big[I-\Delta^- A(t)\big]\ x\big|:\, t\in[a,b],\, x\in\mathbb{C}^n,\, |x|=1\big\}\\ &\quad -\sup\big\{\big|\big[\Delta^-(A_k(t)-A(t))\big]\ x\big|:\, t\in[a,b],\, x\in\mathbb{C}^n,\, |x|=1\big\}.\end{aligned}$$

Since, due to the assumption (1.8),

$$\lim_{k\to\infty}\|\Delta^- A_k-\Delta^- A\|_\infty=0,$$

we conclude that there is a $k_0\geq k_1$ such that

$$(c_{A_k})^{-1}\geq(c_A)^{-1}-(2\,c_A)^{-1}=(2\,c_A)^{-1} \quad \text{for} \quad k\geq k_0.$$

To summarize,

$$c_{A_k}\leq 2\,c_A<\infty \quad \text{for} \quad k\geq k_0. \tag{2.12}$$

Step 3. Set $w_k = (x_k - g_k) - (x - g)$. Then, for $k \geq k_0$,

$$w_k(t) = \widetilde{w}_k + \int_a^t \mathrm{d}[A_k]\, w_k + h_k(t) - h_k(a) \quad \text{for } \ t \in [a,b],$$

where

$$h_k(t) = \int_a^t \mathrm{d}[A_k - A]\,(x-g) + \left(\int_a^t \mathrm{d}[A_k]\, g_k - \int_a^t \mathrm{d}[A]\, g\right) \quad \text{for } \ t \in [a,b]$$

and

$$\widetilde{w}_k = (\widetilde{x}_k - g_k(a)) - (\widetilde{x} - g(a))\,.$$

By (2.8) and (2.9) we can see that

$$\lim_{k\to\infty} \widetilde{w}_k = 0. \tag{2.13}$$

Furthermore, since $x - g \in BV^n[a,b]$ and $\lim_{k\to\infty} \|A_k - A\|_\infty = 0$, by Theorem 1.1 we have

$$\lim_{k\to\infty} \left\| \int_a^t \mathrm{d}[A_k - A]\,(x-g) \right\|_\infty = 0$$

and, by Lemma 1.3,

$$\lim_{k\to\infty} \int_a^t \mathrm{d}[A_k]\, g_k = \int_a^t \mathrm{d}[A]\, g.$$

To summarize,

$$\lim_{k\to\infty} \|h_k\|_\infty = 0. \tag{2.14}$$

On the other hand, applying Theorem 2.2 and taking into account the relation (2.12), we get

$$\|w_k\|_\infty \leq 2\, c_A\, (|\widetilde{w}_k| + 2\, \|h_k\|_\infty)\, \exp(2\, c_A\, \alpha^*) \text{ for } t \in [a,b] \text{ and } k \geq k_0,$$

wherefrom, by virtue of (2.13) and (2.14), the relation

$$\lim_{k\to\infty} \|w_k\|_\infty = 0$$

follows. Finally, having in mind the assumptions (2.8) and (2.9), we conclude that the relation

$$\lim \|x_k - x\|_\infty = 0$$

is true, as well. This completes the proof. □

It is easy to see that the generalized differential equation (2.1) is equivalent with the equation

$$x(t) = \widetilde{x} + \int_a^t \mathrm{d}[B]\, x + f(t) - f(a)$$

whenever $B - A$ and $f - g$ are constant on $[a, b]$. Therefore Theorem 2.5 can be also reformulated as follows.

2.6 . Corollary. *Let the assumptions of Theorem 2.5 be satisfied, but with*

$$\lim_{k\to\infty} \|(A_k - A_k(a)) - (A - A(a))\|_\infty = 0 \tag{2.15}$$

and

$$\lim_{k\to\infty} \|(g_k - g_k(a)) - (g - g(a))\|_\infty = 0 \tag{2.16}$$

instead of (1.8) and (2.8), respectively

Then the conclusion of Theorem 2.5 remains valid.

2.7 . Notation. For a $k \in \mathbb{N}$ denote by $\mathcal{D}_k$ a division of $[a, b]$ given by

$$\begin{cases} \mathcal{D}_k = \{\alpha_0^k, \alpha_1^k, \dots, \alpha_{2^k}^k\}, & \text{where} \\ \alpha_i^k = a + \frac{i\,(b-a)}{2^k} & \text{for } i = 0, 1, \dots, 2^k. \end{cases} \tag{2.17}$$

For given $A \in BV^{n\times n}[a, b]$, $g \in G^n[a, b]$ and $k \in \mathbb{N}$, we define

$$A_k(t) = \begin{cases} A(t) & \text{if } t \in \mathcal{D}_k, \\ A(\alpha_{i-1}^k) + \dfrac{A(\alpha_i^k) - A(\alpha_{i-1}^k)}{\alpha_i^k - \alpha_{i-1}^k}\,(t - \alpha_{i-1}^k) & \text{if } t \in (\alpha_{i-1}^k, \alpha_i^k), \end{cases} \tag{2.18}$$

and

$$g_k(t) = \begin{cases} g(t) & \text{if } t \in \mathcal{D}_k, \\ g(\alpha_{i-1}^k) + \dfrac{g(\alpha_i^k) - g(\alpha_{i-1}^k)}{\alpha_i^k - \alpha_{i-1}^k}\,(t - \alpha_{i-1}^k) & \text{if } t \in (\alpha_{i-1}^k, \alpha_i^k). \end{cases} \tag{2.19}$$

Then, obviously, $\{A_k\} \subset AC^{n\times n}[a, b]$ and $\{g_k\} \subset AC^n[a, b]$. Moreover, we have

2.8 . Lemma. *Let sequences* $\{\mathcal{D}_k\} \subset \mathfrak{D}[a,b]$ *and* $\{A_k\} \subset AC^{n\times n}[a,b]$ *be defined by* (2.17) *and* (2.18), *respectively. Then*

$$\operatorname{var}_a^b A_k \le \operatorname{var}_a^b A \quad \textit{for all} \quad k \in \mathbb{N}.$$

Proof. Since

$$\operatorname{var}_{\alpha_{\ell-1}^k}^{\alpha_\ell^k} A_k = \left|A(\alpha_\ell^k) - A(\alpha_{\ell-1}^k)\right| \le \operatorname{var}_{\alpha_{\ell-1}^k}^{\alpha_\ell^k} A$$

for each $k \in \mathbb{N}$ and $\ell = 1, 2, \dots, 2^k$, we have

$$\operatorname{var}_a^b A_k = \sum_{\ell=1}^{2^k} \operatorname{var}_{\alpha_{\ell-1}^k}^{\alpha_\ell^k} A_k \le \sum_{\ell=1}^{2^k} \operatorname{var}_{\alpha_{\ell-1}^k}^{\alpha_\ell^k} A = \operatorname{var}_a^b A.$$

□

Equations (2.10) with A_k and g_k given by (2.18) and (2.19) are just initial value problems for linear ordinary differential systems

$$x' = A_k'(t)\,x + g_k'(t), \quad x(a) = \widetilde{x}_k. \tag{2.20}$$

In this view, the next Theorem 2.9 says that the solutions of (2.1) can be uniformly approximated by solutions of linear ordinary differential equations provided the functions A and g are continuous.

2.9 . Theorem. *Assume that* $A \in BV^{n\times n}[a,b] \cap C^{n\times n}[a,b]$ *and* $g \in C^n[a,b]$. *Let* $\widetilde{x}$ *and* $\widetilde{x}_k \in \mathbb{C}^n$, $k \in \mathbb{N}$, *be such that* (2.9) *holds. Furthermore, let the sequence* $\{\mathcal{D}_k\}$ *of divisions of the interval* $[a,b]$ *be given by* (2.17) *and let sequences* $\{A_k\} \subset AC^{n\times n}[a,b]$, $\{g_k\} \subset AC^n[a,b]$ *be defined by* (2.18) *and* (2.19), *respectively.*

Then the equation (2.1) *has a unique solution* x *on* $[a,b]$. *Furthermore, for each* $k \in \mathbb{N}$, *the equation* (2.10) *has a solution* x_k *on* $[a,b]$ *and* (2.11) *holds.*

Proof. *Step* 1. Since A is uniformly continuous on $[a,b]$, we have:

$$\begin{cases} \text{for each } \varepsilon > 0 \text{ there is a } \delta > 0 \text{ such that} \\ |A(t) - A(s)| < \frac{\varepsilon}{2} \\ \text{holds for all } t, s \in [a,b] \text{ such that } |t-s| < \delta\,. \end{cases} \tag{2.21}$$

Let $\frac{1}{2^{k_0}} < \delta$ and let t be an arbitrary point of $[a,b]$. Furthermore, let

$$\alpha_{\ell-1}, \alpha_\ell \in \mathcal{P}_{k_0} = \{\alpha_0, \alpha_1, \dots, \alpha_{p_{k_0}}\} \text{ and } t \in [\alpha_{\ell-1}, \alpha_\ell].$$

Then

$$|\alpha_\ell - \alpha_{\ell-1}| = \frac{1}{2^{k_0}} < \delta$$

and, according to (2.17), (2.18) and (2.21), we get for $k \ge k_0$

$$\begin{aligned} |A_k(t) - A(t)| &= |A_k(t) - A_k(\alpha_{\ell-1}) + A(\alpha_{\ell-1}) - A(t)| \\ &\le |A_k(t) - A_k(\alpha_{\ell-1})| + \frac{\varepsilon}{2} \\ &\le \left| A(\alpha_{\ell-1}) + [A(\alpha_\ell) - A(\alpha_{\ell-1})] \left[\frac{t-\alpha_{\ell-1}}{\alpha_\ell - \alpha_{\ell-1}}\right] - A(\alpha_{\ell-1}) \right| + \frac{\varepsilon}{2} \\ &= |A(\alpha_\ell) - A(\alpha_{\ell-1})| \left[\frac{t-\alpha_{\ell-1}}{\alpha_\ell - \alpha_{\ell-1}}\right] + \frac{\varepsilon}{2} \\ &\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

As k_0 was chosen independently of t, we can conclude that (1.8) is true.

Step 2. Analogously we can show that (2.8) holds for $\{g_k\}$ and g.

Step 3. By Lemma 2.8, (1.7) holds. Moreover, as A and A_k, $k \in \mathbb{N}$, are continuous, the equations (2.1) and (2.10) have unique solutions by Theorem 2.1 and we can complete the proof by making use of Theorem 2.5. □

3. Approximated solutions

In this section we will continue the consideration of the topics mentioned at the close of the previous section. Our aim is to disclose the relationship between solutions of generalized linear differential equation and limits of solutions of corresponding approximating sequences of linear ordinary differential equations.

We start by introducing the notions of *piecewise linear approximating sequences* and *approximated solution* to the generalized linear differential equation (2.1). Recall that the divisions $\mathcal{D}_k$ occurring below have been introduced in Notation 2.7. Furthermore, we will use the following notations.

3.1 . Notation. For given $A \in BV^{n\times n}[a,b]$, $g \in G^n[a,b]$ and $k \in \mathbb{N}$, we denote

$$\begin{cases} \mathfrak{S}^+(A;[a,b]) & = \{t \in [a,b]\colon\ \Delta^+A(t) \neq 0\}\,, \\ \mathfrak{S}^-(A;[a,b]) & = \{t \in [a,b]\colon\ \Delta^-A(t) \neq 0\}\,, \\ \mathfrak{S}(A;[a,b]) & = \mathfrak{S}^+(A;[a,b]) \cup \mathfrak{S}^-(A;[a,b]), \\ \mathfrak{S}^+(g;[a,b]) & = \{t \in [a,b]\colon\ \Delta^+g(t) \neq 0\}\,, \\ \mathfrak{S}^-(g;[a,b]) & = \{t \in [a,b]\colon\ \Delta^-g(t) \neq 0\}\,, \\ \mathfrak{S}(g;[a,b]) & = \mathfrak{S}^+(g;[a,b]) \cup \mathfrak{S}^-(g;[a,b]), \\ \mathfrak{S}(A,g;[a,b]) & = \mathfrak{S}(A;[a,b]) \cup \mathfrak{S}(g;[a,b]). \end{cases}$$

and

$$\begin{cases} \mathfrak{U}^+(A,k;[a,b]) & = \{t \in [a,b]\colon\ |\Delta^+A(t)| \geq \frac{1}{k}\}, \\ \mathfrak{U}^-(A,k;[a,b]) & = \{t \in [a,b]\colon\ |\Delta^-A(t)| \geq \frac{1}{k}\}, \\ \mathfrak{U}(A,k;[a,b]) & = \mathfrak{U}^+(A,k;[a,b]) \,\cup\, \mathfrak{U}^-(A,k;[a,b]), \\ \mathfrak{U}^+(g,k;[a,b]) & = \{t \in [a,b]\colon\ |\Delta^+A(t)| \geq \frac{1}{k}\}, \\ \mathfrak{U}^-(g,k;[a,b]) & = \{t \in [a,b]\colon\ |\Delta^-A(t)| \geq \frac{1}{k}\}, \\ \mathfrak{U}(g,k;[a,b]) & = \mathfrak{U}^+(g,k;[a,b]) \,\cup\, \mathfrak{U}^-(g,k;[a,b]), \\ \mathfrak{U}(A,g,k;[a,b]) & = \mathfrak{U}(A,k;[a,b]) \,\cup\, \mathfrak{U}(g,k;[a,b]). \end{cases}$$

3.2 . Remark. In particular, we have

$$\mathfrak{S}(A;[a,b]) = \bigcup_{k=1}^{\infty} \mathfrak{U}(A,k;[a,b]).$$

3.3. Definition. Let $A \in BV^{n\times n}[a,b]$, $g \in G^n[a,b]$ and let $\mathcal{D}_k \in \mathfrak{D}[a,b]$ be given by (2.17). We say that the sequence $\{A_k,\, g_k\} \subset AC^{n\times n}[a,b] \times AC^n[a,b]$ is a *piecewise linear approximation* ($p\,\ell$ -approximation) of $(A,\, g)$ if there exists a sequence $\{\mathcal{P}_k\}$ of divisions of the interval $[a,b]$ such that

$$\mathcal{P}_k \supset \mathcal{D}_k \,\cup\, \mathfrak{U}(A,g,k;[a,b]) \quad \text{for} \quad k \in \mathbb{N} \tag{3.1}$$

and A_k, g_k are for $k \in \mathbb{N}$ defined by (2.18) and (2.19).

3.4 . Remark. Let $\{A_k,\, g_k\}$ be a $p\,\ell$ -approximation of $(A,\, g)$. Then

$$\alpha^* := \sup\{\operatorname{var}_a^b A_k\colon\ k \in \mathbb{N}\} < \infty$$

due to Lemma 2.8. Furthermore, as A_k are continuous, due to (2.3), we have $c_{A_k} = 1$ for $k \in \mathbb{N}$. Hence, Corollary 2.3 yields

$$\operatorname{var}_a^b(x_k - g_k) \le \alpha^* \left(|\widetilde{x}| + 2\,\|g_k\|_\infty\right) < \infty \quad \text{for all} \quad k \in \mathbb{N}$$

and, by Helly's Theorem, there is a subsequence $\{x_{k_m} - g_{k_m}\}$ of $\{x_k - g_k\}$ and $y \in G^n[a,b]$ and such that

$$\lim_{m\to\infty} (x_{k_m}(t) - g_{k_m}(t)) = y(t) + g(t) \quad \text{for each} \quad t \in [a,b].$$

In particular,

$$\lim_{m\to\infty} x_{k_m}(t) = w(t) + g(t)$$

for all $t \in [a,b]$ such that $\lim_{m\to\infty} g_{k_m}(t) = g(t)$.

Notice that if the set $\mathfrak{S}(g;[a,b])$ has at most a finite number of elements, then

$$\lim_{k\to\infty} g_k(t) = g(t) \quad \text{for all} \quad t \in [a,b]. \tag{3.2}$$

3.5 . Definition. Let $A \in BV^{n\times n}[a,b]$, $g \in G^n[a,b]$ and $\widetilde{x} \in \mathbb{C}^n$. We say that $x^*\colon [a,b] \to \mathbb{C}^n$ is an *approximated solution* to equation (2.1) on the interval $[a,b]$ if there is a $p\,\ell$-approximation $\{A_k,\, g_k\}$ of $(A,\, g)$ such that

$$\lim_{k\to\infty} x_k(t) = x^*(t) \quad \text{for} \quad t \in [a,b] \tag{3.3}$$

holds for solutions x_k, $k \in \mathbb{N}$, of the corresponding approximating initial value problems (2.20).

3.6 . Remark. Notice that, using the language of Definitions 3.3 and 3.5, we can translate Theorem 2.9 into the following form:

Assume that $A \in BV^{n\times n}[a,b] \cap C^{n\times n}[a,b]$ and $g \in C^n[a,b]$. Then, the equation (2.1) has a unique approximated solution x^* on $[a,b]$ and x^* coincides on $[a,b]$ with the solution of (2.1).

In the rest of this paper we consider the case when the set $\mathfrak{S}(A,g;[a,b])$ of discontinuities of the coefficients A, g is non empty. We will start with the simplest case $\mathfrak{S}(A,g;[a,b]) = \{b\}$.

3.7. Lemma. *Let* $A \in BV^{n\times n}[a,b]$ *and* $g \in G^n[a,b]$ *be continuous on* $[a,b)$ *and such that*

$$|\Delta^- A(b)|\,|\Delta^- g(b)| = 0 \tag{3.4}$$

and let $\widetilde{x} \in \mathbb{C}^n$.

Then the equation (2.1) *has a unique approximated solution* x^* *on* $[a,b]$. *Furthermore,* x^* *is continuous on* $[a,b)$

$$x^*(b) = \exp\big(\Delta^- A(b)\big)\; x^*(b-) + \Delta^- g(b) \tag{3.5}$$

and x^* *coincides with the solution of* (2.1) *on* $[a,b)$.

Proof. *Step* 1. Let $\{A_k,\, g_k\}$ be an arbitrary $p\,\ell$-approximation of $\{A,\, g\}$ and let $\{\mathcal{P}_k\}$ be the corresponding sequence of divisions of $[a,b]$ fulfilling (2.18) and (2.19). Notice that, under our assumptions, $\mathcal{P}_k = \mathcal{D}_k$ for $k \in \mathbb{N}$. For $k \in \mathbb{N}$, put

$$\tau_k = \max\{t \in \mathcal{D}_k\colon\; t < b\}.$$

By (2.17) we have $b - \frac{b-a}{2^k} < \tau_k < b$ for $k \in \mathbb{N}$, and hence

$$\lim_{k\to\infty} \tau_k = b. \tag{3.6}$$

For $k \in \mathbb{N}$ and $t \in [a,b]$, define

$$\widetilde{A}_k(t) = \begin{cases} A_k(t) & \text{if } t \in [a,\tau_k],\\[2mm] A(\tau_k) + \dfrac{A(b-) - A(\tau_k)}{b-\tau_k}\,(t-\tau_k) & \text{if } t \in (\tau_k, b], \end{cases}$$

$$\widetilde{g}_k(t) = \begin{cases} g_k(t) & \text{if } t \in [a,\tau_k],\\[2mm] g(\tau_k) + \dfrac{g(b-) - g(\tau_k)}{b-\tau_k}\,(t-\tau_k) & \text{if } t \in (\tau_k, b]. \end{cases}$$

Furthermore, let

$$\widetilde{A}(t) = \begin{cases} A(t) & \text{if } t \in [a,b),\\ A(b-) & \text{if } t = b, \end{cases} \qquad \widetilde{g}(t) = \begin{cases} g(t) & \text{if } t \in [a,b),\\ g(b-) & \text{if } t = b. \end{cases} \tag{3.7}$$

We have $\widetilde{A}_k \in AC^{n\times n}[a,b]$, $\widetilde{g}_k \in AC^n[a,b]$ for $k \in \mathbb{N}$, $\widetilde{A} \in BV^{n\times n}[a,b] \cap C^{n\times n}[a,b]$ and $\widetilde{g} \in C^n[a,b]$.

Consider problems (2.1), (2.20) and

$$u_k' = \widetilde{A}_k'(t)\, u_k + \widetilde{g}_k'(t), \quad u_k(a) = \widetilde{x}, \quad k \in \mathbb{N}, \tag{3.8}$$

and

$$u(t) = \widetilde{x} + \int_a^t \mathrm{d}[\widetilde{A}]\, u + \widetilde{g}(t) - \widetilde{g}(a). \tag{3.9}$$

Let $\{x_k\}$ and $\{u_k\}$ be the sequences of solutions on $[a,b]$ of problems (2.20) and (3.8), respectively. We can see that, for each $k \in \mathbb{N}$, u_k coincides with x_k on $[a,\tau_k]$. Furthermore, by Theorem 2.1, equation (3.9) possesses a unique solution u on $[a,b]$ a u is continuous on $[a,b]$. It's easy to see that the relations

$$\lim_{k\to\infty} \|\widetilde{A}_k - \widetilde{A}\|_\infty = 0 \quad \text{and} \quad \lim_{k\to\infty} \|\widetilde{g}_k - \widetilde{g}\|_\infty = 0$$

are true. Therefore, by Theorem 2.5, we get

$$\lim_{k\to\infty} \|u_k - u\|_\infty = 0. \tag{3.10}$$

Since $x_k = u_k$ on $[a,\tau_k]$, and due to (3.6), we have

$$\lim_{k\to\infty} x_k(t) = u(t) \quad \text{for } t \in [a,b). \tag{3.11}$$

Step 2. Next we will prove that

$$\lim_{k\to\infty} x_k(\tau_k) = u(b). \tag{3.12}$$

Indeed, let $\varepsilon > 0$ be given and let $\delta > 0$ be such that

$$|u(t) - u(b)| < \frac{\varepsilon}{2} \quad \text{for } t \in [b-\delta, b]$$

Further, by (3.10), there is a $k_0 \in \mathbb{N}$ such that

$$\tau_k \in [b-\delta, b) \quad \text{and} \quad \|u_k - u\|_\infty < \frac{\varepsilon}{2} \quad \text{whenever } k \ge k_0.$$

Consequently,

$$\begin{aligned} |x_k(\tau_k) - u(b-)| &\le |x_k(\tau_k) - u(\tau_k)| + |u(\tau_k) - x(b-)| \\ &= |u_k(\tau_k) - u(\tau_k)| + |u(\tau_k) - x(b-)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

holds for $k \geq k_0$. This completes the proof of (3.12).

Step 3. On the intervals $[\tau_k, b]$, the equations from (2.20) reduce to the equations with constant coefficients

$$x_k' = B_k\, x_k + e_k,$$

where

$$B_k = \frac{A_k(b) - A_k(\tau_k)}{b - \tau_k} \quad \text{and} \quad e_k = \frac{g_k(b) - g_k(\tau_k)}{b - \tau_k}.$$

Their solutions x_k are on $[\tau_k, b]$ given by

$$x_k(t) = \exp\left(B_k\,(t - \tau_k)\right) x_k(\tau_k) + \left(\int_{\tau_k}^{t} \exp\left(B_k\,(t - r)\right) \mathrm{d}r\right) e_k.$$

In particular,

$$\begin{aligned} x_k(b) &= \exp\left(A(b) - A(\tau_k)\right) x_k(\tau_k) \\ &\qquad + \frac{1}{b - \tau_k} \left(\int_{\tau_k}^{b} \exp\left([A(b) - A(\tau_k)]\,\frac{b - r}{b - \tau_k}\right) \mathrm{d}r\right) [g_k(b) - g_k(\tau_k)]. \end{aligned}$$

By Lemma 1.2, we have

$$\begin{aligned} &\lim_{k\to\infty} \frac{1}{b - \tau_k} \left(\int_{\tau_k}^{b} \exp\left([A(b) - A(\tau_k)]\frac{b - r}{b - \tau_k}\right) \mathrm{d}r\right) [g_k(b) - g_k(\tau_k)] \\ &\qquad = \lim_{k\to\infty} \frac{1}{b - \tau_k} \left(\int_{\tau_k}^{b} \exp\left(\Delta^- A(b)\frac{b - r}{b - \tau_k}\right) \mathrm{d}r\right) \Delta^- g(b). \end{aligned}$$

In particular, having in mind (3.12), we obtain

$$\lim_{k\to\infty} x_k(b) = \begin{cases} \exp\left(\Delta^- A(b)\right) u(b) & \text{if } \Delta^- g(b) = 0, \\ u(b) + \Delta^- g(b) & \text{if } \Delta^- A(b) = 0. \end{cases}$$

So, in view of the assumption (3.4), we can conclude that the relation

$$\lim_{k\to\infty} x_k(b) = \exp\big(\Delta^- A(b)\big)\, u(b) + \Delta^- g(b) \tag{3.13}$$

is true.

Step 4. Define

$$x^*(t) = \begin{cases} u(t) & \text{if } \ t \in [a,b), \\ \exp\left(\Delta^- A(b)\right) u(b) + \Delta^- g(b) & \text{if } \ t = b. \end{cases}$$

Then $x^*(b-) = u(b)$, $x^*(t) = \lim_{k\to\infty} x_k(t)$ for $t \in [a,b)$ due to (3.12) and $x^*(b) = \lim_{k\to\infty} x_k(b)$ due to (3.13). Therefore, x^* is a $p\ell$-approximated solution of (2.1). Since it does not depend upon the choice of the approximating sequence $\{A_k, g_k\}$, we can see that x^* is also the unique approximated solution of (2.1). This completes the proof. □

The following assertion can be related to Lemma 3.7 by introducing new independent variable s by a substitution $s = a + b - t$. Nevertheless, we prefer to give here its direct proof, though little bit more concise.

3.8. Lemma. *Let* $A \in BV^{n\times n}[a,b]$ *and* $g \in G^n[a,b]$ *be continuous on* $(a,b]$ *and such that*

$$|\Delta^+ A(a)|\,|\Delta^+ g(a)| = 0 \tag{3.14}$$

and let $\widetilde{x} \in \mathbb{C}^n$.

Then the equation (2.1) *has a unique approximated solution* x^* *on* $[a,b]$. *Furthermore,* x^* *is continuous on* $(a,b]$,

$$x^*(a+) = \exp\left(\Delta^+ A(a)\right)\,\widetilde{x} + \Delta^+ g(a)$$

and x^* *coincides on* $(a,b]$ *with the solution* y *of the equation*

$$y(t) = \widetilde{y} + \int_a^t \mathrm{d}[\widetilde{A}]\,y + \widetilde{g}(t) - \widetilde{g}(a) \quad \text{on} \ \ [a,b], \tag{3.15}$$

where

$$\widetilde{y} = \exp\left(\Delta^+ A(a)\right)\,\widetilde{x} + \Delta^+ g(a),$$

and

$$\widetilde{A}(t) = \begin{cases} A(a+) & \text{if } \ t = a, \\ A(t) & \text{if } \ t \in (a,b] \end{cases} \quad \text{and} \quad \widetilde{g}(t) = \begin{cases} g(a+) & \text{if } \ t = a, \\ g(t) & \text{if } \ t \in (a,b]. \end{cases}$$

Proof. Step 1. On the intervals $[a,\tau_k]$, the equations from (2.20) reduce to equations with constant coefficients

$$A_k'(t) = \frac{A_k(\tau_k) - A_k(a)}{\tau_k - a}, \quad g_k'(t) = \frac{g_k(\tau_k) - g_k(a)}{\tau_k - a}.$$

Their solutions x_k are on $[a, \tau_k]$ given by

$$x_k(t) = \exp\left(\frac{A_k(\tau_k) - A_k(a)}{\tau_k - a}\,(t-a)\right)\widetilde{x} \\ + \left(\int_a^t \exp\left(\frac{A_k(\tau_k) - A_k(a)}{\tau_k - a}\,(t-r)\right)\mathrm{d}r\right)\frac{g_k(\tau_k) - g_k(a)}{\tau_k - a}.$$

In particular,

$$x_k(\tau_k) = \exp\left(A(\tau_k) - A(a)\right) x_k(\tau_k) \\ + \frac{1}{\tau_k - a}\left(\int_a^{\tau_k} \exp\left([A(\tau_k) - A(a)]\,\frac{\tau_k - r}{\tau_k - a}\right)\mathrm{d}r\right)[g_k(\tau_k) - g_k(\tau_k)].$$

By Lemma 1.2, we have

$$\lim_{k\to\infty} \frac{1}{\tau_k - a}\left(\int_a^{\tau_k} \exp\left([A(\tau_k) - A(a)]\,\frac{\tau_k - r}{\tau_k - a}\right)\mathrm{d}r\right)[g_k(\tau_k) - g_k(a)] \\ = \lim_{k\to\infty} \frac{1}{\tau_k - a}\left(\int_a^{\tau_k} \exp\left(\Delta^+ A(a)\frac{\tau_k - r}{\tau_k - a}\right)\mathrm{d}r\right)\Delta^+ g(b).$$

Thus,

$$\lim_{k\to\infty} x_k(a) = \begin{cases} \exp\left(\Delta^+ A(a)\right)\widetilde{x} & \text{if } \ \Delta^+ g(a) = 0, \\ \widetilde{x} + \Delta^+ g(a) & \text{if } \ \Delta^+ A(a) = 0. \end{cases}$$

With respect to the assumption (3.4), we can conclude that the relation

$$\lim_{k\to\infty} x_k(\tau_k) = \exp\left(\Delta^+ A(a)\right)\widetilde{x} + \Delta^+ g(a) = \widetilde{y} \tag{3.16}$$

is true.

Step 2. Let $\{A_k, g_k\}$ be an arbitrary $p\,\ell$-approximation of $\{A, g\}$ and let $\{\mathcal{D}_k\}$ be the corresponding sequence of divisions of $[a, b]$ fulfilling (2.18) and (2.19). Let $\{x_k\}$ be a sequence of solutions of the approximating initial value problems (2.20) on $[a, b]$. Consider equation (3.15). By Theorem 2.1, it has a unique solution u on $[a, b]$, u is continuous on $[a, b]$ and, by an argument analogous to that used in *Step* 1 of the proof of Lemma L3.6, we can show that the relation

$$\lim_{k\to\infty} x_k(t) = u(t) \quad \text{for } \ t \in (a, b] \tag{3.17}$$

is true. Finally, notice that, with respect to (3.16), we have also

$$\lim_{k\to\infty} x_k(\tau_k) = u(a).$$

Step 3. Analogously to *Step* 4 of the proof of lemma 3.7, we can complete the proof by showing that the function

$$x^*(t) = \begin{cases} \widetilde{x} & \text{if } \ t = a, \\ u(t) & \text{if } \ t \in (a,b], \end{cases}$$

is the unique approximated solution of (2.1). □

3.9 . Remark. First, let us notice that if $a < c < b$ and the functions x_1^* and x_2^* are respectively $p\,\ell$ -approximated solutions to

$$x(t) = \widetilde{x}_1 + \int_a^t \mathrm{d}[A]\,x + g(t) - g(a), t \in [a,c]$$

and

$$x(t) = \widetilde{x}_2 + \int_c^t \mathrm{d}[A]\,x + g(t) - g(c), t \in [c,b],$$

where $\widetilde{x}_2 = x_1^*(c)$, then the function

$$x^*(t) = \begin{cases} x_1^*(t) & \text{if } \ t \in [a,c], \\ x_2^*(t) & \text{if } \ t \in (c,b] \end{cases}$$

is a $p\,\ell$ -approximated solution to (2.1).

3.10. Theorem. *Assume that* $A \in BV^{n\times n}[a,b]$, $g \in G^n[a,b]$, $s_1, s_2, \ldots, s_m \in (a,b)$, $\mathfrak{S}(A,g;[a,b])) = \{s_1, s_2, \ldots, s_m\}$ *and*

$$\begin{cases} \mathfrak{S}^-(A;[a,b])) \cap \mathfrak{S}^-(g;[a,b])) = \emptyset, \\ \mathfrak{S}^+(A;[a,b])) \cap \mathfrak{S}^+(g;[a,b])) = \emptyset. \end{cases} \tag{3.18}$$

Then, for each $\widetilde{x} \in \mathbb{C}^n$, *there is exactly one approximated solution* x^* *of equation* (2.1) *on* $[a,b]$. *Furthermore,* $\mathfrak{S}(x^*;[a,b]) = \mathfrak{S}(A,g;[a,b]))$ *and the relations*

$$\begin{cases} x^*(t) = \exp\left(\Delta^- A(t)\right)\, x^*(t-) + \Delta^- g(t), \\ x^*(t+) = \exp\left(\Delta^+ A(t)\right) x^*(t) + \Delta^+ g(t) \end{cases} \qquad \text{if } t \in \mathfrak{S}(A,g;[a,b]))$$

and

$$x^*(t) = x^*(s_{i-1}+) + \int_{s_{i-1}}^{t} \mathrm{d}[\widetilde{A}^{[i]}]\, x^* + g(t) - g(s_{i-1}+) \quad \textit{if } t \in (s_{i-1}, s_i)$$

hold for $t \in [a, b]$ *and* $i = 1, 2, \ldots, m+1$, *where* $s_0 = a$ *and* $s_{m+1} = b$ *and*

$$\widetilde{A}^{[i]}(t) = \begin{cases} A(s_{i-1}+) & \textit{if } \ t = s_{i-1}, \\ A(t) & \textit{if } \ t \in (s_{i-1}, s_i), \\ A(s_i-) & \textit{if } \ t = s_i. \end{cases}$$

Proof. Having in mind Remark 3.9, we deduce the assertion of Theorem 3.10 by a successive use of Lemmas 3.7 and 3.8. To this aim it is sufficient to choose a division $\mathcal{D} = \{\alpha_0, \alpha_1, \ldots, \alpha_r\}$ of $[a, b]$ such that for each subinterval $[\alpha_{k-1}, \alpha_k]$, $k = 1, 2, \ldots, r$, either the assumptions of Lemma 3.7 or the assumptions of Lemma 3.8 are satisfied with with α_{k-1} in place of a and α_k in place of b. □

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