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Local a posteriori error estimator based on the hypercircle method

Combination of the equilibrated residual method and the method of hypercircle

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Mathematical modeling

Reality — Math. model — Discrete model — Computerized model

↓	↓	↓
Physical assump. Simplifications . . .	Discretization error: $e = u - u_h$	Quadrature error Iteration error Round-off error

A posteriori error estimator $\mathcal{E}$: $\begin{cases} \|e\| \approx \mathcal{E} \\ \|e\| \le \mathcal{E} \quad \text{guaranteed} \end{cases}$

Computable and fast

Linear elliptic model problem

$$
\begin{aligned}
-\nabla \cdot (\mathcal{A}\nabla \bar{u}) &= f && \text{in } \Omega,\\
\bar{u} &= g_{\mathsf{D}} && \text{on } \Gamma_{\mathsf{D}},\\
(\mathcal{A}\nabla \bar{u}) \cdot \nu &= g_{\mathsf{N}} && \text{on } \Gamma_{\mathsf{N}}.
\end{aligned}
$$

Notation:

$\Omega \subset \mathbb{R}^2$ ... polygonal domain,

$\nu = \nu(x_1, x_2)$... unite outer normal to $\partial\Omega$,

$\Gamma_{\mathsf{D}} \subset \partial\Omega$ ... Dirichlet part of $\partial\Omega$,

$\Gamma_{\mathsf{N}} \subset \partial\Omega$ ... Neumann part of $\partial\Omega$.

Model problem – weak formulation

Weak solution $\bar{u} \in H^1(\Omega)$, $\bar{u} = u + g_\mathsf{D}$ and $u \in V$ satisfies

$$(\mathcal{A}\nabla u, \nabla v) = (f, v) - (\mathcal{A}\nabla g_\mathsf{D}, \nabla v) + \langle g_\mathsf{N}, v\rangle \quad \forall v \in V,$$

where

$$V = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_\mathsf{D}\},$$

$\mathcal{A} \in [L^\infty(\Omega)]^{2\times 2}$ ... symmetric, uniformly positive definite matrix,
$g_\mathsf{D} \in H^1(\Omega)$ ... prolongation of values on $\partial\Gamma_\mathsf{D}$ into interior of Ω,
$f \in L^2(\Omega)$ ... the right-hand side,
$g_\mathsf{N} \in L^2(\Gamma_N)$ ... the Neumann boundary condition,
$\Gamma_\mathsf{N} \subset \partial\Omega$ ... Neumann part of $\partial\Omega$.

$$\begin{aligned}
(\mathcal{A}\nabla u, \nabla v) &= \int_\Omega (\mathcal{A}\nabla u)\cdot\nabla v \,\mathrm{d}x, && \forall u, v \in V,\\
(f, v) &= \int_\Omega f v \,\mathrm{d}x && \forall f, v \in L^2(\Omega),\\
\langle g_\mathsf{N}, v\rangle &= \int_{\Gamma_\mathsf{N}} g_\mathsf{N} v \,\mathrm{d}s && \forall g_\mathsf{N}, v \in L^2(\Gamma_\mathsf{N}).
\end{aligned}$$

Finite element method

Finite element solution:

$\bar{u}_h = u_h + g_{\mathsf{D}}$, $\bar{u}_h \in H^1(\Omega)$ and $u_h \in V_h$ satisfies

$$(\mathcal{A}\nabla u_h, \nabla v_h) = (f, v_h) - (\mathcal{A}\nabla g_{\mathsf{D}}, \nabla v_h) + \langle g_{\mathsf{N}}, v_h \rangle \quad \forall v_h \in V_h.$$

T_h ... triangulation of Ω,

$V_h \subset V$ ... finite element space based on T_h

continuous and piecewise polynomial functions of degree p.

Residual equation:

$$(\mathcal{A}\nabla e, \nabla v) = \mathcal{R}(v) \quad \forall v \in V,$$

where

$$\mathcal{R}(v) = (f, v) - (\mathcal{A}\nabla \bar{u}_h, \nabla v) + \langle g_{\mathsf{N}}, v \rangle \text{... residuum,} \quad v \in V.$$

Notation: L^2-norm: $\|v\|^2_{0,\Omega} = (v, v)$, energy norm: $\|v\|^2 = (\mathcal{A}\nabla v, \nabla v)$.

The equilibrated residual method

$$\text{Notation: } (\mathcal{A}\nabla u, \nabla v)_K = \int_K (\mathcal{A}\nabla u)\cdot\nabla v \, \mathrm{d}x, \quad \forall u, v \in H^1(K),$$
$$(f, v)_K = \int_K f v \, \mathrm{d}x \qquad \forall f, v \in L^2(K),$$
$$\langle g_K, v\rangle_{\partial K} = \int_{\partial K} g_K v \, \mathrm{d}s \qquad \forall g_K, v \in L^2(\partial K).$$

Error estimator:

$$\mathcal{E}_{\mathrm{EQ}}^2 = \sum_{K \in T_h} \|\Phi_K\|_K^2 , \qquad \text{where } \|v\|_K^2 = (\mathcal{A}\nabla v, \nabla v)_K.$$

Local Neuman problems on triangles: $\Phi_K \in V(K)$,

$$(\mathcal{A}\nabla\Phi_K, \nabla v)_K = \underbrace{(f, v)_K - (\mathcal{A}\nabla \bar{u}_h, \nabla v)_K + \langle g_K, v\rangle_{\partial K}}_{\mathcal{R}_K^{\mathrm{EQ}}(v)} \quad \forall v \in V(K),$$

Upper bound $\|e\| \le \mathcal{E}_{\mathrm{EQ}}$. $\qquad V(K) = \left\{ v \in H^1(K) : v = 0 \text{ on } \Gamma_{\mathrm{D}} \right\}$

Boundary fluxes g_K

- $g_K \approx (\mathcal{A}\nabla\bar{u}) \cdot \nu_K$, on ∂K.
- If K and K^* denote two adjacent elements then

$$\left.\begin{aligned} g_K + g_{K^*} &= 0 \qquad \text{on } \partial K \cap \partial K^*, \\ g_K &= g_{\mathsf{N}} \qquad \text{on } \partial K \cap \partial \Gamma_{\mathsf{N}}, \end{aligned}\right\} \implies \mathcal{R}(v) = \sum_{K \in T_h} \mathcal{R}_K^{\mathsf{EQ}}(v).$$

- $g_K \in P^p(\gamma)$ ... polynomials of degree p on edges γ of elements K
  if $\gamma \not\subset \Gamma_N$.
- p-th order equilibration condition:

$$\mathcal{R}_K^{\mathsf{EQ}}(\theta_K) = (f, \theta_K)_K - (\mathcal{A}\nabla\bar{u}_h, \nabla\theta_K)_K + \langle g_K, \theta_K \rangle_{\partial K} = 0$$

for all polynomials θ_K of degree p from $V(K)$.

- Boundary fluxes g_K can be computed quickly.

The equilibrated residual method – summary

- Compute boundary fluxes g_K – fast algorithm.

- Find approximate solutions to the local residual problems

$$(\mathcal{A}\nabla\Phi_K, \nabla v)_K = (f, v)_K - (\mathcal{A}\nabla\bar{u}_h, \nabla v)_K + \langle g_K, v\rangle_{\partial K} \quad \forall v \in V(K).$$

- Evaluate the estimator: $\|e\|^2 \leq \sum_{K\in T_h} \|\Phi_K\|_K^2$.

Trouble: This a posteriori error estimator is locally computable, but it is not guaranteed upper bound.

The method of hypercircle

Notation: $\|\mathbf{q}\|^2_{\mathcal{A}^{-1},\Omega} = (\mathcal{A}^{-1}\mathbf{q}, \mathbf{q})$; $\quad H(\mathrm{div}, \Omega) \subset \left[L^2(\Omega)\right]^2$

$$Q(f, g_\mathrm{N}) = \{\mathbf{q} \in H(\mathrm{div}, \Omega) : (\mathbf{q}, \nabla v) = (f, v) + \langle g_\mathrm{N}, v\rangle \quad \forall v \in V\}$$

$$\Downarrow$$

$$\|\mathbf{q} - \mathcal{A}\nabla \bar{u}_h\|^2_{\mathcal{A}^{-1},\Omega} = \|\mathbf{q} - \mathcal{A}\nabla \bar{u}\|^2_{\mathcal{A}^{-1},\Omega} + \|\bar{u} - \bar{u}_h\|^2 \qquad \forall \mathbf{q} \in Q(f, g_\mathrm{N})$$

$$\Downarrow$$

$$\|e\| \le \|\mathbf{q} - \mathcal{A}\nabla \bar{u}_h\|_{\mathcal{A}^{-1},\Omega} \qquad \forall \mathbf{q} \in Q(f, g_\mathrm{N})$$

$$\Downarrow$$

$$\|e\| \le \|\bar{\mathbf{p}} + \mathbf{curl}\, y - \mathcal{A}\nabla \bar{u}_h\|_{\mathcal{A}^{-1},\Omega} \qquad \forall y \in W,$$

where

$Q(f, g_\mathrm{N}) = \bar{\mathbf{p}} + \mathbf{curl}\, W$ and $\bar{\mathbf{p}} \in Q(f, g_\mathrm{N})$ fixed

$$W = \left\{w \in H^1(\Omega) : w = 0 \text{ on } \Gamma_\mathrm{N}\right\}$$

$$\mathbf{curl} = (\partial/\partial x_2, -\partial/\partial x_1)^\top$$

The method of hypercircle – first ⇓

Substituting $v = e = \bar{u} - \bar{u}_h$ into the weak formulation we get:

$$-(\mathcal{A}\bar{u}, \nabla e) = -(f, e) - \langle g_\mathsf{N}, e\rangle .$$

Let us compute for any $\mathbf{q} \in H(\mathrm{div}, \Omega)$:

$$\begin{aligned}
&\|\mathbf{q} - \mathcal{A}\nabla\bar{u}_h\|^2_{\mathcal{A}^{-1},\Omega} \\
&\quad = \left(\mathcal{A}^{-1}\mathbf{q} - \nabla\bar{u} - \nabla\bar{u}_h + \nabla\bar{u}\,,\ \mathbf{q} - \mathcal{A}\nabla\bar{u} - \mathcal{A}\nabla\bar{u}_h + \mathcal{A}\nabla\bar{u}\right) \\
&\quad = \left\|\mathbf{q} - \mathcal{A}\nabla\bar{u}\right\|^2_{\mathcal{A}^{-1},\Omega} + 2\left(\mathbf{q} - \mathcal{A}\nabla\bar{u}, \nabla\bar{u} - \nabla\bar{u}_h\right) + \left\|\bar{u} - \bar{u}_h\right\|^2 \\
&\quad = \left\|\mathbf{q} - \mathcal{A}\nabla\bar{u}\right\|^2_{\mathcal{A}^{-1},\Omega} + 2\,(\mathbf{q}, \nabla e) - 2\,(f, e) - 2\,\langle g_\mathsf{N}, e\rangle + \left\|\bar{u} - \bar{u}_h\right\|^2 .
\end{aligned}$$

$$Q(f, g_\mathsf{N}) = \{\mathbf{q} \in H(\mathrm{div}, \Omega) : (\mathbf{q}, \nabla v) = (f, v) + \langle g_\mathsf{N}, v\rangle \quad \forall v \in V\}$$

$$\Downarrow$$

$$\|\mathbf{q} - \mathcal{A}\nabla\bar{u}_h\|^2_{\mathcal{A}^{-1},\Omega} = \|\mathbf{q} - \mathcal{A}\nabla\bar{u}\|^2_{\mathcal{A}^{-1},\Omega} + \|\bar{u} - \bar{u}_h\|^2 , \quad \forall \mathbf{q} \in Q(f, g_\mathsf{N}).$$

The method of hypercircle – computation

$$\|e\| \leq \|\bar{\mathbf{p}} + \mathbf{curl}\, y - \mathcal{A}\nabla \bar{u}_h\|_{\mathcal{A}^{-1},\Omega} \qquad \forall y \in W$$

$\bar{\mathbf{p}}$ is explicitly computable:

$$\bar{\mathbf{p}} = \mathbf{F} + \mathbf{curl}\, w, \text{ where } \mathbf{F}(x_1, x_2) = \left(-\int_0^{x_1} f(s, x_2)\,\mathrm{d}s, 0\right)^\top,$$

$$\mathbf{curl}\, w \cdot \nu = \nabla w \cdot \tau = g_{\mathrm{N}} - \mathbf{F} \cdot \nu \qquad \text{on } \Gamma_{\mathrm{N}},$$

$$\tau = (-\nu_2, \nu_1)^\top$$

Replace W by a finite dimensional subspace $W_h \subset W$.
The optimal choice $y_h \in W_h$ minimizes the estimator over W_h:

$$(\mathcal{A}^{-1}\,\mathbf{curl}\, y_h, \mathbf{curl}\, v_h) = (\nabla \bar{u}_h - \mathcal{A}^{-1}\bar{\mathbf{p}}, \mathbf{curl}\, v_h) \qquad \forall v_h \in W_h.$$

Trouble: it involves solution of a global problem,
i.e., the estimator is not local – not fast.

THE COMBINED METHOD

- Compute boundary fluxes g_K by the equilibrated residual method.

- Apply the method of hypercircle to the local residual problem

$$(\mathcal{A}\nabla\Phi_K, \nabla v)_K = (f, v)_K - (\mathcal{A}\nabla\bar{u}_h, \nabla v)_K + \langle g_K, v\rangle_{\partial K} \quad \forall v \in V(K).$$

The combined method

$$Q_K(f, g_K, \bar{u}_h) = \Big\{ \mathbf{q} \in H(\mathrm{div}, K) :$$
$$(\mathbf{q}, \nabla v)_K = (f, v)_K - (\mathcal{A}\nabla \bar{u}_h, \nabla v)_K + \langle g_K, v\rangle_{\partial K} \quad \forall v \in V(K) \Big\}$$

$$\Downarrow$$

$$\|\mathbf{q}\|^2_{\mathcal{A}^{-1},K} = \|\mathbf{q} - \mathcal{A}\nabla\Phi_K\|^2_{\mathcal{A}^{-1},K} + \|\Phi_K\|^2_K \quad \forall \mathbf{q} \in Q_K(f, g_K, \bar{u}_h)$$

$$\Downarrow$$

$$\|\Phi_K\|_K \leq \|\mathbf{q}\|_{\mathcal{A}^{-1},K} \quad \forall \mathbf{q} \in Q_K(f, g_K, \bar{u}_h)$$

$$\Downarrow$$

$$\|e\|^2 \leq \sum_{K\in T_h} \|\Phi_K\|^2_K \leq \sum_{K\in T_h} \|\mathbf{q}\|^2_{\mathcal{A}^{-1},K} = \sum_{K\in T_h} \|\bar{\mathbf{p}}_K + \mathbf{curl}\, y_K\|^2_{\mathcal{A}^{-1},K}$$
$$\forall y_K \in W(K) \text{ with } \mathbf{q} = \bar{\mathbf{p}}_K + \mathbf{curl}\, y_K$$

$$Q_K(f, g_K, \bar{u}_h) = \bar{\mathbf{p}}_K + \mathbf{curl}\, W(K),$$
$$W(K) = \Big\{ v \in H^1(K) : v = 0 \text{ on } \partial K \setminus \Gamma_{\mathrm{D}} \Big\},$$
$$\bar{\mathbf{p}}_K \in Q_K(f, g_K, \bar{u}_h) \text{ is fixed.}$$

The combined method – computation

$$\|e\|^2 \leq \sum_{K \in \mathcal{T}_h} \|\bar{\mathbf{p}}_K + \mathbf{curl}\, y_K\|^2_{\mathcal{A}^{-1},K} \quad \forall y_K \in W(K)$$

$\bar{\mathbf{p}}_K$ explicitly computable: $\bar{\mathbf{p}}_K = \mathbf{F} + \mathbf{curl}\, w_K - \mathcal{A}\nabla \bar{u}_h,$ where

$$\mathbf{F}(x_1, x_2) = \left(-\int_0^{x_1} f(s, x_2)\, \mathrm{d}s, 0 \right)^\top,$$

$w_K \in H^1(K)$ and $\mathbf{curl}\, w_K \cdot \nu_K = \dfrac{\partial w_K}{\partial \tau_K} = g_K - \mathbf{F} \cdot \nu_K$ on $\partial K \setminus \Gamma_{\mathrm{D}}$,

$$\tau_K = (-\nu_{K,2}, \nu_{K,1})^\top.$$

Finite dimensional subspace: $W_h(K) \subset W(K)$.
Minimizer $y_{Kh} \in W_h(K)$, over $W_h(K)$ satisfies

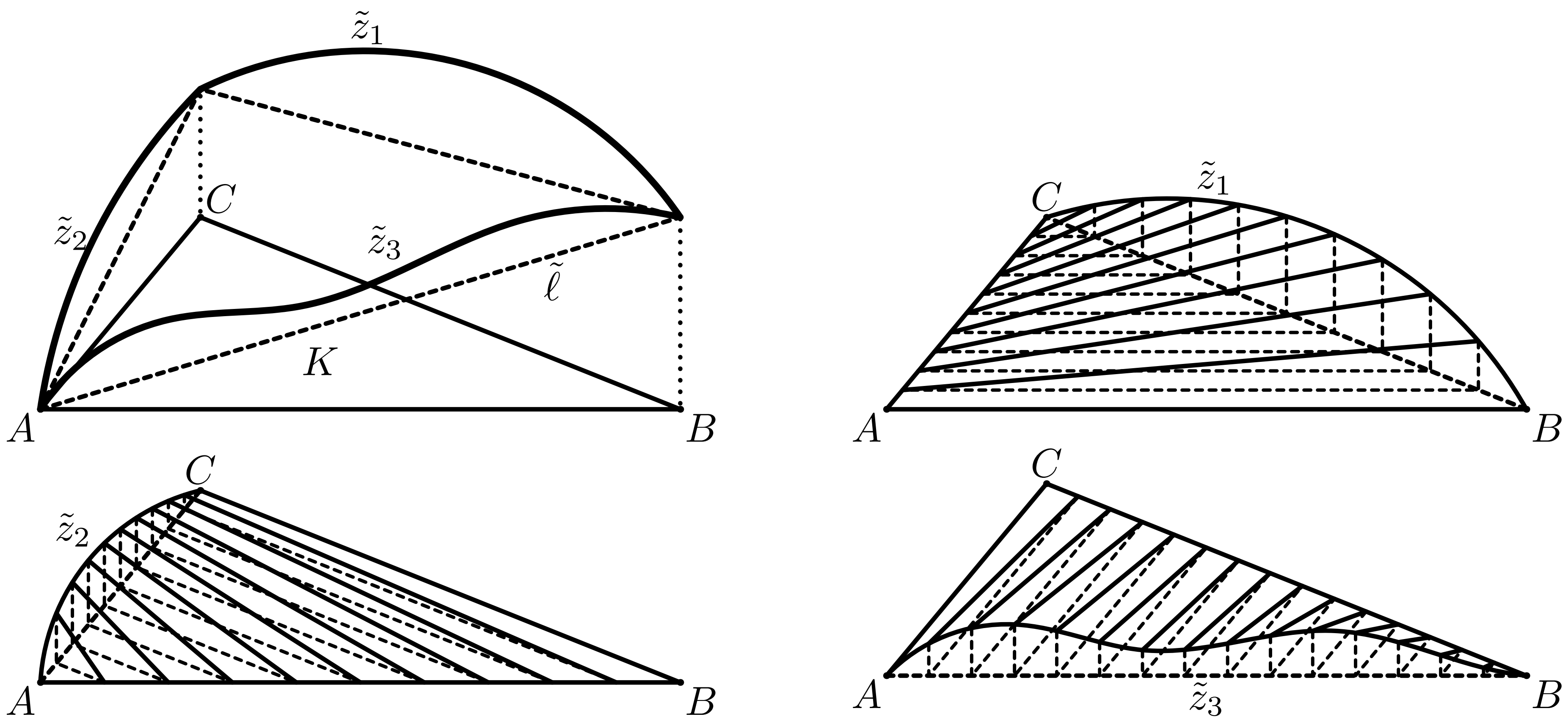
$$\left(\mathcal{A}^{-1}\, \mathbf{curl}\, y_{Kh}, \mathbf{curl}\, v \right)_K = -\left(\mathcal{A}^{-1} \bar{\mathbf{p}}_K, \mathbf{curl}\, v \right)_K \quad \forall v \in W_h(K).$$

The combined method – construction of w_K

$$\frac{\partial w_K}{\partial \tau_K} = g_K - \mathbf{F}\cdot\nu_K \quad \text{on } \partial K \setminus \Gamma_D \quad \Rightarrow$$

values of w_K are given by primitive function to $g_K - \mathbf{F}\cdot\nu_K$.

Prolongation of these values into the interior of K:



The combined method – properties of the prolongation

- If $\omega \in C^0(\partial K)$ and $\omega|_\gamma \in P^p(\gamma)$ for all edges $\gamma \subset \partial K$ then the prolongation $\tilde{\omega} \in P^p(K)$.

- Derivatives of the prolonged w_K – explicitly computable.

- If $u_h \in V_h$ is exact and $\mathcal{A}$ is constant then $\bar{\mathbf{p}}_K + \mathbf{curl}\, y_K = 0$ (estimator is exact).

Notation: $P^p(\Theta)$ – polynomials of degree p defined on the set Θ.

The combined method – summary

- Compute boundary fluxes g_K using residual equilibration method.
- Construct for all triangles K in T_h vector

$$\bar{\mathbf{p}}_K = \mathbf{F} + \mathbf{curl}\, w_K - \mathcal{A}\nabla \bar{u}_h,$$

  where construction of w_K employs the prolongation shown above.

- Find solution $y_{Kh} \in W_h(K)$ of the finite dimensional local problem

$$\left(\mathcal{A}^{-1}\,\mathbf{curl}\, y_{Kh}, \mathbf{curl}\, v\right)_K = -\left(\mathcal{A}^{-1}\bar{\mathbf{p}}_K, \mathbf{curl}\, v\right)_K \quad \forall v \in W_h(K).$$

- Evaluate estimate

$$\|e\|^2 \le \sum_{K \in T_h} \|\bar{\mathbf{p}}_K + \mathbf{curl}\, y_{Kh}\|^2_{\mathcal{A}^{-1},K}\,.$$

NUMERICAL EXPERIMENTS

- Finite element method:

  V_h ... continuous and piecewise quadratic functions with zero on Γ_D.

- Equilibrated residual method:

  $V_h(K)$ ... cubic polynomials with zero on Γ_D.

- The method of hypercircle:

  W_h ... continuous and piecewise quadratic functions with zero on Γ_N.

- The combined methods:

  $W_h(K)$ ... cubic polynomials with zero on $\partial K \setminus \Gamma_\mathrm{D}$.

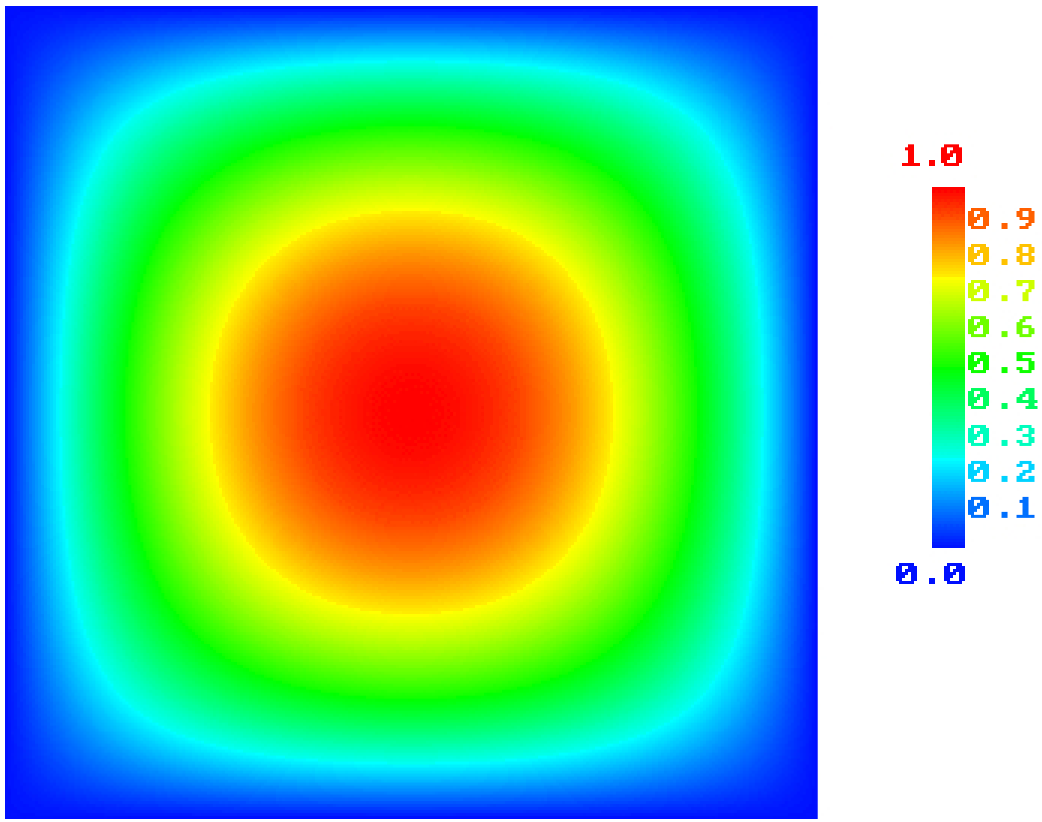
Remark: If we consider interior element K, then $\dim V_h(K) = 10$ and $\dim W_h(K) = 1$. Thus, the combined method performs faster.

Example 1

Data:

$$
\begin{aligned}
\Omega &= [-1,1]^2, \\
\Gamma_D &= \partial\Omega, \\
\Gamma_N &= \emptyset, \\
\mathcal{A} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
g_D &= 0, \\
f(x_1,x_2) &= 2(2 - x_1^2 - x_2^2), \\
u(x_1,x_2) &= (x_1^2 - 1)(x_2^2 - 1).
\end{aligned}
$$

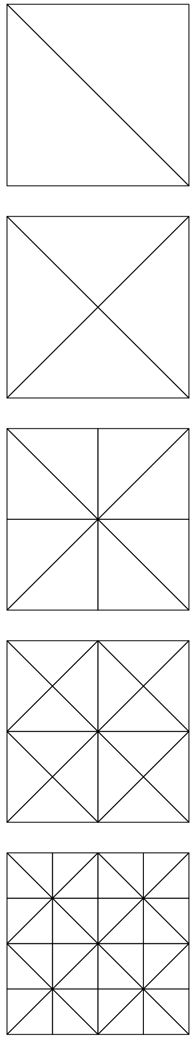
Exact solution:



Comparison of effectivity indices $I_{\text{eff}} = \mathcal{E} / \|e\|$

N_{tri}	equilibrated residua	method of hypercircle	combined method
2	1.43	1.11	1.06
4	1.23	1.25	1.01
8	1.34	1.20	1.00
16	1.30	1.30	1.16
32	1.39	1.23	1.29
64	1.32	1.32	1.27
128	1.41	1.25	1.52
256	1.33	1.34	1.33
512	1.41	1.25	1.64
1024	1.33	1.34	1.36
2048	1.41	1.26	1.71
4096	1.33	1.34	1.38
8192	1.41	1.26	1.74
16384	1.33	1.34	1.38
32768	1.41	1.26	1.75

First five meshes:

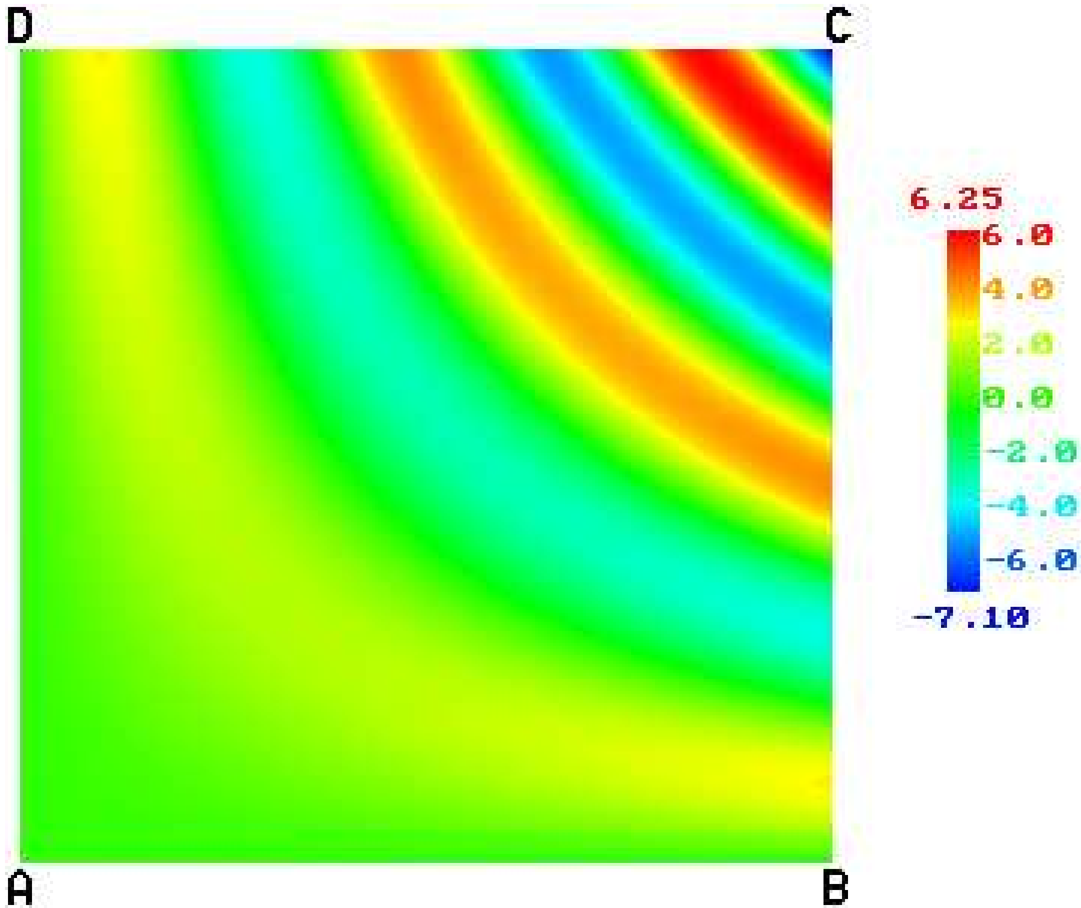


Example 2

Data:

$$\Omega = [0,1]^2,$$
$$\Gamma_D = DA \cup AB,$$
$$\Gamma_N = BC \cup CD,$$
$$\mathcal{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$
$$g_D = 0,$$
$$g_N, f = \text{ such that}$$
$$u(x_1, x_2) = \sin(17 x_1 x_2) e^{x_1 + x_2}$$

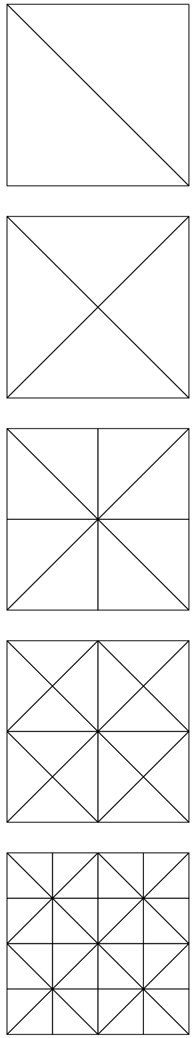
Exact solution:



Comparison of effectivity indices $I_{\text{eff}} = \mathcal{E} / \|e\|$

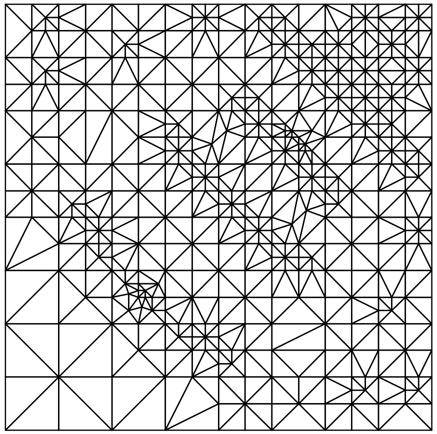
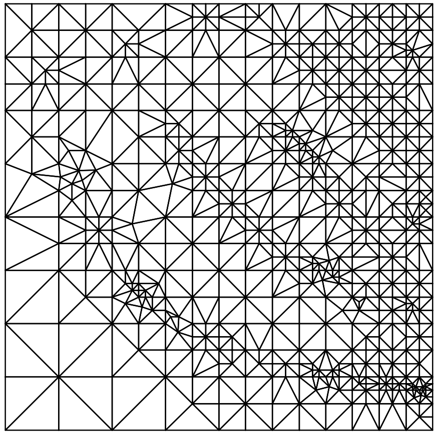
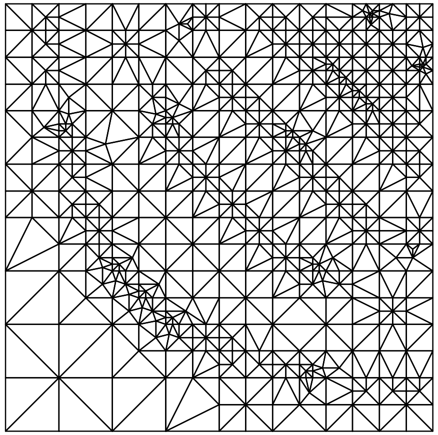
N_{tri}	equilibrated residua	method of hypercircle	combined method
2	0.37	1.93	1.86
4	0.29	1.70	1.71
8	0.49	1.65	1.69
16	0.94	1.51	1.19
32	1.11	1.56	1.37
64	1.06	1.51	1.29
128	1.14	1.68	1.37
256	1.17	1.48	1.28
512	1.30	1.52	1.58
1024	1.22	1.49	1.35
2048	1.34	1.50	1.67
4096	1.24	1.50	1.38
8192	1.35	1.49	1.70
16384	1.25	1.51	1.39
32768	1.35	1.49	1.71

First five meshes:



Adaptivity

Relative error	$> 10\%$ on a triangle	$\Rightarrow$	refine it.
Relative error	$< 10\%$ on all triangles	$\Rightarrow$	stop.

equilibrated residua	hypercircle method	combined method
		
7 refinement steps	8 refinement steps	8 refinement steps
$I_{\text{eff}} = 0.37, 0.49, 1.11, 1.14, 1.30, 1.30, 1.30$	$I_{\text{eff}} = 1.93, 1.65, 1.56, 1.68, 1.54, 1.45, 1.37, 1.37$	$I_{\text{eff}} = 1.86, 1.69, 1.37, 1.37, 1.58, 1.56, 1.57, 1.56$

Thank you for your attention.

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