HERMES_2D: an hp-FEM solver

Hierarchic Modular hp-FEM System

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Hermes_2D

What?

- Elliptic problems – (non)linear systems
- Time-harmonic Maxwell's equations
- Stokes problem
  incompressible Navier-Stokes equations

Hermes_2D

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How? hp-FEM!

- Input: triangulation + distribution of polynomial degrees,
  equation parameters, other settings
- Output: graphical (hp-solution, error, derivatives, . . .)
  error (exact $\times$ reference solution)

Hermes_2D

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Modular structure

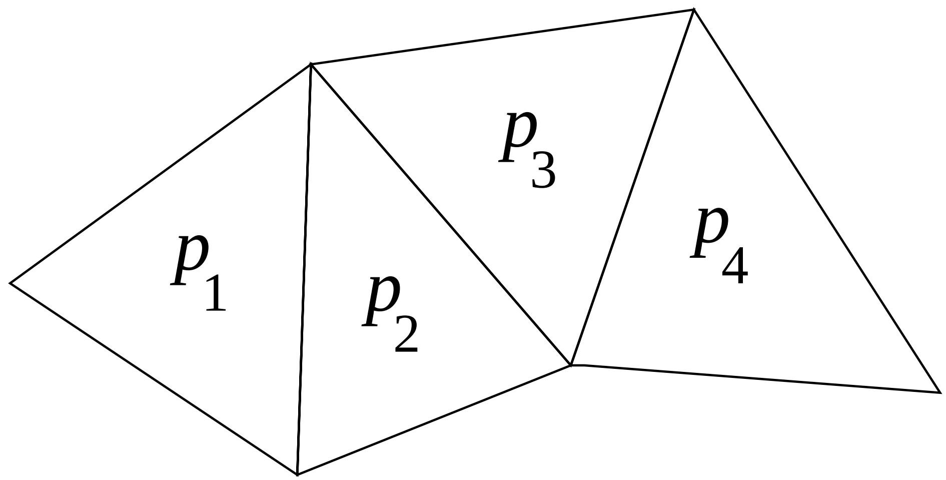
- hp-FEM technology $\times$ concrete PDEs

Under development

- automatic hp-adaptivity (J. Červený)
- parallelization (M. Lazar)
- 3D code (M. Zítka)
- parabolic problems (T. Vejchodský)

hp-FEM technology

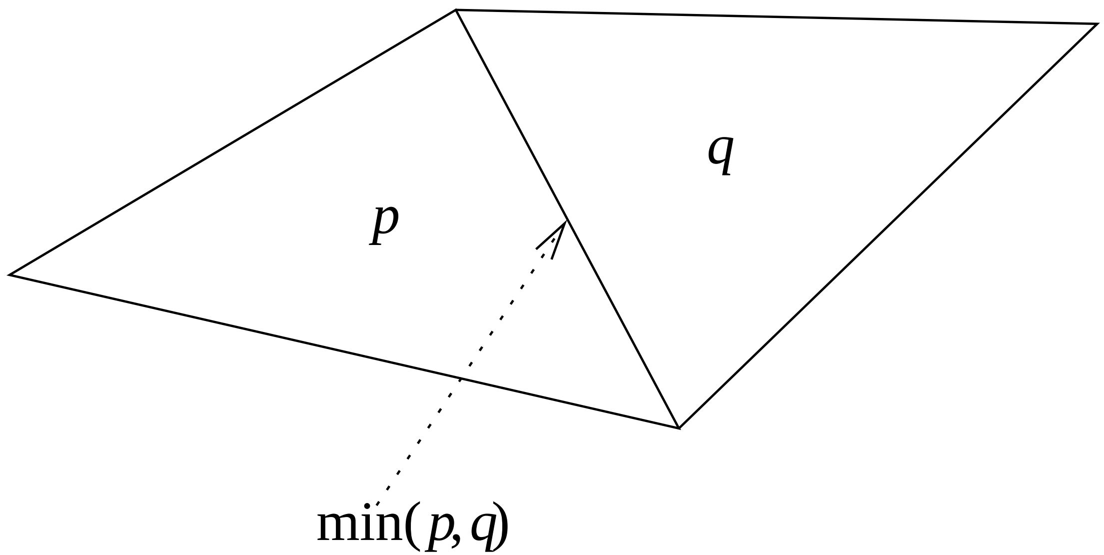
$$u \in V : a(u,v) = F(v) \quad \forall v \in V$$



$$V_{hp} = \left\{ v_{hp} \in V : v_{hp}|_{K_j} \in P^{p_j}(K_j) \right\}$$

Minimum rule

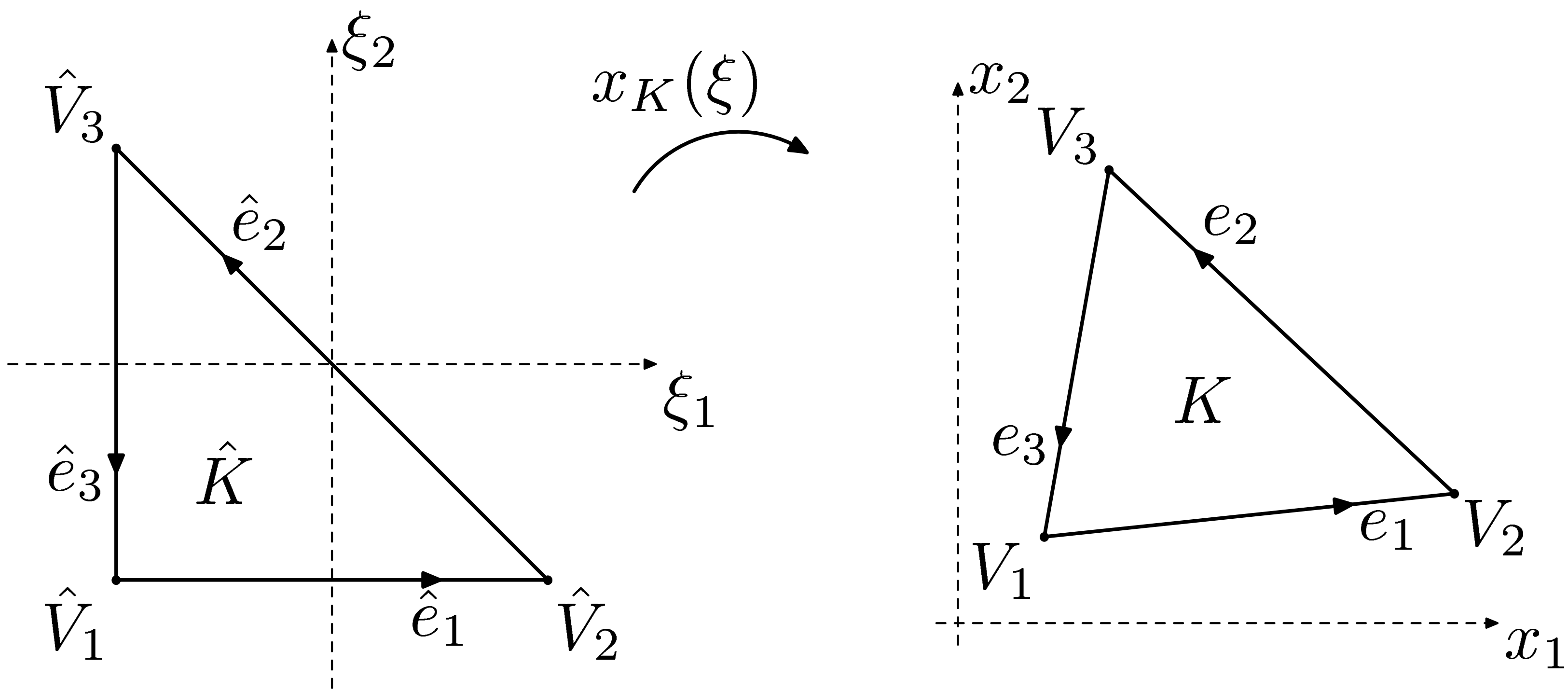
$$V_{hp} = \left\{ v_{hp} \in V : v_{hp}|_{K_j} \in P^{p_j}(K_j) \right\}$$



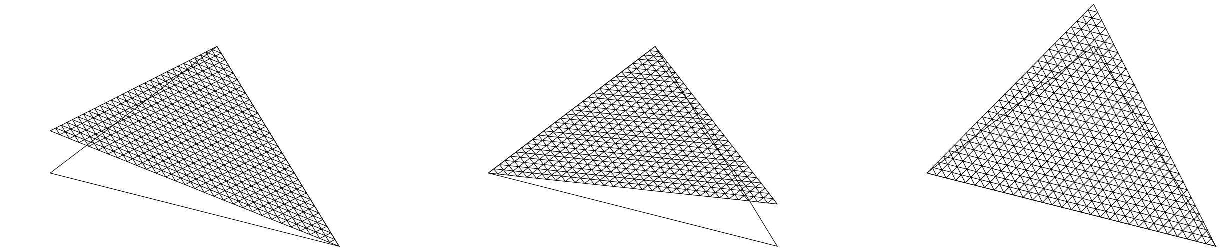
Determination of the polynomial degree of edge nodes.

Orientation of edges.

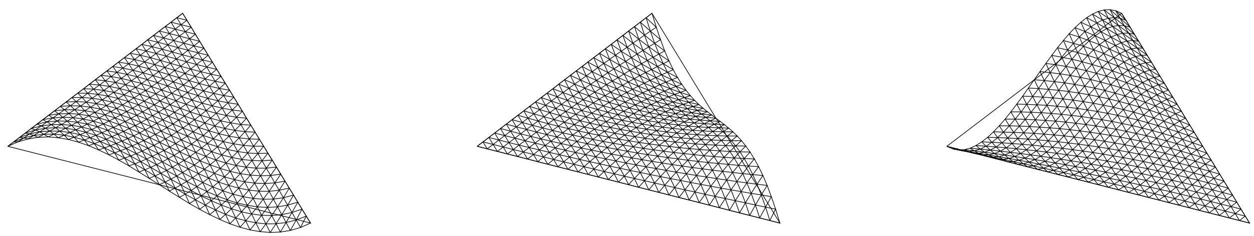
Reference element



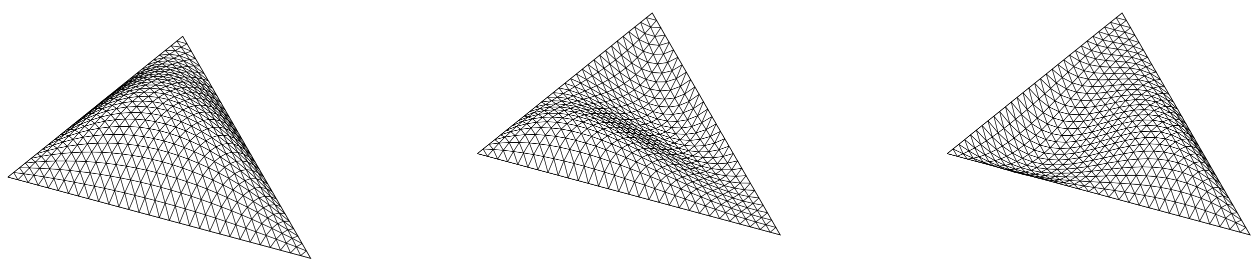
Shape functions



Vertex functions.



Cubic edge functions.



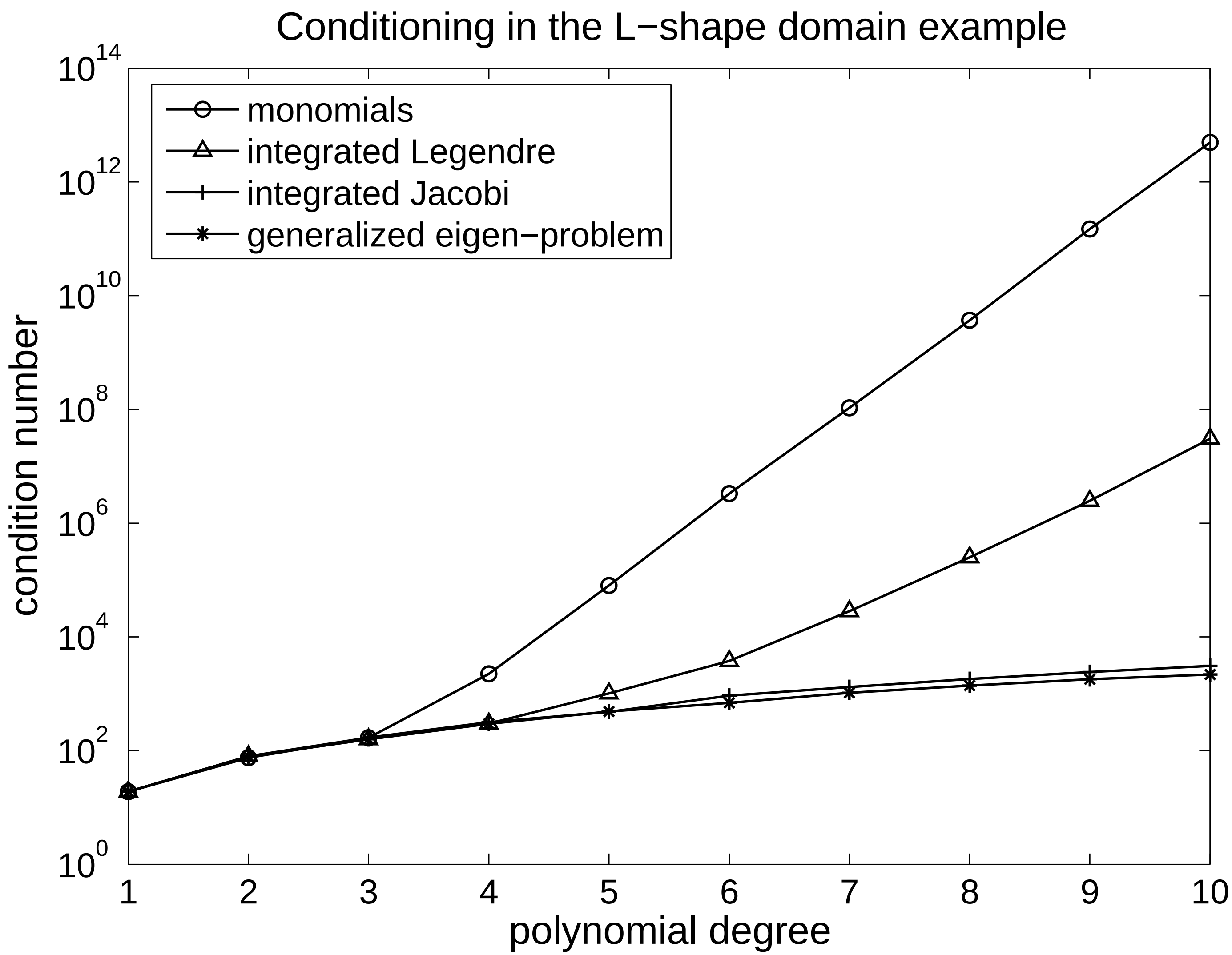
Cubic bubble function and fourth-order bubble functions.

Optimization of shape functions

Shape functions

- Simple: $\lambda_1^i \lambda_2 \lambda_3^j$, $i + j = p - 1$
- Ainsworth: based on integrated Legendre polynomials
- Beuchler: based on integrated Jacobi polynomials
- Eigen-bubbles: $(\nabla u, \nabla v)_{\hat{K}} = \lambda (u, v)_{\hat{K}} \ \forall v \in V_{hp,0}(\hat{K})$

Optimization of shape functions



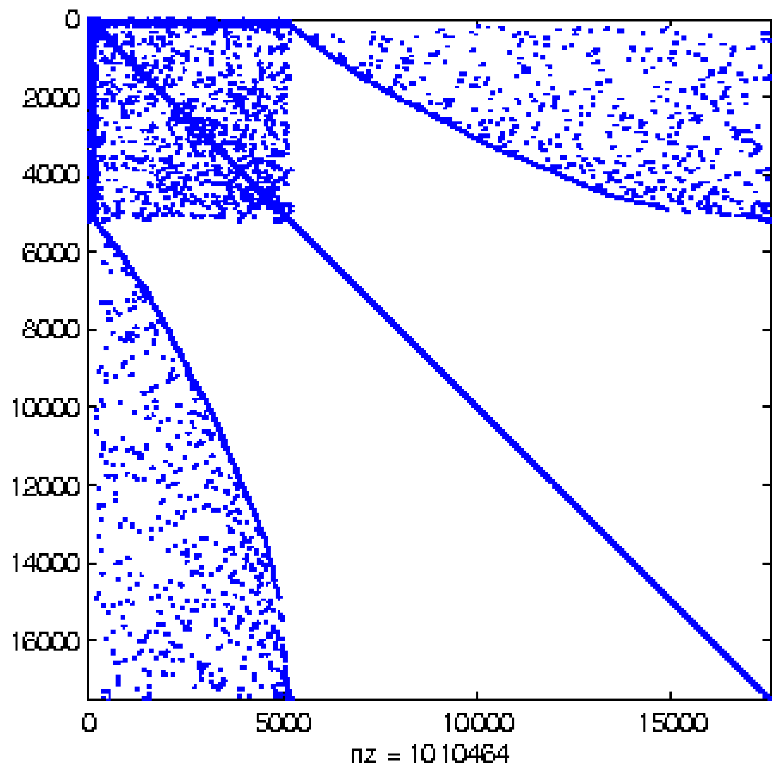
Static condensation of internal DOFs

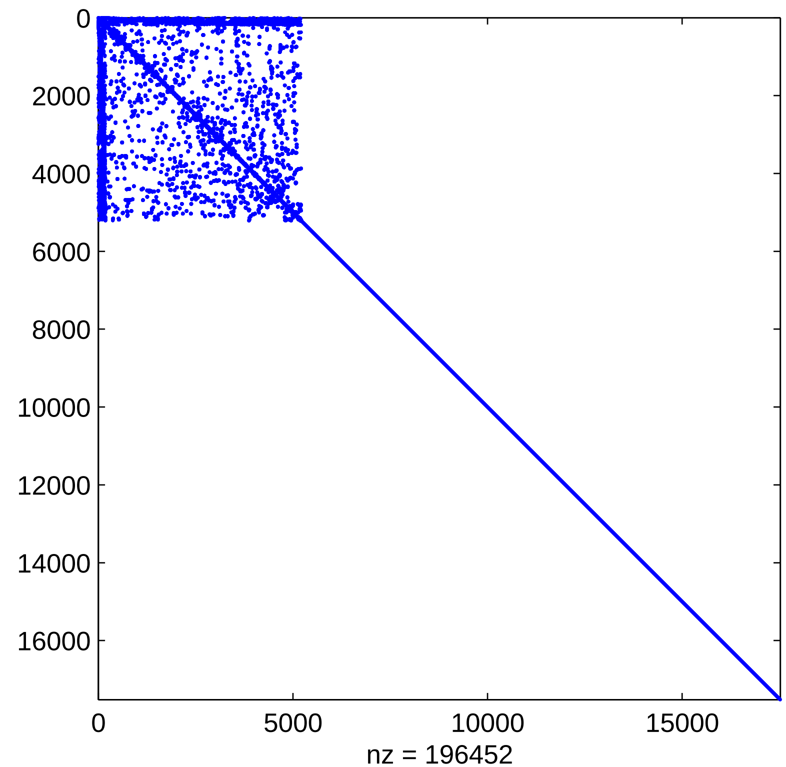
UNDER DEVELOPMENT

$$A = \left(\begin{array}{c|c|c} \mathrm{VV} & \mathrm{VE} & \mathrm{VB} \\ \hline \mathrm{EV} & \mathrm{EE} & \mathrm{EB} \\ \hline \mathrm{BV} & \mathrm{BE} & \mathrm{BB} \end{array}\right)$$

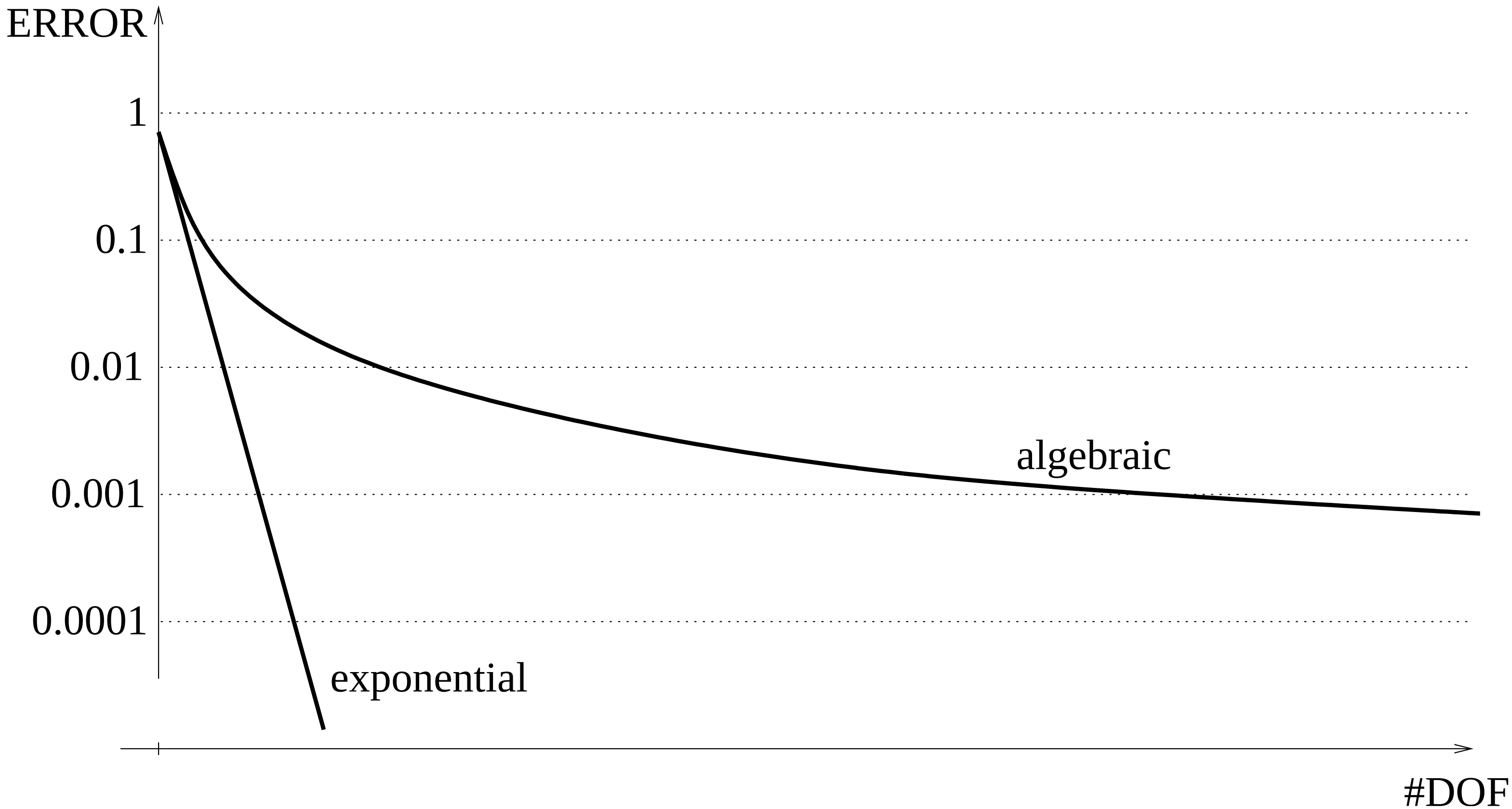
Static condensation of internal DOFs

UNDER DEVELOPMENT





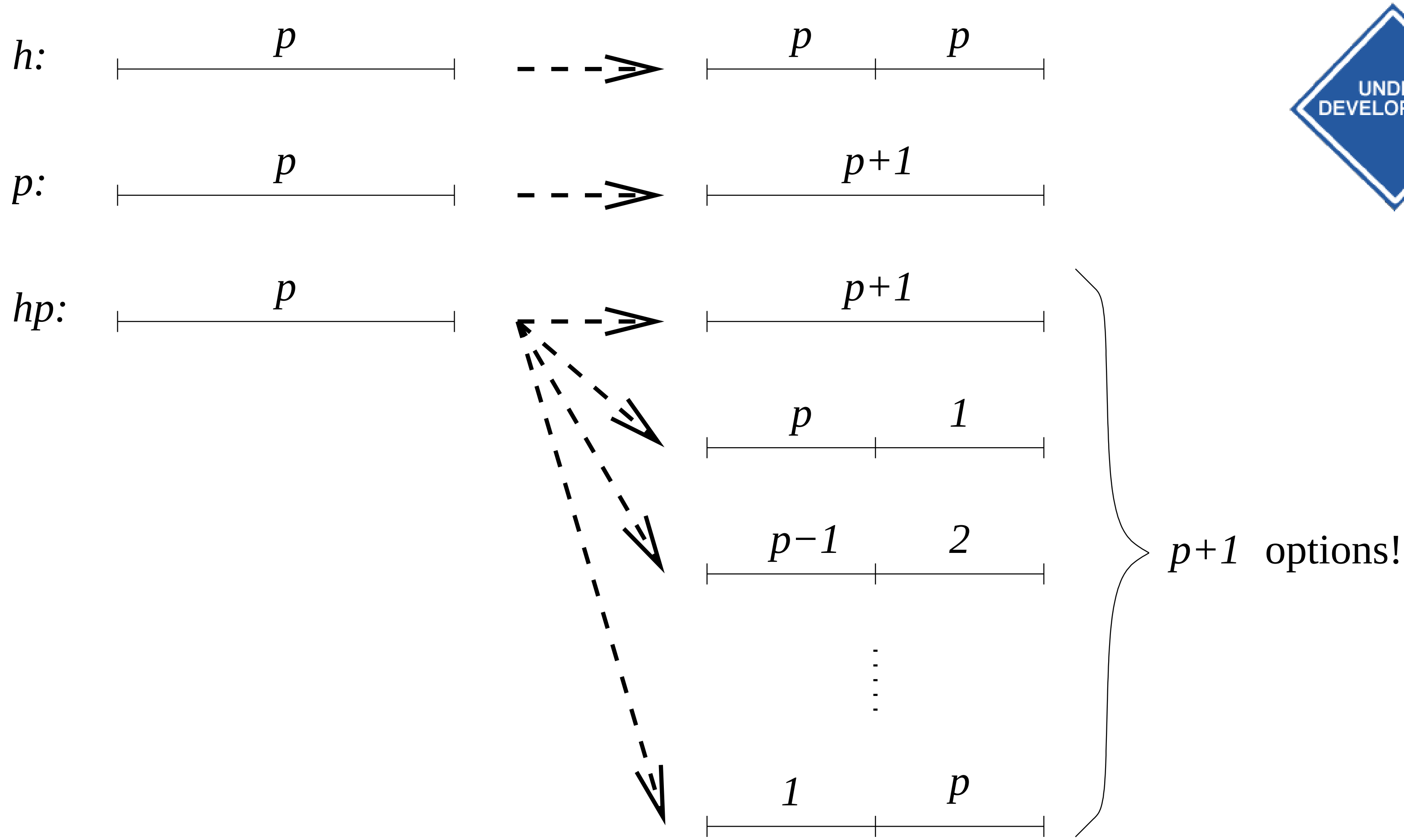
Exponential convergence of hp-FEM



$$\left\| u - u_{hp} \right\| \leq C_1 \exp(-C_2 N_{\mathrm{dof}})$$

Automatic hp-adaptivity

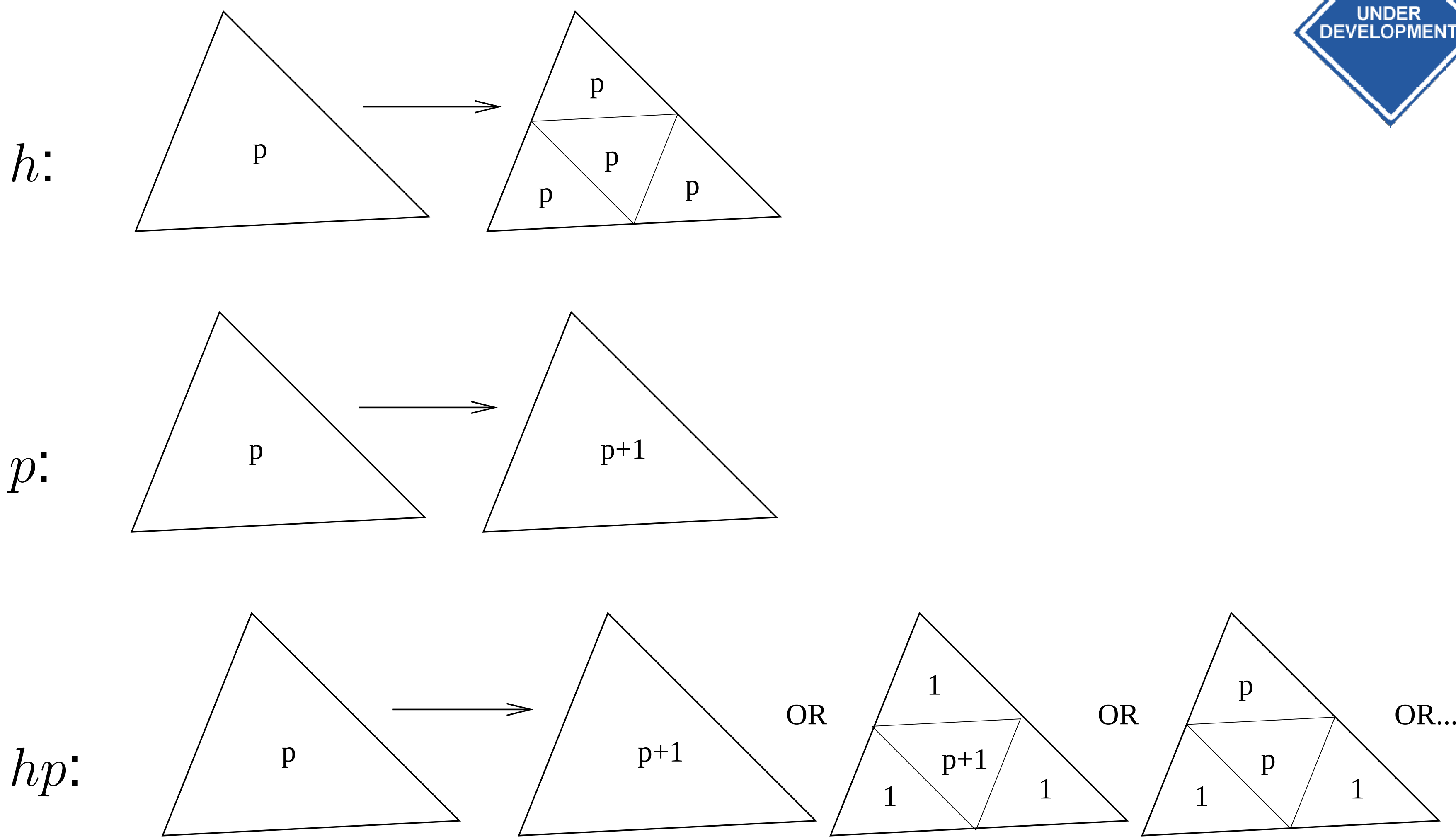
UNDER DEVELOPMENT



h-, p- and hp-refinement in 1D.

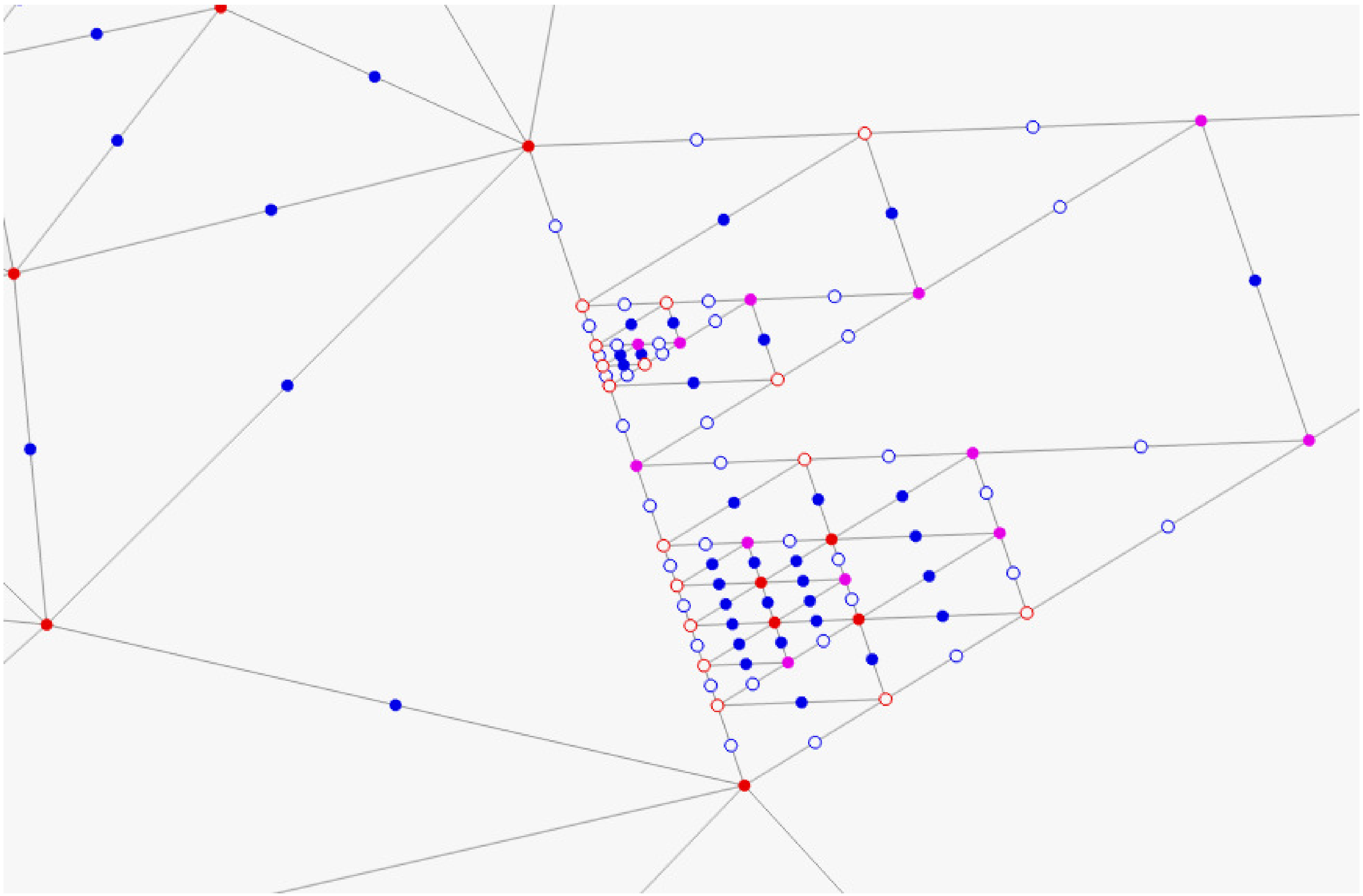
Automatic hp-adaptivity

UNDER DEVELOPMENT



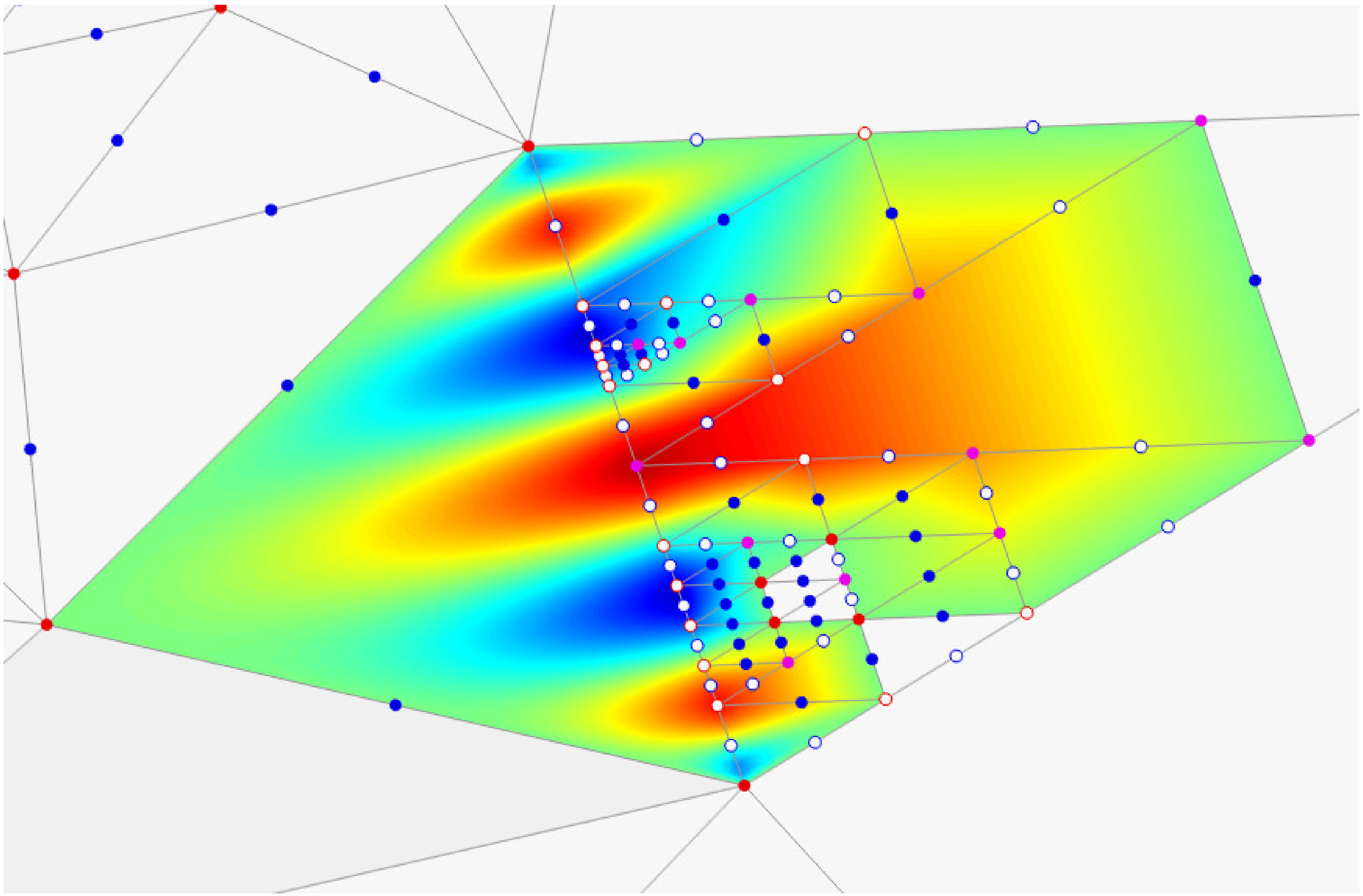
Multilevel constrained approximation

Hanging nodes



Multilevel constrained approximation

Hanging nodes



Parallelization

Supercomputer Felina

- CRAY XD1
- 72 x 2.2 GHz AMD Opteron processor,
- 144 GB RAM,
- 317 GFlops
- ONR Award No. 05PR07548-00

PETSc

UNDER DEVELOPMENT

Elliptic problems

$$\frac{\partial}{\partial x_1}\left(P_1\frac{\partial u}{\partial x_1}\right)+\frac{\partial}{\partial x_1}\left(P_2\frac{\partial u}{\partial x_2}\right)+\frac{\partial}{\partial x_2}\left(P_3\frac{\partial u}{\partial x_1}\right)+\frac{\partial}{\partial x_2}\left(P_4\frac{\partial u}{\partial x_2}\right)$$
$$+\frac{\partial}{\partial x_1}(P_5 u)+\frac{\partial}{\partial x_2}(P_6 u)+P_7 u = F,$$

$$u = u(\boldsymbol{x}) \in \mathbb{R}^{N_{\mathsf{eq}}}$$

$$F = F(\boldsymbol{x}, u, \nabla u) \in \mathbb{R}^{N_{\mathsf{eq}}}$$

$$P_k = P_k(\boldsymbol{x}, u, \nabla u) \in \mathbb{R}^{N_{\mathsf{eq}} \times N_{\mathsf{eq}}}, \quad k = 1, \ldots, 7$$

Elliptic problems

$$\frac{\partial}{\partial x_1}\left(P_1\frac{\partial u}{\partial x_1}\right)+\frac{\partial}{\partial x_1}\left(P_2\frac{\partial u}{\partial x_2}\right)+\frac{\partial}{\partial x_2}\left(P_3\frac{\partial u}{\partial x_1}\right)+\frac{\partial}{\partial x_2}\left(P_4\frac{\partial u}{\partial x_2}\right)$$
$$+\frac{\partial}{\partial x_1}(P_5 u)+\frac{\partial}{\partial x_2}(P_6 u)+P_7 u = F,$$

$$u_i(\boldsymbol{x}) = g_{D,i}(\boldsymbol{x}), \quad \boldsymbol{x}\in\Gamma_{D,i},\ i=1,2,\ldots,N_{\mathsf{eq}},$$

$$n_1\left(P_1\frac{\partial u}{\partial x_1}+P_2\frac{\partial u}{\partial x_2}+P_5 u\right)_i + n_2\left(P_3\frac{\partial u}{\partial x_1}+P_4\frac{\partial u}{\partial x_2}+P_6 u\right)_i = g_{N,i},$$

on $\Gamma_{N,i}$

Time-harmonic Maxwell's equations

$$\mathbf{curl}\left(\mu_r^{-1}\,\mathrm{curl}\,\mathbf{E}\right) - \kappa^2 \epsilon_r \mathbf{E} = \mathbf{F} \quad \text{in } \Omega$$

- $\mathbf{curl} = (\partial/\partial x_2, -\partial/\partial x_1)^\top$
- $\mathrm{curl}\,\mathbf{E} = \partial E_2/\partial x_1 - \partial E_1/\partial x_2$
- $\Omega \subset \mathbb{R}^2$
- $\mu_r = \mu_r(x) \in \mathbb{R}$ relative permeability
- $\epsilon_r = \epsilon_r(x) \in \mathbb{C}^{2\times 2}$ relative permittivity
- $\mathbf{E} = \mathbf{E}(x) \in \mathbb{C}^2$ phasor of the electric field intensity
- $\mathbf{F} = \mathbf{F}(x) \in \mathbb{C}^2$
- $\kappa \in \mathbb{R}$ the wave number

Time-harmonic Maxwell's equations

$$\mathbf{curl}\left(\mu_{\mathrm{r}}^{-1}\,\mathrm{curl}\,\mathbf{E}\right) - \kappa^2\epsilon_{\mathrm{r}}\mathbf{E} = \mathbf{F} \quad \text{in } \Omega$$

$$\begin{aligned} \mathbf{E}\cdot\tau &= 0 && \text{on } \Gamma_P \\ \mu_{\mathrm{r}}^{-1}\,\mathrm{curl}\,\mathbf{E} - \mathrm{i}\kappa\lambda\mathbf{E}\cdot\tau &= \mathbf{g}\cdot\tau && \text{on } \Gamma_I \end{aligned}$$

- $\tau = (-\nu_2, \nu_1)^\top$ positively oriented unit tangent vector
- $\lambda = \lambda(x) > 0$ impedance
- $\mathbf{g} = \mathbf{g}(x) \in \mathbb{C}^2$

Edge elements

Whitney functions:

$$\hat{\psi}_0^{e_1} = \frac{\lambda_3 \mathbf{n}_2}{\mathbf{n}_2 \cdot \mathbf{t}_1} + \frac{\lambda_2 \mathbf{n}_3}{\mathbf{n}_3 \cdot \mathbf{t}_1}$$

$$\hat{\psi}_0^{e_2} = \frac{\lambda_1 \mathbf{n}_3}{\mathbf{n}_3 \cdot \mathbf{t}_2} + \frac{\lambda_3 \mathbf{n}_1}{\mathbf{n}_1 \cdot \mathbf{t}_2}$$

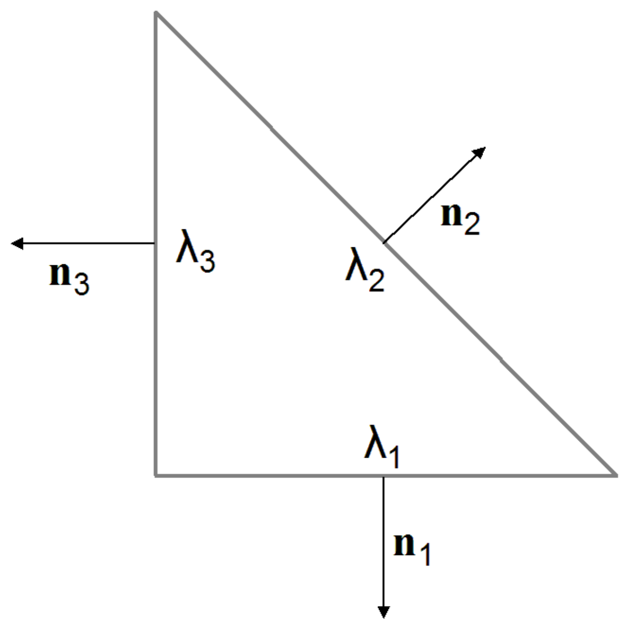
$$\hat{\psi}_0^{e_3} = \frac{\lambda_2 \mathbf{n}_1}{\mathbf{n}_1 \cdot \mathbf{t}_3} + \frac{\lambda_1 \mathbf{n}_2}{\mathbf{n}_2 \cdot \mathbf{t}_3}$$

First order functions:

$$\hat{\psi}_1^{e_1} = \frac{\lambda_3 \mathbf{n}_2}{\mathbf{n}_2 \cdot \mathbf{t}_1} - \frac{\lambda_2 \mathbf{n}_3}{\mathbf{n}_3 \cdot \mathbf{t}_1}$$

$$\hat{\psi}_1^{e_2} = \frac{\lambda_1 \mathbf{n}_3}{\mathbf{n}_3 \cdot \mathbf{t}_2} - \frac{\lambda_3 \mathbf{n}_1}{\mathbf{n}_1 \cdot \mathbf{t}_2}$$

$$\hat{\psi}_1^{e_3} = \frac{\lambda_2 \mathbf{n}_1}{\mathbf{n}_1 \cdot \mathbf{t}_3} - \frac{\lambda_1 \mathbf{n}_2}{\mathbf{n}_2 \cdot \mathbf{t}_3}$$



$$\mathbf{t}_i = \begin{bmatrix} -\mathbf{n}_{i,2} \\ \mathbf{n}_{i,1} \end{bmatrix}$$

Edge elements

Edge functions:

$$\hat{\psi}_k^{e_1} = \frac{2k-1}{k} L_{k-1}(\lambda_3 - \lambda_2)\hat{\psi}_1^{e_1} - \frac{k-1}{k} L_{k-2}(\lambda_3 - \lambda_2)\hat{\psi}_0^{e_1},$$

$$\hat{\psi}_k^{e_2} = \frac{2k-1}{k} L_{k-1}(\lambda_1 - \lambda_3)\hat{\psi}_1^{e_2} - \frac{k-1}{k} L_{k-2}(\lambda_1 - \lambda_3)\hat{\psi}_0^{e_2},$$

$$\hat{\psi}_k^{e_3} = \frac{2k-1}{k} L_{k-1}(\lambda_2 - \lambda_1)\hat{\psi}_1^{e_3} - \frac{k-1}{k} L_{k-2}(\lambda_2 - \lambda_1)\hat{\psi}_0^{e_3},$$

$$k = 2, 3, \ldots$$

Edge elements

Edge based bubble functions:

$$\hat{\psi}_k^{b,e_1} = \lambda_3 \lambda_2 L_{k-2}(\lambda_3 - \lambda_2)\mathbf{n}_1,$$
$$\hat{\psi}_k^{b,e_2} = \lambda_1 \lambda_3 L_{k-2}(\lambda_1 - \lambda_3)\mathbf{n}_2,$$
$$\hat{\psi}_k^{b,e_3} = \lambda_2 \lambda_1 L_{k-2}(\lambda_2 - \lambda_1)\mathbf{n}_3, \quad k = 2, 3, \ldots$$

Genuine bubble functions:

$$\hat{\psi}_{n_1,n_2}^{b,1} = \lambda_1 \lambda_2 \lambda_3 L_{n_1-1}(\lambda_3 - \lambda_2) L_{n_2-1}(\lambda_2 - \lambda_1) \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$\hat{\psi}_{n_1,n_2}^{b,2} = \lambda_1 \lambda_2 \lambda_3 L_{n_1-1}(\lambda_3 - \lambda_2) L_{n_2-1}(\lambda_2 - \lambda_1) \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad 1 \le n_1, n_2$$

Examples

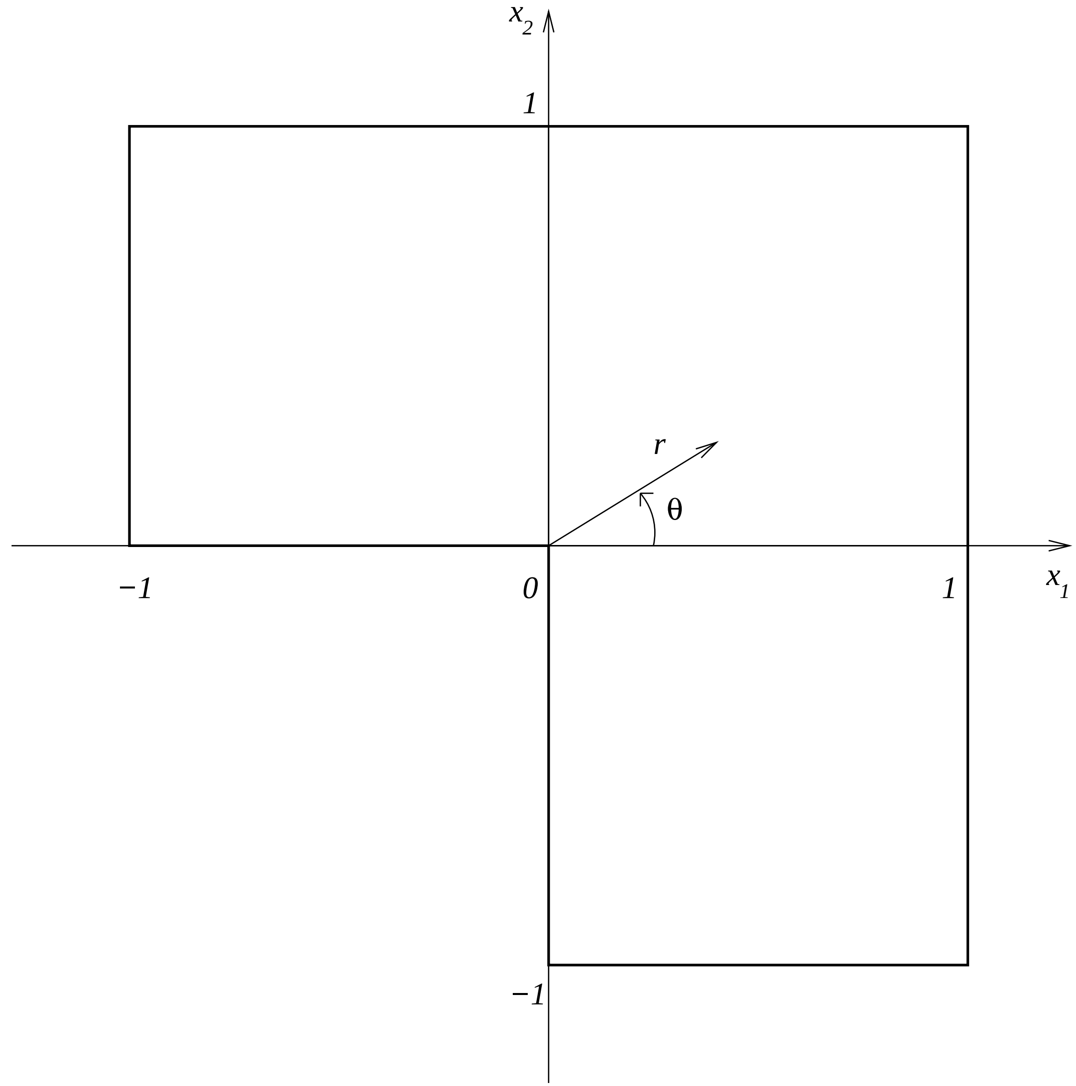
L-shape domain

$$-\Delta u = 0 \qquad \text{in } \Omega$$

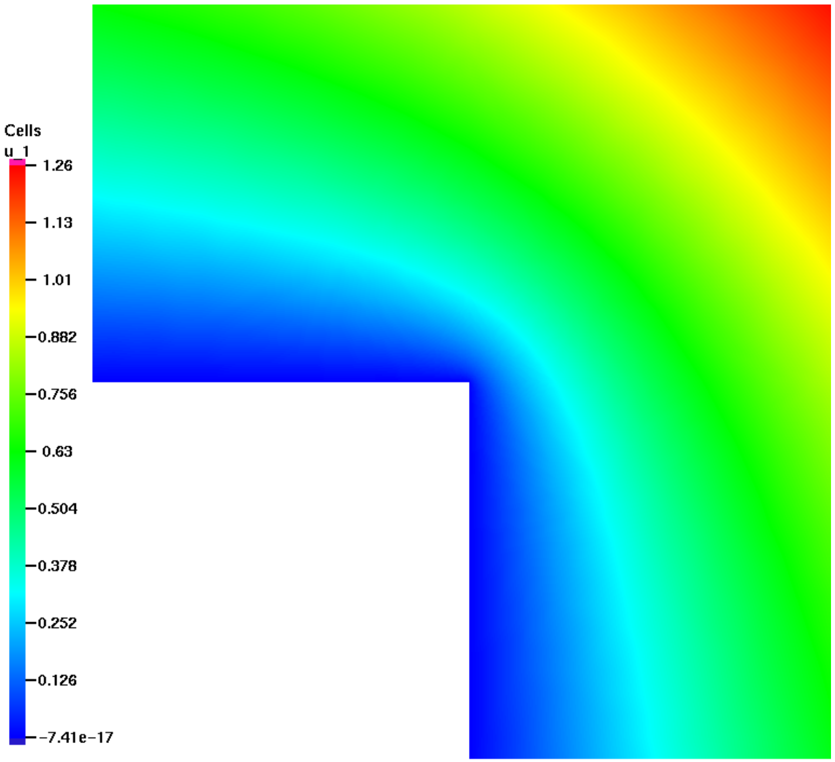
$$u = \varphi \quad \text{on } \partial\Omega$$

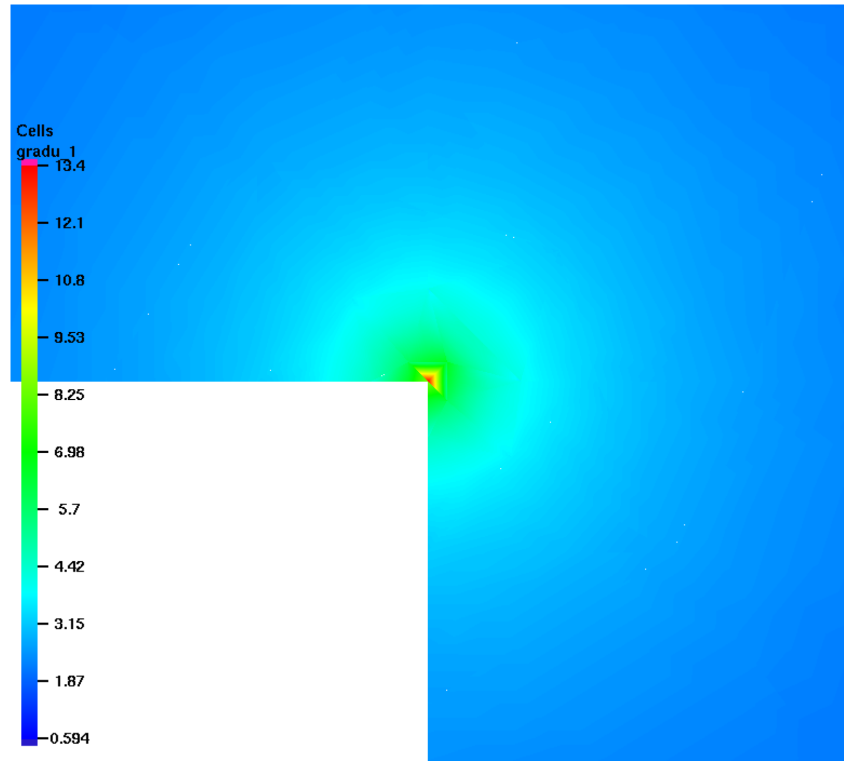
Exact solution:

$$u = r^{2/3} \sin\left(2\theta/3 + \pi/3\right)$$



L-shape domain

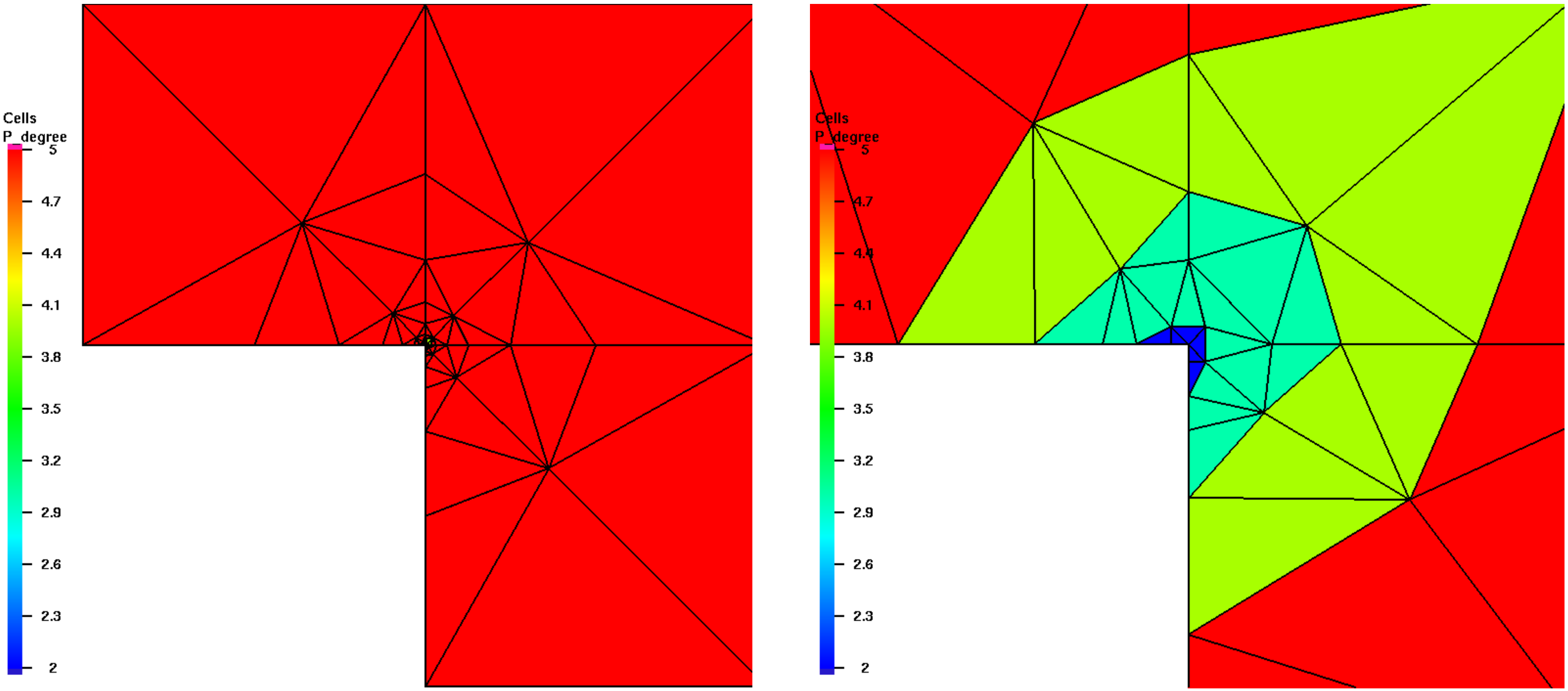




The exact solution.

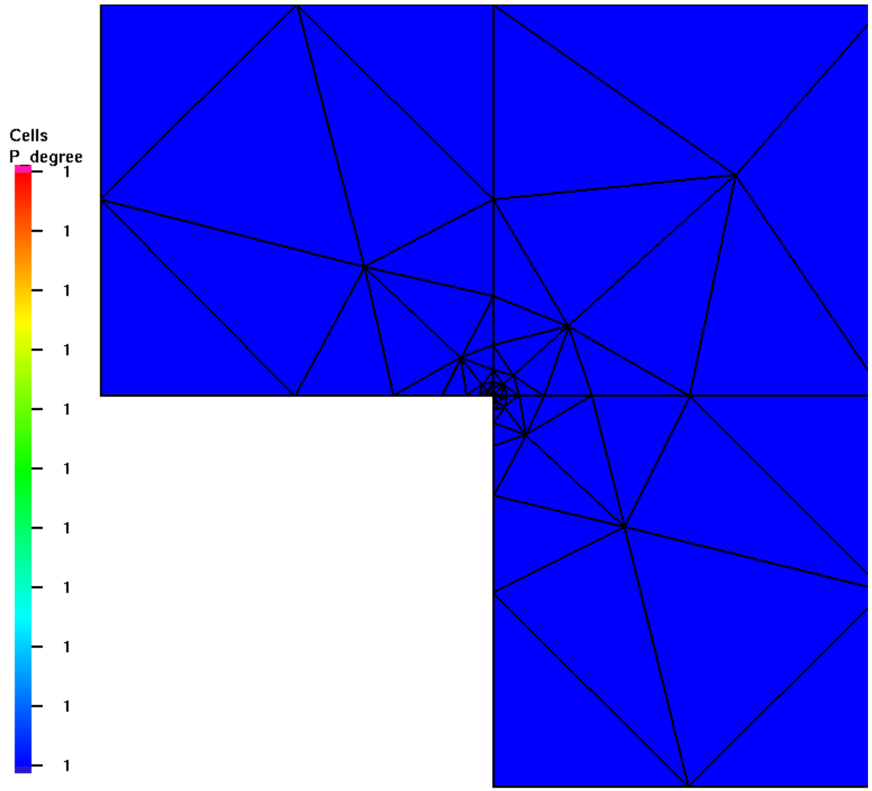
Detail view of $|\nabla u_{h,p}|$ at the re-entrant corner (zoom = 70).

L-shape domain



The hp-mesh.
Red – fifth-order elements.
Blue – second-order elements. (Right: zoom = 70)

L-shape domain

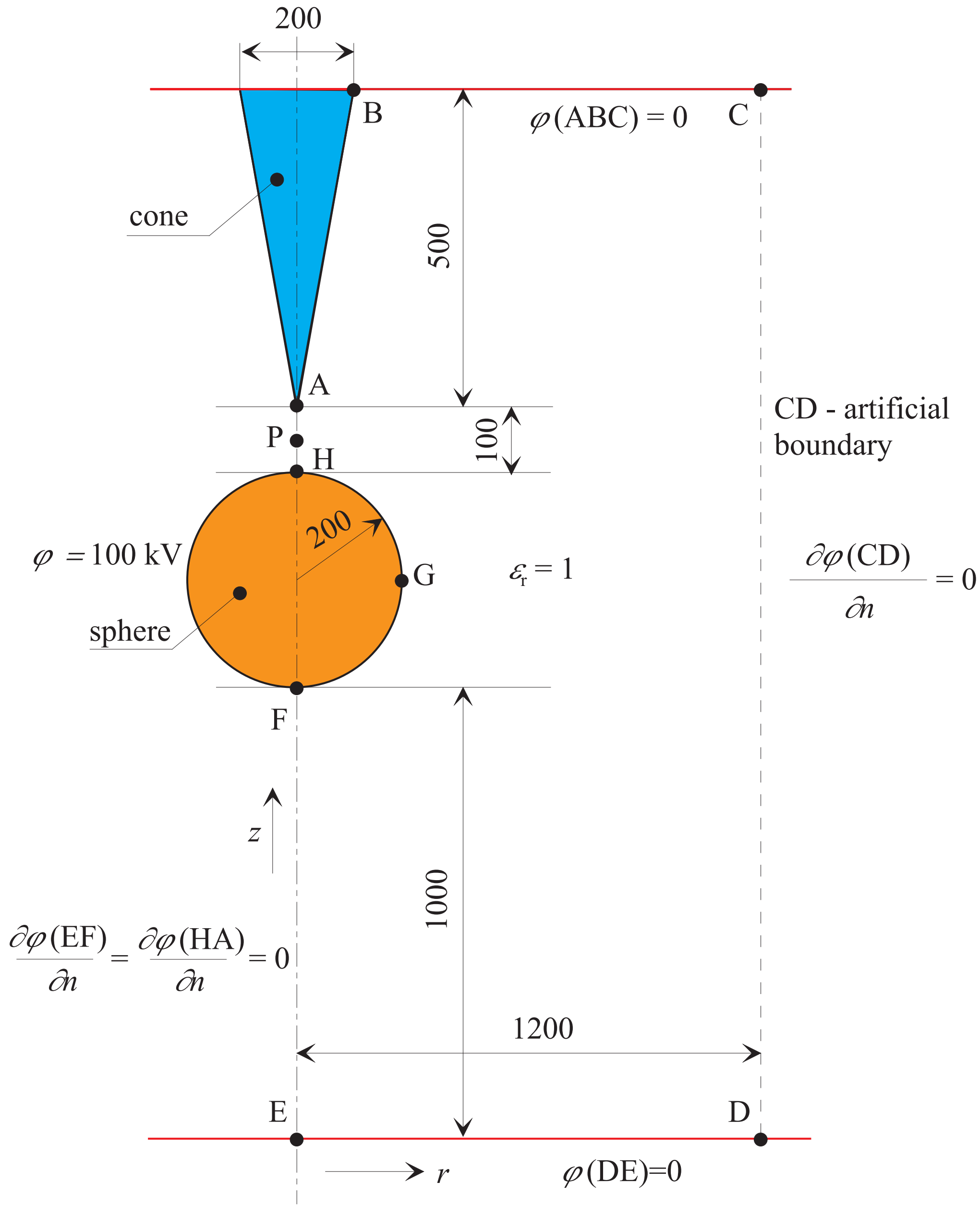


The piecewise-linear mesh. Uniform refinement to reach the accuracy (each edge was subdivided into 60).

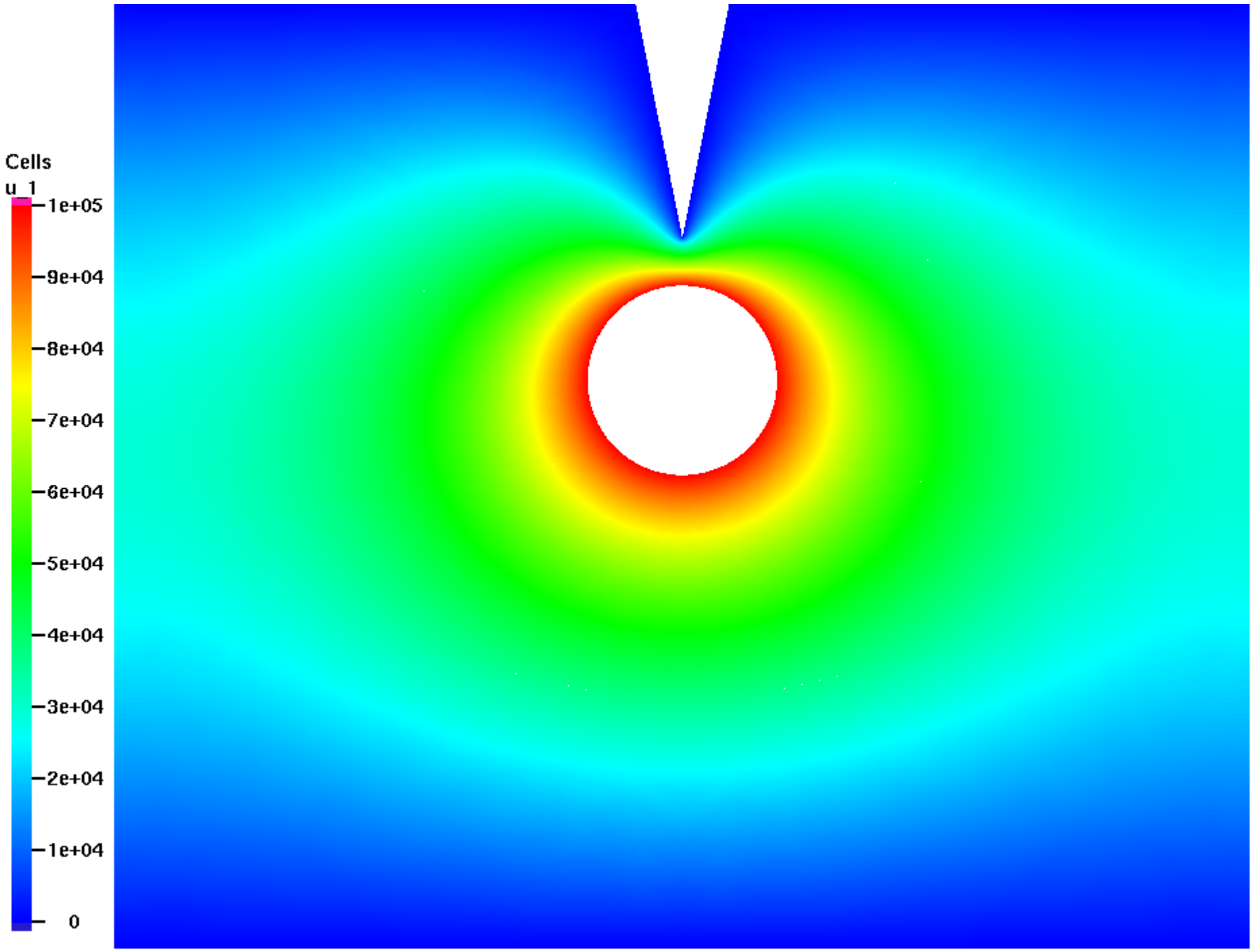
L-shape domain

	linear elements	hp-elements
DOF	143161	839
Error	0.1876 %	0.1603 %
Iterations	421	30
CPU time	**2.1 min.**	**0.35 sec.**

Sphere-Cone problem

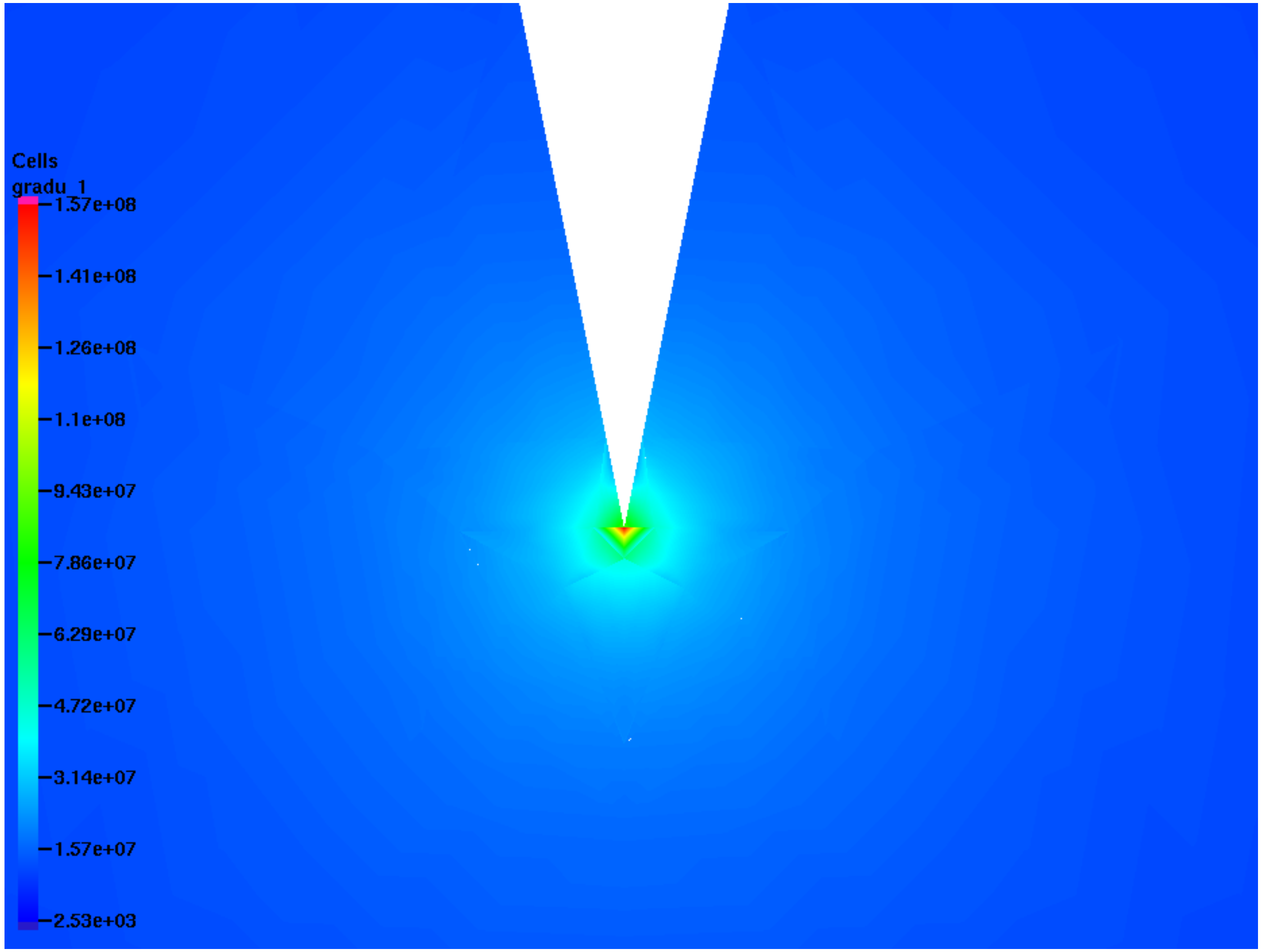


Sphere-Cone problem



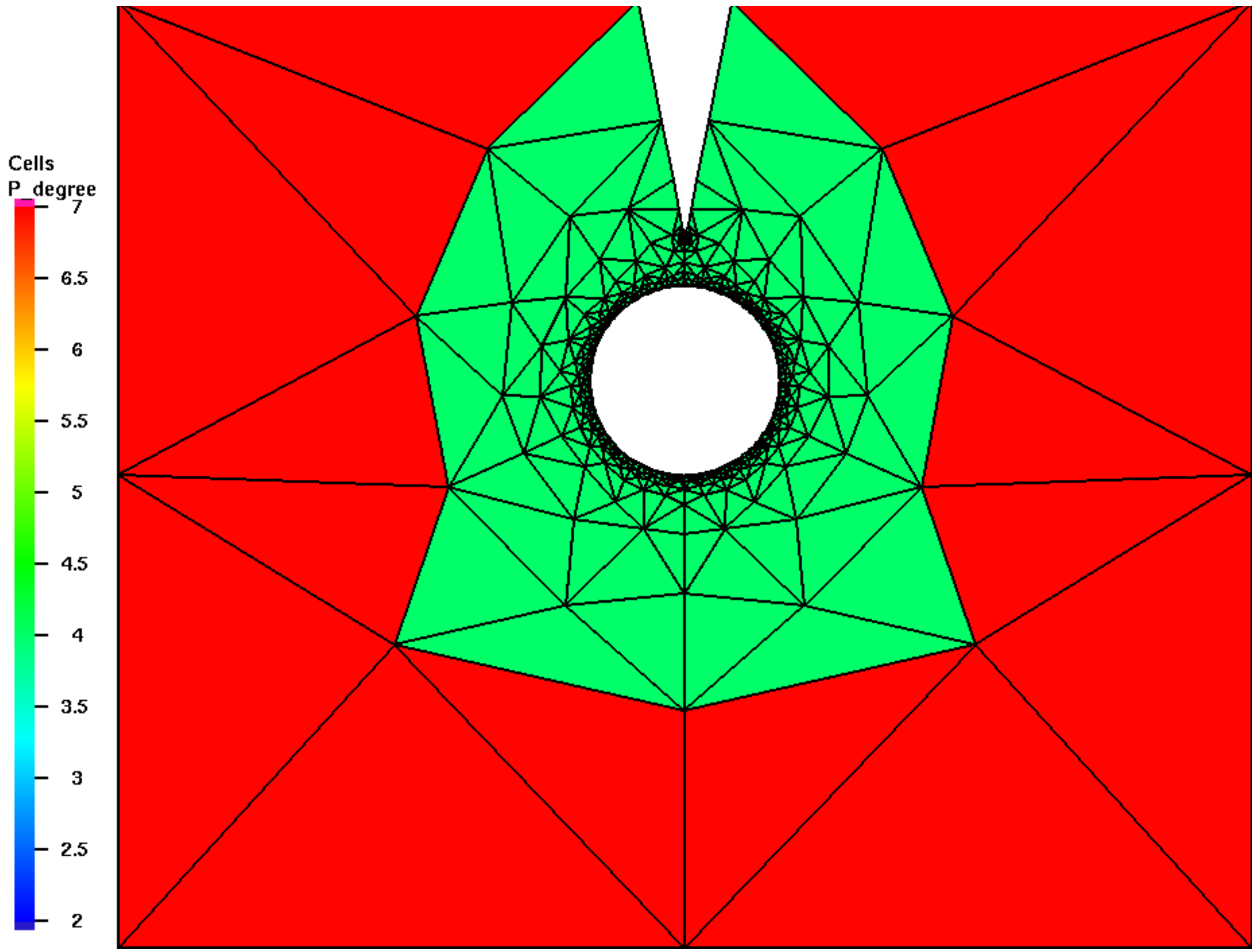
Solution of the cone-sphere problem (the electric potential)

Sphere-Cone problem



Detail of the singularity of $|\mathbf{E}| = |-\nabla\varphi|$ (zoom = 100,000).

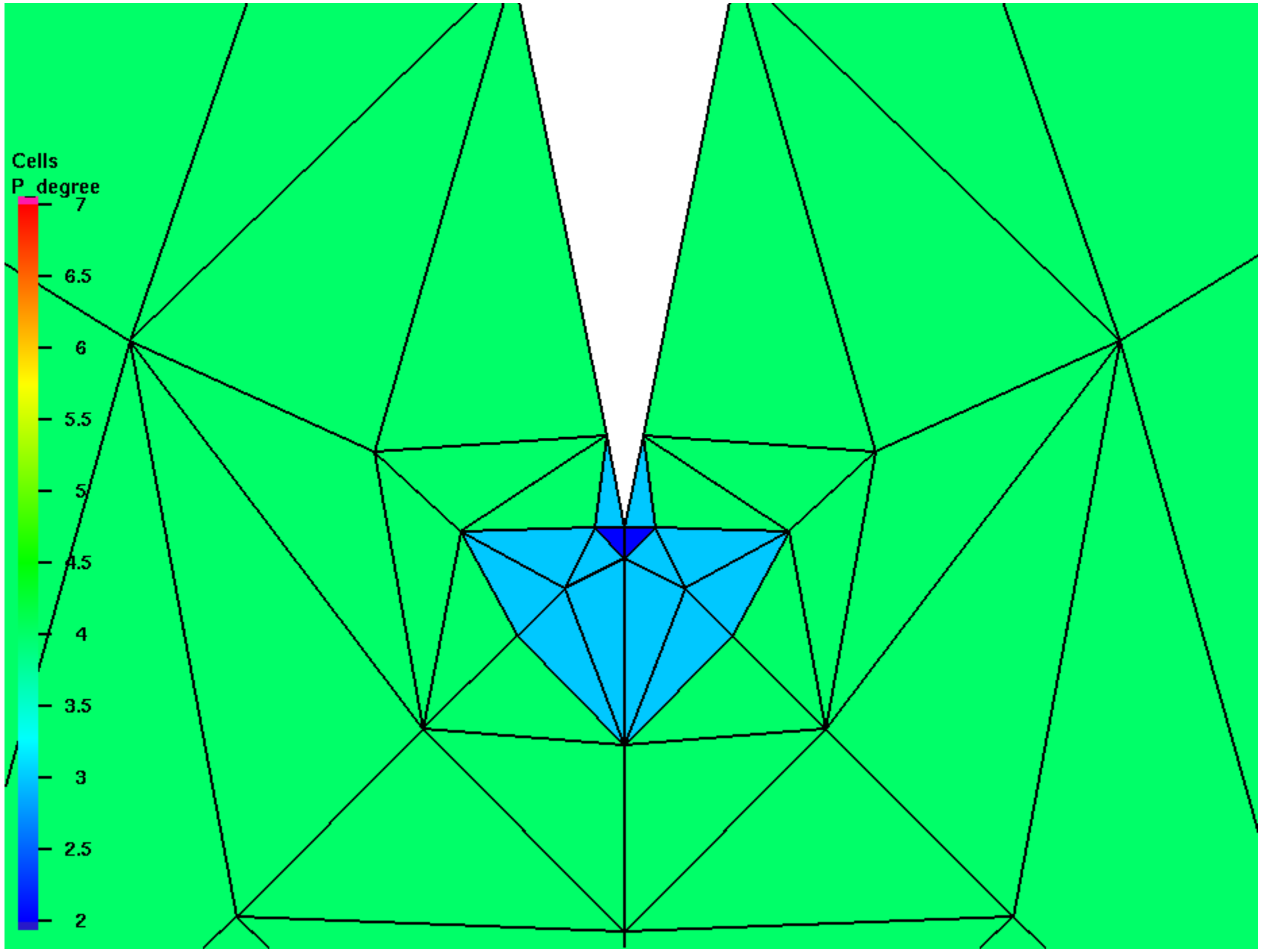
Sphere-Cone problem



The hp-mesh – global view.

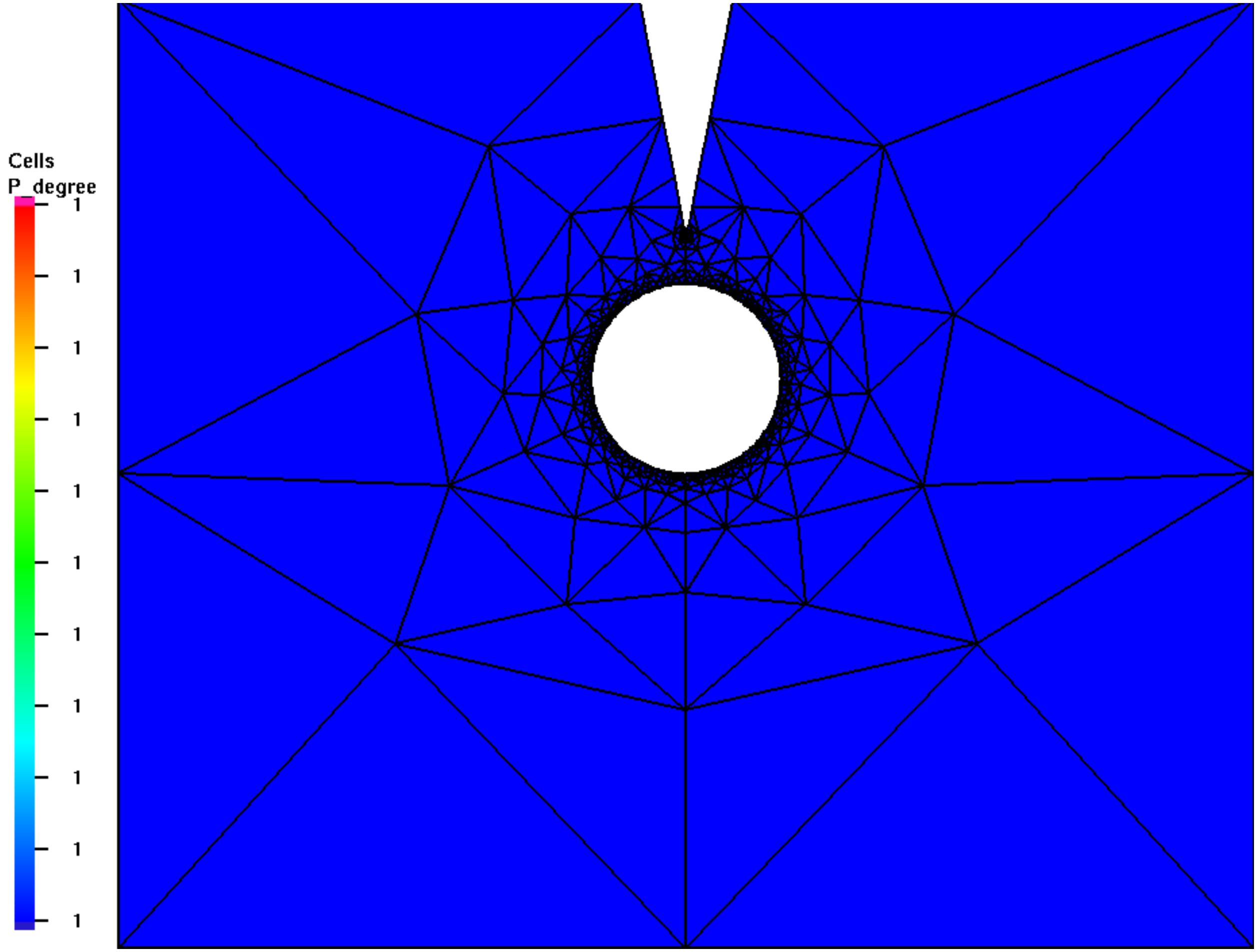
Red – seventh-order elements. Blue – quadratic elements.

Sphere-Cone problem



The hp-mesh – detail of the tip of the cone (zoom = 100,000).

Sphere-Cone problem



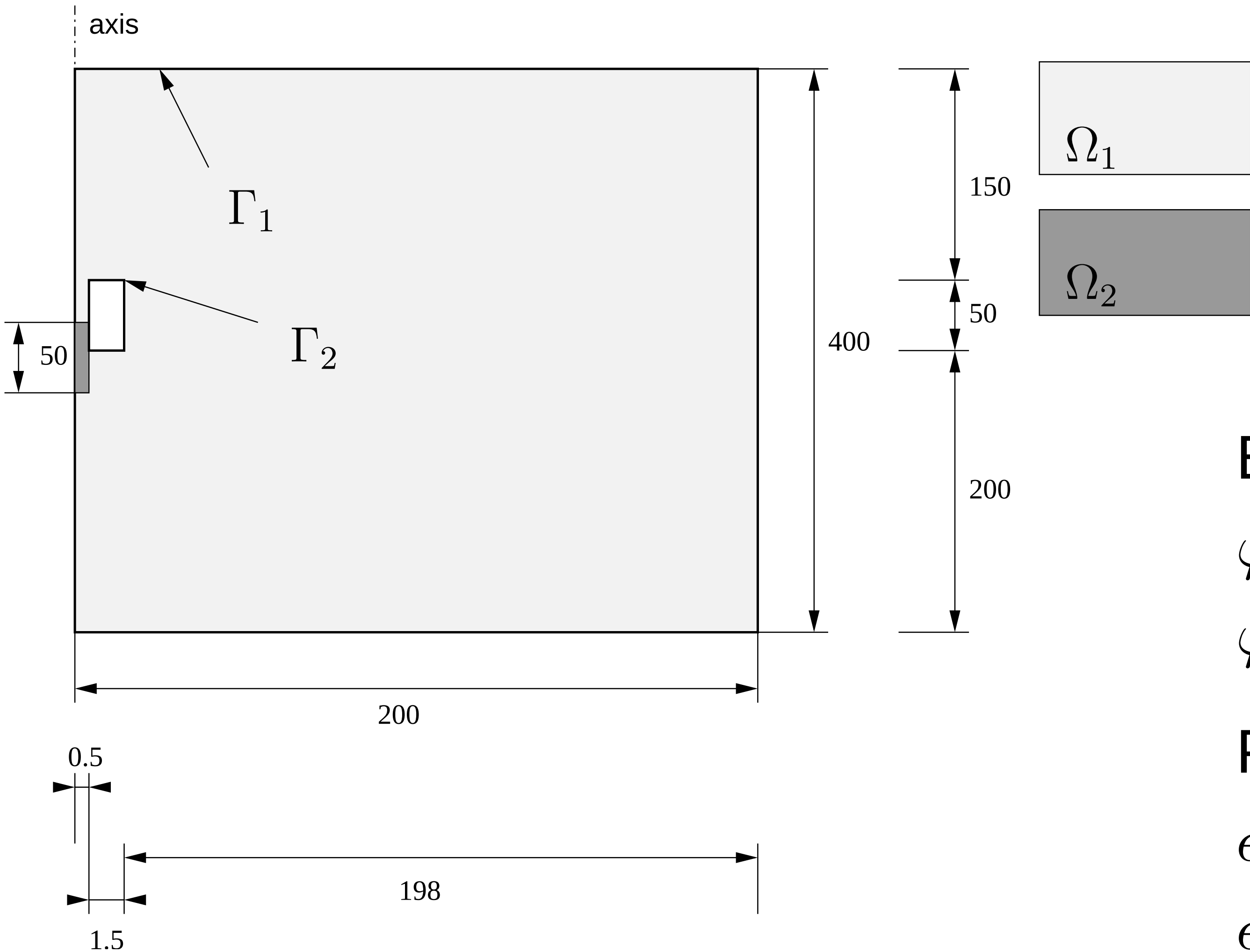
The piecewise-linear mesh. Uniform refinement to reach accuracy (each edge was subdivided into 48).

Sphere-Cone problem

	linear elements	hp-elements
DOF	488542	3317
Error	0.5858 %	0.2804 %
Iterations	859	44
CPU time	**30 min.**	**10.53 sec.**

Electrostatic micromotor

Motors that can resist destructive electromagnetic waves.



El. potential:

$\varphi = 0$V on Γ_1

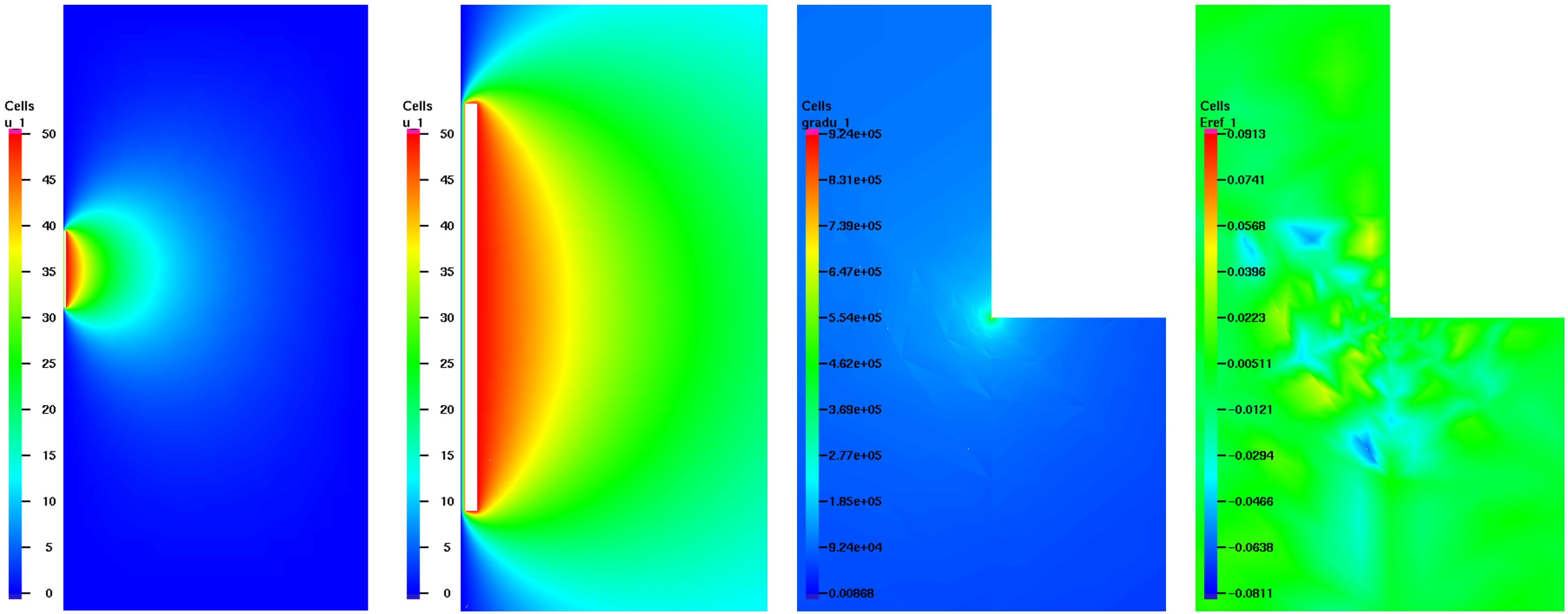
$\varphi = 50$V on Γ_2.

Permittivity:

$\epsilon = 1$ in Ω_1

$\epsilon = 10$ in Ω_2.

Electrostatic micromotor

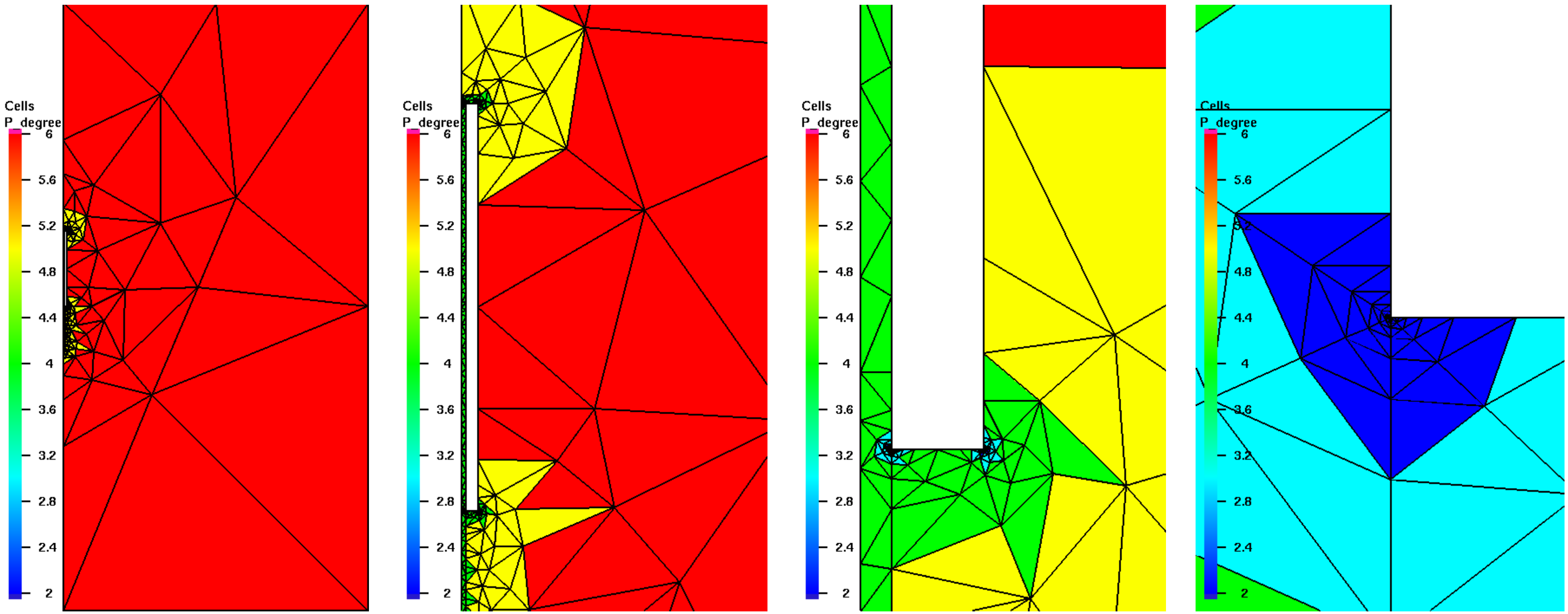


Electric potential φ (zoom = 1 and 6),
singularity of $|\mathbf{E}| = |-\nabla\varphi|$ (zoom = 1000),

error estimate (zoom = 1000)

MATEMATICKÝ ÚSTAV AV ČR

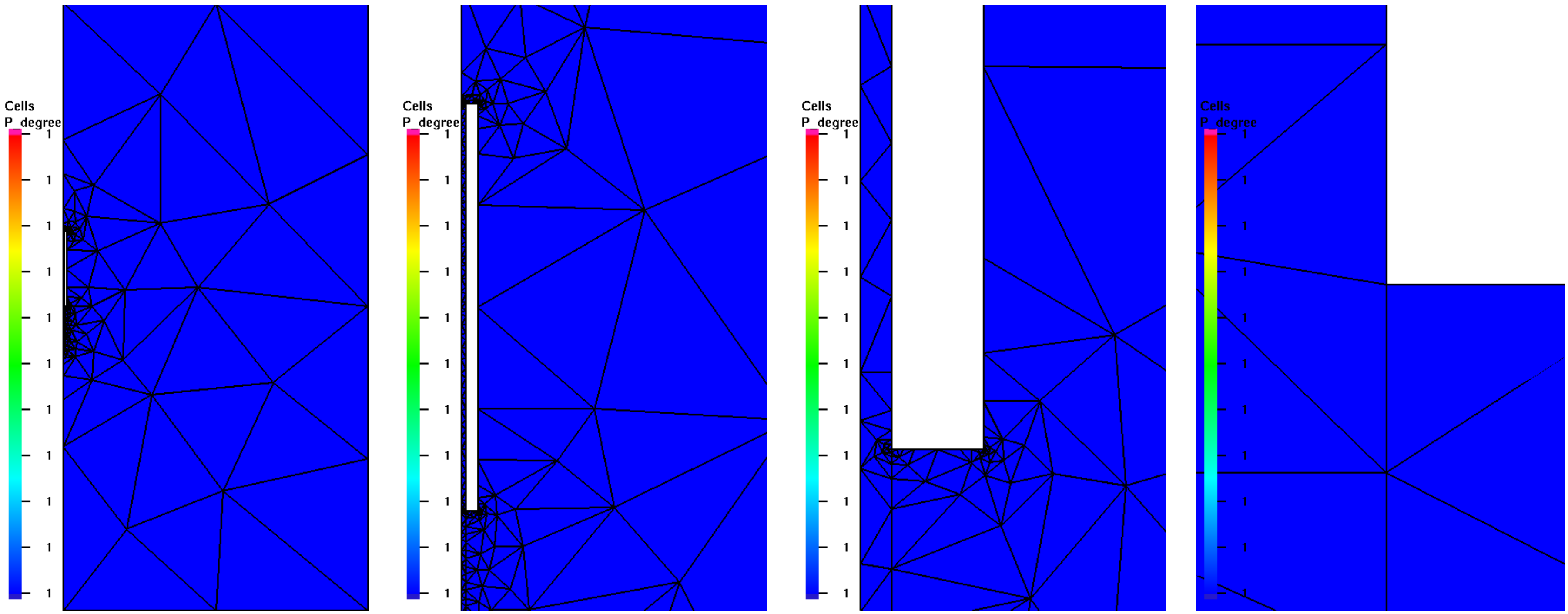
Electrostatic micromotor



The hp-mesh (zoom = 1, 6, 50, 1000).

Red – sixth-order elements. Blue – quadratic elements.

Electrostatic micromotor



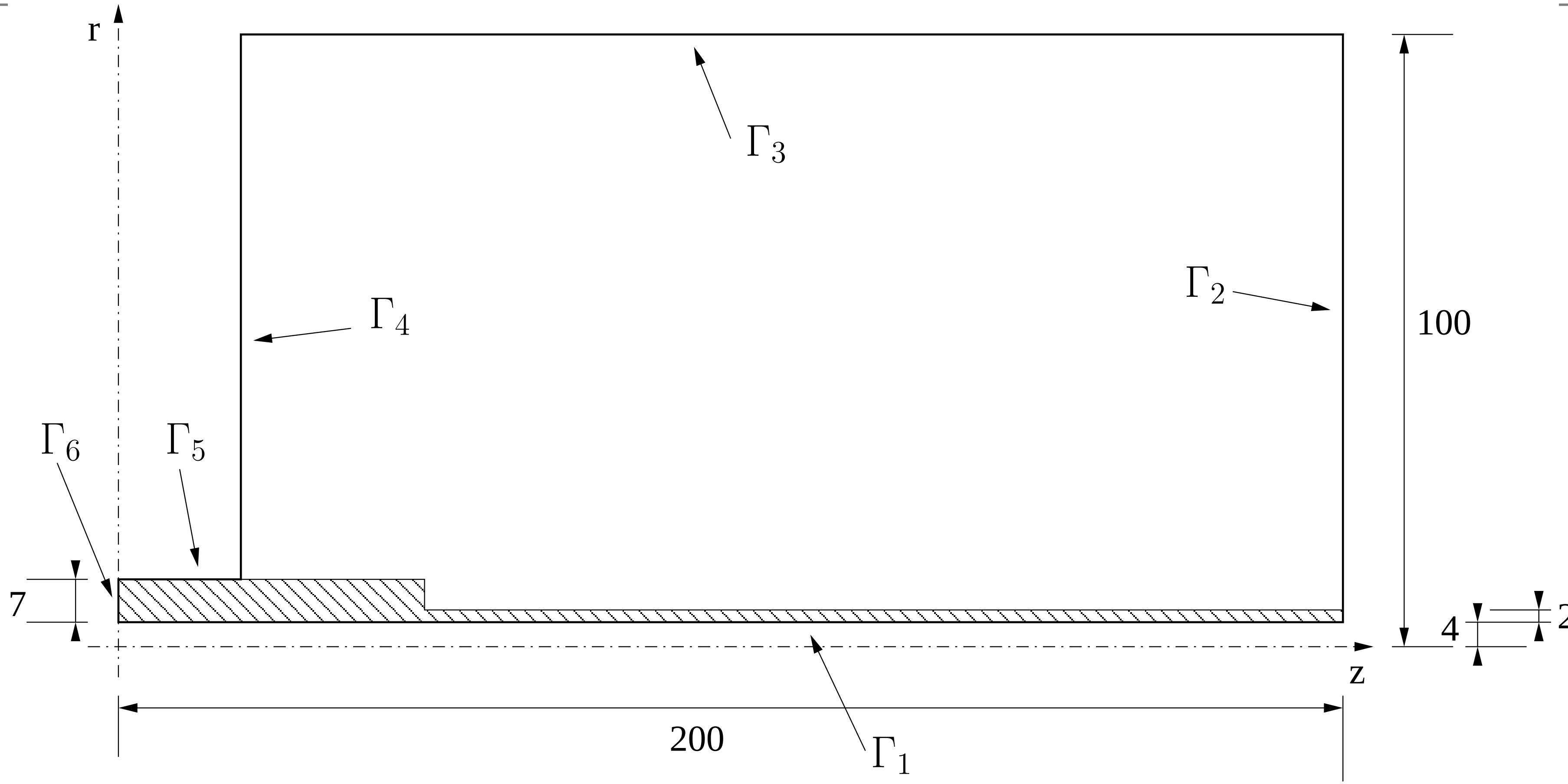
The piecewise-linear mesh.
Uniform refinement (each edge was subdivided into 44).

Zoom = 1, 6, 50, 1000.

Electrostatic micromotor

	linear elements	hp-elements
DOF	472384	4511
Error	0.2024 %	0.173 %
Iterations	387	71
CPU time	**32 min.**	**17 sec.**

Insulator problem

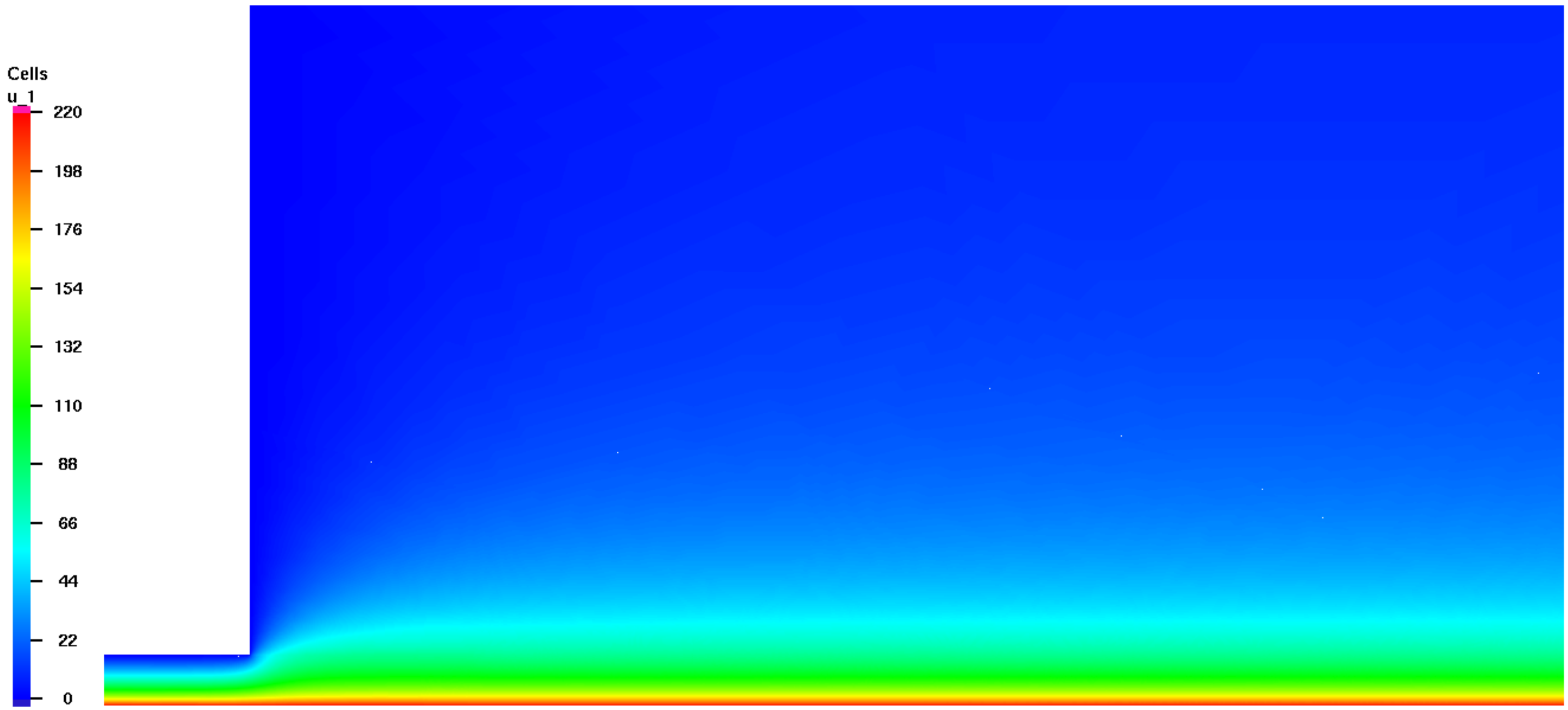


Computational domain (all measures are in millimeters).

El. potential: $\varphi = 220$ V on Γ_1,

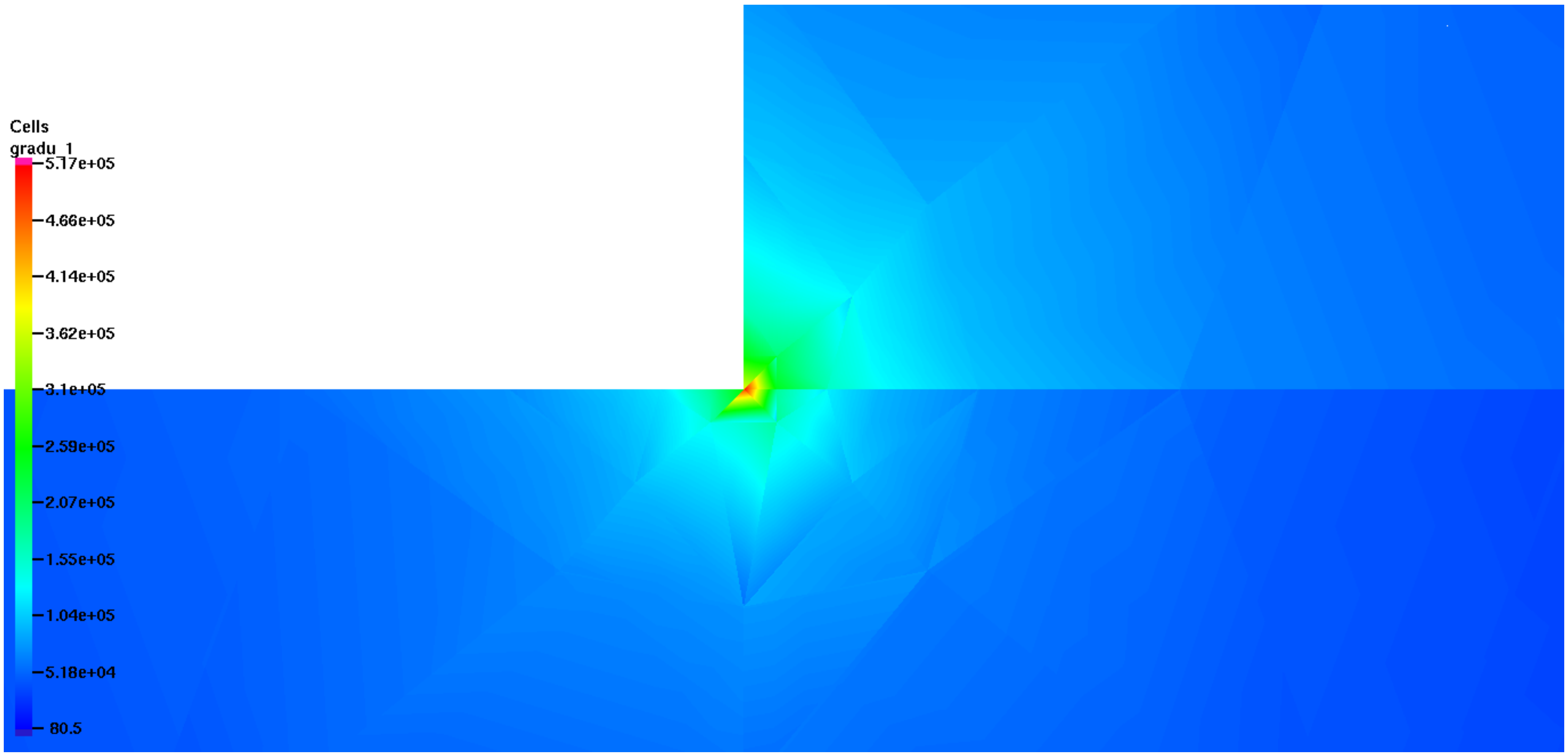
Permittivity: $\epsilon = 1$ in Ω_1 and $\epsilon = 10$ in Ω_2.

Insulator problem



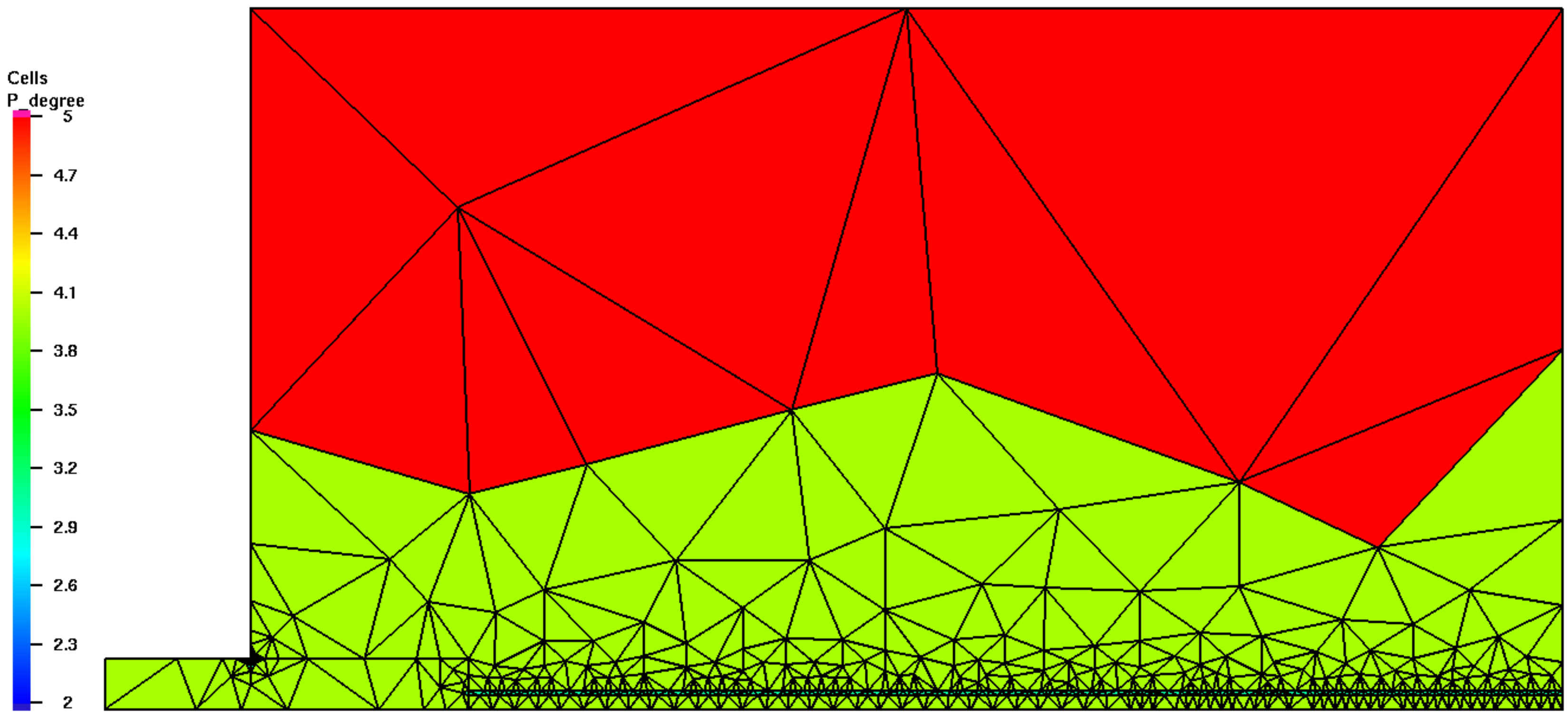
Solution of the insulator problem (electric potential φ).

Insulator problem



Detail of the singularity of $|\mathbf{E}|$ at the re-entrant corner and discontinuity along the material interface (zoom = 1000).

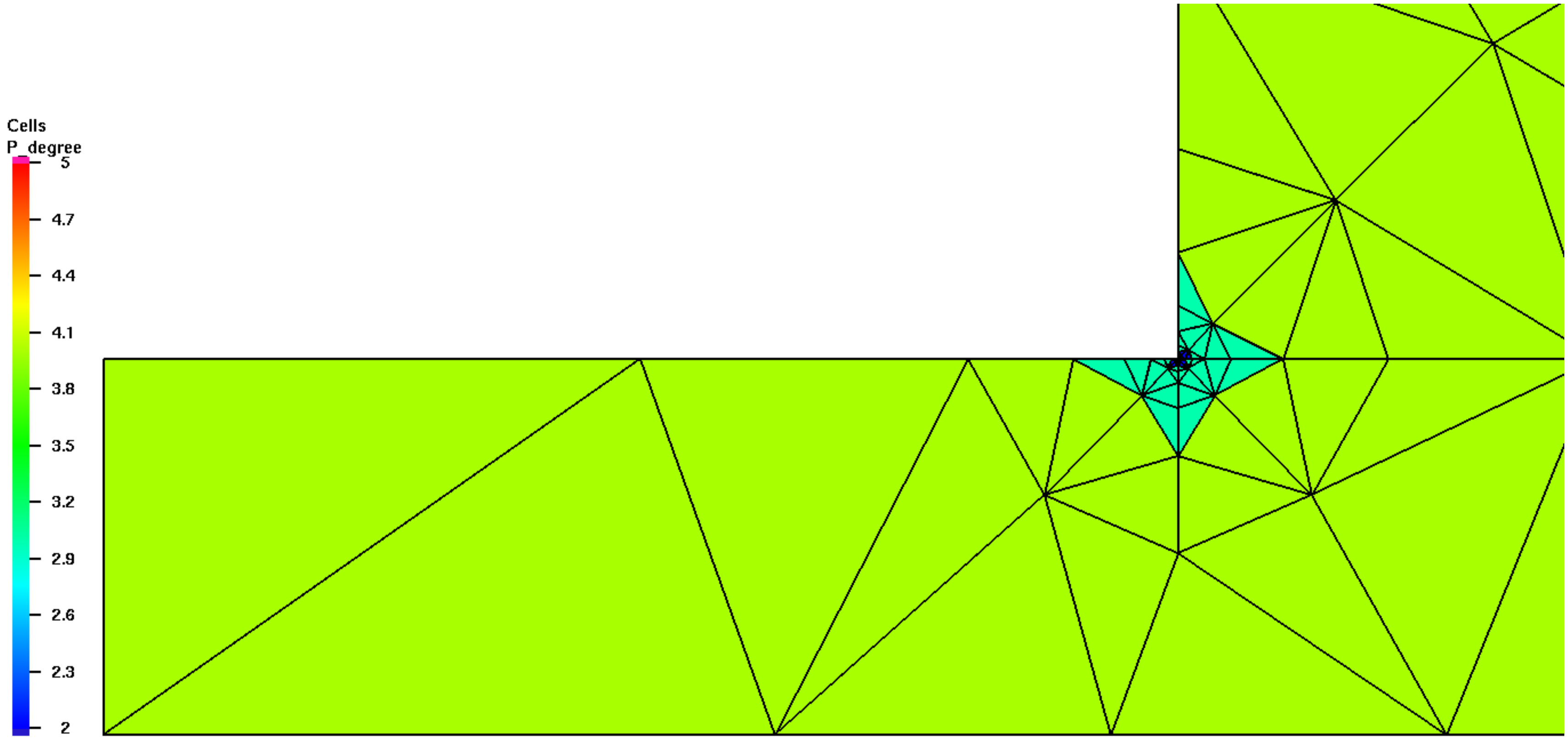
Insulator problem



The hp-mesh – global view.

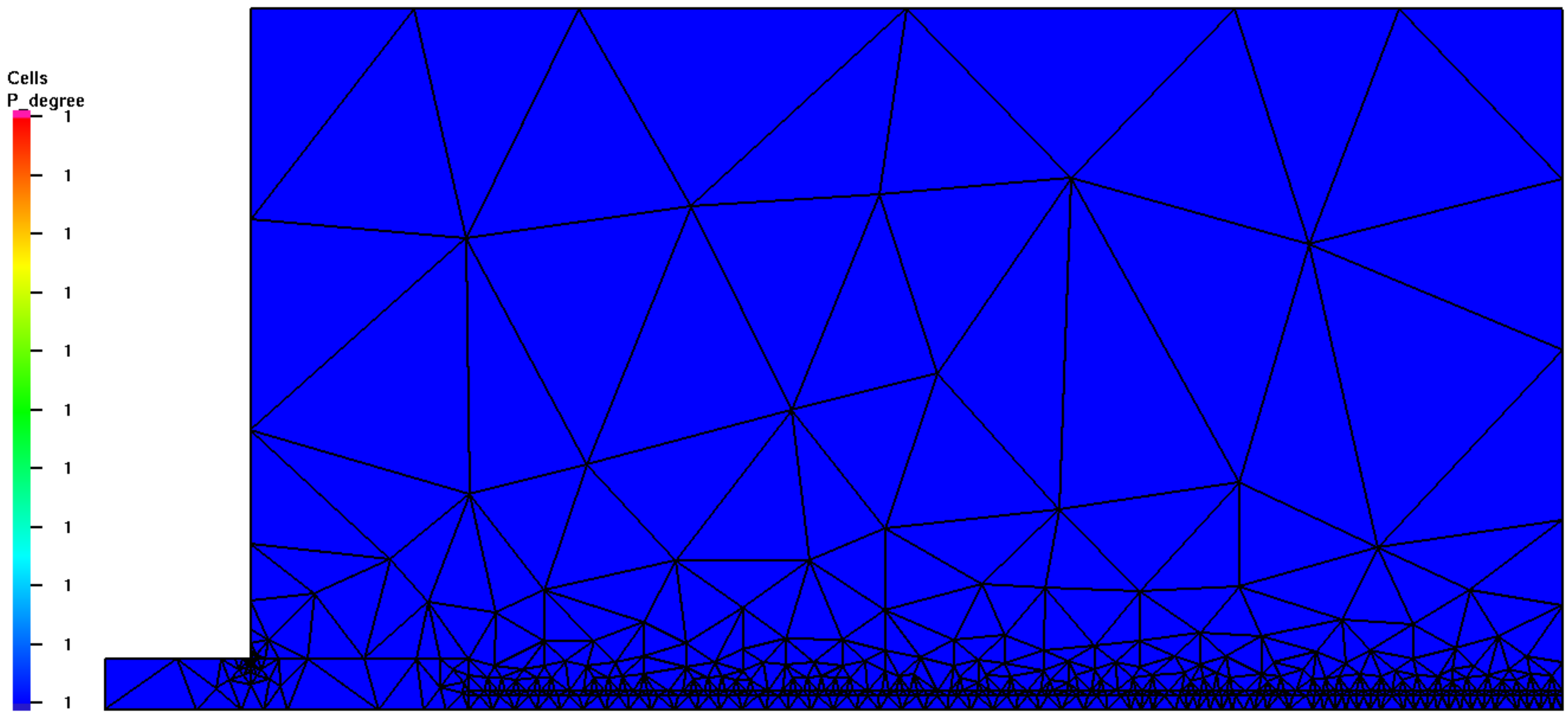
Red – fifth-order elements. Blue – quadratic elements.

Insulator problem



The hp-mesh – detail of the re-entrant corner (zoom = 1000).

Insulator problem



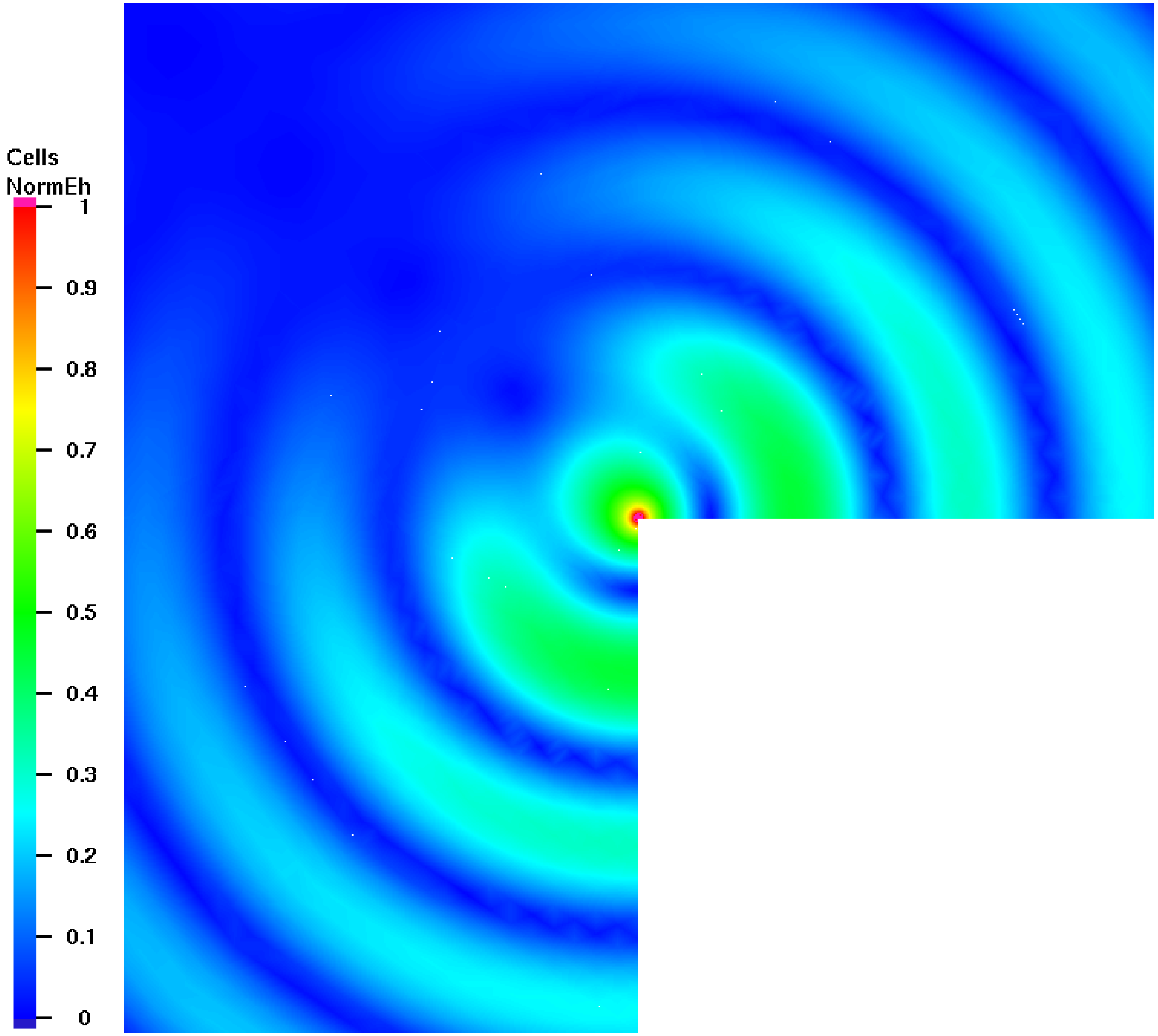
The piecewise-linear mesh.

Uniform refinement (each edge was subdivided into 23).

Insulator problem

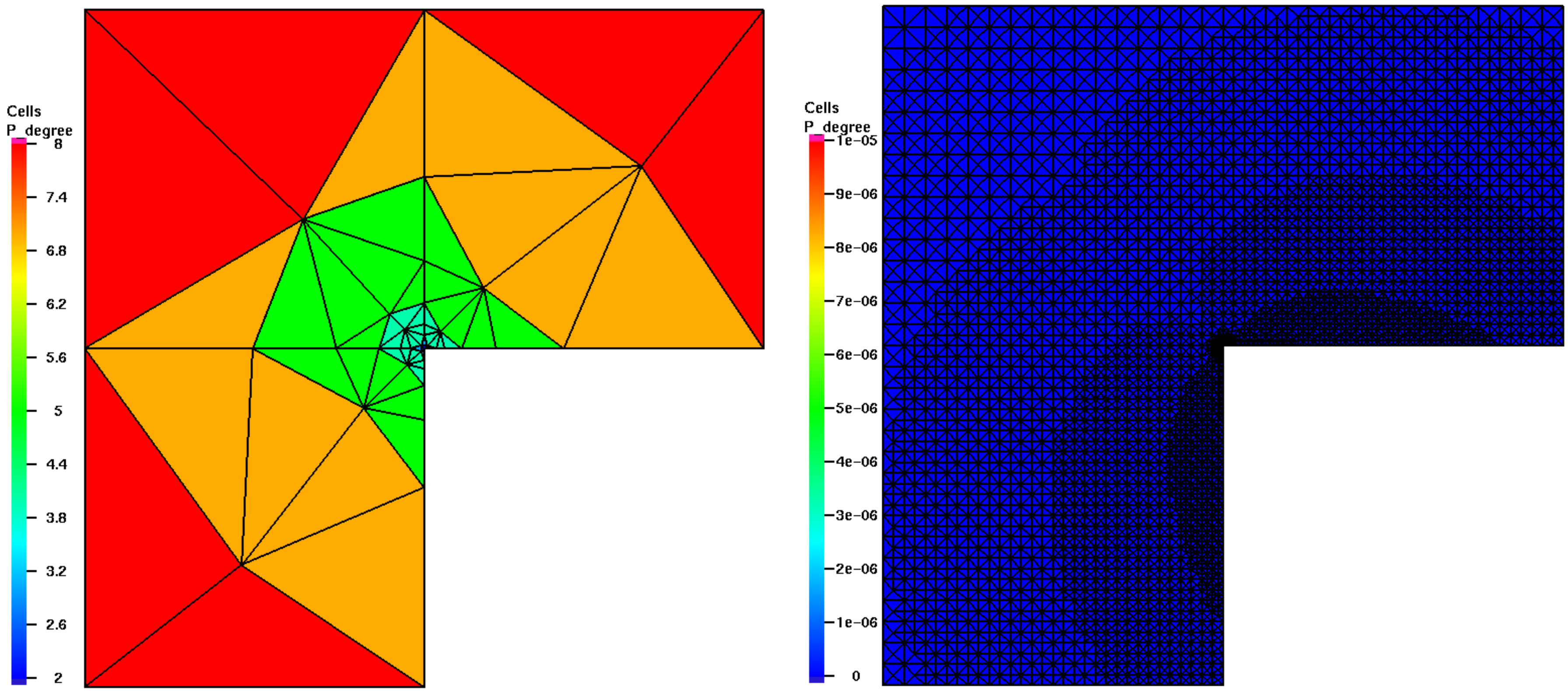
	linear elements	hp-elements
DOF	259393	6331
Error	1.617 %	1.521 %
Iterations	228	60
CPU time	**34 min.**	**11.58 sec.**

Diffraction problem



$$\mathbf{E} = \mathbf{curl}(J_\alpha(r)\cos(\alpha\phi)), \quad r(x) = \sqrt{x_1^2 + x_2^2}, \qquad \alpha = \frac{2}{3}.$$

Diffraction problem



The meshes:

hierarchic hp edge elements and Whitney elements.

Diffraction problem

	Whitney edge elements	hp edge elements
DOF	2586540	4324
Error	0.6445 %	0.6211 %
CPU time	**21.2 min.**	**2.49 sec.**

Thank you for your attention.

http://servac.math.utep.edu/fem_group

http://servac.math.utep.edu/hermes

http://groups.google.com/group/femgroup

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