Archimédův výpočet čísla π

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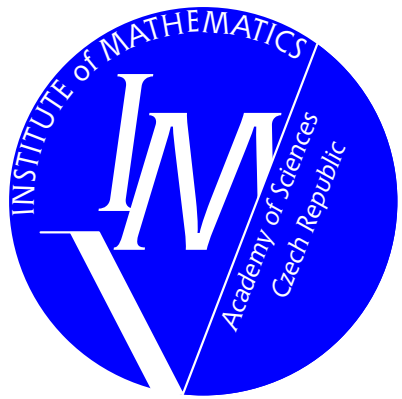
Žitná 25, 115 67 Praha 1



9. listopadu 2007, Praha

Číslo π

- $\pi \approx 3.14\ldots$

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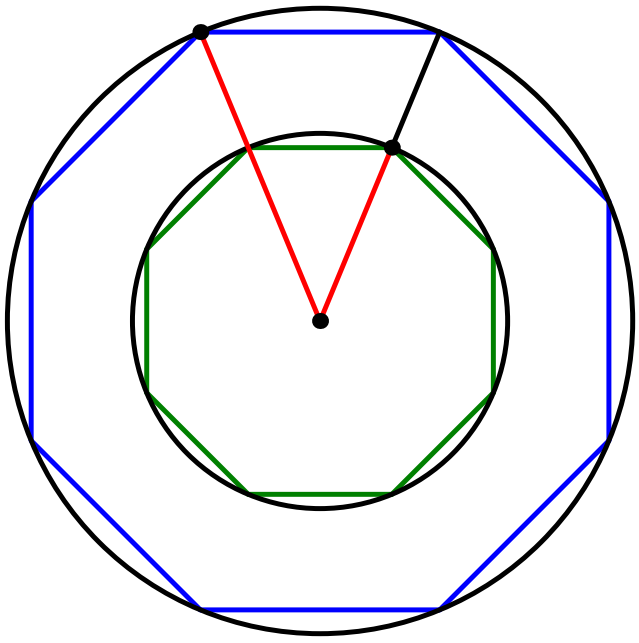
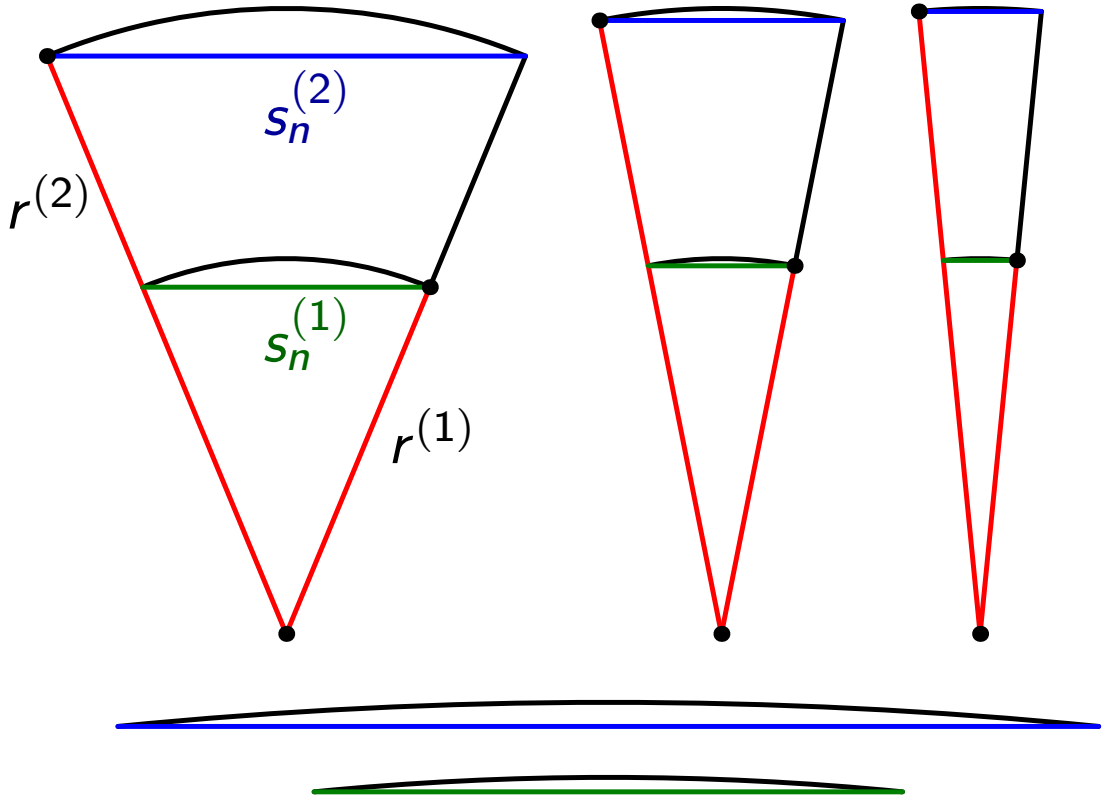
- plocha kruhu: $S = \pi r^2$

$o = 2\pi r$

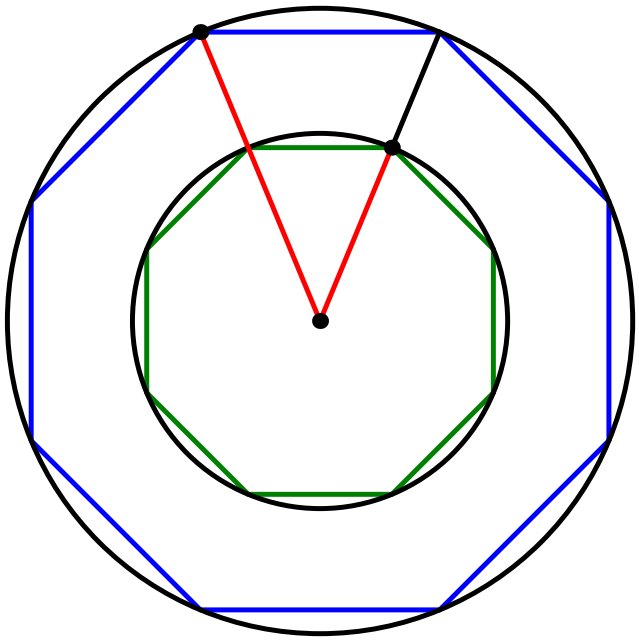


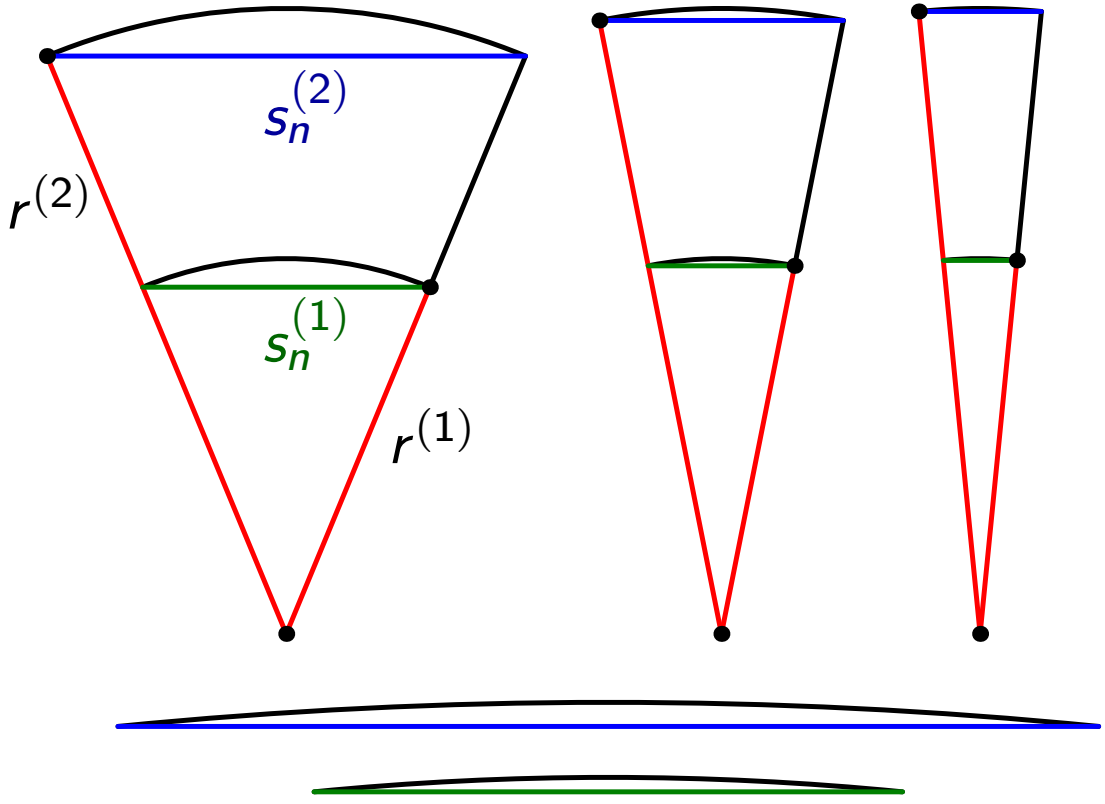


$$\frac{s_n^{(1)}}{r^{(1)}} = \frac{s_n^{(2)}}{r^{(2)}}$$

$$\frac{o^{(1)}}{r^{(1)}} \approx \frac{ns_n^{(1)}}{r^{(1)}} = \frac{ns_n^{(2)}}{r^{(2)}} \approx \frac{o^{(2)}}{r^{(2)}}$$

$o = 2\pi r$





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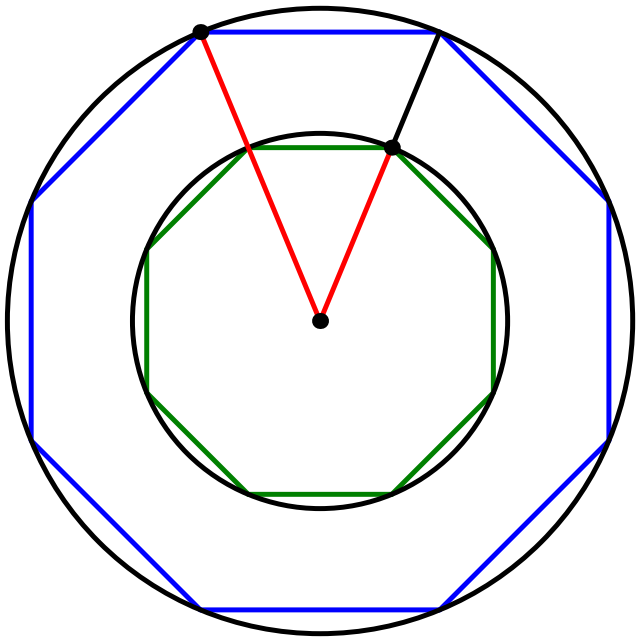

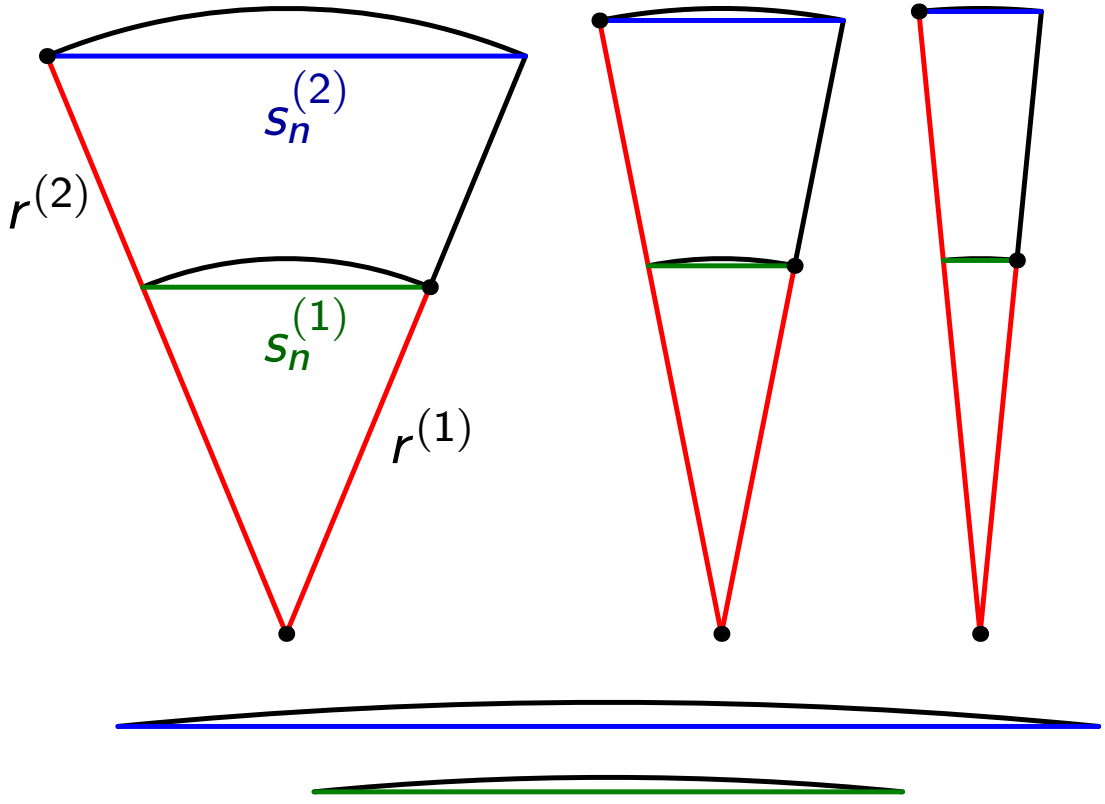

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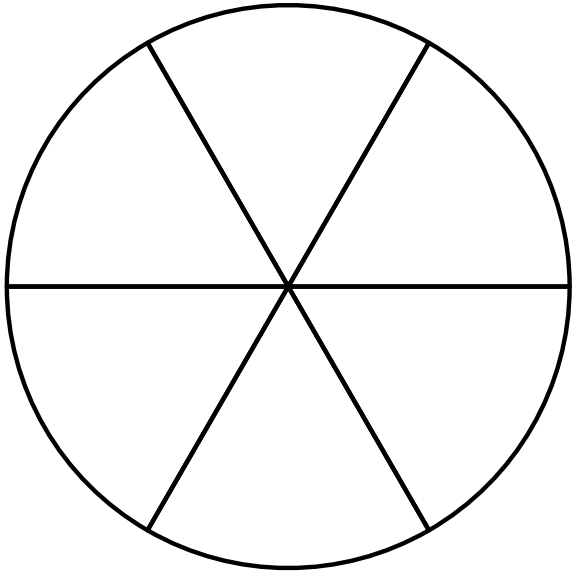

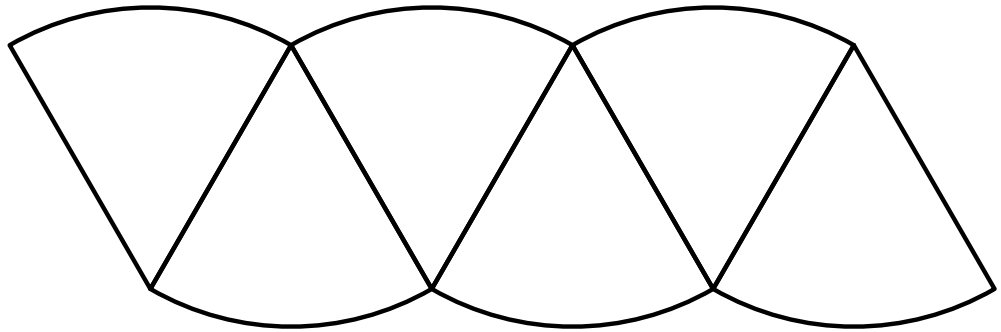

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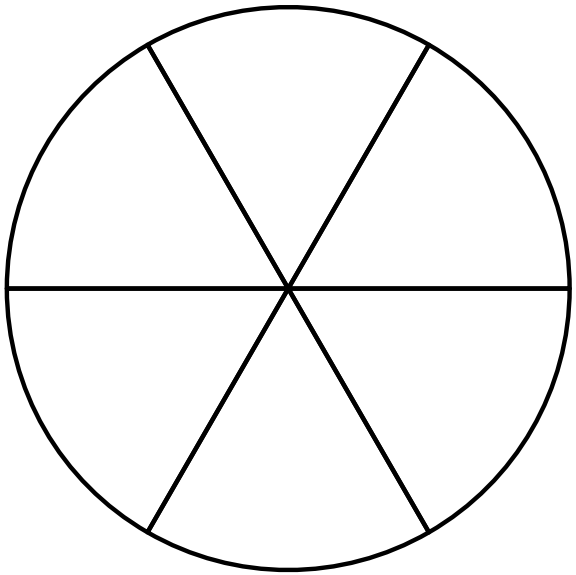

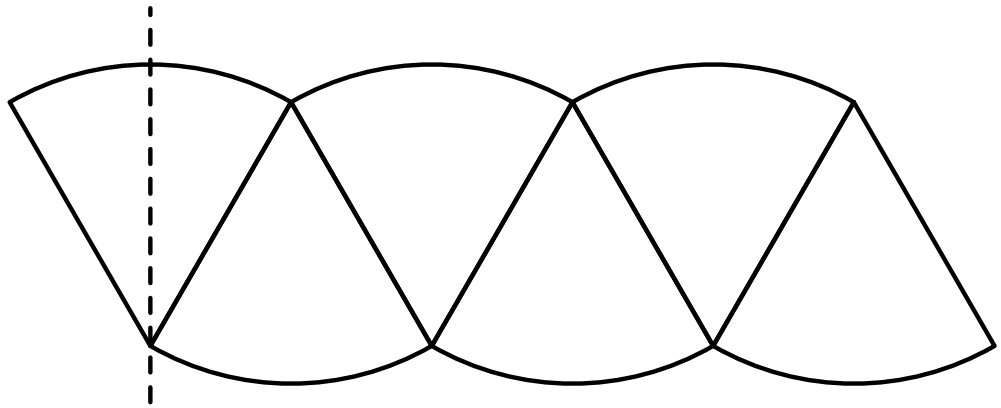

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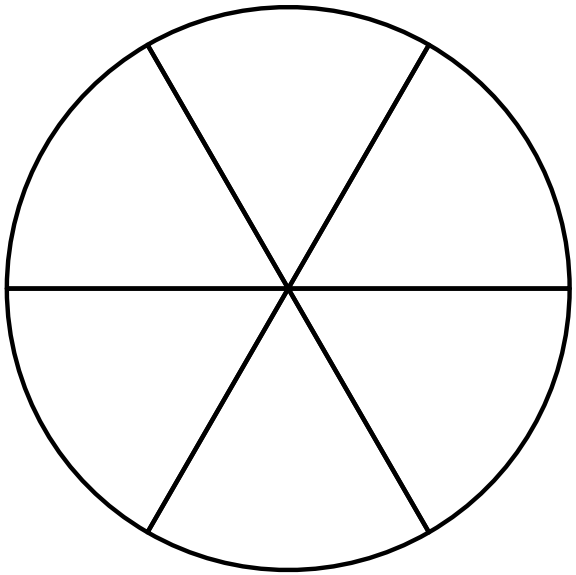

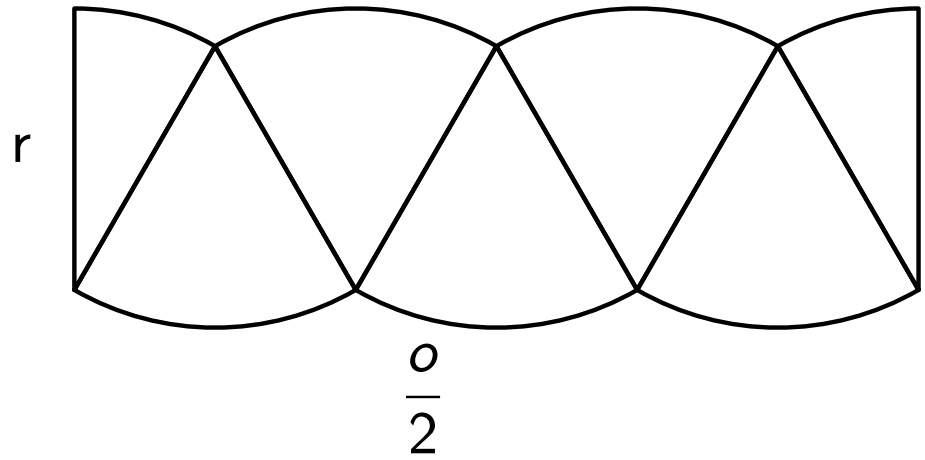

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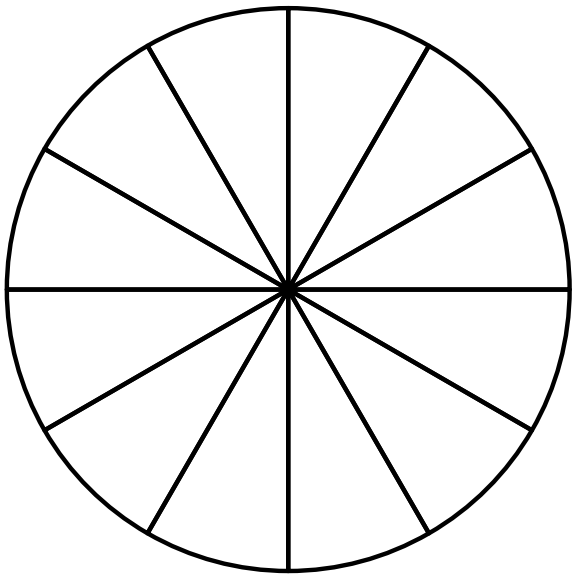

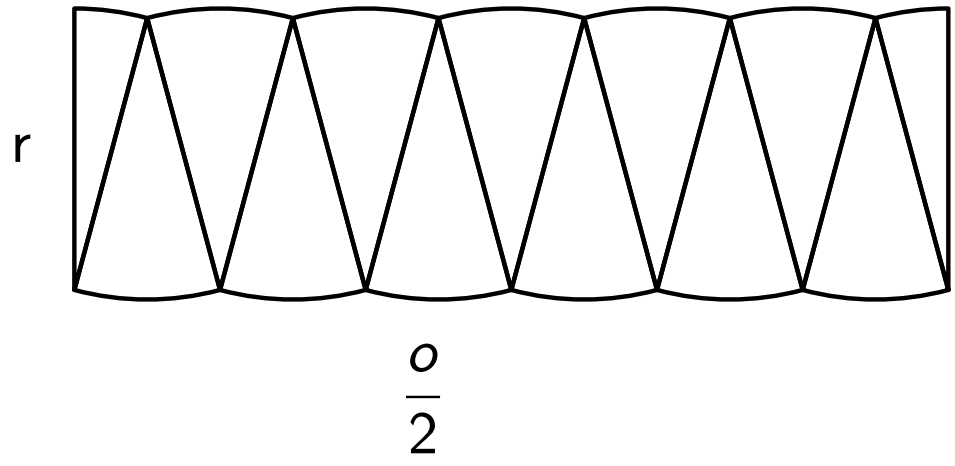

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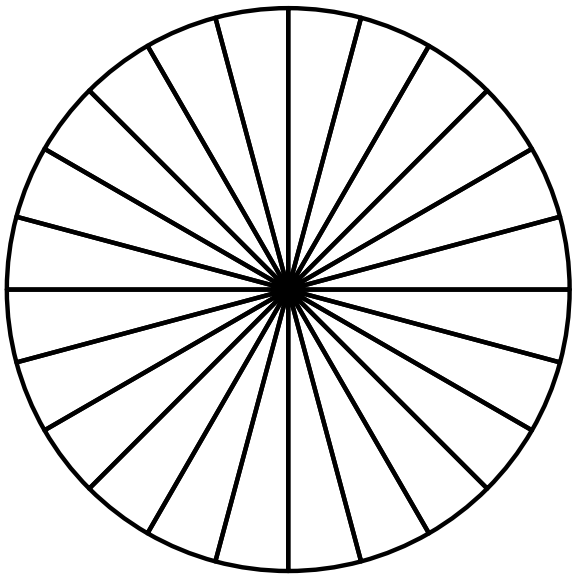

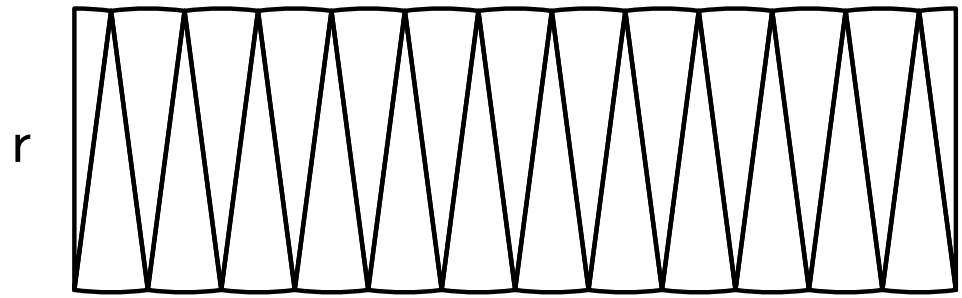

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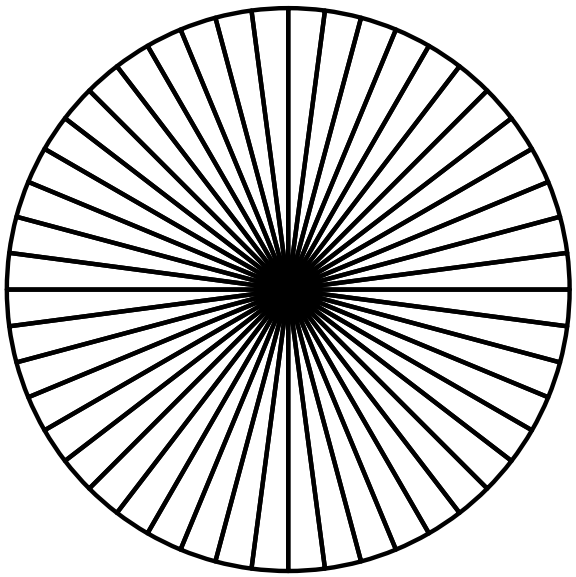

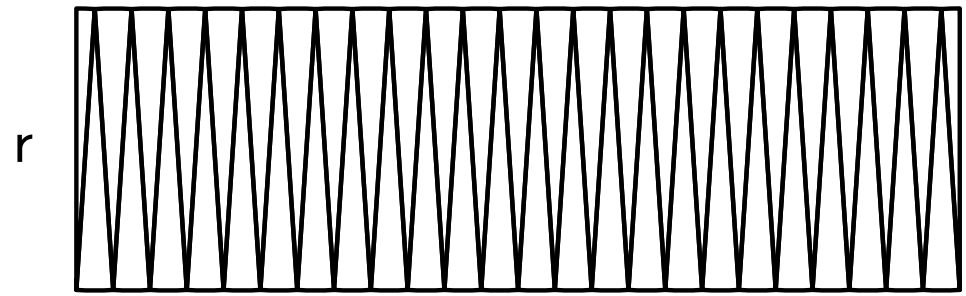

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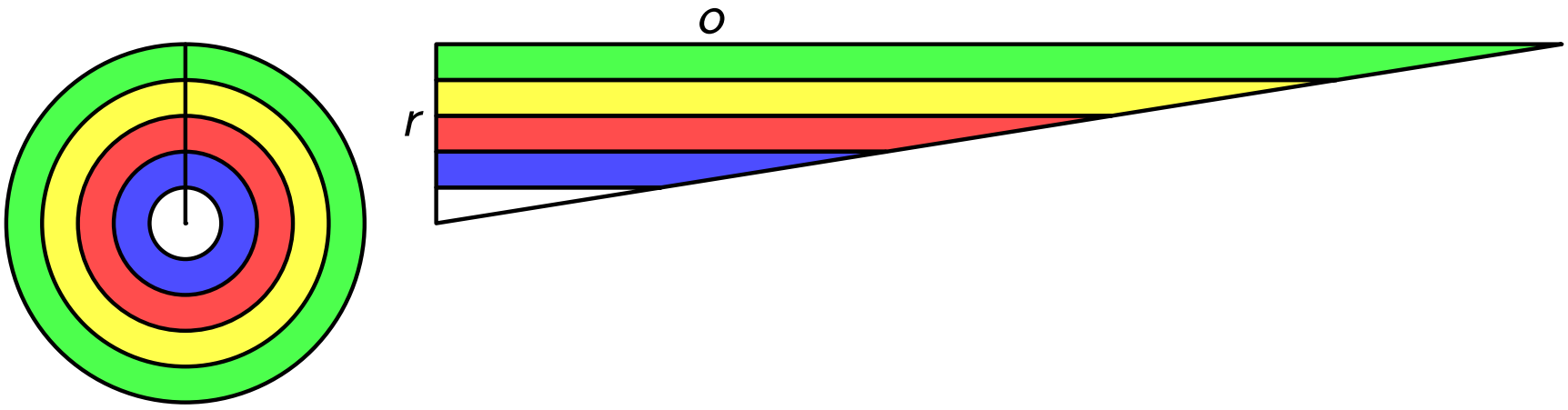

$$S = \frac{1}{2} ro = \frac{1}{2} r 2\pi r = \pi r^2$$

Archimédes (asi 287 - 212 př.n.l., Syrakusy, Sicílie)

$$\frac{223}{71} < \pi < \frac{22}{7} \qquad \Rightarrow \qquad \pi \approx 3.1418$$

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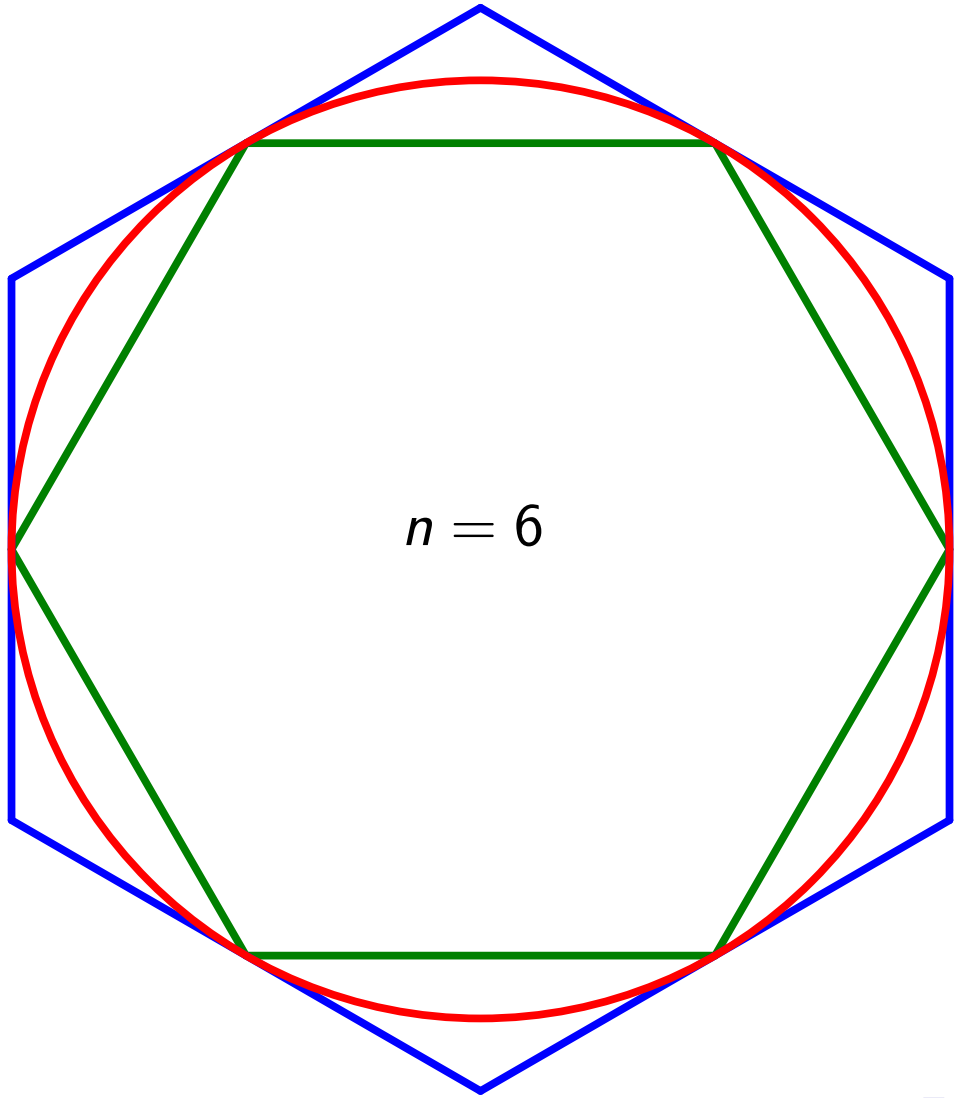

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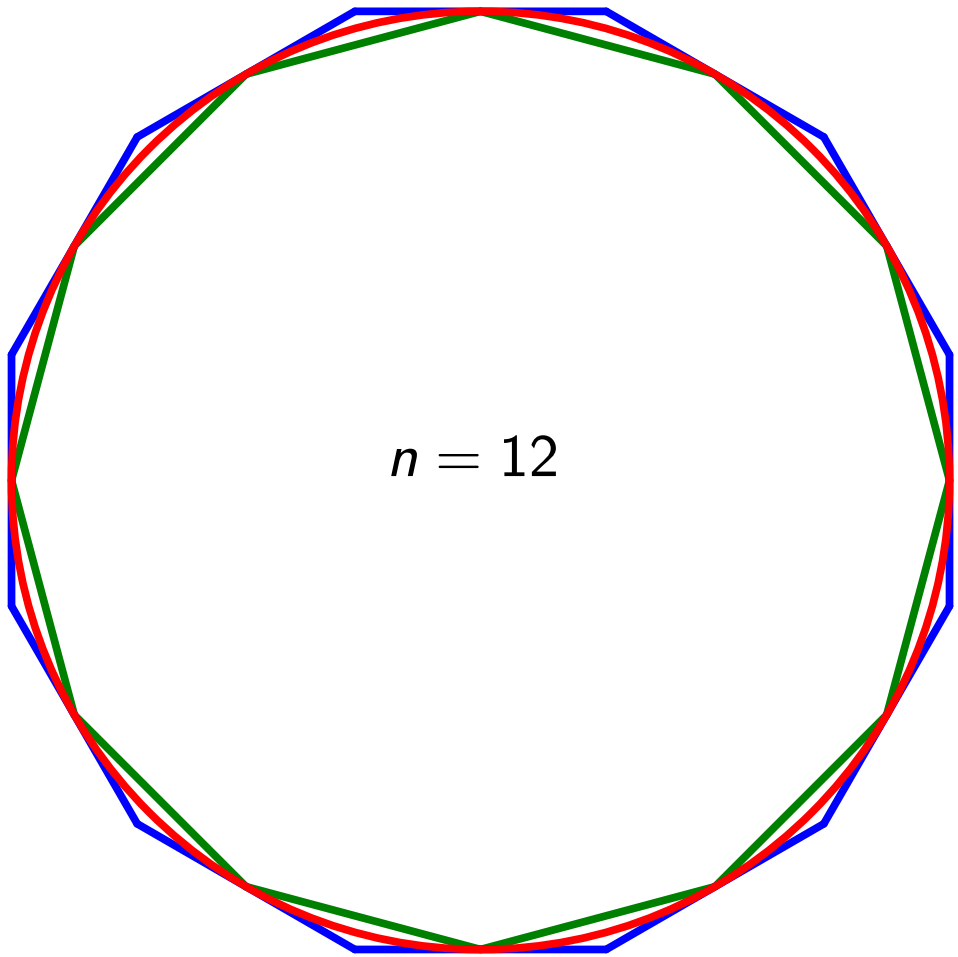

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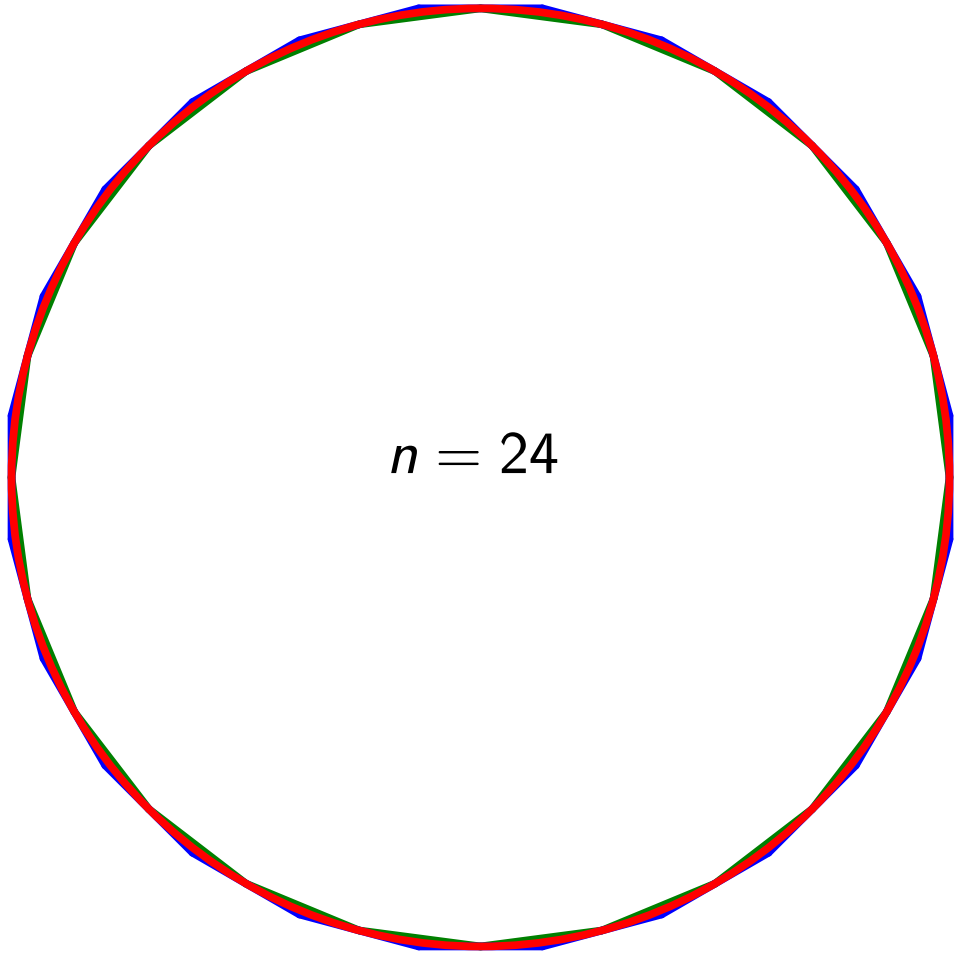

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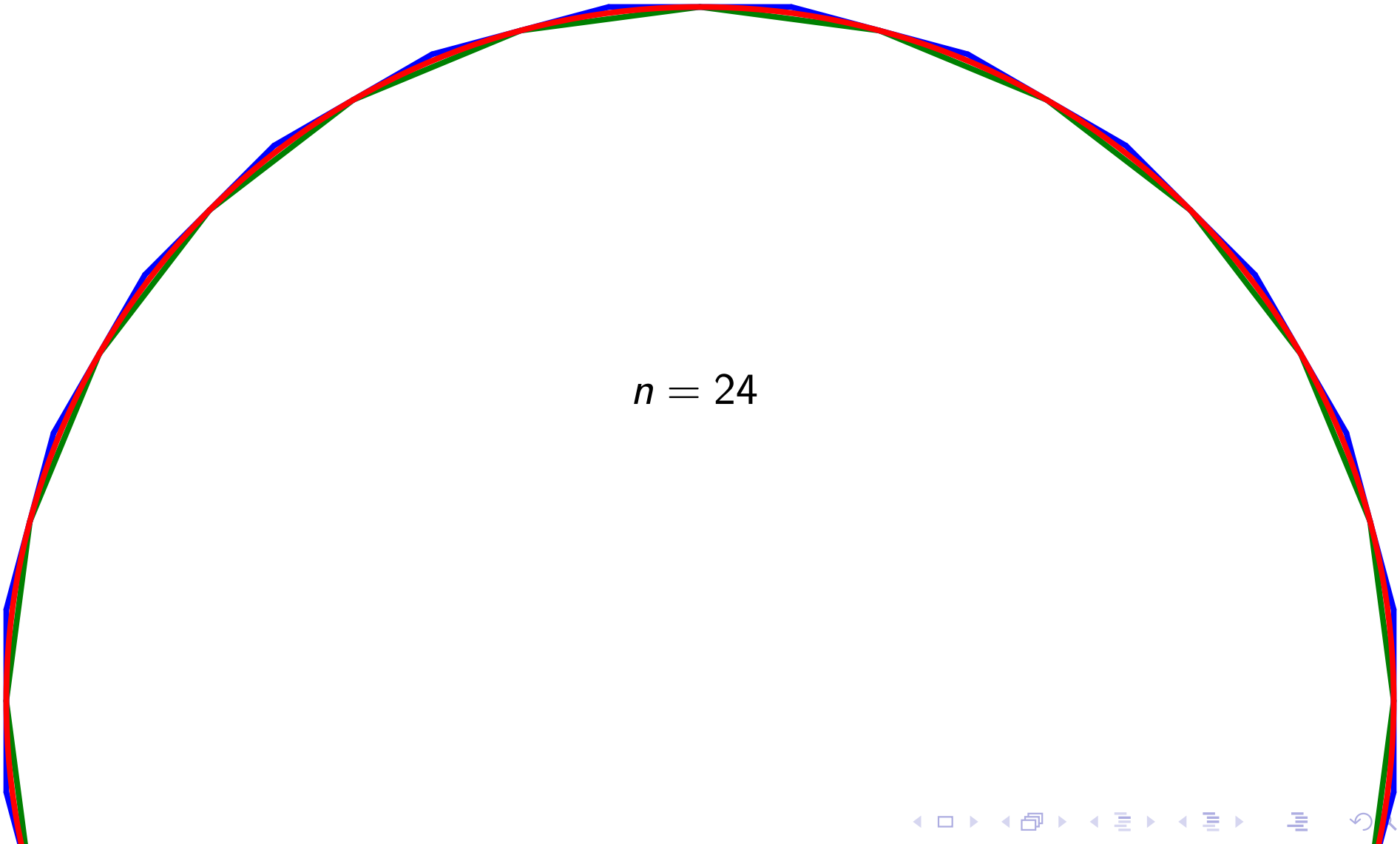

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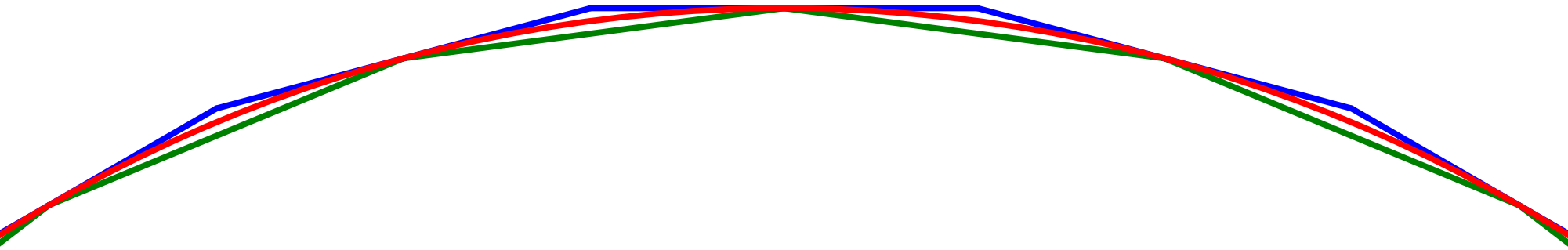

$$n = 24$$

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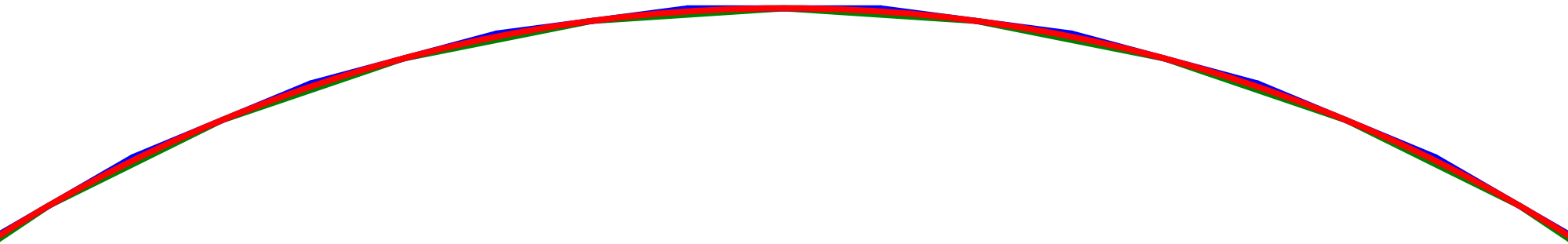

$$n = 48$$

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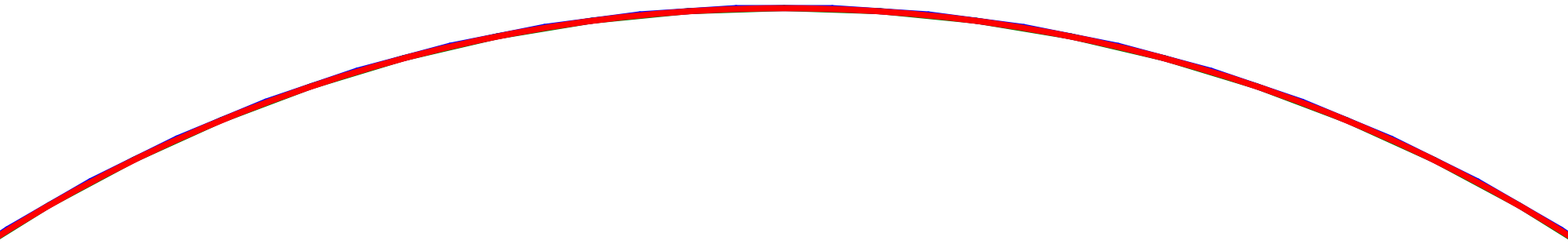

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Metoda:

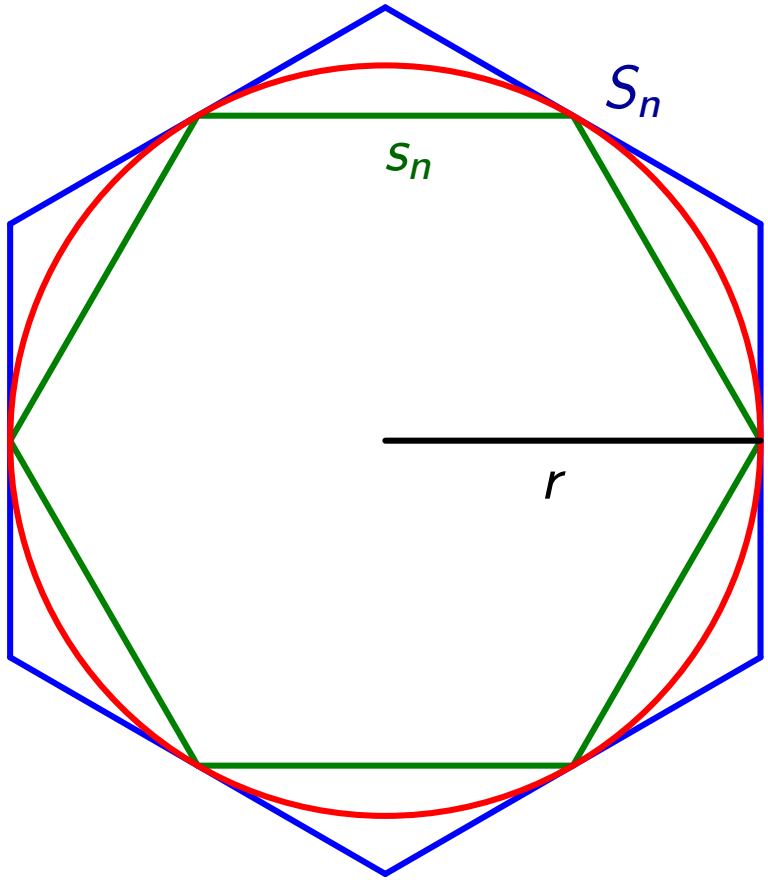
$o = 2\pi r \quad \Leftrightarrow \quad \pi = \frac{o}{2r}$

$o_n = n s_n \quad O_n = n S_n$

$$\frac{o_n}{2r} < \frac{o}{2r} < \frac{O_n}{2r}$$

$$\underline{\pi}_n < \pi < \overline{\pi}_n$$

$r = 1$



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6-ti úhelník: $(n = 6)$

$s_6 = 1$

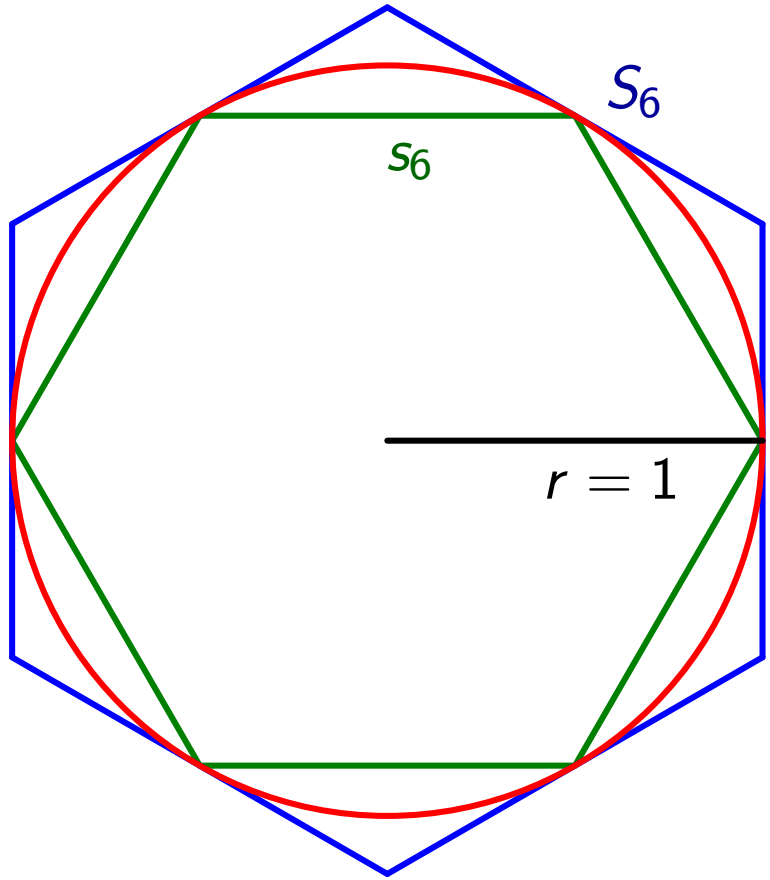
$o_6 = 6$

$\underline{\pi}_6 = 3$

$S_6 = \frac{2}{3}\sqrt{3}$

$O_6 = 4\sqrt{3}$

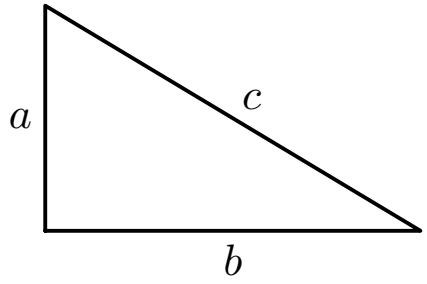
$\overline{\pi}_6 = 2\sqrt{3} \approx 3.46$



Pomocné věty

Věta (Pythagorova):

$a^2 + b^2 = c^2$



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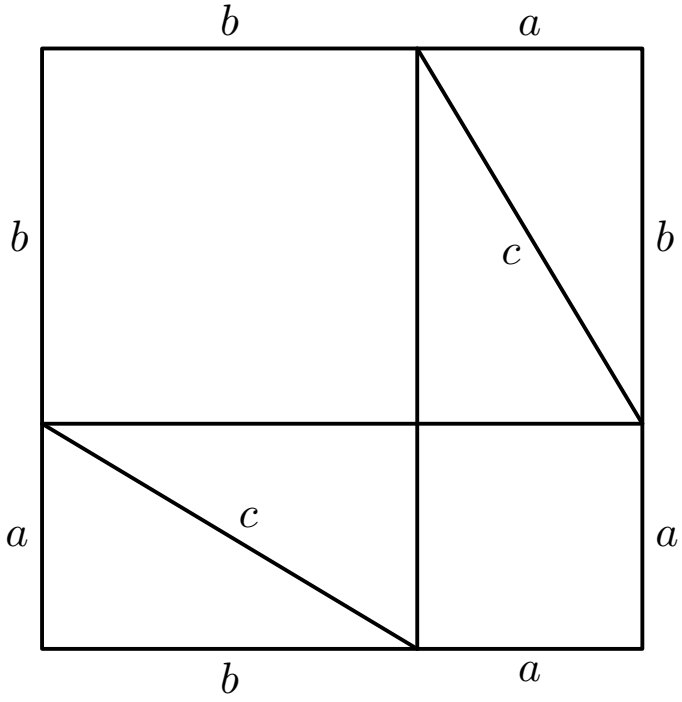

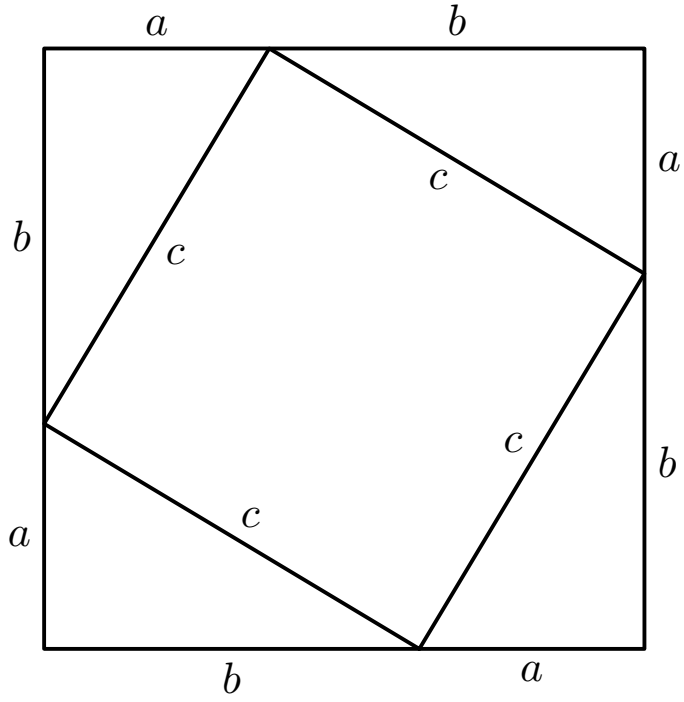

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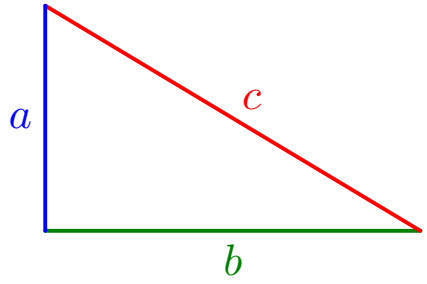

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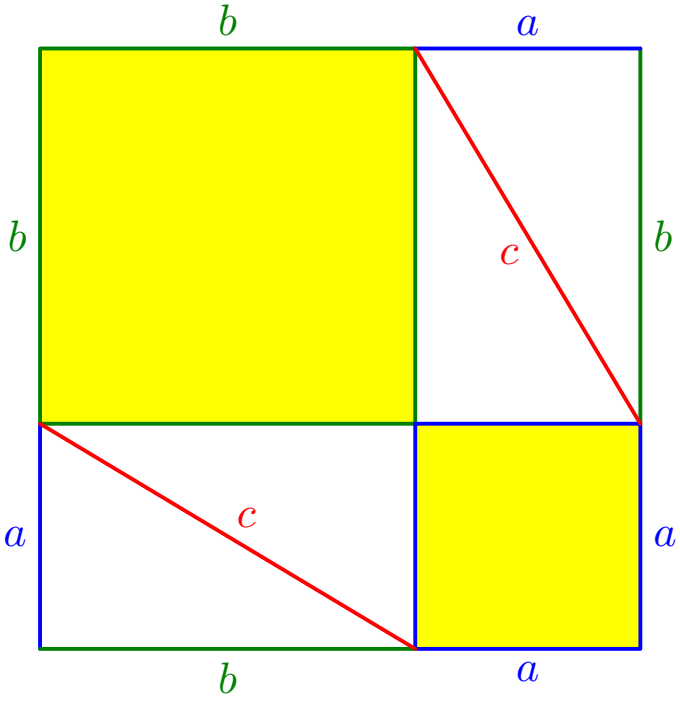

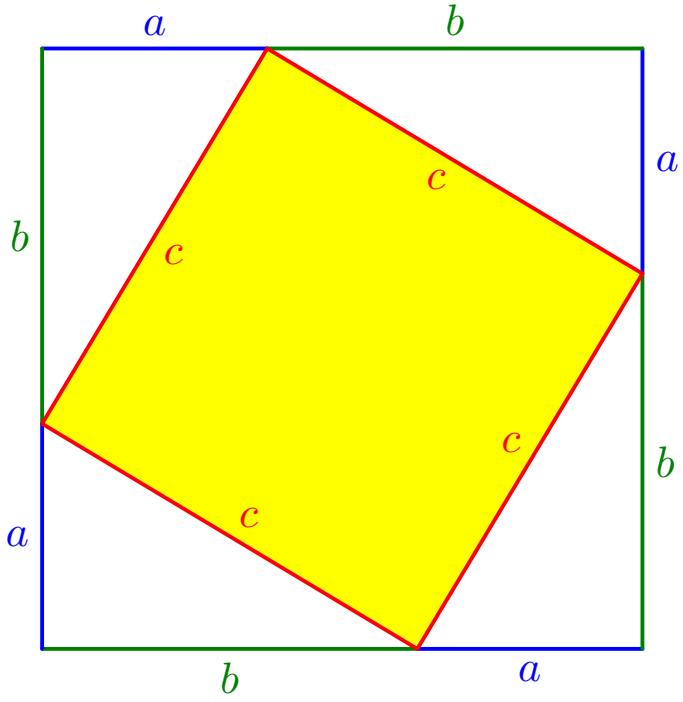

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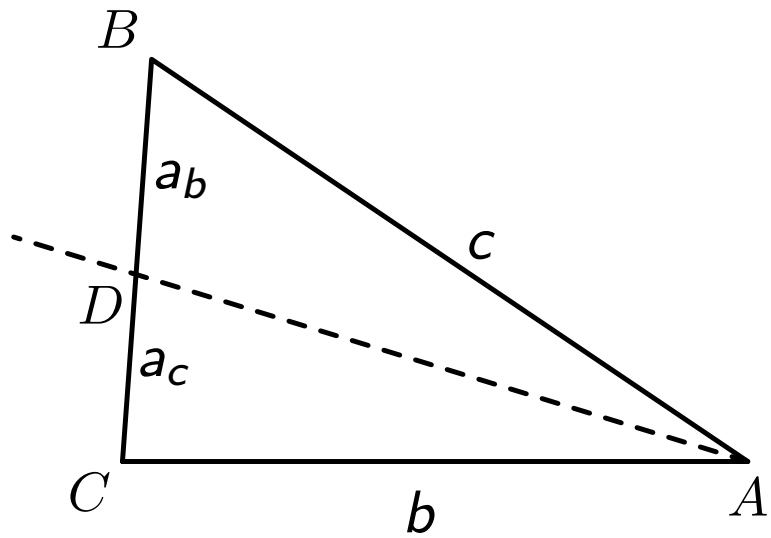
Věta: $\frac{a_c}{a_b} = \frac{b}{c}$

D je průsečík osy úhlu CAB s hranou a



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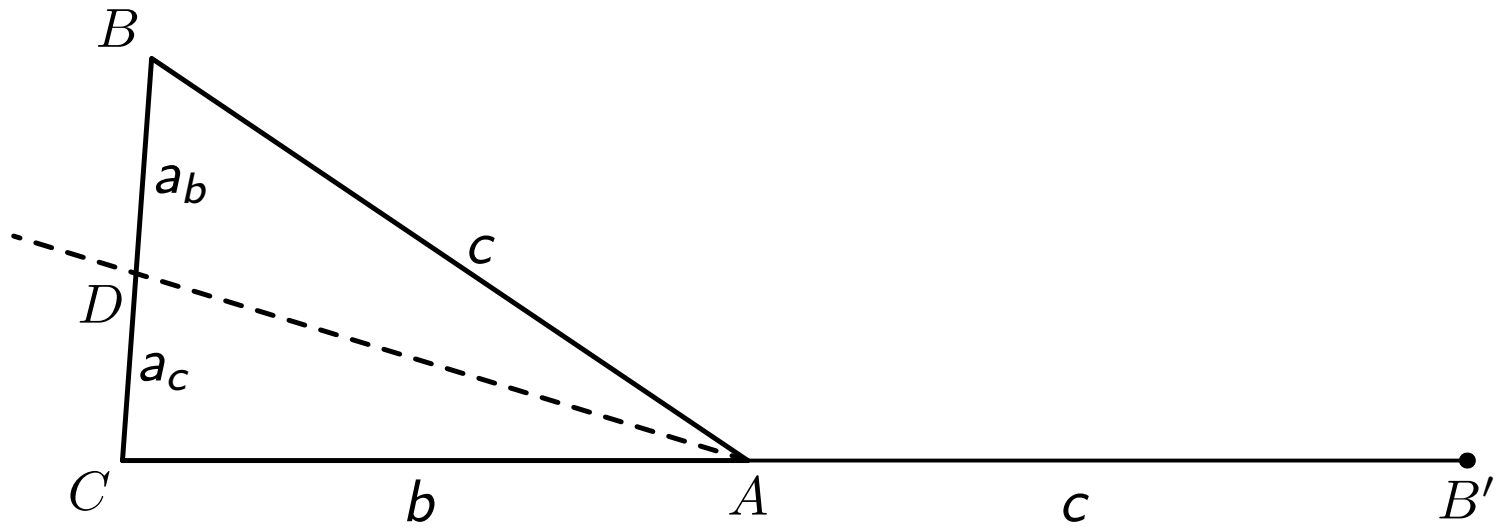

$$\frac{a_c}{b} = \frac{a_c + a_b}{b + c}$$

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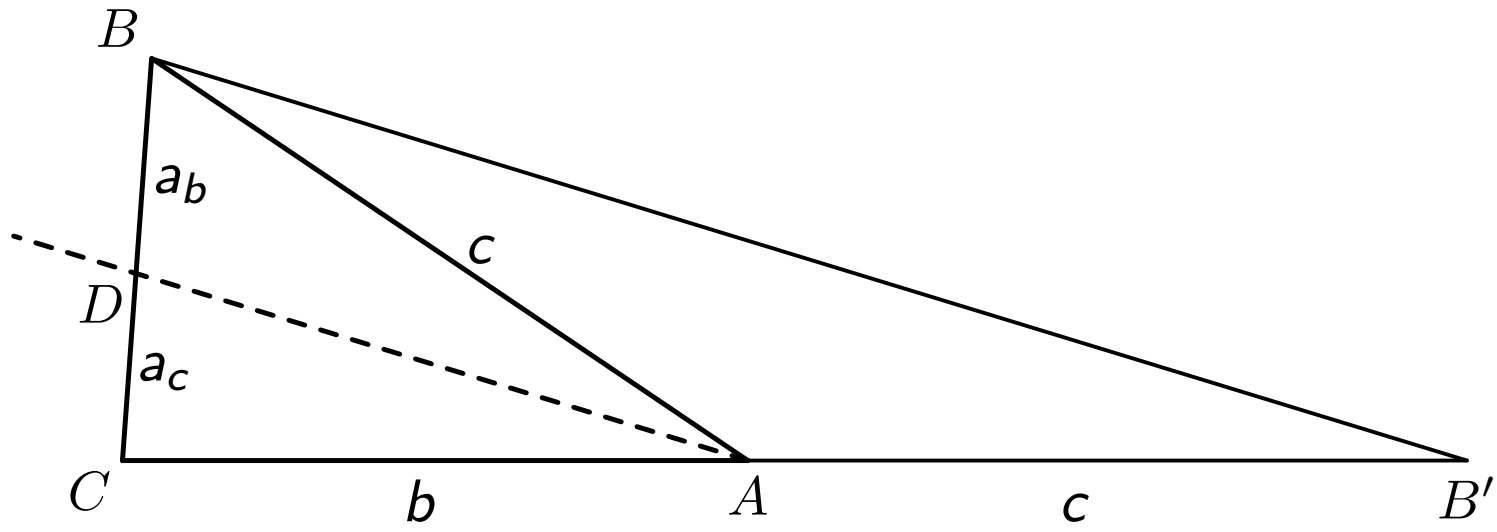

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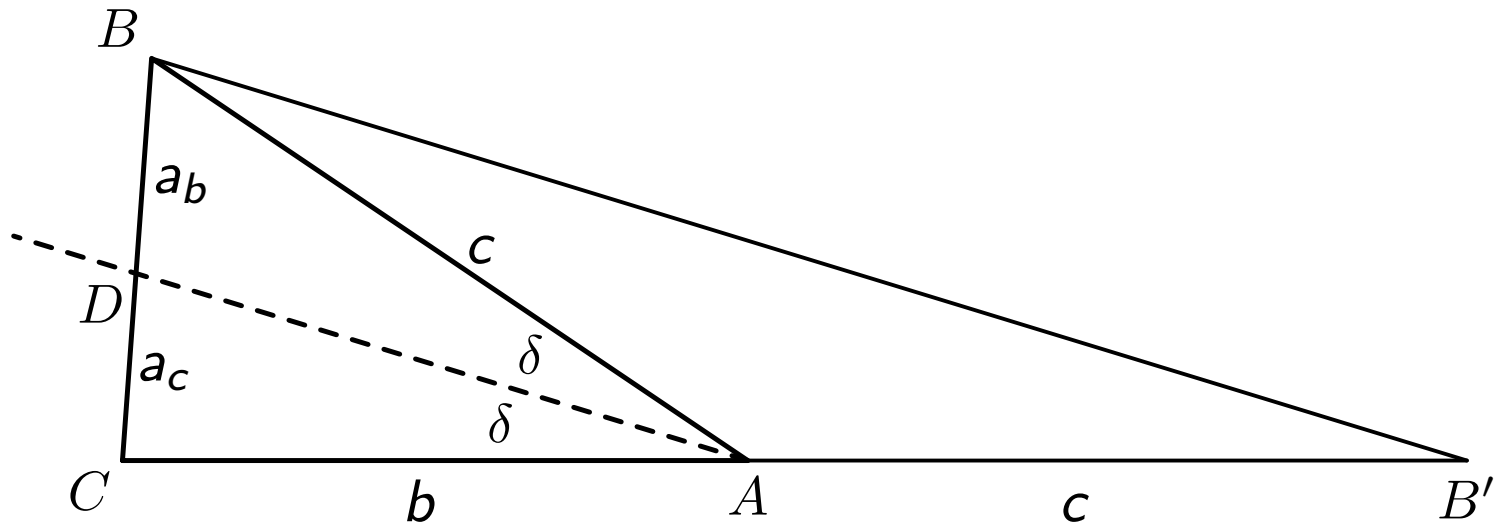

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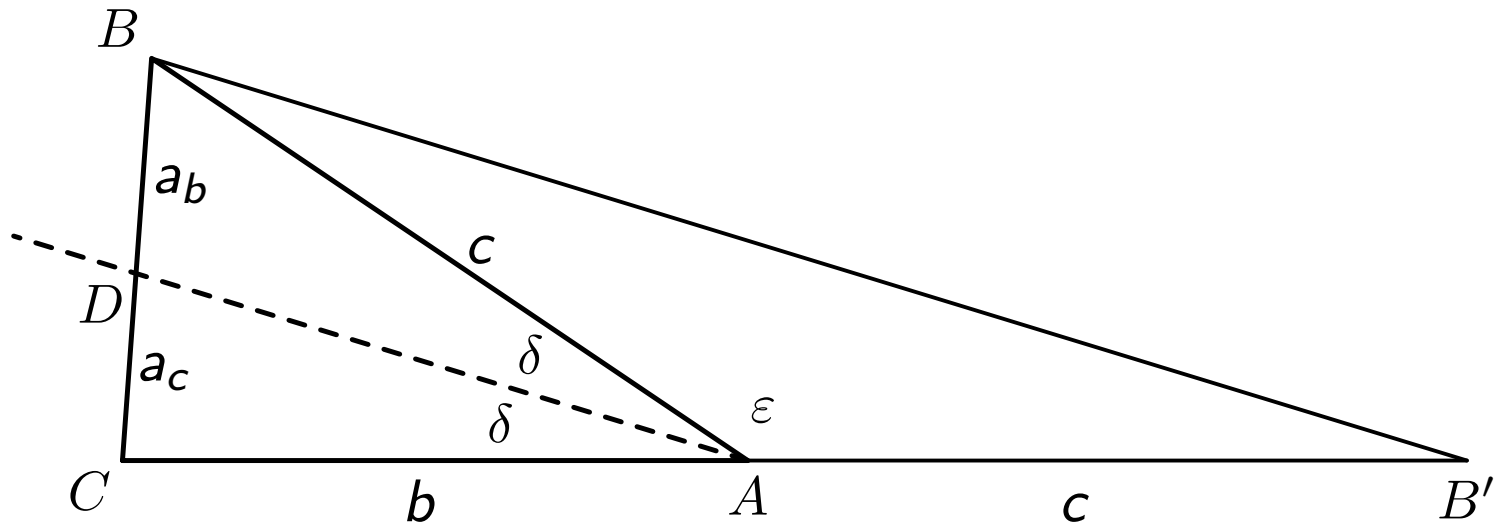

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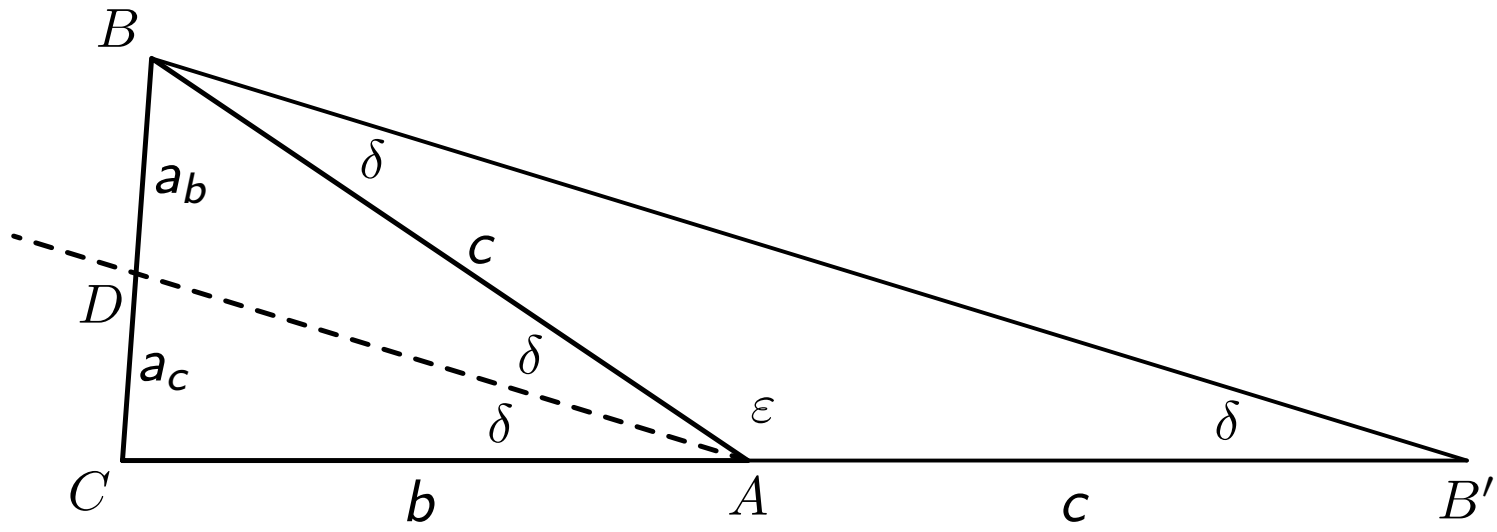

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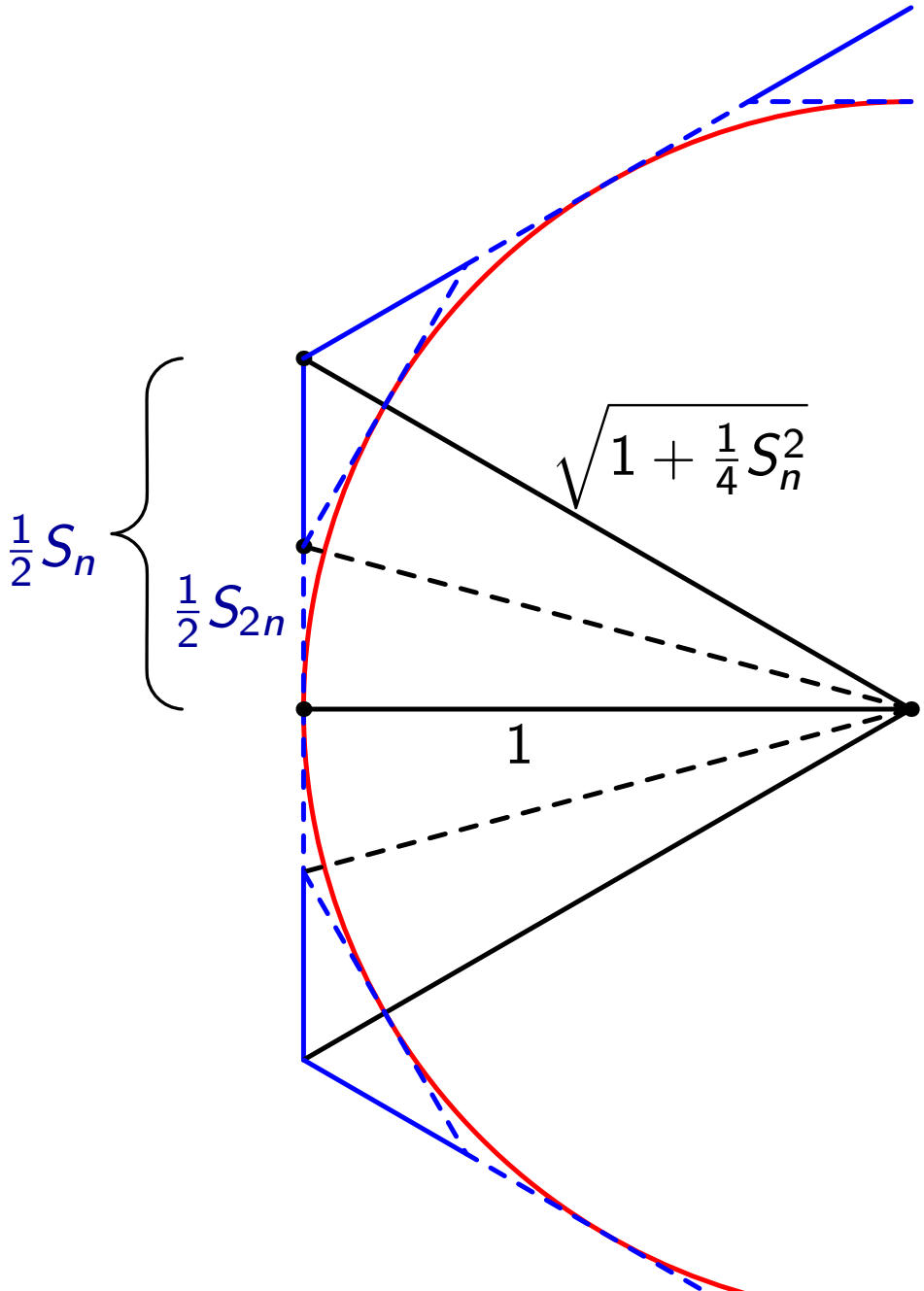
Opsané pravidelné n-úhelníky



$$\frac{\frac{1}{2}S_{2n}}{\frac{1}{2}S_n-\frac{1}{2}S_{2n}}=\frac{1}{\sqrt{1+\frac{1}{4}S_n^2}}$$

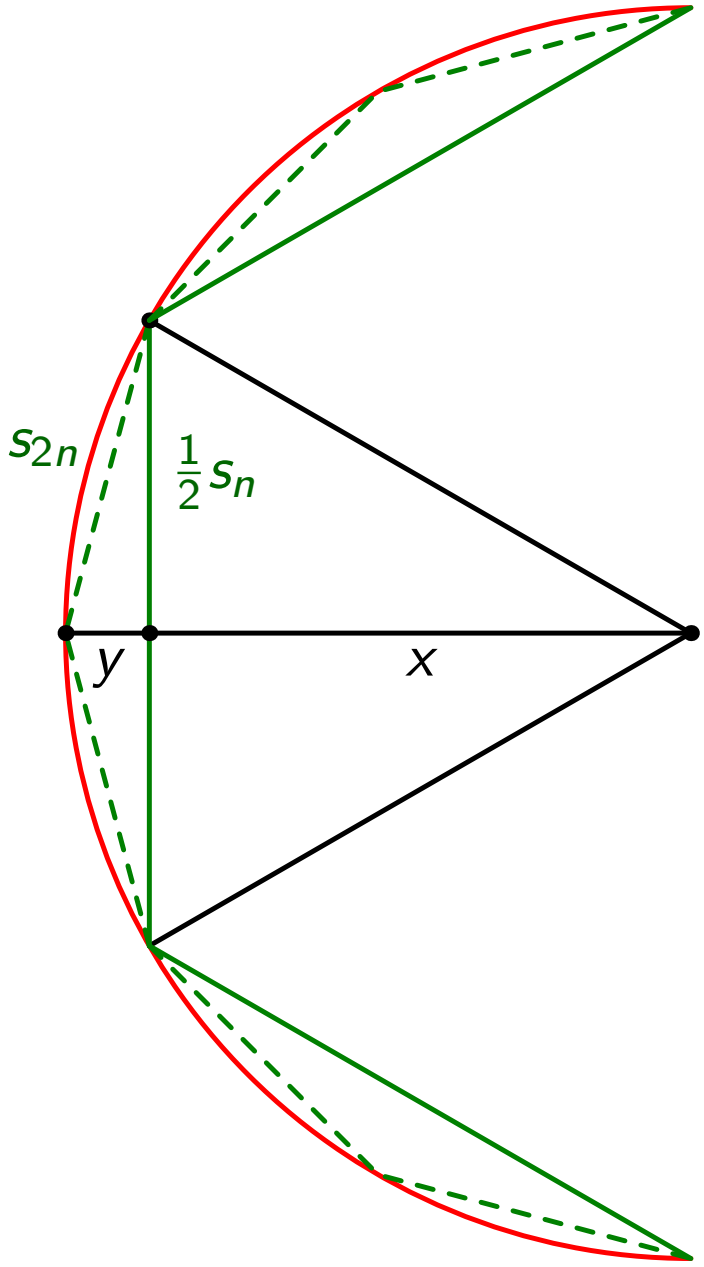
$$\frac{S_{2n}}{S_n-S_{2n}}=\frac{1}{\sqrt{1+\frac{1}{4}S_n^2}}$$

$$\frac{1}{\frac{S_n}{S_{2n}}-1}=\frac{1}{\sqrt{1+\frac{1}{4}S_n^2}}$$

$$\frac{S_n}{S_{2n}}=1+\sqrt{1+\frac{1}{4}S_n^2}$$

$$S_{2n}=\frac{2S_n}{2+\sqrt{4+S_n^2}}$$

Vepsané pravidelné n-úhelníky



$$x + y = 1$$

$$x^2 + \frac{s_n^2}{4} = 1$$

$$y^2 + \frac{s_n^2}{4} = s_{2n}^2$$

$$s_{2n}^2 = 1 - 2x + x^2 + \frac{s_n^2}{4}$$

$$s_{2n}^2 = 2 - 2x$$

$$s_{2n}^2 = 2 - 2\sqrt{1 - \frac{s_n^2}{4}}$$

$$s_{2n} = \sqrt{2 - \sqrt{4 - s_n^2}}$$

Numerický výpočet π

$$s_{2n} = \sqrt{2 - \sqrt{4 - s_n^2}}$$

$$\underline{\pi}_n = n s_n / 2$$

$$S_{2n} = \frac{2S_n}{2 + \sqrt{4 + S_n^2}}$$

$$\overline{\pi}_n = n S_n / 2$$

$$s_6 = 1$$

$$\Rightarrow \quad \underline{\pi}_6 = 3$$

$$s_{12} = \sqrt{2 - \sqrt{4 - 1^2}}$$

$$\approx 0.5176$$

$$\Rightarrow \quad \underline{\pi}_{12} = 3.1058$$

$$s_{24} = \dots$$

$$S_6 = 2\sqrt{3}/3 \approx 1.1547$$

$$\Rightarrow \quad \overline{\pi}_6 = 3.4641$$

$$S_{12} = \frac{2 \cdot 1.1547}{2 + \sqrt{4 + 1.1547^2}}$$

$$\approx 0.5359$$

$$\Rightarrow \quad \overline{\pi}_{12} = 3.2154$$

$$S_{24} = \dots$$

Numerický výpočet $\pi = 3.14159265358979$

krok	n	$\underline{\pi}_n$	$\overline{\pi}_n$
1	6	3.00000000000000	3.46410161513775
2	12	3.10582854123025	3.21539030917347
3	24	3.13262861328124	3.15965994209750
4	48	3.13935020304687	3.14608621513144
5	96	3.14103195089053	3.14271459964537
⋮	⋮	⋮	⋮
13	24576	3.14159264532122	3.14159267070200
14	49152	3.14159264532122	3.14159265786785
15	98304	3.14159264532122	3.14159265465931
16	196608	3.14159264532122	3.14159265385717
17	393216	3.14159366984943	3.14159265365664
⋮	⋮	⋮	⋮
26	201326592	3.35410196624968	3.14159265358980
27	402653184	4.24264068711929	3.14159265358980
28	805306368	6.00000000000000	3.14159265358980
29	1610612736	0.00000000000000	3.14159265358980

Numerický výpočet $\pi = 3.14159265358979$

$$s_{2n} = \sqrt{2 - \sqrt{4 - s_n^2}} \qquad \Rightarrow \quad \underline{\pi}_n = s_n/2$$

$$s_{2n}^2 = 2 - \sqrt{4 - s_n^2}$$

$$s_{2n}^2 = \left(2 - \sqrt{4 - s_n^2}\right) \frac{2 + \sqrt{4 - s_n^2}}{2 + \sqrt{4 - s_n^2}}$$

$$s_{2n}^2 = \frac{4 - (4 - s_n^2)}{2 + \sqrt{4 - s_n^2}}$$

$$s_{2n} = \sqrt{\frac{s_n^2}{2 + \sqrt{4 - s_n^2}}} \qquad \Rightarrow \quad \underline{\pi}_n = s_n/2$$

Numerický výpočet $\pi = 3.14159265358979$

krok	n	$\underline{\pi}_n$	$\overline{\pi}_n$
1	6	3.00000000000000	3.46410161513775
2	12	3.10582854123025	3.21539030917347
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⋮	⋮	⋮	⋮
13	24576	3.14159264503369	3.14159267070200
14	49152	3.14159265145077	3.14159265786785
15	98304	3.14159265305504	3.14159265465931
16	196608	3.14159265345610	3.14159265385717
17	393216	3.14159265355637	3.14159265365664
⋮	⋮	⋮	⋮
26	201326592	3.14159265358979	3.14159265358980
27	402653184	3.14159265358979	3.14159265358980
28	805306368	3.14159265358979	3.14159265358980
29	1610612736	3.14159265358979	3.14159265358980

Děkuji za pozornost

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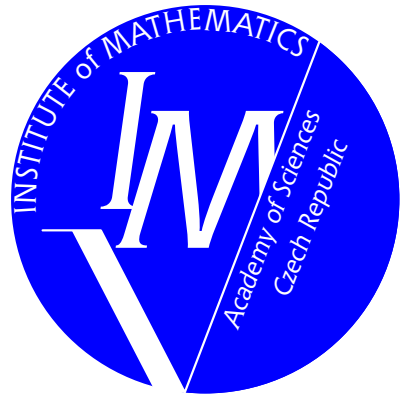

9. listopadu 2007, Praha