# Substitution Frege and extended Frege proof systems in non-classical logics

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**Abstract**

We investigate the substitution Frege (*SF*) proof system and its relationship to extended Frege (*EF*) in the context of modal and superintuitionistic (si) propositional logics. We show that *EF* is p-equivalent to *tree-like SF*, and we develop a "normal form" for *SF*-proofs. We establish connections between *SF* for a logic $L$, and *EF* for certain bimodal expansions of $L$.

We then turn attention to specific families of modal and si logics. We prove p-equivalence of *EF* and *SF* for all extensions of **KB**, all tabular logics, all logics of finite depth and width, and typical examples of logics of finite width and infinite depth. In most cases, we actually show an equivalence with the usual *EF* system for classical logic with respect to a naturally defined translation.

On the other hand, we establish exponential speed-up of *SF* over *EF* for all modal and si logics of infinite branching, extending recent lower bounds by P. Hrubeš. We develop a model-theoretical characterization of maximal logics of infinite branching to prove this result.

## Contents



(**DON'T PANIC.** Sections 4, 5, and 6 are mostly independent of each other.)

# 1 Introduction

The main motivation behind propositional proof complexity theory comes from its connections to fundamental open problems in computational complexity theory: by Cook and Reckhow [8], there exists a polynomially bounded proof system for the classical propositional logic (**CPC**) if and only if $NP = coNP$. The bulk of proof complexity therefore concerns itself with proof systems for **CPC**. From this perspective, proof complexity of *non-classical* logics can be useful by providing broader context to proof complexity questions, which helps to isolate the most intrinsic properties of various proof systems and to develop suitable methods for studying their complexity, as well as by providing variants of notoriously hard problems in classical proof complexity (such as lower bounds for Frege and similar systems) which are potentially easier to solve: non-classical logics are often more complex than **CPC** (e.g., $PSPACE$-complete), thus lower bounds on non-classical proof systems can be thought of as approximations to weaker (hence hopefully more tractable) problems (e.g., $PSPACE \neq NP$) than in the classical case. Anyway, apart from these auxiliary purposes, we believe that proof complexity of non-classical logics is also quite interesting as a subject in its own right: non-classical logics often exhibit features which are either absent or trivialized in the degenerate case of classical logic (such as the disjunction property), and many of these can be recast as proof complexity problems.

Non-classical proof complexity typically deals with Frege systems and equivalent systems like sequent calculi or natural deduction: see e.g. Buss and Mints [5], Buss and Pudlák [6], Ferrari et al. [9], Mints and Kojevnikov [16], Jeřábek [14], Hrubeš [10, 11]. (Though sometimes the number of lines is taken as the basic complexity measure instead of size, which amounts to working with extended Frege in disguise.) In contrast, we will study another two types of proof systems which naturally appear in a non-classical version: *extended Frege* (*EF*) *systems*, which allow introduction of abbreviations for formulas in a proof, and *substitution Frege* (*SF*) *systems*, which admit substitution to be used directly as a rule of inference. In classical logic these two proof systems happen to be polynomially equivalent, but as we will see the situation is much more complicated in non-classical logics. Indeed, the main problem we study in this paper is to find in which modal and superintuitionistic logics the *EF* and *SF* systems are equivalent, in which logics they are not equivalent, and why is it so. We will learn other useful properties of *SF* and *EF* along the way. Notably, we will see that behaviour of logics with respect to issues in *proof complexity* is deeply connected with their *model-theoretic* properties, in particular with parameters such as width, depth, and branching. Moreover, many of our proof manipulations can be interpreted as formalizations of constructive proof-theoretic versions of standard model-theoretic arguments in *EF*.

The paper is organized as follows. We start with extensive preliminaries (Section 2). In Section 3 we mostly deal with easy general properties of *EF* and *SF* where the choice of logic makes little or no difference (e.g., equivalence of tree-like and dag-like Frege, or feasible deduction theorem). We also show that Frege or extended Frege systems for **CPC** and **IPC** are p-equivalent wrt monotone sequents (Theorem 3.9), which improves a result of Atserias et al. [2]. Apart from that, the results worth mentioning are the p-equivalence of *EF* with tree-like *SF* (Theorem 3.12), and most importantly Theorem 3.15, which shows that *SF*-proofs can be brought to a "normal form" consisting of an *EF*-subproof followed (more or less) by substitutions of propositional *constants* arranged in a specific pattern. We also show a linear lower bound on the height of **K**-*F*-proofs, which implies that tree-like Frege (and *SF*) systems can be exponentially sensitive to the choice of their rules (Corollary 3.23).

In Section 4 we show that going from *EF* to *SF* corresponds to strengthening the logic by (conservatively) adding a new modal operator which can "see backwards". More exactly, we prove that *SF* for a logic $L$ is p-simulated by *EF* for a so-called weak tense expansion of $L$ (Theorem 4.7); for transitive logics (including all superintuitionistic), we obtain a nicer result: $L$-*SF* is *p-equivalent* to *EF* for the expansion of $L$ by a universal modality (Theorem 4.12). As a corollary, *EF* and *SF* are p-equivalent in extensions of **KB**.

In Section 5 we construct p-simulations of *SF* in *EF* for several classes of logics. We consider tabular modal and si logics (Theorem 5.10, e.g.: Smetanich logic **SmL**), logics of finite depth and width (Theorem 5.20, e.g.: **S5**), and some logics of finite width and infinite depth: $\mathbf{K4BW}_k$ and its variants (Theorem 5.25, e.g.: **S4.3**), Gödel–Dummett logic **LC** (Theorem 5.26), and with respect to restricted classes of formulas (based on an ad hoc complexity measure), all cofinal subframe logics of finite width (Corollary 5.23). The proofs use a detour via classical logic: given a logic $L$, we consider a poly-time translation of $L$-formulas to classical propositional formulas (essentially, model checking of suitable Kripke $L$-frames), and we show how to transform an $L$-*SF*-proof of a formula to a **CPC**-*SF* = **CPC**-*EF*-proof of its translation, and transform a **CPC**-*EF*-proof of the translation to an $L$-*EF*-proof of the original formula. This implies a result interesting on its own, namely that the *EF* systems for **CPC** and $L$ are for all practical purposes (lower bounds, for instance) equivalent.

We complement these results in Section 6 by showing exponential speed-up of *SF* over *EF* for all modal and si logics of infinite branching (Theorem 6.37). We use (variants of) tautologies introduced by Hrubeš [10, 11], who proved an exponential lower bound on (number of lines in Frege proofs, hence size of) extended Frege proofs for **K**, **IPC**, and some other modal logics. We verify that the tautologies have poly-time constructible *SF*-proofs, and we extend the exponential *EF* lower bound to all logics of infinite branching by reducing it to four special cases, and using the method of propositional valuations from [14] to prove the lower bound for these four cases (in particular, we obtain a simplified proof of the original Hrubeš's results). The reduction (Lemmas 6.30, 6.33) is more generally applicable, it gives e.g. a simulation of Jankov's (De Morgan) logic **KC** in **IPC** wrt negation-free formulas. The model-theoretic part of the reduction (Theorem 6.21) is also of independent interest, it shows that every logic of infinite branching is valid in a frame of a particular shape (thus in a sense, describes the maximal logics of infinite branching). In Section 7 we mention some open problems related to our work.


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## 2 Preliminaries

### 2.1 Modal and superintuitionistic logics

Throughout the paper, we work with (normal mono-) modal and superintuitionistic logics. Generally speaking, we rely on Chagrov and Zakharyaschev [7] as the canonical source for the theory of such logics (the reader may also consult Blackburn et al. [3]). Nevertheless, as the paper is primarily targeted on proof complexity audience rather than modal logicians, we tried to keep the definitions below as self-contained as possible.

Propositional *modal formulas* are built from propositional variables (atoms), a fixed complete set of Boolean connectives (say, $\to, \wedge, \vee, \bot$), and an additional unary connective denoted by $\Box$. (The formula $\Box\varphi$ does not necessarily mean "necessarily $\varphi$"; depending on the logic, it could as well mean "$\varphi$ is obligatory", "$\varphi$ is provable", "$\varphi$ will always hold", etc.) We will usually denote propositional variables by lower-case Latin letters $p$, $q$, $\dots$, and formulas by lower-case Greek letters $\varphi$, $\psi$, $\dots$ (though we will also use capital Latin letters). We will write $\Diamond\varphi$, $\boxdot\varphi$, $\Diamonddot\varphi$, $\Box^n\varphi$, and $\Box^{\le n}\varphi$ for $\neg\Box\neg\varphi$, $\varphi\land\Box\varphi$, $\varphi\lor\Diamond\varphi$, $\Box\cdots\Box\varphi$ (with $n$ boxes), and $\bigwedge_{k\le n}\Box^k\varphi$ respectively. The *modal degree* of a formula $\varphi$, written as $\mathrm{md}(\varphi)$, is the maximal number of nested $\Box$ in $\varphi$. A set $L$ of modal formulas is called a *normal modal logic* (*nml*) if it contains all classical propositional tautologies, the Kripke axiom

$$\text{(K)}\qquad \Box(\varphi\to\psi)\to(\Box\varphi\to\Box\psi),$$

and it is closed under substitution, the detachment (modus ponens) rule

$$\text{(MP)}\qquad \varphi,\varphi\to\psi\vdash\psi,$$

and the necessitation rule

$$\text{(Nec)}\qquad \varphi\vdash\Box\varphi.$$

The smallest nml is called $\mathbf{K}$. If $L$ is a logic, and $X$ a formula or a set of formulas, then

$$L\oplus X$$

is the smallest nml which includes $L$ and $X$. A logic which can be written as $\mathbf{K}\oplus X$ for some finite set $X$ is called *finitely axiomatizable*. A formula $\varphi$ is *provable* in a nml $L$ from a set of assumptions $X$ (written as $X\vdash_L\varphi$), if $\varphi$ is in the closure of $L\cup X$ under MP and Nec. Formulas $\varphi$ such that $\varnothing\vdash_L\varphi$ (i.e., $\varphi\in L$) are called *$L$-tautologies*. Some modal axioms are listed in Table 1, and some of the most common nml are given in Table 2. (The names of the *logics* are mostly standard, whereas naming of *axioms* varies wildly in the literature.) We stress that these are but a few examples; the lattice of normal modal logics has the cardinality of continuum, a quite complex structure of which only several corners are understood, and it does not admit any classification by a transparent list of axioms.

*Intuitionistic formulas* are generated from propositional variables by means of the connectives $\to$, $\land$, $\lor$, and $\bot$. (The choice matters: unlike classical logic, the connectives are not interdefinable in the intuitionistic logic.) Negation is defined as the abbreviation $\neg\varphi=(\varphi\to\bot)$. A set of intuitionistic formulas is called a *superintuitionistic* (*si*) *logic*, if it is closed under substitution and MP, and contains the axioms

$$\begin{gathered}
(\varphi\to(\psi\to\chi))\to((\varphi\to\psi)\to(\varphi\to\chi)),\\
\varphi\to(\psi\to\varphi),\\
\bot\to\varphi,\\
\varphi_1\to(\varphi_2\to\varphi_1\land\varphi_2),\\
\varphi_1\land\varphi_2\to\varphi_i\qquad(i=1,2),\\
\varphi_i\to\varphi_1\lor\varphi_2\qquad(i=1,2),\\
(\varphi_1\to\psi)\to((\varphi_2\to\psi)\to(\varphi_1\lor\varphi_2\to\psi)).
\end{gathered}$$

| symbol | axiom | Kripke frame condition |
|---|---|---|
| T | $\Box p \to p$ | reflexive |
| 4 | $\Box p \to \Box\Box p$ | transitive |
| B | $p \to \Box\Diamond p$ | symmetric |
| D | $\Diamond\top$ | serial ($\forall x\, x{\uparrow} \neq \varnothing$) |
| GL | $\Box(\Box p \to p) \to \Box p$ | transitive, $R^{-1}$ well-founded |
| Grz | $\Box(\Box(p \to \Box p) \to p) \to \Box p$ | $(R \setminus \mathrm{id})^{-1}$ well-founded† |
| 5 | $\Diamond p \to \Box\Diamond p$ | Euclidean ($x\ R\ y, z \Rightarrow y\ R\ z$) |
| .1 | $\boxdot\Diamond p \to \Diamonddot\Box p$ | $\forall x\, \exists y \in x\text{↥}\, (y\text{↥} = \{y\})$† |
| .2 | $\Diamond\boxdot p \to \Box\Diamonddot p$ | directed ($y, z \in x{\uparrow} \Rightarrow y \parallel z$) |
| .3 | $\Box(\Box p \to q) \vee \Box(\boxdot q \to p)$ | connected |
| | | ($y, z \in x{\uparrow} \Rightarrow y\ R\ z \vee y = z \vee z\ R\ y$) |
| $\mathrm{Tra}_k$ | $\Box^{\leq k} p \to \Box^{k+1} p$ | $k$-transitive |
| $\mathrm{Alt}_k$ | $\displaystyle\bigvee_{i=0}^{k} \Box\Big(\bigwedge_{j<i} p_j \to p_i\Big)$ | $\forall x\, \lvert x{\uparrow}\rvert \leq k$ |
| $\mathrm{BD}_k$ | see below | depth at most $k$† |
| $\mathrm{BW}_k$ | $\displaystyle\bigvee_{i=0}^{k} \Box\Big(\bigwedge_{j\neq i} \boxdot p_j \to p_i\Big)$ | width at most $k$† |
| $\mathrm{BB}_k$ | see Remark 6.11 | branching at most $k$†‡ |

† This frame condition is only valid for transitive frames.

‡ This frame condition is only valid for finite frames.

Table 1: Assorted modal axioms

| name | axiomatization |
|---|---|
| **T** | **K** $\oplus$ T |
| **K4** | **K** $\oplus$ 4 |
| **S4** | **K4** $\oplus$ T |
| **GL**, **K4W** | **K** $\oplus$ GL = **K4** $\oplus$ GL |
| **K4Grz** | **K4** $\oplus$ Grz |
| **Grz**, **S4Grz** | **T** $\oplus$ Grz = **S4** $\oplus$ Grz |
| **KB** | **K** $\oplus$ B |
| **KTB** | **T** $\oplus$ B |
| **K5** | **K** $\oplus$ 5 |
| **K45** | **K4** $\oplus$ 5 |
| **K4B** | **K4** $\oplus$ B |
| **S5** | **T** $\oplus$ 5 = **K4** $\oplus$ B $\oplus$ D |

| name | axiomatization |
|---|---|
| **D** | **K** $\oplus$ D |
| **D4** | **K4** $\oplus$ D |
| **D45** | **D4** $\oplus$ 5 |
| **K4.1** | **K4** $\oplus$ .1 |
| **K4.2** | **K4** $\oplus$ .2 |
| **K4.3** | **K4** $\oplus$ .3 |
| **S4.1** | **S4** $\oplus$ .1 |
| **S4.2** | **S4** $\oplus$ .2 |
| **S4.3** | **S4** $\oplus$ .3 |
| **GL.3** | **GL** $\oplus$ .3 |
| $\mathbf{K4BD}_k$ | **K4** $\oplus$ $\mathrm{BD}_k$ |
| $\mathbf{K4BW}_k$ | **K4** $\oplus$ $\mathrm{BW}_k$ |
| $\mathbf{K4BB}_k$ | **K4** $\oplus$ $\mathrm{BB}_k$ |

Table 2: Popular (or otherwise important) normal modal logics

| name | axiomatization | Kripke frame condition |
|---|---|---|
| **CPC** | $\mathbf{IPC} + p \vee \neg p = \mathbf{IPC} + \neg\neg p \to p$ $= \mathbf{IPC} + ((p \to q) \to p) \to p$ | discrete |
| **KC** | $\mathbf{IPC} + \neg p \vee \neg\neg p$ | directed |
| **LC** | $\mathbf{IPC} + (p \to q) \vee (q \to p)$ | connected |
| **KP** | $\mathbf{IPC} + (\neg p \to q \vee r) \to (\neg p \to q) \vee (\neg p \to r)$ | § |
| **SmL** | $\mathbf{IPC} + (\neg q \to p) \to (((p \to q) \to p) \to p)$ | $\forall x\, \lvert x{\uparrow}\rvert \le 2$ |
| $\mathbf{Alt}_k$ | $\mathbf{IPC} + \bigvee_{i=0}^{k} \Big(\bigwedge_{j<i} p_j \to p_i\Big)$ | $\forall x\, \lvert x{\uparrow}\rvert \le k$ |
| $\mathbf{BD}_k$ | see below | depth at most $k$ |
| $\mathbf{BW}_k$ | $\mathbf{IPC} + \bigvee_{i=0}^{k} \Big(\bigwedge_{j\neq i} p_j \to p_i\Big)$ | width at most $k$ |
| $\mathbf{T}_k$ | $\mathbf{IPC} + \bigwedge_{i=0}^{k} \Big(\Big(p_i \to \bigvee_{j\neq i} p_j\Big) \to \bigvee_{j} p_j\Big) \to \bigvee_{i} p_i$ | branching at most $k^{\ddagger}$ |

‡ This frame condition is only valid for finite frames.

§ $y, z \in x{\uparrow} \Rightarrow \exists w \in x{\uparrow}\, (w \le y, z \wedge \forall u \in w{\uparrow}\, (u \parallel y \vee u \parallel z))$

Table 3: Some superintuitionistic logics

The smallest si logic is called the *intuitionistic logic*, and we will denoted it **IPC**. The smallest si logic which contains a logic $L$, and a formula or a set of formulas $X$, is denoted $L + X$. Some si logics are listed in Table 3. Every consistent si logic is contained in the classical logic (**CPC**); for this reason the si logics are also called *intermediate logics*.

An intuitionistic formula $\varphi$ is *essentially negative* if every occurrence of a variable in $\varphi$ is in a scope of some $\neg$. An essentially negative formula is called *negative* if every occurrence of $\vee$ in $\varphi$ is in a scope of some $\neg$ as well. Formulas built from variables, $\wedge$, $\vee$, $\bot$, and $\top$ are called *monotone*.

A (*modal*) *Kripke frame* is a pair $\langle W, R\rangle$, where $R$ is a binary relation on the set $W$ (called the *accessibility relation*). A *valuation* (or *assignment*) in $W$ is a relation $\Vdash$ between elements of $W$, and propositional variables. We can extend any assignment to all modal formulas by evaluating the Boolean connectives locally, and putting

$$x \Vdash \Box\varphi \quad \text{iff} \quad \forall y\, (x\ R\ y \Rightarrow y \Vdash \varphi).$$

The triple $\langle W, R, \Vdash\rangle$ is called a *Kripke model*.

An *intuitionistic Kripke frame* is a Kripke frame $\langle X, \le\rangle$ such that $\le$ is a partial order (i.e., a transitive, reflexive and antisymmetric relation). Intuitionistic valuations $\Vdash$ in intuitionistic frames are required to satisfy the *persistence condition*

$$x \le y \wedge x \Vdash p \Rightarrow y \Vdash p,$$

and they are extended to all intuitionistic formulas by

$$\begin{aligned}
&x \Vdash \varphi \wedge \psi \quad \text{iff} \quad x \Vdash \varphi \wedge x \Vdash \psi,\\
&x \Vdash \varphi \vee \psi \quad \text{iff} \quad x \Vdash \varphi \vee x \Vdash \psi,\\
&x \Vdash \varphi \to \psi \quad \text{iff} \quad \forall y \ge x\,(y \Vdash \varphi \Rightarrow y \Vdash \psi),\\
&x \nVdash \bot.
\end{aligned}$$

Let $\Vdash$ be a valuation in a modal or intuitionistic Kripke frame $W = \langle W, R\rangle$. If $x \Vdash \varphi$, we say that the formula $\varphi$ is *satisfied* by $\Vdash$ in $x$, otherwise it is *refuted*. A formula $\varphi$ is *valid* in the frame $W$, if it is satisfied by every valuation in every point of $W$. The set of all formulas valid in a frame $W$ is always a normal modal (resp., superintuitionistic) logic, which we call the *logic of* $W$, and denote $L(W)$. If $L \subseteq L(W)$, we say that $W$ is an *$L$-frame*, and we write $W \vDash L$.

If $\langle W, R\rangle$ is a frame, and $X \subseteq W$, we define

$$\begin{aligned}
X{\uparrow} &:= R[X] = \{y;\ \exists x \in X\ x\ R\ y\},\\
X{\upharpoonleft} &:= X \cup X{\uparrow},\\
X{\downarrow} &:= R^{-1}[X] = \{y;\ \exists x \in X\ y\ R\ x\},\\
X{\downharpoonleft} &:= X \cup X{\downarrow}.
\end{aligned}$$

We also write $x{\uparrow} := \{x\}{\uparrow}$, etc. If $W' \subseteq W$, the frame $\langle W', R \cap (W' \times W')\rangle$ is called a *subframe* of $\langle W, R\rangle$. If additionally $W'{\uparrow} \subseteq W'$, it is a *generated subframe*, which we write as $W' \subseteq\cdot\, W$. Formulas valid in $W$ are also valid in all its generated subframes. Another frame operations which preserve validity are *disjoint unions* (defined in the obvious way), and p-morphic images: if $\langle W, R\rangle$ and $\langle W', R'\rangle$ are frames, a mapping $f\colon W \to W'$ is a *p-morphism* (aka *bounded morphism*, *reduction*, *pseudo-epimorphism*) if it is surjective, and satisfies the conditions

(i) $x\ R\ y \Rightarrow f(x)\ R'\ f(y)$,

(ii) if $f(x)\ R'\ u$, there exists $y$ such that $x\ R\ y$ and $f(y) = u$

for every $x, y \in W$, $u \in W'$.

A point $x \in W$ is *reflexive* if $x\ R\ x$, otherwise it is *irreflexive*. Points $x$ and $y$ are *compatible* (written $x \parallel y$), if $x{\upharpoonleft} \cap y{\upharpoonleft} \neq \varnothing$.

Let $R^n$ be the $n$-fold composition of $R$ with itself (where $R^0 = \mathrm{id}$), and let $R^* = \bigcup_{n\in\omega} R^n$ be the reflexive and transitive closure of $R$. The frame $W$ is *$k$-transitive*, if $R^* = (R \cup \mathrm{id})^k$, or equivalently, $R^{k+1} \subseteq \bigcup_{n\le k} R^n$. (Usual transitivity is somewhat stronger than 1-transitivity.) A point $r \in W$ is a *root* of $W$, if $W = R^*[r]$. A frame with at least one root is called *rooted*.

Let $\langle W, R\rangle$ be a transitive Kripke frame. The accessibility relation induces a preorder $\le\ = R \cup \mathrm{id}$, an equivalence relation $\sim\ =\ \le \cap \le^{-1}$, and a strict order $<\ = R \smallsetminus R^{-1} =\ \le \smallsetminus \sim$. The equivalence classes of $\sim$ are called *clusters*. A cluster is *proper* if it contains at least two points, otherwise it is *simple*. Notice that every proper cluster is reflexive. A point or cluster is *final* if it is $<$-maximal. Final points are also called *leaves*. The quotient structure

$\varrho W := \langle W, R\rangle/{\sim}$ is an antisymmetric transitive frame, called the *skeleton* of $W$. The *depth* of $W$ is the maximal size of a finite $<$-chain in $W$ if it exists, and $\infty$ otherwise. If $W$ is rooted, its *width* is similarly defined as the maximal size of a finite $\le$-antichain in $W$, or $\infty$ if there is no maximum. In general, the width of $W$ is defined as the supremum of widths of its rooted generated subframes. Notice that $\varrho W$ has the same depth and width as $W$. A point $y$ is an *immediate successor* of $x$, if $x < y$, and there does not exist any $z$ such that $x < z < y$. If $W$ is finite and antisymmetric, then the *branching* of $W$ is the maximal number of immediate successors of any point in $W$. If $W$ is not antisymmetric (but still finite and transitive), we define its branching as the branching of $\varrho W$.

An axiom or logic $L$ *characterizes* a frame condition (i.e., a class of frames) $\mathcal{C}$, if

$$W \in \mathcal{C} \quad \text{iff} \quad W \vDash L$$

for every Kripke frame $W$. The characteristic frame conditions of some modal and si axioms are given in Tables 1 and 3. A logic $L$ is *complete* wrt a class of frames $\mathcal{C}$, if

$$\varphi \in L \quad \text{iff} \quad \forall W\,(W \in \mathcal{C} \Rightarrow W \vDash \varphi).$$

$L$ is *strongly complete* wrt $\mathcal{C}$, if moreover every $L$-consistent set of formulas is satisfiable in a point of a model based on a frame from $\mathcal{C}$. $L$ is *Kripke complete*, if it is complete wrt some class $\mathcal{C}$. Notice that $L$ is Kripke complete iff it is complete wrt the class of Kripke frames it characterizes. $L$ has the *finite model property*, if it is complete wrt a class of finite frames. All logics in Tables 2 and 3 have the finite model property. Notice that the frame conditions of many of these logics can be simplified on finite frames: e.g., a transitive finite frame is converse well-founded iff it is irreflexive (i.e., a strict partial order).

There exist Kripke incomplete modal and si logics, and many Kripke complete logics (e.g., **GL**, **Grz**, $\mathbf{T}_k$) are not strongly complete, thus a more general semantics is needed. A (*modal*) *general frame* is a triple $\langle W, R, V\rangle$, where $\langle W, R\rangle$ is a Kripke frame, and $V$ is a family of subsets of $W$ which is closed under Boolean operations (complement, binary intersection), and under the operation

$$\Box A := \{x \in W;\ \forall y \in W\,(x\ R\ y \Rightarrow y \in A)\} = W \smallsetminus (W \smallsetminus A){\downarrow}.$$

We can identify a Kripke frame $\langle W, R\rangle$ with the general frame $\langle W, R, \mathcal{P}(W)\rangle$.

An *intuitionistic general frame* is a triple $\langle W, \le, V\rangle$, where $\langle W, \le\rangle$ is an intuitionistic Kripke frame, and $V$ is a family of upper subsets of $W$ which contains $\varnothing$, and is closed under (binary) intersection, union, and the operation

$$A \to B := \{x \in W;\ \forall y \in W\,(x \le y \wedge y \in A \Rightarrow y \in B)\} = W \smallsetminus (A \smallsetminus B){\downarrow}.$$

A valuation $\Vdash$ is *admissible* in a (modal or intuitionistic) general frame $\langle W, R, V\rangle$, if

$$\{x \in W;\ x \Vdash \varphi\} \in V$$

for every formula $\varphi$, or equivalently, for every propositional variable. A formula $\varphi$ is *valid* in $\langle W, R, V\rangle$ if it holds under every admissible valuation. A frame is *refined* if

$$\forall A \in V\,(x \in A \Leftrightarrow y \in A) \Rightarrow x = y,$$
$$\forall A \in V\,(x \in \Box A \Rightarrow y \in A) \Rightarrow x\ R\ y$$

for every $x, y \in W$ (which is equivalent to

$$\forall A \in V\,(x \in A \Rightarrow y \in A) \Rightarrow x \leq y$$

in the intuitionistic case). A frame is *compact* if every subset of $V$ (or $V \cup \{W \setminus A;\ A \in V\}$ in the intuitionistic case) with the finite intersection property (fip) has a nonempty intersection. A refined compact frame is called *descriptive*. (Kripke frames are refined, but infinite Kripke frames are never descriptive. All finite refined frames are Kripke.)

Every normal modal or si logic is strongly complete wrt a class of descriptive frames.

A frame $\langle W', R', V' \rangle$ is a *subframe* of $\langle W, R, V \rangle$, if $\langle W', R' \rangle$ is a subframe of $\langle W, R \rangle$, and $V' \subseteq V$ (which implies $W' \in V$). A *generated subframe* of $\langle W, R, V \rangle$ is a frame $\langle W', R', V' \rangle$ such that $\langle W', R' \rangle$ is a generated subframe of $\langle W, R \rangle$, and $V' = \{A \cap W';\ A \in V\}$. Notice that unlike the Kripke case, a generated subframe is not necessarily a subframe. A *p-morphism* $f \colon \langle W, R, V \rangle \to \langle W', R', V' \rangle$ is a Kripke p-morphism $f \colon \langle W, R \rangle \to \langle W', R' \rangle$ such that

$$A \in V' \Rightarrow f^{-1}[A] \in V$$

for every $A \subseteq W'$.

A modal logic $L$ is *reflexive*, *transitive*, or *$k$-transitive* if all refined $L$-frames have the same property; this is equivalent to $L \supseteq \mathbf{T}$, $L \supseteq \mathbf{K4}$, and $L \ni \mathrm{Tra}_k$ respectively. All si logics are also considered transitive. Logics which are $k$-transitive for some $k$ (including all si logics) are called *weakly transitive*.

Let $L$ be a transitive logic, and $k \in \omega$. $L$ has *depth $k$* (*width $k$*), if all refined $L$-frames have depth (width, resp.) at most $k$. If $L$ has depth (width) $k$ for some $k$, then it has *finite* (or: *bounded*) *depth* (*width*, resp.). $L$ has width $k$ if and only if it proves the axiom $\mathrm{BW}_k$ given in Tables 1, 3. $L$ has bounded depth iff it proves the axiom $\mathrm{BD}_k$, which is defined by

$$\begin{aligned} \mathrm{BD}_0 &:= \bot, \\ \mathrm{BD}_{k+1} &:= p_k \vee \Box(\Box p_k \to \mathrm{BD}_k) \end{aligned}$$

in the modal case; the intuitionistic version is obtained by deleting all boxes. *Segerberg's theorem* [17] states that every logic of finite depth has the finite model property.

The si logic $\mathbf{T}_k$ is defined as the set of formulas valid in all finite frames of branching at most $k$; it can be axiomatized as in Table 3. (The letter T stands for "tree", as the logic is complete wrt finite $k$-ary trees.) We say that a si logic $L$ has *branching $k$* if $L \supseteq \mathbf{T}_k$, and all such logics are said to have *finite* (or *bounded*) *branching*. The bounded branching logics $\mathbf{T}_k$ are usually defined only in the si case, however we will find it useful to study also the corresponding modal logics. We thus introduce $\mathbf{K4BB}_k$ as the logic of all finite (transitive) frames of branching at most $k$, and we call extensions of $\mathbf{K4BB}_k$ the logics of branching $k$. As we will show in Lemma 6.10, the logics $\mathbf{K4BB}_k$ are finitely axiomatizable, which justifies the appearance of an "axiom" $\mathrm{BB}_k$ in Table 1.

We have to define logics of branching $k$ in the indirect way as above, because the concept of finite branching is not well-behaved (or even well-defined) for infinite frames. For example, the full infinite binary tree of height $\omega$ appears to have branching 2, but its (si) logic is just

**IPC**, which has infinite branching. Cf. also Remark 6.11. Nevertheless, if $L$ is complete wrt a class $\mathcal{C}$ of *finite* transitive frames, then $L$ has branching $k$ iff all frames in $\mathcal{C}$ have branching at most $k$, as expected. Branching is related to width: logics of width $k$ also have branching $k$, and all logics of finite depth and finite branching have finite width. Also, cofinal subframe logics of branching $k$ have width $k$.

A modal or si logic is *tabular*, if there exists a finite Kripke frame $F$ such that $L = L(F)$. A modal logic is tabular iff it proves $\mathrm{Alt}_k \wedge \mathrm{Tra}_k$ for some $k$. A si logic is tabular iff it proves some $\mathrm{Alt}_k$ iff it has finite depth and finite width. A logic has the finite modal property iff it is an intersection of a family of tabular logics.

A transitive logic $L$ is called a *subframe* (*sf*) *logic*, if it is complete wrt a class of general frames closed under subframes. More generally, $L$ is a *cofinal subframe* (*csf*) *logic*, if it is complete wrt a class of general frames closed under cofinal subframes, where a subframe $\langle W', R', V' \rangle$ of a transitive frame $\langle W, R, V \rangle$ is called *cofinal* if $W'{\uparrow} \subseteq W'{\downarrow}$. An important feature of csf logics, proved by Zakharyaschev [18], is that all of them have the finite model property. Most of the standard modal and si logics are csf (e.g., all transitive logics in Tables 2, 3, save $\mathbf{K4BB}_k$, $\mathbf{T}_k$, and **KP**, are csf). A si logic is csf if and only if it can be axiomatized over **IPC** by $\vee$-free formulas.

Let $\vec{p}$, $\vec{q}$, and $\vec{r}$ be pairwise disjoint sequences of variables, and $L$ a modal or si logic. An *interpolant* of an $L$-tautology

$$\alpha(\vec{p}, \vec{q}) \to \beta(\vec{p}, \vec{r})$$

is a formula (or circuit) $I(\vec{p})$ such that $L$ proves

$$\alpha(\vec{p}, \vec{q}) \to I(\vec{p}),$$
$$I(\vec{p}) \to \beta(\vec{p}, \vec{r}).$$

The class of normal modal logics can be generalized in various ways, we will occasionally need the following.

A *quasi-normal modal logic* is a set of modal formulas extending **K**, which is closed under substitution and modus ponens (but not necessarily under necessitation). Every normal modal logic is quasi-normal. The smallest quasi-normal logic which contains a logic $L$ and a formula $\varphi$ will be denoted by

$$L + \varphi$$

to distinguish it from the normal closure $L \oplus \varphi$. If $W$ is a general frame, and $x$ is a distinguished point of $W$, then the set of formulas valid in $x$ under every admissible valuation is a quasi-normal logic, which we will denote by $L_{qn}(W, x)$. If $W$ is a rooted frame, we will implicitly assume that the distinguished point is a root of $W$, and write just $L_{qn}(W)$.

A (*normal*) *polymodal* logic is a logic in a language with several modal operators $\Box_1$, $\dots$, $\Box_k$, each of them obeying the Kripke axiom and the necessitation rule. The semantics of polymodal logics can be developed similarly to the monomodal case, using $k$ accessibility relations $R_1$, $\dots$, $R_k$. As a special case, polymodal logics with two boxes are called *bimodal*.

We will only target on superintuitionistic and normal monomodal logics (which we will call just “modal logics”), the other will be used for auxiliary purposes.

## 2.2 Propositional proof systems

We will need the following basic concepts from propositional proof complexity. More background can be found in Krajíček [15].

Let $L$ (a "logic") be a set of strings ("formulas") in a finite alphabet. A *proof system* for $L$ is a polynomial-time function $P$ such that $\mathrm{rng}(P) = L$ (Cook, Reckhow [8]). A $P$*-proof* of a formula $\varphi$ is any $\pi$ such that $P(\pi) = \varphi$.

A proof system $P_1$ *p-simulates* a proof system $P_2$, written as $P_2 \le_p P_1$, if there exists a poly-time function $f$ such that $P_2 = P_1 \circ f$. We will use this definition even when $P_1$ and $P_2$ are proof systems for different logics $L_1$, $L_2$; notice however that $P_2 \le_p P_1$ implies $L_2 \subseteq L_1$. (To avoid confusion, we will not apply the term "simulation" to yet more general interpretations which also involve translation of formulas.) Proof systems $P_1$ and $P_2$ (necessarily for the same logic) are *p-equivalent*, written as $P_1 \equiv_p P_2$, if $P_1 \le_p P_2$ and $P_2 \le_p P_1$. We introduce a restricted variant of p-simulation as follows: if $\Gamma$ is a set of formulas, then $P_1$ *p-simulates* $P_2$ *wrt* $\Gamma$, written as $P_2 \le_{p,\Gamma} P_1$, if there exists a poly-time function $f$ such that $P_2(\pi) = P_1(f(\pi))$ whenever $P_2(\pi) \in \Gamma$.

If $P$ is a proof system for $L$, and $\varphi$ is an $L$-tautology, we define the basic complexity measure

$$s_P(\varphi) := \min\{|\pi|;\, P(\pi) = \varphi\},$$

the minimal *size* of a $P$-proof of $\varphi$. A proof system $P_1$ *simulates* $P_2$, written as $P_2 \le P_1$, if there exists a polynomial $p(n)$ such that

$$s_{P_1}(\varphi) \le p(s_{P_2}(\varphi))$$

for every tautology $\varphi$. Notice that $P_2 \le_p P_1$ implies $P_2 \le P_1$.

Let $L$ be a modal or si logic. An *inference system* $P$ is given by a set of rules of the form $\varphi_1, \dots, \varphi_m \vdash \varphi_0$, where $\varphi_i$ are formulas. A $P$*-proof* of a formula $\varphi$ from a set $X$ of assumptions is a sequence $\pi$ of formulas $\varphi_1, \dots, \varphi_k$ such that $\varphi_k = \varphi$, and each $\varphi_i$ belongs to $X$, or is inferred from some of the formulas $\varphi_j$, $j < i$, by a substitution instance of a $P$-rule. The formulas $\varphi_i$ are called the *lines* (or *steps*) of $\pi$. We write $\varphi_1, \dots, \varphi_m \vdash_P \varphi$ if there exists a $P$-proof of $\varphi$ from assumptions $\{\varphi_1, \dots, \varphi_m\}$. An inference system $P$ is called a *Frege system* for $L$, if

(i) $P$ uses only finitely many rules,

(ii) $P$ is sound: if $\vdash_P \varphi$, then $\varphi \in L$,

(iii) $P$ is strongly complete: if $\varphi_1, \dots, \varphi_m \vdash_L \varphi$, then $\varphi_1, \dots, \varphi_m \vdash_P \varphi$.

A Frege system $P$ is *standard*, if

(ii') $P$ is strongly sound: if $\varphi_1, \dots, \varphi_m \vdash_P \varphi$, then $\varphi_1, \dots, \varphi_m \vdash_L \varphi$.

While the study of nonstandard Frege systems is an interesting topic closely related to investigation of admissible rules of inference (cf. [16, 14]), it is rather tangential to the purpose of this paper. We therefore adopt the convention that *all Frege systems are assumed to be standard*. An immediate consequence (Lemma 3.1) is that all Frege systems for the same logic $L$ are p-equivalent. We will thus speak of *the* Frege system for $L$, and denote it $L$-$F$. In particular, we may assume that $L$-$F$ is given by a finite set of axioms (i.e., rules with no assumptions), and the rules of modus ponens and (in the modal case) necessitation. However, we will also consider standard Frege systems with more general sets of rules to allow ourselves more flexibility. Notice that a (standard) Frege system for a logic $L$ exists if and only if $L$ is finitely axiomatizable. We thus introduce another convention: *whenever we speak about proof systems for a logic L, we tacitly assume L to be finitely axiomatizable.*

A *substitution Frege proof* of $\varphi$ is a sequence of formulas $\varphi_1, \dots, \varphi_k = \varphi$ such that each $\varphi_i$ is derived by a Frege rule, or it is obtained from some $\varphi_j$, $j < i$, by simultaneous substitution of some formulas for variables. We denote the substitution Frege system for $L$ by $L$-$SF$. Notice that the substitution rule is unsound in the presence of non-tautological assumptions; we thus only define $L$-$SF$-proofs of formulas with no assumptions.

An *extended* (or *extension*) *Frege proof* of $\varphi$ is a sequence of formulas $\varphi_1, \dots, \varphi_k = \varphi$ such that each $\varphi_i$ is inferred by a Frege rule, or it is an *extension axiom*

$$q_i \leftrightarrow \psi_i,$$

where $q_i$ (called an *extension variable*) is a variable which does not occur in $\psi_i$, in $\varphi_j$ for $j < i$, or in $\varphi$. The extended Frege proof system for $L$ is denoted by $L$-$EF$.

A *circuit* is a directed acyclic graph (dag) with one node of out-degree 0, whose nodes are labelled by variables or connectives of the same arity as the in-degree of the node. (Formulas can be identified with circuits whose underlying graph is a tree.) Circuits which represent the same formula are called *similar*. (Similarity of circuits is recognizable in polynomial time, or even non-deterministic logarithmic space.) A *circuit*[1] *Frege proof* of $\varphi$ is a sequence of circuits $\varphi_1, \dots, \varphi_k = \varphi$ such that each $\varphi_i$ is inferred by a Frege rule, or it is similar to some $\varphi_j$, $j < i$. We denote the circuit Frege system for $L$ by $L$-$CF$. (This treatment of $CF$ follows [12].) The circuit Frege system is p-equivalent to $EF$ (Proposition 3.3), and we will not usually distinguish them. We will often find it more convenient to work with $CF$ in proofs, however we will formulate theorems using $EF$, as it is the more standard of the two proof systems.

If $P$ is a Frege, extended Frege, circuit Frege, or substitution Frege proof system, and $\varphi$ a tautology, we denote by $k_P(\varphi)$ the minimal number of lines of a $P$-proof of $\varphi$.

Let $P$ be a Frege, circuit Frege, or substitution Frege system, formulated using axioms, MP, and Nec. A $P$-proof is called *tree-like*, if each formula in the proof is used at most once as an assumption to a rule. We denote the tree-like version of $P$ by $P^*$.

## 3 General properties

We begin with a review of simple properties of Frege and similar systems. We will often skip the proofs where they are identical to the classical case. The first one is an elementary but

[1] The classical circuit Frege system may be also called $P/poly$-Frege (or simply $P$-Frege). This is inappropriate for non-classical logics.

very important observation, originally due to Cook and Reckhow [8].

**Lemma 3.1** *If $L' \supseteq L$, then any (extended, circuit, substitution) Frege system for $L'$ p-simulates any (extended, circuit, substitution) Frege system for $L$. In particular, any two (extended, circuit, substitution) Frege systems for $L$ are p-equivalent.*

*Proof:* We replace instances of a Frege rule by instances of its fixed derivation in the other proof system. $\square$

We will use Lemma 3.1 extensively throughout the paper without explicit mention, usually in the following form: substitution instances of any fixed $L$-tautology have uniform linear-size $L$-$F$-proofs.

**Proposition 3.2** *Let $L$ be a modal or si logic, and $\varphi$ an $L$-tautology. Then*

$$k_{L\text{-}EF}(\varphi) \le k_{L\text{-}F}(\varphi) = k_{L\text{-}CF}(\varphi) \le O(k_{L\text{-}EF}(\varphi)).$$

*Proof:* An $L$-$F$-proof of $\varphi$ is also an $L$-$EF$-proof and an $L$-$CF$-proof. Given an $L$-$EF$-proof of $\varphi$, we can construct an $L$-$F$-proof by substitution of their defining formulas for the extension variables, and supplying fixed-length derivations of the formulas $\psi \leftrightarrow \psi$ resulting from the extension axioms. Given an $L$-$CF$-proof of $\varphi$, we replace all circuits with equivalent formulas to obtain an $L$-$F$-proof. $\square$

**Proposition 3.3** *If $L$ is a modal or si logic, then $L$-$EF \equiv_p L$-$CF$, and*

$$s_{L\text{-}EF}(\varphi) = O(k_{L\text{-}EF}(\varphi) + |\varphi|^2).$$

*Proof:* As in the classical logic, see [15, L. 4.5.7], [12]. $\square$

After Proposition 3.3, we generally will not hesitate to confuse *EF* and *CF*. Another consequence of 3.2 and 3.3 is that the size of *EF*-proofs is roughly the same measure as the number of lines in $F$-proofs. We could indeed formulate most of our results in terms of lines in Frege systems, and avoid *EF* altogether, but there are at least two good reasons not to do that. For one, it allows to formulate the results in a stronger form, as the concept of p-simulation does not make sense for line counting. The second reason is practical: we can usually recognize a polynomial-size object (*EF*-proof) instantly, whereas it is not always obvious that a particular parameter of an exponential-size object ($F$-proof) is in fact polynomial.

**Definition 3.4** Let $L$ be a modal logic, and $P$ a Frege, circuit Frege, or substitution Frege system for $L$ using axioms, modus ponens, and necessitation. The proof system $P_{-}$ is defined similarly to $P$, except that it does not use the necessitation rule, and it includes extra axioms of the form

$$\Box^k(\varphi \to \psi) \to (\Box^k\varphi \to \Box^k\psi),$$

and

$$\Box^k\alpha,$$

where $\alpha$ is (an instance of) an axiom of $P$, and $k \in \omega$.

**Lemma 3.5** *Let $L$ be a modal logic, and let $P$ be $L$-$F$, $L$-$CF$, or $L$-$SF$. Then $P \equiv_p P_-$.*

*Proof:* It is easy to see that the extra axioms of $P_-$ have poly-time constructible proofs in $L$-$F$, thus $P_- \leq_p P$. Let $\pi\colon \varphi_0, \dots, \varphi_n = \varphi$ be a $P$-proof. We construct the sequence $\Box^{\leq n}\varphi_0, \Box^{\leq(n-1)}\varphi_1, \dots, \boxdot\varphi_{n-1}, \varphi_n$, and complete it to a $P_-$-proof. For example, if $\varphi_i$ was derived in $\pi$ by modus ponens from $\varphi_j$ and $\varphi_k = (\varphi_j \to \varphi_i)$, we use $\Box^{\leq(n-j)}\varphi_j$, $\Box^{\leq(n-k)}(\varphi_j \to \varphi_i)$, and a short **CPC**-$F$ subproof to derive the formulas $\Box^\ell\varphi_j$ and $\Box^\ell(\varphi_j \to \varphi_i)$ for every $\ell \leq n - i$, we use the new axioms and modus ponens to get $\Box^\ell\varphi_i$, and then derive their conjunction $\Box^{\leq(n-i)}\varphi_i$ by another **CPC**-$F$ subproof. □

**Proposition 3.6 (Feasible deduction theorem)** *Let $L$ be a modal or si logic, and let $P$ be $L$-$F$, $L$-$CF$, or $L$-$EF$. Given a $P$-derivation $\pi$ of a formula (circuit, in the case of $CF$) $\varphi$ using assumptions $\varphi_0, \dots, \varphi_k$ (which do not contain any extension variables in the case of $EF$) as extra axioms, we can construct in polynomial time a $P$-proof of*

$$\bigwedge_{i \leq k} \Box^{\leq m}\varphi_i \to \varphi$$

*for some $m \in \omega$ ($m = 0$ in the si case).*

*Proof:* If $P$ if $L$-$F$ or $L$-$CF$, and $L$ is a si logic, we can use the same proof as for the classical logic (see [15, L. 4.4.10]). If $L$ is a modal logic, we construct a $P_-$-derivation of $\varphi$ from assumptions of the form $\Box^{\leq m}\varphi_i$ for some $m$ as in Lemma 3.5, apply the classical proof of the deduction theorem, and replace the resulting $P_-$-proof by an equivalent $P$-proof.

If $P$ is $L$-$EF$, we apply the feasible deduction for $L$-$F$ to obtain an $L$-$F$-proof of a formula

$$(*) \qquad \bigwedge_j \Box^{\leq m}(q_j \leftrightarrow \psi_j) \wedge \bigwedge_{i \leq k} \Box^{\leq m}\varphi_i \to \varphi,$$

where $q_j \leftrightarrow \psi_j$ are the extension axioms used in the original $L$-$EF$-derivation. We reintroduce these extension axioms, and use necessitation and modus ponens to eliminate them from $(*)$. □

**Lemma 3.7** *We can construct in polynomial time* **K**-$F$*-proofs of*

$$\bigwedge_i \Box^{\leq m}(\alpha_i \leftrightarrow \beta_i) \to (\varphi(\vec{\alpha}) \leftrightarrow \varphi(\vec{\beta})),$$

*where $m$ is the modal degree of $\varphi$.*

*Proof:* By straightforward induction on the complexity of $\varphi$. □

We will use two kinds of interpretations of **CPC** in **IPC**. The first is the well-known double-negation translation.

**Proposition 3.8 (Glivenko translation)** *Let $L$ be a consistent si logic, and $\Gamma$ be the set of negative formulas. Then* **CPC**-$F \equiv_{p,\Gamma} L$-$F$*,* **CPC**-$EF \equiv_{p,\Gamma} L$-$EF$*, and* **CPC**-$SF \equiv_{p,\Gamma} L$-$SF$*.*

*Proof:* Wlog $L =$ **IPC**. Let $\pi$ be a classical proof of $\varphi$. We prefix $\neg\neg$ to every formula in $\pi$, and fill the gaps to make it an intuitionistic proof of $\neg\neg\varphi$ (e.g., we use instances of the intuitionistic tautology $\neg\neg(p \to q) \to (\neg\neg p \to \neg\neg q)$ to fix modus ponens). Then we construct a short **IPC**-$F$-proof of $\neg\neg\varphi \to \varphi$ by induction on the complexity of $\varphi$. □

The second is a feasible version of conservativity of **CPC** over **IPC** wrt implications between monotone formulas. Atserias et al. [2] give a simulation of Frege systems for formulas of this form, with a polynomial bound on the number of lines, and a quasi-polynomial bound on the size of the proof. We improve the size bound to polynomial. (Note that our proof essentially uses intuitionistic implication, it does not apply to monotone sequent calculus considered in [2].)

**Theorem 3.9** *Let $\Gamma$ be the set of all formulas*

$$\alpha \to \beta$$

*such that $\alpha$ and $\beta$ are monotone. Then* **CPC**-$F \le_{p,\Gamma}$ **IPC**-$F$*, and* **CPC**-$EF \le_{p,\Gamma}$ **IPC**-$EF$*.*

*Proof:* Assume that $\vec{p}$ are all variables which appear in $\varphi = \alpha \to \beta$, and pick new variables $\vec{q}$. Let $\pi$ be a **CPC**-$F$-proof of $\varphi$ (the case of $EF$ is similar, we only use circuits instead of formulas). By [2, Thm. 2], we can construct in polynomial time an **IPC**-$F$-proof of the formula

$$\bigwedge_i (p_i \vee q_i) \wedge \alpha \to \beta \vee \bigvee_i (p_i \wedge q_i).$$

We substitute the formulas $p_i \to \beta$ for $q_i$ in the entire proof. There are easily constructible **IPC**-$F$-proofs of

$$\bigvee_i (p_i \wedge (p_i \to \beta)) \to \beta,$$

hence after rearranging the formula we obtain a proof of

$$\alpha \to \Big(\bigwedge_i (p_i \vee (p_i \to \beta)) \to \beta\Big).$$

We construct **IPC**-$F$-proofs of

$$(*) \qquad \Big(\bigwedge_{i<m} (p_i \vee (p_i \to \beta)) \to \beta\Big) \to \beta$$

by induction on $m$. If $m \le 1$, then $(*)$ is an instance of a fixed tautology, hence it has a proof of size $O(|\beta|)$. The induction step goes as follows:

$$\begin{aligned}\Big(\bigwedge_{i<m+1} (p_i \vee (p_i \to \beta)) \to \beta\Big) &\to \Big((p_m \vee (p_m \to \beta)) \to \Big(\bigwedge_{i<m} (p_i \vee (p_i \to \beta)) \to \beta\Big)\Big)\\ &\to \Big((p_m \vee (p_m \to \beta)) \to \beta\Big)\\ &\to \beta,\end{aligned}$$

using the induction hypothesis for $m$ and 1. $\square$

**Proposition 3.10** *Let $L$ be a consistent modal logic, and $\Gamma$ the set of $\Box$-free formulas. Then* **CPC**-$F \equiv_{p,\Gamma} L$-$F$*,* **CPC**-$EF \equiv_{p,\Gamma} L$-$EF$*, and* **CPC**-$SF \equiv_{p,\Gamma} L$-$SF$*.*

*Proof:* Let $\pi$ be an $L$-proof of $\varphi$. By Makinson's theorem, $L$ is contained in **Verum** $= \mathbf{K} \oplus \Box\bot$ or **Triv** $= \mathbf{K} \oplus (p \leftrightarrow \Box p)$. In the former case we replace every boxed formula $\Box\psi$ in $\pi$ by $\top$, in the latter case by $\psi$. We obtain a classical proof of $\varphi$. $\square$

**Proposition 3.11** *If $L$ is a modal or si logic, then $L$-$F \equiv_p L$-$F^*$, and $L$-$CF \equiv_p L$-$CF^*$.*

*Proof:* Let $P$ be $L$-$F$ or $L$-$CF$. If $L$ is a si logic, then $P \equiv_p P^*$ by the same proof as for **CPC**-$F$ (see [15, L. 4.4.8]). If $L$ is a modal logic, we transform a $P$-proof into a $P_-$-proof by Lemma 3.5, we make it tree-like by the classical argument, and we transform it back into a $P$-proof: the extra axioms of $P_-$ have short tree-like derivations. $\Box$

We will see later that $EF$ is not in general p-equivalent to $SF$. Nevertheless we can extract some useful information from the classical proof of the equivalence (cf. [15, L. 4.5.4, 4.5.5]); the next theorem is of that kind.

**Theorem 3.12** *If $L$ is a modal or si logic, then $L$-$EF \equiv_p L$-$SF^*$.*

*Proof:* Let $\pi$ be a tree-like $L$-$SF$-proof of $\varphi$, we construct an $L$-$EF$-proof of $\varphi$ of size $O(|\pi|^2)$ by induction on the length of $\pi$. If $\varphi = \alpha(\vec{\psi})$ was derived from $\alpha(\vec{q})$ by substitution, we take an $L$-$EF$-proof of $\alpha$ by the induction hypothesis, and possibly rename the variables $\vec{q}$ in the proof so that they do not appear in $\vec{\psi}$. We introduce an $O(|\varphi|^2)$-size subproof of $\alpha(\vec{q}) \leftrightarrow \alpha(\vec{\psi})$ from new extension axioms $q_i \leftrightarrow \psi_i$, and derive $\alpha(\vec{\psi})$ by modus ponens. If $\varphi$ was derived by necessitation or modus ponens, we use the induction hypothesis and apply the same rule; the size bound follows from tree-likeness of $\pi$.

Given an $L$-$EF$-proof $\pi$ of $\varphi$, we use Propositions 3.6 and 3.11 to construct a tree-like $L$-$F$-proof of a formula of the form

$$\bigwedge_{i<k} \Box^{\leq m}(q_i \leftrightarrow \psi_i) \to \varphi,$$

where $q_i \leftrightarrow \psi_i$ are extension axioms from $\pi$. We may assume that $q_i$ does not appear in $\psi_j$, $j \geq i$. Then we successively eliminate the conjuncts $\Box^{\leq m}(q_i \leftrightarrow \psi_i)$ in the following way: we substitute $\psi_0$ for $q_0$, construct a short tree-like $L$-$F$-proof of $\Box^{\leq m}(\psi_0 \leftrightarrow \psi_0)$, and derive

$$\bigwedge_{0<i<k} \Box^{\leq m}(q_i \leftrightarrow \psi_i) \to \varphi$$

by modus ponens. $\Box$

**Proposition 3.13** *Let $L$ be a modal or si logic, and assume that a formula $\varphi$ has an $L$-$SF$-proof of size $s$ with $\ell$ lines. Then $\varphi$ has an $L$-$F^*$-proof of size $(s/\ell)^\ell < 2^s$ with $2^\ell$ lines.*

*Proof:* Let $\varphi_1, \dots, \varphi_\ell = \varphi$ be an $L$-$SF$-proof, and put $s_i = |\varphi_i|$. We construct tree-like $L$-$F$-proofs $\pi_i$ of $\varphi_i$ by induction on $i$. If $\varphi_i$ was derived by a Frege rule, we take the proofs of its assumptions given by the induction hypothesis, and apply the same rule. If $\varphi_i = \varphi_j(\vec{\psi})$ was derived by substitution from $\varphi_j(\vec{q})$, we take $\pi_j$, and substitute $\vec{\psi}$ for $\vec{q}$ in the whole proof.

The number of lines in $\pi_i$ at most doubles at each step, thus the number of lines in $\pi_\ell$ is less than $2^\ell$. The size of the proof increases most in the substitution steps, where it is multiplied by the size of $\vec{\psi}$, which is bounded by $s_i$. Thus the size of $\pi_\ell$ is bounded by $\prod_{i=1}^{\ell} s_i$. For fixed $\ell$ and $s = \sum_i s_i$, this product is maximized when $s_i = s/\ell$, which gives it the value $(s/\ell)^\ell$. $\Box$

Note that the exponential size upper bound of Proposition 3.13 is nontrivial: unlike classical logic, it is not true in general that any formula has an exponential-size Frege proof. In fact, there exist $\Sigma^0_1$-complete (finitely axiomatizable) modal and si logics $L$, for which there is no recursive upper bound on the length of proofs in any proof system for $L$.

**Definition 3.14** $(\varphi \,?\, \psi_0 : \psi_1) := (\varphi \wedge \psi_0) \vee (\neg\varphi \wedge \psi_1)$.

**Theorem 3.15 (Normal form for *SF* proofs)** *Let $L$ be a modal or si logic. The following are constructible from each other in polynomial time:*

*(i)* *an $L$-$SF$-proof of a formula $\varphi$,*

*(ii)* *a proof of $\varphi$ in $L$-$CF$ augmented by substitution of (Boolean) propositional constants,*

*(iii)* *an $L$-$EF$-proof of a formula of the form*

$$\bigwedge_i (\psi_i \,?\, \Box^{\le m} r_i : \Box^{\le m} \neg r_i) \to \varphi$$

*for some $m \in \omega$ in the modal case, or*

$$\bigwedge_i ((\psi_i \wedge r_i) \vee \neg r_i) \to \varphi \vee \bigvee_i (\psi_i \wedge \neg r_i)$$

*in the intuitionistic case, where $r_i$ are distinct variables which do not appear in $\varphi$, in the formulas $\psi_j$ for $j \ge i$, or in any extension axioms for variables which appear in them (recursively).*

*Proof:* (ii) $\mapsto$ (i) is an easy extension of the simulation of $L$-$CF$ by $L$-$SF$.

(iii) $\mapsto$ (ii): First we transform the proof to an $L$-$CF$-proof in the usual way. (The formulas $\psi_i$ get extension variables replaced with their definitions, which turns them into circuits.) In the si case, we substitute $\bot$ and $\top$ for $r_0$ to obtain

$$\bigwedge_{i>0} ((\psi_i \wedge r_i) \vee \neg r_i) \to \varphi \vee \bigvee_{i>0} (\psi_i \wedge \neg r_i) \vee \psi_0,$$
$$\psi_0 \wedge \bigwedge_{i>0} ((\psi_i \wedge r_i) \vee \neg r_i) \to \varphi \vee \bigvee_{i>0} (\psi_i \wedge \neg r_i),$$

from which we derive

$$\bigwedge_{i>0} ((\psi_i \wedge r_i) \vee \neg r_i) \to \varphi \vee \bigvee_{i>0} (\psi_i \wedge \neg r_i).$$

We continue in the same way to eliminate all $r_i$. The modal case is similar.

(i) $\mapsto$ (iii): We consider the modal case, the intuitionistic one being similar. First we eliminate necessitation by Lemma 3.5 to obtain an $L$-$SF_{-}$-proof $\pi\colon \varphi_1(\vec{p}), \dots, \varphi_n(\vec{p}) = \varphi$. We are going to construct an $L$-$EF$-proof $\pi'$ which contains the formulas $\xi \to \varphi_i(\vec{q^i})$ for each $i$, where $\xi$ is a formula of the form

$$\bigwedge_t (\psi_t \,?\, \Box^{\le m} r_t : \Box^{\le m} \neg r_t),$$

and $q^i_j$ are new extension variables. We define $m$ as the maximal modal degree of a formula from $\pi$.

For any $i = 1, \dots, n$, let $i_1, \dots, i_\ell$ be the list of all indices $i_t > i$ such that $\varphi_{i_t}$ was derived in $\pi$ by a rule which used $\varphi_i$ as an assumption. If $\varphi_{i_t}$ was derived by modus ponens, we put $\psi^{i,t}_j = q^{i_t}_j$. If $\varphi_{i_t}(\vec{p}) = \varphi_i(\vec{\chi}(\vec{p}))$ was derived by the substitution rule, we define $\psi^{i,t}_j = \chi_j(\vec{q^{i_t}})$. Let $r^i_t$ be fresh variables. We put in $\pi'$ the extension axioms

$$q^i_j \leftrightarrow \neg r^i_1 \mathbin{?} \psi^{i,1}_j : \neg r^i_2 \mathbin{?} \psi^{i,2}_j : \dots : \neg r^i_{\ell-1} \mathbin{?} \psi^{i,\ell-1}_j : \psi^{i,\ell}_j,$$

and define

$$\begin{aligned} \xi^i &= \bigwedge_t (\varphi_i(\vec{\psi^{i,t}}) \mathbin{?} \Box^{\le m} r^i_t : \Box^{\le m} \neg r^i_t), \\ \xi &= \bigwedge_i \xi^i. \end{aligned}$$

We can derive

$$\neg r^i_t \wedge \bigwedge_{u<t} r^i_u \to \bigwedge_j (q^i_j \leftrightarrow \psi^{i,t}_j)$$

from the extension axioms, hence also

$$\begin{aligned} \Box^{\le m} \neg r^i_t \wedge \bigwedge_{u<t} \Box^{\le m} r^i_u &\to \bigwedge_j \Box^{\le m} (q^i_j \leftrightarrow \psi^{i,t}_j) \\ &\to (\varphi_i(\vec{q^i}) \leftrightarrow \varphi_i(\vec{\psi^{i,t}})) \end{aligned}$$

using Lemma 3.7, as $m \ge \mathrm{md}(\varphi_i)$. Hence we can include in $\pi'$ short Frege subproofs of

$$\begin{aligned} \xi^i \wedge \bigvee_t \neg \varphi_i(\vec{\psi^{i,t}}) &\to \xi^i \wedge \bigvee_t \Big( \neg \varphi_i(\vec{\psi^{i,t}}) \wedge \bigwedge_{u<t} \varphi_i(\vec{\psi^{i,u}}) \Big) \\ &\to \bigvee_t \Big( \neg \varphi_i(\vec{\psi^{i,t}}) \wedge \Box^{\le m} \neg r^i_t \wedge \bigwedge_{u<t} \Box^{\le m} r^i_u \Big) \\ &\to \neg \varphi_i(\vec{q^i}), \end{aligned}$$

thus

$$(*) \qquad \xi \wedge \varphi_i(\vec{q^i}) \to \varphi_i(\vec{\psi^{i,t}}).$$

We include in $\pi'$ the formulas $\xi \to \varphi_i(\vec{q^i})$, and complete the proof as follows. If $\varphi_i$ is an instance of an axiom of $L$-$SF_-$, then $\xi \to \varphi_i(\vec{q^i})$ follows from an instance of the same axiom, hence it has a short $L$-$F$-proof. If $\varphi_i$ was derived from $\varphi_j$ by substitution, then $\varphi_i(\vec{q^i}) = \varphi_j(\vec{\psi^{j,t}})$ for some $t$, thus $\xi \to \varphi_i(\vec{q^i})$ can be derived from $\xi \to \varphi_j(\vec{q^j})$ and $(*)$. If $\varphi_i$ was derived from $\varphi_j$ and $\varphi_k = (\varphi_j \to \varphi_i)$ by modus ponens, we can prove $\xi \to \varphi_j(\vec{q^i})$ and $\xi \to (\varphi_j(\vec{q^i}) \to \varphi_i(\vec{q^i}))$ by $\xi \to \varphi_j(\vec{q^j})$, $\xi \to \varphi_k(\vec{q^k})$, and $(*)$, hence $\xi \to \varphi_i(\vec{q^i})$ follows. $\Box$

## 3.1 More on tree-like systems

We defined tree-like Frege (and *CF*, *SF*) systems in a more restrictive way than the corresponding dag-like systems, with a fixed set of non-axiom rules (modus ponens, necessitation). As we will explain shortly, the reason for this inconsistency is that neither Proposition 3.11 nor Theorem 3.12 hold for tree-like versions of standard (circuit, substitution) Frege systems in general. In other words, tree-like (circuit, substitution) Frege systems are not in general p-equivalent even if their dag-like counterparts are.

For the rest of this subsection, let us temporarily lift the restriction on allowed rules in tree-like systems.

The size of tree-like proofs is intimately related to *proof height*, which we define as the maximal length of a derivation path in the proof. If $P$ is a (circuit, substitution) Frege proof system, and $\varphi$ a formula, let $h_P(\varphi)$ be the minimal height of a $P$-proof of $\varphi$. We observe that this measure is very robust.

**Lemma 3.16** *For any tautology $\varphi$, $h_P(\varphi)$ is the same for dag-like or tree-like Frege, circuit Frege, and substitution Frege systems $P$ which use the same set of rules, and*

$$k_P(\varphi) \leq 2^{O(h_P(\varphi))}.$$

*If $P$ and $P'$ are two Frege systems for $L$, then $h_P(\varphi) = \Theta(h_{P'}(\varphi))$.*

*Proof:* The height of a dag-like proof does not change if we unravel it in a tree. Given a *CF*-proof, we replace all circuits by equivalent formulas to get an $F$-proof of the same height. If we have a tree-like *SF*-proof, we construct a Frege proof by going from the axioms downward, and applying the substitutions used in the original proof to the whole subtree above the instance of the substitution rule. Again, this does not increase the height of the proof. If we switch to a different set of Frege rules, the simulation in Lemma 3.1 increases the height only by a multiplicative constant factor. □

First, we mention some cases where the tree-like systems *are* equivalent.

**Proposition 3.17** *Let $L$ be a weakly transitive logic. Then*

$$h_F(\varphi) = \Theta(\log k_F(\varphi)),$$

*and given a (circuit) Frege proof of $\varphi$ with $k$ lines, we can construct in polynomial time a (circuit) Frege proof of $\varphi$ of height $O(\log k)$.*

*Proof:* If $L$ is a si logic, then the proof of Proposition 3.11 gives directly a proof of logarithmic height. Let $L$ be an $n$-transitive logic. The only parts of the proof constructed in 3.11 which can be deeper than $O(\log k)$ are the subproofs of the extra axioms

$$(*) \qquad \begin{gathered} \Box^m(\varphi \to \psi) \to (\Box^m\varphi \to \Box^m\psi), \\ \Box^m\alpha \end{gathered}$$

of $P_-$, where $\alpha$ is an axiom of $P$. However, as $L$ is $n$-transitive, it suffices to use a restricted version of $P_-$ where $(*)$ are included only for $m \leq n$. Then these axioms are instances of a fixed finite number of tautologies, hence they have proofs of height $O(1)$. □

**Corollary 3.18** *If $L$ is weakly transitive, then all tree-like (circuit) Frege systems for $L$ are p-equivalent.*

*Proof:* We transform the proof to logarithmic height by Proposition 3.17, apply the construction of Lemma 3.1, and unravel the proof in a tree. The result has height logarithmic in the number of lines of the original proof, hence its size is polynomial. $\square$

**Proposition 3.19** *Let $P$ be a (circuit) Frege system for a modal logic $L$, and let $Q$ be a (circuit) Frege system for $L$ which uses axioms,* MP, *and* Nec. *Then $P^* \equiv_p Q^*$ (or equivalently, $P^* \equiv_p P$) if and only if there are poly-time constructible $P^*$-proofs of $(*)$.*

*Proof:* By inspection of the proof of Proposition 3.11. $\square$

We are going to show an exponential separation between two tree-like systems for **K**. We first establish a sort of converse to the relation between height and number of lines from Lemma 3.16.

**Definition 3.20** Let $P$ be a Frege, circuit Frege, or substitution Frege system. We define $P_2$ as the corresponding $F$, $CF$, or $SF$ system in which every $P$-rule of the form

$$\frac{\alpha_1, \alpha_2, \dots, \alpha_n}{\beta}$$

is replaced by the rule

$$\frac{\alpha_1, \alpha_1, \alpha_2, \alpha_2, \dots, \alpha_n, \alpha_n}{\beta}.$$

**Lemma 3.21** *Let $P$ be an $F$, $CF$, or $SF$ system. Then*

$$h_P(\varphi) = h_{P_2^*}(\varphi) = O(\log k_{P_2^*}(\varphi)).$$

*Proof:* Let $\pi$ be a $P_2^*$-proof. By induction on $k$, we will show that each subproof of $\pi$ with $k$ lines of a formula $\psi$ contains a $P$-proof of $\psi$ of height at most $\log_2(k+1)$. Assume that the proof ends with

$$\frac{\alpha_1, \alpha_1, \dots, \alpha_n, \alpha_n}{\psi},$$

let $\pi_i$ and $\pi_i'$ be the two subproofs of $\alpha_i$ in $\pi$, and let $k_i$ and $k_i'$ be their number of lines. Without loss of generality, we may assume that $\sum_i k_i \le \sum_i k_i'$, hence $k_i \le (k-1)/2$. By the induction hypothesis, $\pi_i$ contains a $P$-proof of $\alpha_i$ of height $h_i \le \log_2(k_i+1)$. Their union augmented by an application of the $P$-rule

$$\frac{\alpha_1, \dots, \alpha_n}{\psi}$$

is thus a $P$-proof of $\psi$ of height

$$1 + \max_{i=1,\dots,n} h_i \le 1 + \log_2((k-1)/2+1) = \log_2(k+1). \qquad \square$$

**Proposition 3.22** *Let* $\mathbf{K}$-$F$ *be the usual Frege system for* $\mathbf{K}$ *consisting of* $\mathbf{CPC}$ *axioms,* K*,* MP*, and* Nec*. If* $\beta$ *is any tautology in* $(*)$*, then* $h_{\mathbf{K}\text{-}F}(\beta) \geq m$*.*

*Proof:* For every formula $\varphi$, let $p_\varphi$ be a fresh variable. For each formula $\varphi$ and a number $h$, we define a formula $\varphi^h$ so that $(\cdot)^h$ commutes with Boolean connectives, preserves variables, and

$$(\Box\varphi)^0 := p_\varphi,$$
$$(\Box\varphi)^{h+1} := \Box(\varphi^h).$$

Notice that $\varphi$ and $\varphi^{h+1}$ are substitution instances of $\varphi^h$. Let $\pi$ be an $F$-proof of height $h$ of a formula $\varphi$, we will show that $\varphi^h$ is a $\mathbf{K}$-tautology by induction on $h$. The induction steps for classical rules are trivial, as $\varphi^h$ preserves Boolean connectives. If $\varphi$ is an instance of K, and $h \geq 1$, then $\psi^h$ is also an instance of K. If $\varphi = \Box\psi$ was derived from $\psi$ by Nec, then $\varphi^h = \Box\psi^{h-1}$ follows by Nec from $\psi^{h-1}$, which is a tautology by the induction hypothesis.

Assume for contradiction that $\beta$ has a proof of height $h < m$. Then $\beta^h$ is (up to renaming of variables) one of the formulas

$$\Box^h p \to (\Box^h q \to \Box^h r),$$
$$\Box^h p,$$

neither of which is a $\mathbf{K}$-tautology. □

**Corollary 3.23** *Let* $\mathbf{K}$-$F$ *be as in Proposition 3.22, and* $\mathbf{K}$-$CF$*,* $\mathbf{K}$-$SF$ *be the corresponding circuit and substitution Frege systems. Then the formulas* $(*)$ *have poly-time constructible* $\mathbf{K}$-$F^*$*-proofs, but require* $\mathbf{K}$-$CF_2^*$ *and* $\mathbf{K}$-$SF_2^*$*-proofs with exponentially many lines.*

# 4 Bimodal expansions

The purpose of this section is to establish connections of $L$-*SF* to *EF* for certain bimodal expansions of $L$. Roughly speaking, we could say that the difference between *EF* and *SF* is that $L$-*SF* can reason "globally" about the entire Kripke model, whereas $L$-*EF* has only access to whatever is visible from the "current world".

The main tool is the next lemma, which essentially describes what we can salvage from the usual proof of $\mathbf{CPC}$-$SF \leq_p \mathbf{CPC}$-$EF$. (This may not be quite apparent, because here we prove Lemma 4.2 as a corollary to Theorem 3.15. We invite the reader to give a direct proof of the lemma along the lines of [15, L. 4.5.5].)

**Definition 4.1** Let $L$ be a modal or si logic, and $L'$ an extension of $L$, possibly in a language with more modal operators. We say that $L'$ has the *property* $(S)$ with respect to $L$, if the following holds: for every $L$-formula $\varphi(r)$, there exists an $L'$-formula $\alpha$, and $L'$-$F$-proofs of $\varphi(\alpha) \vdash_{L'} \varphi(\bot)$, and $\varphi(\alpha) \vdash_{L'} \varphi(\top)$, constructible from $\varphi$ in polynomial time.

**Lemma 4.2** *If* $L'$ *has the property* $(S)$ *wrt* $L$*, then* $L$-$SF \leq_p L'$-$EF$*.*

*Proof:* Take an $L$-$EF$-derivation of a formula

$$\varphi_0 = \bigwedge_{i<k} (\psi_i \mathbin{?} \Box^{\le m} r_i : \Box^{\le m} \neg r_i) \to \varphi$$

as in Theorem 3.15. We construct an $L'$-formula $\alpha_0$, and $L'$-$F$-proofs of $\varphi_0(r_0/\alpha_0) \vdash \varphi_0(r_0/\bot)$, $\varphi_0(r_0/\top)$. We introduce a new extension axiom $r_0 \leftrightarrow \alpha_0$, derive $\varphi_0(r_0/\alpha_0)$ by a poly-size $L'$-$F$-proof, infer $\varphi_0(r_0/\bot)$ and $\varphi_0(r_0/\top)$ by assumption, and conclude

$$\bigwedge_{0<i<k} (\psi_i \mathbin{?} \Box^{\le m} r_i : \Box^{\le m} \neg r_i) \to \varphi$$

as in the proof of Theorem 3.15. We continue to eliminate $r_1, \dots, r_{k-1}$ in the same way. $\Box$

**Definition 4.3** Let $L$ be a modal logic. The *weak tense expansion* $L^{wt}$ *of* $L$ is a bimodal logic which has an extra modal operator $H$, and the axioms and rules

$$H(\varphi \to \psi) \to (H\varphi \to H\psi),$$
$$\varphi \vdash H\varphi,$$
$$\Diamond H\varphi \to \varphi.$$

$L'$ has a *definable weak past modality* for $L$, if $L' \supseteq L$, and there exists an $L'$-formula $H(p)$ which interprets $L^{wt}$ in $L'$.

$L$ is *weakly symmetric*, if it proves the axiom

$$(\mathrm{B}_k) \qquad \varphi \to \Box\Diamond^{\le k}\varphi$$

for some $k$.

**Remark 4.4** A *tense* (or *temporal*) logic is a normal bimodal logic with two modal operators $G$ and $H$, which includes the axioms

$$FHp \to p,$$
$$PGp \to p,$$

where $F\varphi = \neg G \neg\varphi$ and $P\varphi = \neg H \neg \varphi$ are the dual ($\Diamond$-like) operators to $G$ and $H$. The intended meaning of $F\varphi$ and $P\varphi$ is "$\varphi$ holds at some time in the **f**uture", and "$\varphi$ holds at some time in the **p**ast", respectively. If $L$ is a (mono)modal logic, its *minimal tense expansion* $L^t$ is the smallest temporal logic which includes $L$, where we identify $\Box = G$. Our $L^{wt}$ is thus a weakening of $L^t$, which omits one of the two characteristic axioms of temporal logics. This should explain the terminology of Definition 4.3.

**Example 4.5** Any extension of **KB** is weakly symmetric.

**Lemma 4.6** *There are $L^{wt}$-$F$-proofs of $\neg\varphi \to \Box^{\le m}\neg H^{\le m}\varphi$ constructible in time polynomial in $|\varphi|$ and $m$, where $H^{\le m}\varphi = \bigwedge_{k\le m} \underbrace{H\cdots H}_{k}\varphi$.*

*Proof:* Exercise. $\Box$

**Theorem 4.7** *Let $L$ be a modal logic.*

*(i) $L$-$SF \leq_p L^{wt}$-$EF$.*

*(ii) If $L$ is weakly symmetric, then $L$-$SF \equiv_p L$-$EF$.*

*Proof:* (i): We will show that $L^{wt}$ has the property (S) wrt $L$. Let $\varphi$ be an $L$-formula, and put $m := \mathrm{md}(\varphi)$, $\alpha := H^{\leq m}\varphi(\bot)$. Using Lemmas 4.6 and 3.7, we can construct short $L^{wt}$-$F$-proofs of

$$\begin{aligned}\neg\varphi(\bot) &\to \neg\varphi(\bot) \wedge \Box^{\leq m}\neg H^{\leq m}\varphi(\bot)\\ &\to \neg\varphi(\bot) \wedge \Box^{\leq m}\neg\alpha\\ &\to \neg\varphi(\alpha),\end{aligned}$$

thus

$$\varphi(\alpha) \vdash \varphi(\alpha) \wedge \varphi(\bot) \vdash \varphi(\alpha) \wedge \Box^{\leq m}H^{\leq m}\varphi(\bot) \vdash \varphi(\alpha) \wedge \Box^{\leq m}\alpha \vdash \varphi(\top)$$

by necessitation.

(ii): If $\vdash_L B_k$, then $L$ has a definable weak past modality $H(p) = \Box^{\leq k}p$. Given an $L$-$SF$-proof of $\varphi$, we construct an $L^{wt}$-$CF$-proof of $\varphi$ by (i) and Proposition 3.3, replace all subcircuits $H\psi$ by $H(\psi)$ in such a way that the size increases only polynomially, and include short $L$-proofs of the axioms of $L^{wt}$ to obtain an $L$-$CF$-proof of $\varphi$. $\Box$

We remark that the derivation of Theorem 4.7 from Lemma 4.2 is tight:

**Proposition 4.8** *Let $L$ be a modal logic, and $L'$ its (possibly polymodal) extension.*

*(i) $L'$ satisfies the property (S) wrt $L$ if and only if $L'$ has a definable weak past modality for $L$.*

*(ii) $L$ has a definable weak past modality for itself if and only if $L$ is weakly symmetric.*

*Proof:* (i): Let $\Box_*$ be the conjunction of all modal operators of $L'$, and put

$$\varphi(r,p,q) = (p\,?\,\Box r : \Box\neg r) \to q.$$

By (S), there exists an $L'$-formula $\alpha(p,q)$ such that

$$(p\,?\,\Box\alpha : \Box\neg\alpha) \to q \vdash_{L'} ((p\,?\,\Box\bot : \Box\neg\bot) \to q) \wedge ((p\,?\,\Box\top : \Box\neg\top) \to q) \vdash_{L'} q,$$

hence by the deduction theorem, there exists an $m$ such that $L'$ proves

$$\Box_*^{\leq m}(p \wedge \Box\alpha \to q) \wedge \Box_*^{\leq m}(\neg p \wedge \Box\neg\alpha \to q) \to q.$$

We substitute $p/\bot$ and $p/\neg q$ to obtain

$$\begin{gathered}\Box_*^{\leq m}(\Box\neg\alpha(\bot,q) \to q) \to q,\\ \Box_*^{\leq m}(\Box\alpha(\neg q,q) \to q) \to q,\end{gathered}$$

thus $L'$ proves

$$(*)\qquad \psi \to \Diamond_*^{\le m}(\Box\neg\alpha(\bot,\neg\psi)\wedge\psi)\wedge\Diamond_*^{\le m}(\Box\alpha(\psi,\neg\psi)\wedge\psi)$$

for every formula $\psi$. We put $k = \mathrm{md}(\alpha)$, and

$$\begin{aligned}\psi &= \neg p\wedge\Diamond(\Box_*^{\le k}p\wedge\alpha(\bot,\top)),\\ \psi' &= \neg p\wedge\Diamond(\Box_*^{\le k}p\wedge\neg\alpha(\bot,\top)).\end{aligned}$$

We have

$$\begin{aligned}\neg p\wedge\Diamond(\Box_*^{\le k}p\wedge\alpha(\bot,\top)) &\to \Diamond_*^{\le m}(\Box\neg\alpha(\bot,\neg\psi)\wedge\neg p\wedge\Diamond(\Box_*^{\le k}p\wedge\alpha(\bot,\top)))\\ &\to \Diamond_*^{\le m}(\Box\neg\alpha(\bot,\neg\psi)\wedge\Diamond(\Box_*^{\le k}\neg\psi\wedge\alpha(\bot,\top)))\\ &\to \Diamond_*^{\le m}(\Box\neg\alpha(\bot,\neg\psi)\wedge\Diamond\alpha(\bot,\neg\psi)))\\ &\to \Diamond_*^{\le m}\bot\\ &\to \bot\end{aligned}$$

by $(*)$ and Lemma 3.7, and similarly

$$\begin{aligned}\neg p\wedge\Diamond(\Box_*^{\le k}p\wedge\neg\alpha(\bot,\top)) &\to \Diamond_*^{\le m}(\Box\alpha(\psi',\neg\psi')\wedge\neg p\wedge\Diamond(\Box_*^{\le k}p\wedge\neg\alpha(\bot,\top)))\\ &\to \Diamond_*^{\le m}(\Box\alpha(\psi',\neg\psi')\wedge\Diamond(\Box_*^{\le k}\neg\psi'\wedge\neg\alpha(\bot,\top)))\\ &\to \Diamond_*^{\le m}(\Box\alpha(\psi',\neg\psi')\wedge\Diamond\neg\alpha(\psi',\neg\psi')))\\ &\to \bot,\end{aligned}$$

thus

$$\begin{aligned}\Diamond\Box_*^{\le k}p &\to \Diamond(\Box_*^{\le k}p\wedge\alpha(\bot,\top))\vee\Diamond(\Box_*^{\le k}p\wedge\neg\alpha(\bot,\top))\\ &\to p.\end{aligned}$$

It follows that $H(p) = \Box_*^{\le k}p$ is a definable weak past modality for $L$.

(ii): Let $H(p)$ be a definable weak past modality, and $k = \mathrm{md}(H)$. As $L$ proves $H(\top)$, we have $\Box^{\le k}p \to H(p)$, thus $p \to \Box\neg H(\neg p) \to \Box\Diamond^{\le k}p$. $\Box$

If the logic $L$ is (weakly) transitive, we can give a sort of converse to Theorem 4.7: $L$-*SF* p-simulates $L^{wt}$-*EF* (on $L$-formulas). In fact, we can simulate a stronger expansion of $L$, which is also much more natural, and often used in the literature. The result also applies to si logics, which are all transitive by definition. (The reason we did not mention si logics in Theorem 4.7 is that Theorem 4.12 will give a stronger result, and the only consistent "weakly symmetric si logic" is **CPC**.)

**Definition 4.9** Let $L$ be a modal or si logic. The *expansion* $L^A$ *of* $L$ *with universal modality* is a logic which has an extra modal operator $A$, the axioms and rules

$$\begin{gathered}A(\varphi\to\psi)\to(A\varphi\to A\psi),\\ A\varphi\to\varphi,\\ A\varphi\vee A\neg A\varphi,\\ \varphi\vdash A\varphi,\end{gathered}$$

and if $L$ is modal, the axiom

$$A\varphi \to \Box\varphi.$$

**Remark 4.10** If $\langle W, R, V\rangle$ is a general $L$-frame, we may define satisfaction of $L^A$-formulas in $W$ by

$$x \Vdash A\varphi \quad \text{iff} \quad \forall u \in W\, u \Vdash \varphi.$$

It is easy to check that $L^A$ is complete wrt this semantics, which gives the universal modality its name; cf. [3].

**Lemma 4.11** *There are poly-time constructible $L^A$-$F$-proofs of*

$$\bigwedge_i A(\psi_i \leftrightarrow \chi_i) \to (\varphi(\vec{\psi}) \leftrightarrow \varphi(\vec{\chi})).$$

*If every variable in $\varphi$ is in the scope of $A$, there are poly-time constructible $L^A$-$F$-proofs of $\varphi \to A\varphi$.*

*Proof:* Easy. $\square$

**Theorem 4.12** *Let $L$ be a weakly transitive logic. Then $L$-SF $\equiv_p$ $L^A$-EF wrt $L$-formulas.*

*Proof:* $L$-*SF* $\leq_p$ $L^A$-*EF*: by Lemma 4.2, it suffices to show that $L^A$ satisfies the property (S) wrt $L$. Given a formula $\varphi(r)$, we define $\alpha = A\varphi(\bot)$, and leave the verification of $\varphi(\alpha) \vdash_{L^A} \varphi(\bot), \varphi(\top)$ as an exercise, using Lemma 4.11.

$L^A$-*EF* $\leq_p$ $L$-*SF*: we assume $L \supseteq \mathbf{K} \oplus \mathrm{Tra}_k$, the intuitionistic case is similar. Let $\pi$ be an $L^A$-*EF*-proof of an $L$-formula $\varphi$, and let $A\psi_1, \dots, A\psi_m$ be all subformulas with topmost connective $A$ which appear in $\pi$, ordered so that $j < i$ whenever $A\psi_i$ is a subformula of $\psi_j$ or of the definition for any extension variable which recursively appears in $\psi_j$. We choose new variables $r_i$, and for any formula $\alpha$, let $\overline{\alpha}$ be the result of replacing each subformula $A\psi_i$ in $\alpha$ by $r_i$. By induction on the length of $\pi$, we will construct an $L$-*EF*-proof which contains the formula $\xi \to \Box^{\leq k}\overline{\alpha}$ for every $\alpha \in \pi$, where

$$\xi = \bigwedge_i (\Box^{\leq k}\overline{\psi_i} \,?\, \Box^{\leq k} r_i : \Box^{\leq k}\neg r_i).$$

The induction steps for axioms of $L$, modus ponens, $\Box$-necessitation, and extension axioms are straightforward.

Consider an axiom $\alpha = A\psi_i \vee A\neg A\psi_i \in \pi$. We have $\overline{\alpha} = r_i \vee r_j$, and $\overline{\psi_j} = \neg r_i$, where $\psi_j = \neg A\psi_i$. We can prove

$$\Box^{\leq k} r_i \to \Box^{\leq k}(r_i \vee r_j),$$

and

$$\begin{aligned} \xi \wedge \neg\Box^{\leq k} r_i &\to \xi \wedge \Box^{\leq k}\neg r_i \\ &\to \Box^{\leq k} r_j \\ &\to \Box^{\leq k}(r_i \vee r_j), \end{aligned}$$

thus $\xi \to \Box^{\leq k}(r_i \vee r_j)$. The other axioms and rules of $L^A$ are treated in a similar way.

We obtain an $L$-*EF*-proof of the formula $\xi \to \Box^{\leq k}\varphi$, from which we construct an $L$-*SF*-proof of $\varphi$ by Theorem 3.15. $\square$

**Problem 4.13** *Does $L$-SF simulate $L^{wt}$-EF or even $L^{A}$-EF for all modal logics $L$?*

# 5 Simulations

We have already seen some logics such that $L$-*EF* $\equiv_p$ $L$-*SF* in Theorem 4.7. In this section we will prove the same for several more classes of logics. We will concentrate on transitive logics, partly because of their overall importance, and partly because their relatively well-behaved model theory ensures various useful structural properties. (More or less) all the logics we consider have finite width; if we continue the handwaving from the lead of Section 4, the intuition could be that in narrow models the "global access" does not give *SF* any significant advantage over *EF*.

All the proofs use a simulation of $L$ by classical logic: we translate an $L$-*SF*-proof to **CPC**-*SF*, apply the classical simulation of *SF* by *EF*, and translate the **CPC**-*EF*-proof to $L$-*EF*. The benefit of this method is that it establishes much stronger results than just simulation of $L$-*SF* by $L$-*EF*, the drawback is that it in principle cannot be applied to logics which are not *coNP*. The device is formally defined below.

**Definition 5.1** Let $P$ be a proof system for a logic $L$, and $P'$ a proof system for a logic $L'$. We say that $P$ is *interpretable* in $P'$, if there exists a poly-time function $f$ such that $\varphi \in L$ iff $f(\varphi) \in L'$, and such that $P$-proofs of $\varphi$ and $P'$-proofs of $f(\varphi)$ are poly-time constructible from each other.

**Observation 5.2** *Let $P$ be a proof system for a modal or si logic $L$, and $P'$ a proof system for* **CPC**. *If $P$ is interpretable in $P'$, then $L$ is in coNP, and lower bounds on the size of $P$-proofs imply lower bounds on the size of $P'$-proofs.*

*Proof:* Clearly $L$ is reducible to **CPC** via $f$, thus $L \in coNP$. If $\varphi$ requires $P$-proofs of size $s(|\varphi|)$, then $\varphi' = f(\varphi)$ requires $P'$-proofs of size $s(|\varphi|)^{\Omega(1)} \geq s(|\varphi'|^{\Omega(1)})^{\Omega(1)}$. □

## 5.1 Tabular logics

The class of tabular logics contains only a few logics of independent interest (**CPC**, **SmL**, finitely valued Gödel logics), but it is conceptually rather important: it comprises the discrete upper part of the lattice of normal modal (resp. si) logics, which can in a sense approximate all logics with the finite model property. It is very well-behaved and easily described.

The idea of the simulation is as follows. If $L$ is the logic of a finite frame $F$, we can define a classical formula $\varphi^F$ which describes the satisfaction of $\varphi$ in $F$ in an obvious way. The problem is to extract a proof of $\varphi$ from a classical proof of $\varphi^F$. If we have "labels" $c_x$ for every point $x \in F$ such that $c_x$ holds only in $x$, we can easily construct $\varphi$ from $\varphi^F$ by a simple substitution. In fact, we do not quite have such labels; but as $c_x$ do not appear in the final formula $\varphi$, we may imagine that we are free to assign them if it is possible at all. If it is *not* possible, then we do not actually live in $F$, but in some smaller frame $G$, which satisfies a proper extension $L'$ of $L$. This suggests that we can construct an $L$-*EF*-proof of $\varphi$ by induction on $L$, where the induction hypothesis would be a proof of $\varphi$ in $L'$-*EF*. The soundness of such induction (among others) follows from the next theorem, which summarizes some well-known properties of tabular logics.

**Theorem 5.3 (see [7])** *Let $L = L(F)$ be a tabular modal or si logic. Then $L$ is finitely axiomatizable, it has only finitely many extensions, and all of them are also tabular. A rooted finite frame $G$ is an $L$-frame if and only if it is a p-morphic image of a (rooted) generated subframe of $F$.*

**Definition 5.4** Let $F = \langle F, R\rangle$ be a finite modal Kripke frame, $x \in F$, and $\varphi(\vec{p})$ a modal formula. We define a classical formula $\varphi^x$ by induction on the complexity of $\varphi$:

$$\begin{aligned} p_i^x &:= p_{i,x}, \\ (\varphi \circ \psi)^x &:= \varphi^x \circ \psi^x, \qquad \circ \in \{\to, \wedge, \vee, \bot\}, \\ (\Box\varphi)^x &:= \bigwedge_{xRy} \varphi^y. \end{aligned}$$

If $F = \langle F, \leq\rangle$ is an intuitionistic finite frame, $x \in F$, and $\varphi$ is an intuitionistic formula, we define

$$\begin{aligned} p_i^x &:= \bigwedge_{y \geq x} p_{i,y}, \\ (\varphi \circ \psi)^x &:= \varphi^x \circ \psi^x, \qquad \circ \in \{\wedge, \vee, \bot\}, \\ (\varphi \to \psi)^x &:= \bigwedge_{y \geq x} (\varphi^y \to \psi^y). \end{aligned}$$

In both cases, we put

$$\varphi^F := \bigwedge_{x \in F} \varphi^x.$$

Notice that even though the formulas $\varphi^x$ and $\varphi^F$ may have exponential size, they can be expressed as (poly-time constructible) circuits of size $O(|F|\,|\varphi|)$.

**Lemma 5.5** *Let $F$ be a finite modal or intuitionistic frame, and $\varphi$ a formula. Then $\varphi \in L(F)$ iff $\varphi^F \in$ **CPC**.*

*Proof:* Let $F$ be a modal frame. Classical assignments $e$ to the variables $p_{i,x}$ are in an obvious 1-1 correspondence with assignments $\Vdash$ to the variables $p_i$ in $F$, and a straightforward induction on the complexity of $\varphi$ shows that $e(\varphi^x) = 1$ iff $x \Vdash \varphi$, thus $e(\varphi^F) = 1$ iff $\varphi$ holds in $F$ under $\Vdash$. The intuitionistic case is similar. $\Box$

**Lemma 5.6** *Let $F$ and $G$ be finite modal or intuitionistic frames such that $L(F) \subseteq L(G)$. Given a **CPC**-$CF$-proof of $\varphi^F$, we can construct in polynomial time a **CPC**-$CF$-proof of $\varphi^G$.*

*Proof:* Let $G = \bigcup_{i<k} G_i$, where each $G_i$ is rooted. By Theorem 5.3, each $G_i$ is a p-morphic image of a generated subframe of $F$, hence $G$ is a p-morphic image of a disjoint union $H = \sum_{i<k} H_i$ of generated subframes $H_i \subseteq\cdot\, F$. Given a **CPC**-$CF$-proof of $\varphi^F$, we construct proofs of $\varphi^{H_i}$ by omitting some of the conjuncts, we rename the variables to make them disjoint, and

we join the proofs to obtain $\varphi^H = \bigwedge_{i<k} \varphi^{H_i}$ (the size increases only polynomially, as $k \le |G|$). Let $f\colon H \twoheadrightarrow G$ be a p-morphism, and let $\sigma$ be the substitution defined by $\sigma p_{i,x} = p_{i,f(x)}$. Then $\sigma\varphi^H$ is almost identical to $\varphi^G$, except that some of the subformulas may be duplicated. We thus apply $\sigma$ to the whole proof, and fix the result by a short **CPC**-*CF*-proof to obtain $\varphi^G$. □

**Lemma 5.7** *Let $F$ be a finite modal or intuitionistic frame. Given an $L(F)$-SF-proof of $\varphi$, we can construct in polynomial time a* **CPC**-*CF-proof of $\varphi^F$.*

*Proof:* Let $P$ be the substitution circuit Frege system for **CPC**, which is p-equivalent to **CPC**-*CF*. If $\varphi_1, \dots, \varphi_m = \varphi$ is an $L$-*SF*-proof, we construct the sequence $\varphi_1^F, \dots, \varphi_m^F$, and fix it up to a $P$-proof in a straightforward way. For example, if $\varphi_i$ was inferred from $\varphi_j$ and $\varphi_k = (\varphi_j \to \varphi_i)$ by modus ponens, then we can prove $\varphi_i^F = \bigwedge_x \varphi_i^x$ from $\varphi_j^F = \bigwedge_x \varphi_j^x$ and $\varphi_k^F = \bigwedge_x (\varphi_j^x \to \varphi_i^x)$ by a poly-size **CPC**-*CF*-subproof. □

We intend to construct a simulation of $L$-*SF* in $L$-*EF* for tabular $L$, which goes via interpretation in the classical logic. In the modal case, we may take a **CPC**-*EF*-proof of $\varphi^F$, substitute some modal formulas inside, and hopefully proceed to construct an $L$-*EF*-proof of $\varphi$. This does not work in the intuitionistic case, as $L$ does not contain **CPC**; we thus need to use some kind of interpretation of **CPC** in **IPC**. Glivenko double-negation translation does not help us, as substitution inside a negative formula can only produce another negative formula. We will use the feasible conservativity from Theorem 3.9 instead.

**Definition 5.8** Let $\varphi$ be a classical formula in variables $\vec{p}$. We pick new variables $\vec{q}$, and define monotone formulas $\varphi^{\mathbf{m}}$, $\varphi_{\mathbf{m}}$ by induction on the complexity of $\varphi$:

$$
\begin{aligned}
(p_i)^{\mathbf{m}} &= p_i & (p_i)_{\mathbf{m}} &= q_i \\
(\varphi \wedge \psi)^{\mathbf{m}} &= \varphi^{\mathbf{m}} \wedge \psi^{\mathbf{m}} & (\varphi \wedge \psi)_{\mathbf{m}} &= \varphi_{\mathbf{m}} \vee \psi_{\mathbf{m}} \\
(\varphi \vee \psi)^{\mathbf{m}} &= \varphi^{\mathbf{m}} \vee \psi^{\mathbf{m}} & (\varphi \vee \psi)_{\mathbf{m}} &= \varphi_{\mathbf{m}} \wedge \psi_{\mathbf{m}} \\
(\neg\varphi)^{\mathbf{m}} &= \varphi_{\mathbf{m}} & (\neg\varphi)_{\mathbf{m}} &= \varphi^{\mathbf{m}} \\
(\varphi \to \psi)^{\mathbf{m}} &= \varphi_{\mathbf{m}} \vee \psi^{\mathbf{m}} & (\varphi \to \psi)_{\mathbf{m}} &= \varphi^{\mathbf{m}} \wedge \psi_{\mathbf{m}} \\
\bot^{\mathbf{m}} &= \bot & \bot_{\mathbf{m}} &= \top
\end{aligned}
$$

(In other words, we push all negations down by De Morgan rules, and introduce new variables for $\neg p_i$.)

**Lemma 5.9** *Given a* **CPC**-*F-proof* (**CPC**-*EF-proof*) *a formula $\varphi(\vec{p})$, we can construct in polynomial time an* **IPC**-*F-proof* (**IPC**-*EF-proof, resp.*) *of the formula*

$$\bigwedge_i (p_i \vee q_i) \to \varphi^{\mathbf{m}}(\vec{p}, \vec{q}).$$

*Proof:* We can construct a **CPC**-*F*-proof of $\psi \leftrightarrow \psi^{\mathbf{m}}(\vec{p}, \neg\vec{p})$ by a straightforward induction on the complexity of $\psi$, hence we have a proof of

$$\varphi^{\mathbf{m}}(\vec{p}, \neg\vec{p}).$$

As $\varphi^{\mathbf{m}}$ is monotone, we can construct a short proof of

$$\bigwedge_i (p_i \vee q_i) \to \bigwedge_i (\neg p_i \to q_i) \\ \to (\varphi^{\mathbf{m}}(\vec{p}, \neg\vec{p}) \to \varphi^{\mathbf{m}}(\vec{p}, \vec{q})),$$

and obtain a **CPC**-$F$-proof (**CPC**-$EF$-proof) of

$$\bigwedge_i (p_i \vee q_i) \to \varphi^{\mathbf{m}}(\vec{p}, \vec{q}).$$

Then it suffices to apply Theorem 3.9. $\square$

**Theorem 5.10** *If $L$ is a tabular modal or si logic, then $L$-EF $\equiv_p$ $L$-SF, and $L$-EF is interpretable in* **CPC**-*EF*.

*Proof:* By Lemmas 5.5 and 5.7, it suffices to show that we can construct an $L(F)$-$CF$-proof of $\varphi$ from a **CPC**-$CF$-proof of $\varphi^F$ in polynomial time. We will consider the modal case first.

Theorem 5.3 and Lemma 5.6 imply that we may proceed by reverse induction on $L$; i.e., we assume that we are given $L'$-$CF$-proofs of $\varphi$ for every proper extension $L'$ of $L = L(F)$. Let $F = \bigcup_{i<m} F_i$, where each $F_i$ is a rooted generated subframe of $F$. Assume first that $L(F) \subsetneq L(F_i)$ for every $i < m$, and let $\alpha_i$ be formulas in pairwise disjoint sets of variables such that $L(F_i) = \mathbf{K} \oplus \alpha_i$. Notice that $L(F)$ (hence every its extension) is $k$-transitive, where $k = |F|$. By the induction hypothesis and Proposition 3.6, we can construct $(\mathbf{K} \oplus \mathrm{Tra}_k)$-$CF$-proofs of formulas of the form

$$\bigwedge_{j<n} \Box^{\leq k} \sigma_j \alpha_i \to \varphi$$

for every $i < m$, where $\sigma_j$ are some substitutions. We may combine them to form a $(\mathbf{K} \oplus \mathrm{Tra}_k)$-$CF$-proof of

$$\bigwedge_{j_0, \dots, j_{m-1} < n} \bigvee_{i<m} \Box^{\leq k} \sigma_{j_i} \alpha_i \to \varphi,$$

from which we construct a $(\mathbf{K} \oplus \mathrm{Tra}_k)$-$CF$-proof of $\varphi$ using $n^m$ instances of the axiom

$$\alpha = \bigvee_{i<m} \Box^{\leq k} \alpha_i.$$

This is an $L(F)$-$CF$-proof of $\varphi$, as $\mathbf{K} \oplus \mathrm{Tra}_k \oplus \alpha \subseteq L(F)$. The size of the proof increases only polynomially, as $m$ and $k$ are constant (for a fixed $F$).

The other case is $L(F) = L(F_i)$ for some $i$, we may thus assume that $F = F_i$ is rooted. Let $X$ be the set of all roots of $F$. We pick fresh variables $c_x$ for every $x \in F$, and put

$$\alpha := \bigwedge_{x \neq y} (c_x \to \neg c_y) \wedge \bigvee_x c_x \wedge \bigwedge_{xRy} (c_x \to \Diamond c_y) \wedge \bigwedge_{x \not R y} (c_x \to \Box \neg c_y),$$
$$\alpha_x := c_x \wedge \Box^{\leq k} \alpha.$$

Let $\Vdash^c$ be the valuation given by $x \Vdash^c c_y$ iff $x = y$.

**Claim 1**

(*i*) *If $G$ is a $k$-transitive frame, and $u \in G$, then $\alpha_x$ is satisfiable in $u$ iff there exists a p-morphism $f\colon G_u \twoheadrightarrow F_x$ such that $f(u) = x$.*

(*ii*) *If $\psi$ is a formula, and $0 \in X$, then $\alpha_0 \to \psi \in L(F)$ iff $F, 0 \Vdash \psi$ holds under every valuation $\Vdash$ which extends $\Vdash^c$.*

(*iii*) *If $0 \in X$, then $L(F)$ proves*

$$\Diamond^{\leq k}\alpha_0 \to \bigvee_{x \in X} \alpha_x.$$

*Proof:* (i): Easy. The p-morphism is defined by $f(v) = y$ iff $v \Vdash c_y$, and vice versa.

(ii): The left-to-right implication follows from (i), as identity is a p-morphism. Right-to-left: let $x \in F$, and $\Vdash$ be a valuation in $F$ such that $x \Vdash \alpha_0$, we need to show $x \Vdash \psi$. By (i), there exists a p-morphism $f\colon F_x \twoheadrightarrow F_0 = F$. As $f$ is onto, we have $|F_x| \geq |F|$ and $F_x \subseteq F$, thus $F = F_x$. By the same argument $f$ is injective, hence it is an isomorphism. If $\Vdash^*$ is the valuation induced by $f$ from $\Vdash$, then $\Vdash^* \supseteq \Vdash^c$ by the proof of (i), hence $0 \Vdash^* \psi$, which implies $x \Vdash \psi$.

(iii): Let $\Vdash$ be a valuation in $F$, and $x \in F$ be such that $x \Vdash \Diamond^{\leq k}\alpha_0$. There exists $y \in F_x$ such that $y \Vdash \alpha_0$. By the proof of (ii), there exists an automorphism $f$ of $F$ such that $u \Vdash c_v$ iff $f(u) = v$. In particular $y \in X$, thus $x \in X$, $f(x) \in X$, and $x \Vdash \alpha_{f(x)}$. □ (Claim 1)

Let $\sigma$ be the substitution such that

$$\sigma p_{i,x} = \Box^{\leq k}(c_x \to p_i).$$

**Claim 2** *Let $0 \in X$. For all formulas $\psi$ in variables $p_i$, there are poly-time constructible $L(F)$-$CF$-proofs of*

$$\alpha_0 \to \bigwedge_x (\sigma\psi^x \leftrightarrow \Box^{\leq k}(c_x \to \psi)).$$

*Proof:* By induction on the complexity of $\psi$. The steps for variables and $\wedge$ are trivial. The steps for $\neg$ and $\Box$ follow from substitution instances of the formulas

$$\alpha_0 \to (\Box^{\leq k}(c_x \to \neg q) \leftrightarrow \neg\Box^{\leq k}(c_x \to q)),$$
$$\alpha_0 \to \Big(\Box^{\leq k}(c_x \to \Box q) \leftrightarrow \bigwedge_{xRy} \Box^{\leq k}(c_y \to q)\Big),$$

which are provable in $L(F)$ by Claim 1. □ (Claim 2)

We take a **CPC**-$CF$-proof of $\varphi^F$, and apply the substitution $\sigma$ to the whole proof to obtain a proof of $\sigma\varphi^F$. By Claim 2 we construct an $L(F)$-$CF$-proof of

$$\bigwedge_{x \in X} (\alpha_x \to \varphi),$$

thus

$$(*) \qquad \Diamond^{\leq k}\alpha_0 \to \bigvee_{x \in X} \alpha_x \to \varphi$$

for any $0 \in X$ by Claim 1. The formula $\alpha_0$ is satisfiable in $F$, thus $L' := L(F) \oplus \neg\alpha_0$ is a proper extension of $L(F)$. The induction hypothesis gives an $L'$-$CF$-proof of $\varphi$, which we transform into a $(\mathbf{K} \oplus \mathrm{Tra}_k)$-$CF$-proof of

$$(**) \qquad \bigwedge_{j<n} \Box^{\leq k} \neg\sigma_j \alpha_0 \to \varphi$$

for some substitutions $\sigma_j$ by Proposition 3.6. We take $n$ substitution instances of $(*)$ to construct an $L(F)$-$CF$-proof of

$$\bigwedge_{j<n} \Box^{\leq k} \neg\sigma_j \alpha_0 \vee \varphi,$$

which we combine with $(**)$ to make an $L(F)$-$CF$-proof of $\varphi$.

In the intuitionistic case we use the same strategy, we will only indicate below the differences. Let $L = L(F)$ with rooted $F$, and assume we are given a **CPC**-$CF$-proof of $\varphi^F$, and $L'$-$CF$-proofs of $\varphi$ in proper extensions $L'$ of $L$. Let $0$ be the (unique) root of $F$, and put

$$\beta := \bigwedge_{x \leq y} \Big( \Big( \bigwedge_{z \not\geq y} c_z \to c_y \Big) \to c_x \Big) \to c_0.$$

Let $\Vdash^c$ be the valuation in $F$ given by $x \Vdash^c c_y$ iff $x \not\leq y$. By Lemma 5.9, we can construct an **IPC**-$CF$-proof of the circuit

$$(***) \qquad \bigwedge_{i,x} (p_{i,x} \vee q_{i,x}) \to (\varphi^0)^{\mathbf{m}}.$$

Let $\sigma$ be the substitution defined by

$$\sigma p_{i,x} := \bigwedge_{y \not\geq x} c_y \to p_i,$$

$$\sigma q_{i,x} := p_i \to c_x.$$

We prove

**Claim 3**

*(i) If $G$ is an intuitionistic frame, and $u \in G$, then $\beta$ is refutable in $u$ iff there exists a p-morphism $f\colon H \twoheadrightarrow F$ from a subframe $H \subseteq G_u$.*

*(ii) If $\psi$ is a formula, then $\beta \vee \psi \in L(F)$ iff $F \Vdash \psi$ holds under every valuation $\Vdash$ which extends $\Vdash^c$.*

*(iii) For all formulas $\psi$ in variables $p_i$ and $x \in F$, there are poly-time constructible $L(F)$-$CF$-proofs of*

$$\beta \vee \big(\sigma(\psi^x)^{\mathbf{m}} \wedge \sigma(\psi^x)_{\mathbf{m}} \to c_x\big),$$

$$\beta \vee \Big(\sigma(\psi^x)^{\mathbf{m}} \wedge \Big(\bigwedge_{y \not\geq x} c_y \to \psi\Big)\Big) \vee \big(\sigma(\psi^x)_{\mathbf{m}} \wedge (\psi \to c_x)\big).$$

similarly to Claims 1 and 2. Then we proceed as in the modal case: we construct a proof of $\beta \vee \varphi$ using $\sigma(***)$ and Claim 3, we derive

$$\bigwedge_{j<n} \sigma_j \beta \to \varphi$$

from the induction hypothesis and Proposition 3.6, and we conclude $\varphi$. $\square$

**Remark 5.11** The exponent of the polynomial simulation from Theorem 5.10 is quite large, and grows rapidly with $F$. It can be shown by a more careful analysis that the exponent is exponential in $|F|$.

### 5.2 Logics of finite depth and width

A transitive logic is tabular if and only if it has finite depth, finite width, and finite cluster size. However, Theorem 4.7 suggests that infinite clusters should not impede simulation of *SF* by *EF*. We are indeed going to generalize Theorem 5.10 to all logics of finite depth and width (including e.g., **S5** and **K45**). (Recall that logics of finite width or depth are transitive by definition.) Notice that a si logic of finite depth and width is tabular, hence we will only consider modal logics in this subsection.

Logics of finite depth and finite width behave very similarly to tabular logics, except that they admit infinite clusters (see Theorem 5.16). A simple model-theoretic argument (Lemma 5.13) shows that we may replace infinite clusters by finite ones of size bounded by the length of the formula we are trying to refute; the basic idea of our simulation, apart from what we used in the tabular case, will be to formalize a proof-theoretic version of Lemma 5.13 in extended Frege.

**Definition 5.12** Let $F$ be a transitive Kripke frame, and $m \in \mathbb{N}$. We define $F^m$ as the frame obtained from $F$ by reducing all clusters to size at most $m$. (In particular, $F^1 = \varrho F$ is the skeleton of $F$.)

**Lemma 5.13** *Let $F$ be a transitive frame, and $\varphi$ a formula. Then $F \vDash \varphi$ iff $F^{|\varphi|} \vDash \varphi$. In particular, $L(F) = \bigcap_{m \in \mathbb{N}} L(F^m)$.*

*Proof:* $F^m$ is a p-morphic image of $F$, thus $L(F) \subseteq L(F^m)$. Let $\Vdash$ be a valuation in $F$ which refutes $\varphi$, let $\varphi_1, \dots, \varphi_k$ be the list of all formulas such that $\Box\varphi_i$ is a subformula of $\varphi$, and put $\varphi_0 = \varphi$, $m = k + 1 \le |\varphi|$. If $C$ is a cluster in $F$ such that $|C| > m$, we pick a subset $C' \subseteq C$ of size $m$ such that

$$\exists x \in C \, x \nVdash \varphi_i \Rightarrow \exists x \in C' \, x \nVdash \varphi_i$$

for all $i$, otherwise we put $C' = C$. Let $F'$ be the union of all $C'$. Then $F' \simeq F^m$, and a straightforward induction on the complexity of $\psi$ shows that

$$F, x \Vdash \psi \quad \text{iff} \quad F', x \Vdash \psi$$

for every $x \in F'$ and $\psi$ a subformula of $\varphi$, thus $\varphi$ is refuted in $F'$. $\square$

**Definition 5.14** Let $F$ be a finite transitive frame with root 0. The *frame formula* $\alpha^\sharp(F,\bot)$ is defined as

$$\bigwedge_{x\neq y}\boxdot(p_x\to\neg p_y)\wedge\boxdot\bigvee_x p_x\wedge\bigwedge_{xRy}\boxdot(p_x\to\Diamond p_y)\wedge\bigwedge_{x\not R y}\boxdot(p_x\to\Box\neg p_y)\to\neg p_0.$$

**Lemma 5.15** *Let $G$ be a transitive frame, and $u\in G$. Then $\alpha^\sharp(F,\bot)$ is refutable in $u$ iff there exists a p-morphism $f\colon G_u\twoheadrightarrow F$ such that $f(u)=0$. In particular, $G\nvDash\alpha^\sharp(F,\bot)$ iff $F$ is a p-morphic image of a generated subframe of $G$.*

*Proof:* Easy, cf. [7]. $\Box$

**Theorem 5.16** *Let $L$ be a transitive modal logic of finite depth and width.*

*(i) There exists a countable Kripke frame $F$ such that $L=L(F)$, and $\varrho F$ is finite.*

*(ii) $L$ is finitely axiomatizable.*

*Proof:* Let $G$ be the disjoint union of skeletons of all rooted $L$-frames. $G$ is finite, as there are only finitely many non-isomorphic rooted frames of a given finite depth and width without proper clusters. Let $R$ be the set of reflexive points of $G$. For any $s\colon R\to\mathbb{N}\cup\{\omega\}$, let $G_s$ be the Kripke frame such that $\varrho G_s=G$, and for every $x\in R$ the size of the cluster of $G_s$ which collapses to $x$ is $s(x)$. Put $X=\{s;\,G_s\vDash L\}$. As $L$ has finite depth, it has the finite model property by Segerberg's theorem, hence it is complete wrt $\{G_s;\,s\in X\}$. We endow $\mathbb{N}\cup\{\omega\}$ with the natural linear order, and the corresponding ordinal topology. The set $S:=(\mathbb{N}\cup\{\omega\})^R$ is then partially ordered by the product order, and it carries the product topology.

**Claim 1** *$X$ is a clopen lower set.*

*Proof:* $X$ is a lower set, as the class of all $L$-frames is closed under p-morphisms. This also implies that $X$ is open, as $[\leftarrow,a]:=\{s\in S;\,s\le a\}=\prod_{x\in R}[1,a(x)]$ is an open set in the product topology for any $a\in S$. Let $s\in\overline{X}$. For any $m\in\mathbb{N}$, define $s^m\in S$ by $s^m(x)=\min\{s(x),m\}$. As $[s^m,s]$ is an open neighbourhood of $s$, and $X$ is a lower set, we must have $s^m\in X$. Then $s\in X$ by Lemma 5.13. $\Box$ (Claim 1)

As $X$ is lower, we have $X=\bigcup_{s\in X}[\leftarrow,s]$. The sets $[\leftarrow,s]$ are open, and $X$ is compact (being a closed subset of $S$, which is compact by Tychonoff's theorem), thus there exists a finite set $\{s_i;\,i<k\}$ such that $X=\bigcup_{i<k}[\leftarrow,s_i]$. This implies $L=\bigcap_{i<k}L(G_{s_i})$, thus $F:=\sum_{i<k}G_{s_i}$ satisfies the requirements of (i).

The set $S\smallsetminus X$ is upper and clopen, hence $S\smallsetminus X=\bigcup_{i<\ell}[s_i,\to]$ for some $\ell\in\omega$ and $s_i\in\mathbb{N}^R$ by a similar compactness argument. For every $i<\ell$ pick a rooted generated subframe $H_i\subseteq\cdot\, G_{s_i}$ such that $H_i\nvDash L$. Assume that $L$ has depth and width at most $k$, and let $Y$ be the (finite) set of all rooted frames of depth and width at most $k$ with no proper clusters, which are not models of $L$. We claim that $L$ can be finitely axiomatized as

$$L=\mathbf{K4BD}_k\mathbf{BW}_k\oplus\{\alpha^\sharp(H,\bot);\,H\in Y\}\oplus\{\alpha^\sharp(H_i,\bot);\,i<\ell\}=:L'.$$

The inclusion $L' \subseteq L$ follows immediately from the definition. As a logic of finite depth, $L'$ has the finite model property, thus to prove $L \subseteq L'$ it suffices to show that any rooted finite $L'$-frame $H$ is an $L$-frame. Clearly $H$ has depth and width at most $k$, and $\varrho H \notin Y$, hence $\varrho H \vDash L$. It follows that $\varrho H$ is (isomorphic to) a disjoint summand of $G$. Let $s \in S$ be such that $H$ is a disjoint summand of $G_s$, and $G_s \smallsetminus H$ has no proper clusters. As $G$ is an $L'$-frame, we have $G_s \vDash L'$. We cannot have $s \in S \smallsetminus X$ as $H_i \nvDash L'$, thus $G_s$ and $H \subseteq\cdot\, G_s$ are $L$-frames. $\square$

**Corollary 5.17** *Inclusion is converse well-founded on the set of all transitive logics of finite depth and width.*

*Proof:* The union of an infinite strictly increasing chain of logics cannot be finitely axiomatizable. $\square$

**Lemma 5.18** *Let $F$ and $L$ be as in Theorem 5.16. Given an $L$-SF-proof of a formula $\varphi$, and $m \in \mathbb{N}$ (in unary), we can construct in polynomial time a* **CPC**-*CF-proof of the circuit* $\varphi^{F^m}$.

*Proof:* We construct a substitution circuit Frege proof of $\varphi^{F^m}$ by induction on the length of the proof of $\varphi$ as in Lemma 5.7. The induction steps for modus ponens, necessitation, and substitution are handled in the same way as in 5.7, hence it suffices to deal with axioms $\alpha$ of $L$. As we work with a *SF* system, we only need to consider the constant-size base form of the axioms, not their substitution instances. It thus suffices to show the following: for any fixed $L$-tautology $\alpha$, there are **CPC**-*CF*-proofs of $\alpha^{F^m}$ constructible in time polynomial in $m$.

**Claim 1** *Given a formula $\varphi$, and $m \geq |\varphi|$, we can construct in polynomial time a* **CPC**-*CF-proof of*

$$\sigma\varphi^{F^{|\varphi|}} \to \varphi^{F^m}$$

*for some substitution $\sigma$.*

*Proof:* Put $k = |\varphi|$, let $\Box\varphi_1, \dots, \Box\varphi_{k-1}$ include all boxed subformulas of $\varphi$, and define $\varphi_k = \varphi$. If $x$ belongs to a cluster of size at most $k$ in $F$, we put $\sigma p_{i,x} = p_{i,x}$. Let $C$ be a cluster of $F$ such that $|C| > k$ (a *large cluster*). We fix an enumeration $\{x_{C,1}, \dots, x_{C,k}\}$ of its image $C^k$ in $F^k$, and an enumeration (possibly with repetitions) $\{y_{C,1}, \dots, y_{C,m}\}$ of its image $C^m$ in $F^m$. Then we define

$$\sigma p_{i,x_{C,j}} = \neg\varphi_j^{y_{C,1}} \mathbin{?} p_{i,y_{C,1}} : \neg\varphi_j^{y_{C,2}} \mathbin{?} p_{i,y_{C,2}} : \dots : \neg\varphi_j^{y_{C,m-1}} \mathbin{?} p_{i,y_{C,m-1}} : p_{i,y_{C,m}}.$$

By induction on the complexity of a subformula $\psi$ of $\varphi$, we construct proofs of

$$(*) \qquad \sigma\psi^{x,F^k} \leftrightarrow \psi^{x,F^m}$$

for $x$ whose cluster in $F$ has size at most $k$, and

$$(**) \qquad \sigma\psi^{x_{C,j}} \leftrightarrow \neg\varphi_j^{y_{C,1}} \mathbin{?} \psi^{y_{C,1}} : \neg\varphi_j^{y_{C,2}} \mathbin{?} \psi^{y_{C,2}} : \dots : \neg\varphi_j^{y_{C,m-1}} \mathbin{?} \psi^{y_{C,m-1}} : \psi^{y_{C,m}}$$

for any large cluster $C$. The steps for variables and Boolean connectives are straightforward. Let $\psi = \Box\varphi_j$. We first prove

$$\bigwedge_{x \in C^k} \sigma\varphi_j^x \leftrightarrow \bigwedge_{y \in C^m} \varphi_j^y$$

for any large cluster $C$. The right-to-left implication is immediate from the induction hypothesis. The left-to-right implication follows by

$$\begin{aligned}
\neg \bigwedge_{y \in C^m} \varphi_j^y &\to \bigvee_{i=1}^{m} \Big(\neg\varphi_j^{y_{C,i}} \wedge \bigwedge_{i'=1}^{i-1} \varphi_j^{C,i'}\Big) \\
&\to \bigvee_{i=1}^{m} (\neg\varphi_j^{y_{C,i}} \wedge (\sigma\psi^{x_{C,j}} \leftrightarrow \varphi_j^{y_{C,i}})) \\
&\to \neg\sigma\varphi_j^{x_{C,j}} \\
&\to \neg \bigwedge_{x \in C^k} \sigma\varphi_j^x
\end{aligned}$$

from the induction hypothesis. Then for any $x \in F^k$, and $y \in F^m$ which belong to the images of the same cluster of $F$, we have

$$\sigma(\Box\varphi_j)^x = \bigwedge_{xRx'} \sigma\varphi_j^{x'} \leftrightarrow \bigwedge_{yRy'} \varphi_j^{y'} = (\Box\varphi_j)^y,$$

which implies the induction statement for $\Box\varphi_j$.

We take $(*)$ and $(**)$ for $\psi = \varphi$, and derive

$$\bigwedge_{x \in F^k} \sigma\varphi^x \leftrightarrow \bigwedge_{y \in F^m} \varphi^y$$

as in the induction step for $\Box$, using $\varphi = \varphi_k$. We obtain $\sigma\varphi^{F^k} \to \varphi^{F^m}$. $\Box$ (Claim 1)

We finish the proof of Lemma 5.18 as follows. If $m \le |\alpha| = O(1)$, we take any proof of $\alpha^{F^m}$. If $m \ge |\alpha|$, we take a proof of $\alpha^{F^{|\alpha|}}$, and apply the claim. $\Box$

**Remark 5.19** Readers familiar with bounded arithmetic may substitute the proof of Lemma 5.18 with the following high-level argument. Lemma 5.13 is obviously provable in the theory $V^1$, and then a straightforward induction (inside $V^1$) on the length of an $L$-$SF$-proof $\pi$ of a formula $\varphi$ shows that $\varphi$ is valid in the frame $F^m$, hence $\varphi^{F^m}$ is a tautology. The propositional translation of this variant of a reflection principle thus has poly-time constructible proofs in **CPC**-$CF$, and implies $\varphi^{F^m}$ if we plug in constants describing a concrete proof $\pi$.

**Theorem 5.20** *If $L$ is a (transitive) modal logic of finite depth and width, then $L$-$EF \equiv_p$ $L$-$SF$, and $L$-$EF$ is interpretable in* **CPC**-$EF$.

*Proof:* The proof goes by reverse induction on $L$, which is possible by Corollary 5.17. Let $L = L(F)$, where $F$ is as in Theorem 5.16. By the same reasoning as in Theorem 5.10, we may assume that $F$ is rooted. Put $G := F^K$, where $K := 1 + \max\{|C|;\, C$ is a finite cluster of $F\}$.

Let $\alpha := \alpha^\sharp(G, \perp)$ in variables $c_x$ instead of $p_x$, and let 0 be the root of $G$ used in the definition of $\alpha$. Fix a p-morphism $h\colon F \twoheadrightarrow G$ such that $h(x) = x$ if $|\mathrm{cl}(x)| < \omega$, and for every infinite cluster $C$ in $F$, $h(C) = C^K$, and $h^{-1}[x]$ is infinite for every $x \in C^K$. Let $\Vdash^c$ be the valuation of the variables $c_x$ in $F$ such that $y \Vdash c_x$ iff $h(y) = x$, and fix $\overline{0} \in h^{-1}[0]$.

**Claim 1**

*(i)* $\alpha \vee \psi \in L$ *iff* $\overline{0} \Vdash \psi$ *under every valuation* $\Vdash$ *in* $F$ *which extends* $\Vdash^c$.

*(ii)* $L$ *proves*

$$\bigwedge_{x \sim 0} \alpha(c_0/c_x, c_x/c_0) \to \boxdot\alpha.$$

*Proof:* (i): The left-to-right implication is obvious, as $\overline{0} \nVdash^c \alpha$. Right-to-left: let $u \in F$, and let $\Vdash$ be a valuation such that $u \nVdash \alpha$, we will show $u \Vdash \psi$. Using Lemma 5.15 there exists a p-morphism $f\colon F_u \twoheadrightarrow G$ such that $f(u) = 0$, and $y \Vdash c_x$ iff $f(y) = x$ for every $y \in F_u$. Let $I$ be the set of infinite clusters of $F$, and $F' = F \smallsetminus \bigcup I$. If $C \in I$, there exists a cluster $D$ in $F$ such that $f(D) = C^K$; by the definition of $K$, $D$ must be infinite. Hence $f$ induces a partial surjection $f^\infty$ from $I$ onto itself. As $I$ is finite, $f^\infty$ is total, and it is a bijection. In particular $f(\bigcup I) \subseteq \bigcup I$, thus $f^{-1}[F'] \subseteq F'$, i.e., $f \restriction F'$ is a partial surjection of $F'$ onto itself. Again, $F'$ is finite, thus $f \restriction F'$ is total, and it is a bijection. In particular, $F_u = \mathrm{dom}(f) = F$, thus $u$ is a root of $F$.

It follows that there exists a function $g\colon F \to F$ such that $f^{-1}[x] = \{g(x)\}$ if $x$ is finite, and for every $C \in I$ and $x \in C^K$, $g$ is a surjection of $h^{-1}[x]$ onto $f^{-1}[x]$. Without loss of generality we may assume $g(\overline{0}) = u$. It is easy to see that $g$ is a p-morphism, and $f \circ g = h$. Let $\Vdash^*$ be the valuation defined by $y \Vdash^* \chi$ iff $g(y) \Vdash \chi$. Then $\Vdash^*$ extends $\Vdash^c$, hence $\overline{0} \Vdash^* \psi$, and $u = g(\overline{0}) \Vdash \psi$.

(ii): Assume that $u \nVdash \boxdot\alpha$ for some valuation $\Vdash$, and $u \in F$. There exists a $v \geq u$ such that $v \nVdash \alpha$, hence there exists a p-morphism $f\colon F_v \twoheadrightarrow F$ such that $f(v) = 0$, and $y \Vdash c_x$ iff $f(y) = x$. By the proof of (i), $\mathrm{dom}(f) = F$, hence $v$ and $u$ are roots of $F$, and $x := f(u)$ is a root of $G$, thus $u \nVdash \alpha(c_0/c_x, c_x/c_0)$. □ (Claim 1)

As $\alpha$ is refutable in $F$, $L' := L \oplus \alpha$ is a proper extension of $L$. By the induction hypothesis, there exists a poly-time interpretation $(\cdot)^\#$ of $L'$-*SF* $\equiv_p$ $L'$-*CF* in **CPC**-*CF*. We define

$$\varphi^* := \varphi^\# \wedge \varphi^{F^N},$$

where $N = \max\{K, |\varphi|\}$. Given an $L$-*SF*-proof of $\varphi$, we can construct in polynomial time a **CPC**-*CF*-proof of $\varphi^*$ by Lemma 5.18. Given a **CPC**-*CF*-proof of $\varphi^*$, we can construct an $L'$-*CF*-proof of $\varphi$ by the induction hypothesis. Using the same reasoning as in Theorem 5.10, it thus suffices to construct an $L$-*CF*-proof of $\boxdot\alpha \vee \varphi$ from a **CPC**-*CF*-proof of $\varphi^{F^N}$. Moreover, part (ii) of Claim 1 implies that it suffices to construct a proof of $\alpha \vee \varphi$.

We denote the accessibility relations in $F^K$ and $F^N$ as $R^K$ and $R^N$, respectively. Assume that $\varphi$ uses the variables $p_0, \ldots, p_{k-1}$. Let $\Box\varphi_1, \ldots, \Box\varphi_{N-1}$ include all boxed subformulas of $\varphi$, and put $\varphi_0 = \varphi$. Let $C$ be any infinite cluster of $F$. We fix enumerations $C^K = \{y_{C,j};\ j < K\}$, $C^N = \{x_{C,j};\ j < N\}$, and we put

$$c_C := \bigvee_{y \in C^K} c_y.$$

For any $i < k$ we define the circuits $P_i^{C,-1}$ as

$$P_i^{C,-1} := \Box\Big(c_C \wedge \bigwedge_{i'<i} (p_{i'} \leftrightarrow P_{i'}^{C,-1}) \to p_i\Big)$$

by induction on $i$, and we define the circuits $P_i^{C,j}$ for $j < N$ as

$$P_i^{C,j} := \Box(c_C \to \varphi_j) \mathbin{?} P_i^{C,j-1} : \Box\Big(c_C \wedge \bigwedge_{i'<i} (p_{i'} \leftrightarrow P_{i'}^{C,j}) \to p_i \vee \varphi_j\Big)$$

by induction on $jk + i$. We define

$$c^{x_{C,j}} := c_C \wedge \bigwedge_{i<k} (p_i \leftrightarrow P_i^{C,j}).$$

If $x \in F$ is such that $\mathrm{cl}(x)$ is finite, we put $c^x := c_x$. Let $\sigma$ be the substitution such that

$$\sigma p_{i,x} = \boxdot(c^x \to p_i).$$

**Claim 2** *There are poly-time constructible $L$-$CF$-proof of $\alpha \vee \beta$, where $\beta$ is*

*(i)* $\boxdot(c_C \to P_i^{C,j}) \vee \boxdot(c_C \to \neg P_i^{C,j})$,

*(ii)* $\boxdot(c^x \to \psi) \vee \boxdot(c^x \to \neg\psi)$ *for any subformula* $\psi$ *of* $\varphi$,

*(iii)* $\Box(c^{x_{C,j}} \to \varphi_j) \to \Box(c_C \to \varphi_j)$,

*(iv)* $\diamonddot c^x$,

*(v)* $\boxdot(c^x \to \Diamond c^y)$, *if* $x \mathrel{R^N} y$,

*(vi)* $\sigma\psi^x \leftrightarrow \boxdot(c^x \to \psi)$ *for any subformula* $\psi$ *of* $\varphi$.

*Proof:* (i): Notice that $P_i^{C,j}$ is a Boolean combination of boxed circuits. We can thus prove the statement by induction on $jk + i$, using instances of the formula

$$(*) \qquad \alpha \vee \boxdot(c_C \to \Box q) \vee \boxdot(c_C \to \neg\Box q),$$

which is provable in $L$ by Claim 1.

(ii): If $\mathrm{cl}_F(x)$ is finite, we use an instance of

$$\alpha \vee \boxdot(c_x \to q) \vee \boxdot(c_x \to \neg q),$$

which follows from Claim 1. If $x = x_{C,j}$, we proceed by induction on the complexity of $\psi$. We use (i) for variables, $(*)$ for boxed formulas, and the steps for Boolean connectives are obvious.

(iii): We prove

$$\alpha \vee \Big(\Diamond(c_C \wedge \neg\varphi_j) \to \Diamond\Big(c_C \wedge \bigwedge_{i'<i}(p_{i'} \leftrightarrow P^{C,j}_{i'}) \wedge \neg\varphi_j\Big)\Big)$$

by induction on $i$, using the definition of $P^{C,j}_i$, and instances of

$$\alpha \vee \Box(c_C \wedge q \to \Diamond q).$$

(iv): If $\mathrm{cl}_F(x)$ is finite, we have $\alpha \vee \dot\Diamond c_x$ by Claim 1. Let $C$ be an infinite cluster. We have $\alpha \vee \Diamond c_C$ by Claim 1, and we derive

$$\alpha \vee \Diamond\Big(c_C \wedge \bigwedge_i (p_i \leftrightarrow P^{C,-1}_i)\Big)$$

as in (iii). Then we prove $\alpha \vee \Diamond c^{x_{C,j}}$ by induction on $j$, using (iii) and the definition of $P^{C,j}_i$.

(v): If $\mathrm{cl}_F(y)$ is finite, we have $\alpha \vee \boxdot(c^x \to \Diamond c_y)$ from Claim 1. If $y \in C^N$ for an infinite cluster $C$, we have $\alpha \vee \boxdot(c^x \to \Diamond c_C)$ from Claim 1, and $\alpha \vee \boxdot(c_C \to \Diamond c^y)$ from (iv) and $(*)$.

(vi): By induction on the complexity of $\psi$. The steps for variables and $\wedge$ are trivial, and the step for $\neg$ follows from (ii) and (iv). Let $\psi = \Box\varphi_j$. We construct a proof of

$$\alpha \vee \Big(\sigma(\Box\varphi_j)^x \leftrightarrow \bigwedge_{xR^N y} \boxdot(c^y \to \varphi_j)\Big)$$

by the induction hypothesis, and the definition of $(\Box\varphi_j)^x$. It thus suffices to prove

$$\alpha \vee \Big(\bigwedge_{xR^N y} \boxdot(c^y \to \varphi_j) \leftrightarrow \boxdot(c^x \to \Box\varphi_j)\Big).$$

Right-to-left: we have

$$\alpha \vee (\boxdot(c^x \to \Box\varphi_j) \to \dot\Diamond(c^y \wedge \varphi_j))$$

by (iv) and (v), thus

$$\alpha \vee (\boxdot(c^x \to \Box\varphi_j) \to \boxdot(c^y \to \varphi_j))$$

by (ii). Left-to-right: we have

$$\alpha \vee \Big(\bigwedge_{xR^N y} \boxdot(c^y \to \varphi_j) \to \bigwedge_{xR^K y} \boxdot(c_y \to \varphi_j)\Big)$$

by (iii), and

$$\alpha \vee \boxdot\Big(c^x \to \Box \bigvee_{xR^K y} c_y\Big)$$

by Claim 1. $\Box$ (Claim 2)

We apply $\sigma$ the **CPC**-$CF$-proof of $\varphi^{F^N}$ we are given. By Claim 2, we construct an $L$-$CF$-proof of

$$\alpha \vee \bigwedge_x \boxdot(c^x \to \varphi).$$

If the root cluster $C$ of $F$ is finite, we conclude $\alpha \vee \varphi$ from $\alpha \vee c_0$. If $C$ is infinite, we obtain

$$\alpha \vee \Box(c_C \to \varphi)$$

from (iii) of Claim 2, we use an instance of

$$\alpha \vee (\Box q \to q)$$

(which is provable by Claim 1, and reflexivity of the root cluster) to derive

$$\alpha \vee (c_C \to \varphi),$$

and we conclude $\alpha \vee \varphi$ from $\alpha \vee c_C$, which is provable by Claim 1. $\square$

### 5.3 Some logics of finite width

Typical modal and si logics do *not* have finite depth, and the purpose of the present subsection is to attempt simulations of *SF* by *EF* in some logics of finite width, but unbounded depth. As our strategy only applies to *coNP* logics, we need additional restrictions on the logic. A convenient requirement is to consider only *cofinal subframe logics*: on the one hand, csf logics of finite width have the poly-size model property, and are decidable in *coNP* (if finitely axiomatizable); on the other hand, most of the standard transitive logics are csf, including combinations of $\mathbf{K4BW}_k$ with **S4**, **GL**, **Grz**, **K4.1**, **K4.2**, etc. Of particular importance are csf logics of width 1: the Gödel–Dummett logic **LC** is one of the fundamental fuzzy logics, and $\mathbf{K4.3} = \mathbf{K4BW}_1$ and its variants (**S4.3**, **GL.3**, **Grz.3**, ...) are used as logics of time. However, we will need more restrictions, either on the set of formulas, or on the logic.

The idea is to find a proof-theoretic version of the following argument, which explains why csf logics of finite width have the poly-size model property. Consider a model of fixed width $k$ refuting a formula $\varphi$, and define its submodel as follows: for each boxed subformula $\Box\psi$ of $\varphi$, pick a point from each maximal cluster where $\psi$ is refuted. (It is actually sufficient to consider only formulas $\Box\psi$ occurring *positively* in $\varphi$.) For fixed $\psi$ these points make an antichain, hence there are at most $k$ of them. We thus obtain a model of size at most $k|\varphi|$, which also refutes $\varphi$.

There are serious obstacles in feasible proof-theoretic formulation of the argument: we cannot define maximal points satisfying a given formula (save in **GL**), and we cannot identify (label) the individual points of the new model from inside.

Our first result in this subsection will apply to all csf logics, but only to restricted classes of formulas: those where the argument above uses only a constant number $c$ of antichains. Notice that the argument then gives nontrivial information even for logics of infinite width: it reduces the depth of the model to a constant. We will in fact formalize this more general statement, and obtain the simulation of *SF* in *EF* in the special case of finite width logics as a corollary using the results of the previous subsection.

**Definition 5.21** The *width* of a modal formula $\varphi$ is the number of distinct positively occurring boxed subformulas of $\varphi$. The width of an intuitionistic formula is the number of distinct succedents of implications occurring positively in $\varphi$. For any $c \geq 0$, let $\Gamma_c$ be the set of all formulas of width at most $c$.

**Theorem 5.22** *If $c \geq 0$, and $L$ is a cofinal subframe logic, then $(L \oplus \mathrm{BD}_{c+2})$-EF $\leq_{p,\Gamma_c}$ $L$-EF.*

*Proof:* We consider the modal case first. Assume $\varphi \in \Gamma_c$, and let $\Box\varphi_1, \dots, \Box\varphi_c$ be all boxed formulas occurring positively in $\varphi$. Put $\varphi_0 := \varphi$, $\varphi_{c+1} := \bot$, and $m := c + 2$. Recall that

$$\begin{aligned} \mathrm{BD}_0 &= \bot, \\ \mathrm{BD}_{n+1} &= p_n \vee \Box(\Box p_n \to \mathrm{BD}_n). \end{aligned}$$

Let $\pi$ be an $(L \oplus \mathrm{BD}_m)$-*CF*-proof of $\varphi$, and $\Theta$ the set of all subcircuits of $\pi$. For any circuit $\alpha$, let $\psi \mapsto \psi^\alpha$ be the translation which preserves propositional variables, commutes with Boolean connectives, and

$$(\Box\psi)^\alpha = \Box(\alpha \to \psi^\alpha).$$

**Claim 1** *If $\psi \in L$, then $\alpha \wedge \Box\Diamond\alpha \to \psi^\alpha \in L$.*

*Proof:* Let $W$ be an $L$-frame such that $x \Vdash \alpha \wedge \Box\Diamond\alpha \wedge \neg\psi^\alpha$ for some $x \in W$, and a valuation $\Vdash$. Let $W'$ be the cofinal subframe of $W$ defined by

$$W' = \{y \in W;\ y \Vdash \alpha\}.$$

Then $W'$ is an $L$-frame as $L$ is csf, and a straightforward induction on the complexity of $\psi$ shows that

$$W', y \Vdash \psi \quad \text{iff} \quad W, y \Vdash \psi^\alpha$$

for any $y \in W'$, hence $W', x \nVdash \psi$. $\Box$ (Claim 1)

For any subsets $Y \subseteq X \subseteq m$ such that $Y \neq \varnothing$, we define the circuits

$$\begin{aligned} \alpha &:= \bigvee_{X \subseteq m} \alpha_X, \\ \alpha_X &:= \bigvee_{\varnothing \neq Y \subseteq X} \alpha_{X,Y}, \\ \alpha_{X,Y} &:= \beta_{X,Y} \wedge \bigvee_{i \in Y} \neg\varphi_i, \\ \beta_{X,Y} &:= \gamma_{X,Y} \wedge \bigwedge_{\substack{\vartheta \in \Theta \\ i \in Y}} \big(\boxdot(\xi_{X,Y} \to \vartheta^{\xi_{X,Y}}) \vee \Box(\Box(\xi_{X,Y} \to \vartheta^{\xi_{X,Y}}) \to \varphi_i)\big), \\ \gamma_{X,Y} &:= \bigwedge_{i \notin X} \boxdot\varphi_i \wedge \bigwedge_{i,j \in Y} \boxdot(\boxdot\varphi_j \to \varphi_i) \wedge \bigwedge_{\substack{i \in Y \\ j \in X \smallsetminus Y}} \boxdot(\boxdot(\boxdot\varphi_i \to \varphi_j) \to \varphi_i), \\ \xi_{X,Y} &:= \bigvee_{i \in Y} \neg\varphi_i \vee \bigvee_{Z \subseteq X \smallsetminus Y} \alpha_Z. \end{aligned}$$

**Claim 2** *There are poly-time constructible* **K4**-*CF-proofs of*

(*i*) $\boxdot(\alpha \to \varphi_i) \to \varphi_i$,

(*ii*) $\psi^\alpha \to \psi$,

*for any positively occurring subformula* $\psi$ *of* $\varphi$.

*Proof:* (i): We first show that the formula is a tautology. If $W$ is a finite transitive frame, $x \in W$, and $x \nVdash \varphi_i$, let $y \geq x$ be a maximal point where $\varphi_i$ fails, and

$$\begin{aligned} X &:= \{j \in m;\, y \nVdash \boxdot\varphi_j\}, \\ Y &:= \{j \in X;\, (y{\uparrow} \smallsetminus y{\downarrow}) \Vdash \varphi_j\}. \end{aligned}$$

Then it is easy to see that $y \Vdash \alpha_{X,Y}$, hence $x \nVdash \boxdot(\alpha \to \varphi_i)$.

We construct a short proof of $\boxdot(\alpha \to \varphi_i) \to \varphi_i$ as follows. The circuit

$$(*) \qquad \neg\varphi_i \to \Diamond\!\!\cdot\Big(\neg\varphi_i \wedge \bigvee_{X \supseteq Y \ni i} \gamma_{X,Y}\Big)$$

is an instance of a fixed tautology, as $m$ is a constant. For any $Y \subseteq X \subseteq m$, $i \in Y$, we can construct a proof of

$$\neg\varphi_i \to \Diamond\!\!\cdot\Big(\neg\varphi_i \wedge \bigwedge_{\vartheta \in \Theta} \big(\boxdot(\xi_{X,Y} \to \vartheta^{\xi_{X,Y}}) \vee \Box(\Box(\xi_{X,Y} \to \vartheta^{\xi_{X,Y}}) \to \varphi_i)\big)\Big)$$

by induction on $|\Theta|$, using instances of $\neg p \to \Diamond\!\!\cdot(\neg p \wedge (\boxdot q \vee \Box(\Box q \to p)))$. We combine it with $(*)$ to obtain a proof of

$$\neg\varphi_i \to \Diamond\!\!\cdot\Big(\neg\varphi_i \wedge \bigvee_{X \supseteq Y \ni i} \beta_{X,Y}\Big) \to \Diamond\!\!\cdot(\neg\varphi_i \wedge \alpha),$$

using $\gamma_{X,Y} \to \boxdot\gamma_{X,Y}$.

(ii): By induction on the complexity of $\psi$. The steps for variables, negated variables, $\wedge$, $\vee$, and $\Diamond$ are easy, and the step for $\Box$ follows from (i). □ (Claim 2)

**Claim 3** *There are poly-time constructible* **K4**-*CF-proofs of*

(*i*) $\beta_{X,Y} \to \boxdot(\alpha \leftrightarrow \xi_{X,Y})$,

(*ii*) $\alpha \to \bigvee_{i<m} \Big(\neg\varphi_i \wedge \bigwedge_{\vartheta \in \Theta} \big(\vartheta^\alpha \vee \Box(\Box(\alpha \to \vartheta^\alpha) \to \varphi_i)\big)\Big)$.

*Proof:* (ii) follows from (i), and the definition of $\alpha_{X,Y}$.

(i): It suffices to prove $\beta_{X,Y} \to (\alpha \leftrightarrow \xi_{X,Y})$, as $\beta_{X,Y} \to \boxdot\beta_{X,Y}$. The implication

$$\beta_{X,Y} \wedge \xi_{X,Y} \to \alpha$$

is straightforward: $\xi_{X,Y}$ gives either $\bigvee_{Z \subseteq X \smallsetminus Y} \alpha_Z$, which implies $\alpha$ by definition, or $\bigvee_{i \in Y} \neg\varphi_i$, which implies $\alpha_{X,Y}$ (hence $\alpha$) by $\beta_{X,Y}$.

Let $\varnothing \neq W \subseteq Z$, we have to show

$$\beta_{X,Y} \wedge \alpha_{Z,W} \to \xi_{X,Y}.$$

The cases $Z \subseteq X \smallsetminus Y$, or $W \subseteq Y$, are clear. We distinguish three other cases.

$Z \nsubseteq X$: pick $i \in Z \smallsetminus X$. We have $\boxdot\varphi_i$ by $\gamma_{X,Y}$, and $\boxdot(\boxdot\varphi_i \to \varphi_j)$ for every $j \in W$ by $\gamma_{Z,W}$, hence $\bigwedge_{j\in W} \varphi_j$, and $\neg\alpha_{Z,W}$.

$Z \subsetneq X$, $Z \nsubseteq X \smallsetminus Y$: let $i \in Z \cap Y$, and $j \in X \smallsetminus Z$. We have $\boxdot\varphi_j$ by $\gamma_{Z,W}$, and $\boxdot(\boxdot\varphi_j \to \varphi_i)$ by $\gamma_{X,Y}$, thus $\boxdot\varphi_i$. This implies $\bigwedge_{k\in W} \boxdot\varphi_k$ by $\gamma_{Z,W}$, hence $\neg\alpha_{Z,W}$.

$Z = X$, $W \nsubseteq Y$: let $i \in W \smallsetminus Y$, and $j \in Y$. We have $\boxdot(\boxdot\varphi_j \to \varphi_i)$ by $\gamma_{Z,W}$, hence $\boxdot\varphi_j$ by $\gamma_{X,Y}$. This implies $\bigwedge_{k\in W} \varphi_k$, and $\neg\alpha_{Z,W}$. □ (Claim 3)

**Claim 4** *If $\vartheta_0, \ldots, \vartheta_m \in \Theta$, there are poly-time constructible* **K4**-*CF-proofs of*

$$\alpha \to (\mathrm{BD}_m(\vartheta_0, \ldots, \vartheta_m))^\alpha.$$

*Proof:* We will construct proofs of

$$\alpha \wedge \bigwedge_{i \notin X} \boxdot\varphi_i \to (\mathrm{BD}_{|X|}(\vartheta_0, \ldots, \vartheta_{|X|}))^\alpha$$

for every $X \subseteq m$ by induction on $|X|$. The base case $X = \varnothing$ is clear, as we have

$$\bigwedge_{i<m} \boxdot\varphi_i \to \neg\alpha.$$

Assume $k = |X| > 0$. We have

$$\begin{aligned}
\alpha \wedge \bigwedge_{i \notin X} \boxdot\varphi_i &\to \bigvee_{i\in X} \big(\vartheta_k^\alpha \vee \Box(\Box(\alpha \to \vartheta_k^\alpha) \to \varphi_i)\big) \\
&\to \vartheta_k^\alpha \vee \bigvee_{i\in X} \Box\Big(\Box(\alpha \to \vartheta_k^\alpha) \to \bigwedge_{j \notin X \smallsetminus \{i\}} \boxdot\varphi_j\Big) \\
&\to \vartheta_k^\alpha \vee \Box\big(\Box(\alpha \to \vartheta_k^\alpha) \wedge \alpha \to (\mathrm{BD}_{k-1}(\vartheta_0, \ldots, \vartheta_{k-1}))^\alpha\big) \\
&\to (\mathrm{BD}_k(\vartheta_0, \ldots, \vartheta_k))^\alpha
\end{aligned}$$

by Claim 3, and the induction hypothesis. □ (Claim 4)

We take the proof $\pi\colon \psi_1, \ldots, \psi_n = \varphi$, and construct the sequence

$$\alpha \to \psi_1^\alpha, \ldots, \alpha \to \psi_n^\alpha.$$

We complete it to an $L$-*CF*-proof as follows. If $\psi_i$ is an instance of an axiom $\chi$ of $L$, then $\alpha \wedge \Box\diamondsuit\alpha \to \psi_i^\alpha \in L$ by Claim 1; moreover, it has a proof of linear size, as it is an instance of the constant-size tautology $q \wedge \Box\diamondsuit q \to \chi^q$. We construct a proof of $\diamondsuit\alpha$ by Claim 2 (i) (recall $\varphi_{c+1} = \bot$), and we derive $\alpha \to \psi_i^\alpha$.

If $\psi_i$ was derived by necessitation or modus ponens, we construct a subproof of $\alpha \to \psi_i^\alpha$ easily. If $\psi_i$ is an instance of $\mathrm{BD}_m$, we use Claim 4.

We thus obtain a $L$-$CF$-proof of $\alpha \to \varphi^\alpha$. We derive $\alpha \to \varphi$ by Claim 2 (ii), hence $\varphi$ by Claim 2 (i).

The proof in the intuitionistic case is quite analogous, we sketch it below. If $\alpha = \{\langle \beta_i, \delta_i \rangle;\ i < n\}$ is a set of pairs of circuits, we define the translation $\psi \mapsto \psi^\alpha$ which preserves variables and $\bot$, commutes with $\vee$ and $\wedge$, and

$$(\psi \to \chi)^\alpha = \bigwedge_i (\beta_i \wedge \psi^\alpha \to \chi^\alpha \vee \delta_i).$$

As in Claim 1, we can show

$$\psi \in L \quad \Rightarrow \quad \neg(\top \to \bot)^\alpha \to (\top \to \psi)^\alpha \in L.$$

Let $\varphi \in \Gamma_c$, and let $\varphi_1, \dots, \varphi_c$ be all succedents of implications occurring positively in $\varphi$. We put $\varphi_0 := \varphi$, $\varphi_{c+1} := \bot$, and $m := c + 2$. Let $\pi\colon \psi_1, \dots, \psi_n$ be an $(L + \mathrm{BD}_m)$-$CF$-proof of $\varphi$, and $\Theta$ the set of all subcircuits of $\pi$. We define

$$\begin{aligned}
\alpha &:= \bigcup_{X \subseteq m} \alpha_X, \\
\alpha_X &:= \{\langle \beta_{X,Y}, \varphi_i \rangle;\ i \in Y \subseteq X\}, \\
\beta_{X,Y} &:= \gamma_{X,Y} \wedge \bigwedge_{\substack{\vartheta \in \Theta \\ i \in Y}} (\vartheta^{\xi_{X,Y}} \vee (\vartheta^{\xi_{X,Y}} \to \varphi_i)), \\
\gamma_{X,Y} &:= \bigwedge_{i \notin X} \varphi_i \wedge \bigwedge_{i,j \in Y} (\varphi_j \to \varphi_i) \wedge \bigwedge_{\substack{i \in Y \\ j \in X \smallsetminus Y}} ((\varphi_i \to \varphi_j) \to \varphi_i), \\
\xi_{X,Y} &:= \{\langle \top, \varphi_i \rangle;\ i \in Y\} \cup \bigcup_{Z \subseteq X \smallsetminus Y} \alpha_Z.
\end{aligned}$$

Analogously to Claim 2, we can construct **IPC**-$CF$-proofs of

$$\bigwedge_{X \supseteq Y \ni i} (\beta_{X,Y} \to \varphi_i) \to \varphi_i$$

for every $i < m$, and

$$\begin{cases} \psi^\alpha \to \psi & \text{if } \psi \text{ occurs positively,} \\ \psi \to \psi^\alpha & \text{if } \psi \text{ occurs negatively,} \end{cases}$$

for every subformula $\psi$ of $\varphi$. Then we construct **IPC**-$CF$-proofs of

$$\beta_{X,Y} \to (\vartheta^\alpha \leftrightarrow \vartheta^{\xi_{X,Y}}),$$

hence

$$\beta_{X,Y} \to \vartheta^\alpha \vee (\vartheta^\alpha \to \varphi_i),$$

for every $i \in Y \subseteq X$, and $\vartheta \in \Theta$, as in Claim 3. We derive

$$(\top \to \mathrm{BD}_m(\vec{\vartheta}))^\alpha$$

for every $\vec{\vartheta} \in \Theta$ as in Claim 4. Finally, we construct an $L$-$CF$-proof including the circuits

$$(\top \to \psi_1)^\alpha, \dots, (\top \to \psi_n)^\alpha,$$

and we conclude $\varphi$. $\square$

**Corollary 5.23** *If $c \geq 0$, and $L$ is a cofinal subframe logic of finite width, then $L$-EF $\equiv_{p,\Gamma_c}$ $L$-SF.*

*Proof:* Given an $L$-*SF*-proof of a formula $\varphi \in \Gamma_c$, we construct an $(L \oplus \mathrm{BD}_{c+2})$-*EF*-proof of $\varphi$ by Theorem 5.20 (or 5.10 in the intuitionistic case), and then an $L$-*EF*-proof by Theorem 5.22. $\square$

**Example 5.24** Hrubeš tautologies from Definitions 6.23 and 6.28 have constant width, hence they have poly-time constructible *EF* proofs in all logics of finite width by Corollary 5.23, and Lemmas 6.26, 6.29.

Our second result will give a full simulation for all formulas, but it will only apply to a selected list of modal logics: $\mathbf{K4BW}_k$ (including $\mathbf{K4.3}$) and friends. The idea is to avoid the labelling problem mentioned above by employing a variant of selective filtration. In general, filtration produces frames with points labelled by sets of formulas, hence there are potentially exponentially many labels. The bound on width however enables to restrict our attention to sets of special form, of which there are only polynomially many. (Basically, points from an antichain of constant size can be separated by using only constantly many formulas.)

**Theorem 5.25** *Let $L$ be $\mathbf{K4} \oplus \mathrm{BW}_K$, $\mathbf{S4} \oplus \mathrm{BW}_K$, $\mathbf{GL} \oplus \mathrm{BW}_K$, $\mathbf{K4Grz} \oplus \mathrm{BW}_K$, or $\mathbf{S4Grz} \oplus \mathrm{BW}_K$ for some $K > 0$. Then $L$-EF $\equiv_p$ $L$-SF, and $L$-EF is interpretable in $\mathbf{CPC}$-EF.*

*Proof:* Let $\varphi$ be a formula in the variables $p_0, \dots, p_{M-1}$, and put $N := (2|\varphi|)^K$. We define the translations $\psi^i$ for all $i < N$ by induction on the complexity of $\psi$: $(p_\ell)^i = p_{\ell,i}$, $(\cdot)^i$ commutes with Boolean connectives, and

$$(\Box\psi)^i = \bigwedge_{j<N} (r_{i,j} \to \psi^j).$$

Let $\chi_{N,M}$ be the conjunction of the formulas

$$\begin{gathered}
e_{i,j} \to e_{j,i} \\
e_{i,j} \wedge e_{j,k} \to e_{i,k} \\
r_{i,j} \to e_{i,i} \wedge e_{j,j} \\
e_{i,j} \wedge r_{j,k} \to r_{i,k} \\
r_{i,j} \wedge e_{j,k} \to r_{i,k} \\
e_{i,j} \wedge p_{\ell,i} \to p_{\ell,j} \\
r_{i,j} \wedge r_{j,k} \to r_{i,k} \\
\neg r_{i,i} \qquad \text{if } L \supseteq \mathbf{GL} \\
e_{i,i} \to r_{i,i} \qquad \text{if } L \supseteq \mathbf{S4} \\
r_{i,j} \wedge r_{j,i} \to e_{i,j} \qquad \text{if } L \supseteq \mathbf{K4Grz} \\
\bigwedge_{u \leq K} e_{i_u,i_u} \to \bigvee_{u \neq v} (e_{i_i,i_v} \vee r_{i_u,i_v})
\end{gathered}$$

for all $i, j, k < N$, $\ell < M$, and $i_0, \dots, i_K < N$. We define

$$\varphi^* := \chi_{N,M} \to \bigwedge_{i<N} (e_{i,i} \to \varphi^i).$$

**Claim 1** *Given an $L$-SF-proof of $\varphi$, we can construct in polynomial time a* **CPC**-*CF-proof of $\varphi^*$.*

*Proof (sketch):* Fix an assignment $v$ such that $v(\chi_{N,M}) = 1$, and put $E = \{\langle i, j\rangle; v(e_{i,j}) = 1\}$, $X = \{i; E(i, i)\}$, $R = \{\langle i, j\rangle; v(r_{i,j}) = 1\}$, and $P_\ell = \{i; v(p_{\ell,i}) = 1\}$. Then $E$ is an equivalence relation on $X$, congruent wrt $R$ and $P_\ell$. $W := \langle X, R\rangle/E$ is a transitive Kripke frame of width at most $K$ (reflexive, irreflexive, or antisymmetric, as appropriate), hence $W \vDash L$. If we define $i/E \Vdash p_\ell$ iff $P_\ell(i)$, we have

$$i/E \Vdash \psi \quad \text{iff} \quad v(\psi^i) = 1$$

by induction on the complexity of $\psi$, thus $v(\varphi^*) = 1$.

The argument above can be easily formalized in the theory $V^1$, and its propositional translation yields a **CPC**-*CF*-proof of $\varphi^*$. The details are left to the reader. □ (Claim 1)

Assume we are given a **CPC**-*CF*-proof $\pi$ of $\varphi^*$, we will construct an $L$-*CF*-proof of $\varphi$. Let $S$ be the set of subformulas of $\Box\varphi$, and $W$ be the set of pairs $\langle\psi, h\rangle$, where $\Box\psi \in S$, and $h$ is a partial function from $S$ to 2 such that $|\mathrm{dom}(h)| < K$. For each $\Box\psi \in S$, let

$$M(\psi) := \begin{cases} \neg\psi \wedge \Box\psi & \text{if } L \supseteq \mathbf{GL}, \\ \neg\psi \wedge \bigwedge\limits_{\vartheta\in S} \big(\boxdot(\vartheta \vee \Box\psi) \vee \boxdot(\neg\vartheta \vee \Box\psi)\big) & \text{if } L \supseteq \mathbf{K4Grz}, L \not\supseteq \mathbf{GL}, \\ \neg\psi & \text{otherwise.} \end{cases}$$

We fix an enumeration $S = \{\vartheta_j; j < m\}$, and $W = \{\langle\psi_i, h_i\rangle; i < N\}$. For each $i < N$ and $j \le m$, we define the circuits $X_{i,j}$ and $X^{i,j}$ by induction on $j$:

$$X_{i,0} = X^{i,0} := M(\psi_i),$$

$$X_{i,j+1} := X_{i,j} \wedge \begin{cases} \boxdot(X^{i,j} \to \vartheta_j) & \text{if } h_i(\vartheta_j) = 1, \\ \boxdot(\boxdot(X^{i,j} \to \vartheta_j) \to \psi_i) & \text{if } h_i(\vartheta_j) = 0, \\ \boxdot(X^{i,j} \to \vartheta_j) \vee \boxdot(\boxdot(X^{i,j} \to \vartheta_j) \to \psi_i) & \text{if } \vartheta_j \notin \mathrm{dom}(h_i), \end{cases}$$

$$X^{i,j+1} := X^{i,j} \wedge X_{i,j+1} \wedge (\vartheta_j \to \boxdot(X^{i,j} \to \vartheta_j)).$$

We also define

$$\overline{X}_{i,j} := X_{i,j} \wedge \begin{cases} \boxdot(\boxdot(X^{i,j} \to \vartheta_j) \to \psi_i) & \text{if } h_i(\vartheta_j) = 1, \\ \boxdot(X^{i,j} \to \vartheta_j) & \text{if } h_i(\vartheta_j) = 0 \end{cases}$$

whenever $\vartheta_j \in \mathrm{dom}(h_i)$. We put

$$\xi_i := X^{i,m},$$

and we define

$$\begin{aligned}
\gamma_i &:= \Diamond\!\!\!\cdot\,\xi_i \wedge \bigwedge_{\vartheta\in S} (\boxdot(\xi_i \to \vartheta) \vee \boxdot(\xi_i \to \neg\vartheta)),\\
E_{i,j} &:= \gamma_i \wedge \gamma_j \wedge \bigwedge_{\vartheta\in S} (\boxdot(\xi_i \to \vartheta) \leftrightarrow \boxdot(\xi_j \to \vartheta)),\\
R^0_{i,j} &:= \gamma_i \wedge \gamma_j \wedge \bigwedge_{\Box\vartheta\in S} (\boxdot(\xi_i \to \Box\vartheta) \to \boxdot(\xi_j \to \Box\vartheta)),\\
R_{i,j} &:= \begin{cases} R^0_{i,j} \wedge (R^0_{j,i} \to E_{i,j}) & \text{if } L \supseteq \mathbf{K4Grz}, L \not\supseteq \mathbf{GL}\\ R^0_{i,j} & \text{otherwise,}\end{cases}\\
P_{\ell,i} &:= \boxdot(\xi_i \to p_\ell).
\end{aligned}$$

Notice that $E_{i,i}$ is equivalent to $\gamma_i$. Let $\sigma$ be the substitution such that $\sigma e_{i,j} = E_{i,j}$, $\sigma r_{i,j} = R_{i,j}$, and $\sigma p_{\ell,i} = P_{\ell,i}$.

**Claim 2** *There are poly-time constructible $L$-CF-proofs of*

*(i)* $X_{i,j} \to \Diamond\!\!\!\cdot\,\big(X_{i,j} \wedge \big(\boxdot(X^{i,j} \to \vartheta_j) \vee \boxdot(\boxdot(X^{i,j} \to \vartheta_j) \to \psi_i)\big)\big)$,

*(ii)* $X_{i,j} \to \Diamond\!\!\!\cdot\,X^{i,j}$,

*(iii)* $\neg\psi \to \Diamond\!\!\!\cdot\,M(\psi)$,

*(iv)* $\gamma_i \wedge \gamma_j \wedge \Diamond\!\!\!\cdot\,(\xi_i \wedge \xi_j) \to E_{i,j}$,

*(v)* $\gamma_i \wedge \gamma_j \wedge \Diamond\!\!\!\cdot\,(\xi_i \wedge \Diamond\xi_j) \to R_{i,j}$.

*Proof:* (i): Notice that $X_{i,j}$ is a conjunction of $\neg\psi_i$, and a monotone combination of formulas starting with $\boxdot$, hence

$$(*) \qquad X_{i,j} \to \boxdot(\neg\psi_i \leftrightarrow X_{i,j}).$$

It thus suffices to show

$$\neg\psi_i \to \Diamond\!\!\!\cdot\,\big(\neg\psi_i \wedge \big(\boxdot(X^{i,j} \to \vartheta_j) \vee \boxdot(\boxdot(X^{i,j} \to \vartheta_j) \to \psi_i)\big)\big),$$

which we prove as follows:

$$\begin{aligned}
&\boxdot\big(\big(\boxdot(X^{i,j} \to \vartheta_j) \vee \boxdot(\boxdot(X^{i,j} \to \vartheta_j) \to \psi_i)\big) \to \psi_i\big)\\
&\qquad \to \boxdot(\boxdot(X^{i,j} \to \vartheta_j) \to \psi_i) \wedge \boxdot\big(\boxdot(\boxdot(X^{i,j} \to \vartheta_j) \to \psi_i) \to \psi_i\big)\\
&\qquad \to \boxdot\psi_i.
\end{aligned}$$

(ii): By induction on $j$. Trivially

$$X^{i,j} \wedge X_{i,j+1} \wedge \boxdot(X^{i,j} \to \vartheta_j) \to X^{i,j+1},$$

and

$$\begin{aligned}X^{i,j} \wedge X_{i,j+1} \wedge \Diamonddot(X^{i,j} \wedge \neg\vartheta_j) &\to \Diamonddot(X^{i,j} \wedge X_{i,j+1} \wedge \neg\vartheta_j)\\ &\to \Diamonddot X^{i,j+1}\end{aligned}$$

by $(*)$, hence

$$X^{i,j} \wedge X_{i,j+1} \to \Diamonddot X^{i,j+1}.$$

Moreover,

$$\begin{aligned}X_{i,j+1} &\to X_{i,j+1} \wedge \Diamonddot X^{i,j}\\ &\to \Diamonddot(X^{i,j} \wedge X_{i,j+1})\end{aligned}$$

by the induction hypothesis, and $(*)$.

(iii): There is nothing to do unless $L \supseteq \mathbf{K4Grz}$. If $L \supseteq \mathbf{GL}$, then

$$\neg\psi \to \Diamonddot(\neg\psi \wedge \Box\psi)$$

follows from the Löb's axiom. If $L \not\supseteq \mathbf{GL}$, we have to construct a proof of

$$\boxdot\Big(\bigwedge_{j<m}\big(\boxdot(\vartheta_j \vee \Box\psi) \vee \boxdot(\neg\vartheta_j \vee \Box\psi)\big) \to \psi\Big) \to \psi.$$

If $m \leq 1$, the formula is an instance of a constant-size $\mathbf{K4Grz}$-tautology. We proceed by induction on $m$; the step for $m+1$ is

$$\begin{aligned}&\boxdot\Big(\bigwedge_{j<m+1}\big(\boxdot(\vartheta_j \vee \Box\psi) \vee \boxdot(\neg\vartheta_j \vee \Box\psi)\big) \to \psi\Big)\\ &\qquad\to \boxdot\Big(\bigwedge_{j<m}\big(\boxdot(\vartheta_j \vee \Box\psi) \vee \boxdot(\neg\vartheta_j \vee \Box\psi)\big)\\ &\qquad\qquad\to \boxdot\big((\boxdot(\vartheta_m \vee \Box\psi) \vee \boxdot(\neg\vartheta_m \vee \Box\psi)) \to \psi\big)\Big)\\ &\qquad\to \boxdot\Big(\bigwedge_{j<m}\big(\boxdot(\vartheta_j \vee \Box\psi) \vee \boxdot(\neg\vartheta_j \vee \Box\psi)\big) \to \psi\Big)\\ &\qquad\to \psi,\end{aligned}$$

using the induction hypothesis for 1 and $m$.

(iv) is clear from the definitions.

(v): We have

$$\begin{aligned}\gamma_i \wedge \gamma_j \wedge \Diamonddot(\xi_i \wedge \Diamond\xi_j) \wedge \boxdot(\xi_i \to \Box\vartheta) &\to \Diamonddot(\Diamond\xi_j \wedge \Box\vartheta)\\ &\to \Diamond(\xi_j \wedge \boxdot\vartheta)\\ &\to \boxdot(\xi_j \to \boxdot\vartheta)\end{aligned}$$

for any $\Box\vartheta \in S$, hence

$$\gamma_i \wedge \gamma_j \wedge \Diamonddot(\xi_i \wedge \Diamond\xi_j) \to R^0_{i,j}.$$

We are thus done, unless $L = \mathbf{K4Grz} \oplus \mathrm{BW}_K$ or $L = \mathbf{S4Grz} \oplus \mathrm{BW}_K$. In this case the definition of $M(\psi)$ gives

$$
\begin{aligned}
&\gamma_i \wedge \gamma_j \wedge \diamonddot(\xi_i \wedge \Diamond \xi_j) \wedge \neg E_{i,j} \\
&\qquad\qquad \to \bigvee_{\vartheta \in \pm S} \big(\boxdot(\xi_j \to \vartheta) \wedge \boxdot\big(\xi_i \to \neg\vartheta \wedge (\boxdot(\vartheta \vee \Box\psi_i) \vee \boxdot(\neg\vartheta \vee \Box\psi_i))\big)\big) \\
&\qquad\qquad \to \bigvee_{\vartheta \in \pm S} \big(\boxdot(\xi_j \to \vartheta) \wedge \boxdot(\xi_i \to \Box(\vartheta \to \Box\psi_i))\big) \\
&\qquad\qquad \to (\boxdot(\xi_j \to \Box\psi_i) \wedge \boxdot(\xi_i \to \neg\psi_i)) \\
&\qquad\qquad \to \neg R^0_{j,i},
\end{aligned}
$$

where $\pm S = S \cup \{\neg\vartheta;\ \vartheta \in S\}$. $\Box$ (Claim 2)

**Claim 3** *There are poly-time constructible $L$-CF-proofs of*

$$
\begin{aligned}
\diamonddot(\xi \wedge \neg\Box\psi) \to \bigvee_{\substack{i<N \\ \psi_i = \psi}} \Big(\diamonddot(\xi \wedge \Diamond X^{i,k}) \wedge \bigwedge_{j \in \mathrm{dom}(h_i)} \diamonddot \overline{X}_{i,j} \\
\wedge \bigwedge_{j<k} \big(\boxdot(X^{i,k} \to \vartheta_j) \vee \boxdot(X^{i,k} \to \neg\vartheta_j)\big)\Big)
\end{aligned}
$$

*for any $k \le m$, $\Box\psi \in S$, and a formula $\xi$.*

*Proof:* By induction on $k$. If $k = 0$, we can take $i$ such that $\langle \psi_i, h_i \rangle = \langle \psi, \varnothing \rangle$ by Claim 2 (iii). Assume we have a proof of the statement for $k$, we will prove it for $k+1$. Let $W'$ be defined similarly to $W$, except that we relax the domain size condition to $|\mathrm{dom}(h)| \le K$. We extend the enumeration of $W$ to $W' = \{\langle \psi_i, h_i \rangle;\ i < N'\}$, and define $X_{i,j}$, $X^{i,j}$, and $\overline{X}_{i,j}$ for $i < N'$ as before. We first construct a proof of

$$
\begin{aligned}
(**)\quad \diamonddot(\xi \wedge \neg\Box\psi) \to \bigvee_{\substack{i<N' \\ \psi_i = \psi}} \Big(\diamonddot(\xi \wedge \Diamond X^{i,k+1}) \wedge \bigwedge_{j \in \mathrm{dom}(h_i)} \diamonddot \overline{X}_{i,j} \\
\wedge \bigwedge_{j<k+1} \big(\boxdot(X^{i,k+1} \to \vartheta_j) \vee \boxdot(X^{i,k+1} \to \neg\vartheta_j)\big)\Big)
\end{aligned}
$$

by formalization of the following argument: "fix $i < N$ which witnesses the statement for $k$. W.l.o.g., assume $\mathrm{dom}(h_i) \subseteq k$. If

$$\boxdot(X_{i,k} \to \neg\boxdot(X^{i,k} \to \vartheta_k)) \vee \boxdot\big(X_{i,k} \to \neg\boxdot(\boxdot(X^{i,k} \to \vartheta_k) \to \psi_i)\big)$$

holds, then also $\boxdot(X^{i,k+1} \to \neg\vartheta_k) \vee \boxdot(X^{i,k+1} \to \vartheta_k)$ by the definition of $X_{i,k+1}$ and $X^{i,k+1}$. Moreover $\diamonddot(\xi \wedge \Diamond X^{i,k+1})$ by Claim 2 (i,ii), hence $i$ witnesses the statement for $k+1$. On the other hand, if we have

$$\diamonddot(X_{i,k} \wedge \boxdot(X^{i,k} \to \vartheta_k)) \wedge \diamonddot\big(X_{i,k} \wedge \boxdot(\boxdot(X^{i,k} \to \vartheta_k) \to \psi_i)\big),$$

let $i_0, i_1 < N'$ be such that $\psi_{i_\varepsilon} = \psi$, and $h_{i_\varepsilon} = h_i \cup \{\langle \vartheta_k, \varepsilon\rangle\}$. Then

$$\bigwedge_{j\in\mathrm{dom}(h_{i_\varepsilon})} \diamond\!\!\!\cdot\, \overline{X}_{i_\varepsilon,j} \wedge \bigwedge_{j<k+1} \big(\boxdot(X^{i_\varepsilon,k+1} \to \vartheta_j) \vee \boxdot(X^{i_\varepsilon,k+1} \to \neg\vartheta_j)\big)$$

holds for both $\varepsilon = 0,1$, and $\diamond\!\!\!\cdot\,(\xi \wedge \Diamond X_{i_\varepsilon,k+1})$ (hence $\diamond\!\!\!\cdot\,(\xi \wedge \Diamond X^{i_\varepsilon,k+1})$) holds for some $\varepsilon$ by Claim 2, thus $i_0$ or $i_1$ is a witness for $k+1$."

We derive the statement for $k+1$ from $(**)$ by constructing a proof of

$$\neg\Big(\diamond\!\!\!\cdot\,(\xi \wedge \Diamond X^{i,k+1}) \wedge \bigwedge_{j\in\mathrm{dom}(h_i)} \diamond\!\!\!\cdot\, \overline{X}_{i,j}\Big)$$

for every $i \geq N$. By the definition of $W$ and $W'$, we have $|\mathrm{dom}(h_i)| = K$, hence there exists an increasing enumeration $\mathrm{dom}(h_i) = \{j_0 < j_1 < \cdots < j_{K-1}\}$. Put

$$\alpha_u = \begin{cases} \overline{X}_{i,j_u} & \text{if } u < K, \\ X_{i,k+1} & \text{if } u = K. \end{cases}$$

Then we have

$$\begin{aligned}
\diamond\!\!\!\cdot\,(\xi \wedge \Diamond X^{i,k+1}) \wedge \bigwedge_{j\in\mathrm{dom}(h_i)} \diamond\!\!\!\cdot\, \overline{X}_{i,j} &\to \bigwedge_{u\leq K} \diamond\!\!\!\cdot\,\alpha_u \\
&\to \bigvee_{u\neq v} \diamond\!\!\!\cdot\,(\alpha_u \wedge \diamond\!\!\!\cdot\,\alpha_v) \\
&\to \bigvee_{\substack{u\neq v\\ j=j_{\min(u,v)}}} \diamond\!\!\!\cdot\,\big(\alpha_v \wedge \boxdot(X^{i,j} \to \vartheta_j) \wedge \boxdot\big(\boxdot(X^{i,j} \to \vartheta_j) \to \psi_i\big)\big) \\
&\to \diamond\!\!\!\cdot\,(\neg\psi_i \wedge \boxdot\psi_i) \\
&\to \bot
\end{aligned}$$

using an instance of $\mathrm{BW}_K$. □ (Claim 3)

**Claim 4** *There is a poly-time constructible L-CF-proof of* $\sigma\chi_{N,M}$.

*Proof:* We have

$$\begin{aligned}
\bigwedge_{u\leq K} E_{i_u,i_u} &\to \bigwedge_{u\leq K} \gamma_{i_u} \\
&\to \bigwedge_{u\leq K} \diamond\!\!\!\cdot\,\xi_{i_u} \\
&\to \bigvee_{u\neq v} \diamond\!\!\!\cdot\,(\xi_{i_u} \wedge \diamond\!\!\!\cdot\,\xi_{i_v}) \\
&\to \bigvee_{u\neq v} (E_{i_u,i_v} \vee R_{i_u,i_v})
\end{aligned}$$

using an instance of $\mathrm{BW}_K$, and Claim 2 (iv,v). The other conjuncts of $\sigma\chi_{N,M}$ are straightforward. □ (Claim 4)

**Claim 5** *For any $\psi \in S$, and $i < N$, there are poly-time constructible $L$-$CF$-proofs of*

$$\gamma_i \to (\sigma\psi^i \leftrightarrow \boxdot(\xi_i \to \psi)).$$

*Proof:* By induction on the complexity of $\psi$. The statement holds for variables by definition, and the steps for Boolean connectives are straightforward. We will show the step for $\Box\psi$. By the induction hypothesis and the definition of $(\Box\psi)^i$, we have a proof of

$$\gamma_i \to \Big(\sigma(\Box\psi)^i \leftrightarrow \bigwedge_j (R_{i,j} \to \boxdot(\xi_j \to \psi))\Big).$$

Clearly

$$\boxdot(\xi_i \to \Box\psi) \to (R_{i,j} \to \boxdot(\xi_j \to \psi))$$

from the definition of $R_{i,j}$. On the other hand, we have

$$\begin{aligned}\gamma_i \wedge \neg\boxdot(\xi_i \to \Box\psi) &\to \bigvee_j \Big(\diamonddot(\xi_i \wedge \Diamond(\xi_j \wedge \neg\psi)) \wedge \bigwedge_u \big(\boxdot(\xi_j \to \vartheta_u) \vee \boxdot(\xi_j \to \neg\vartheta_u)\big)\Big)\\ &\to \bigvee_j (R_{i,j} \wedge \neg\boxdot(\xi_j \to \psi))\end{aligned}$$

by Claims 3 and 2 (v). □ (Claim 5)

We finish the proof of the theorem as follows. We apply $\sigma$ to $\pi$ to get an $L$-$CF$-proof of $\sigma\varphi^*$, and use Claims 4 and 5 to derive

$$\bigwedge_i (\gamma_i \to \boxdot(\xi_i \to \varphi)).$$

We construct a proof of

$$\neg\varphi \to \bigvee_i (\gamma_i \wedge \diamonddot(\xi_i \wedge \neg\varphi))$$

by the same reasoning as in the proof of Claim 5, and we conclude $\varphi$. □

It is not clear how to modify the proof of Theorem 5.25 so that it applies to the si logics $\mathbf{BW}_k$, which lack classical negation. Nevertheless, we give an ad hoc simulation for the most important case of the Gödel–Dummett logic $\mathbf{LC} = \mathbf{BW}_1$. The idea is that in a linearly ordered finitely generated refined frame, we can recognize the point where we are by counting the number of generators which are satisfied. Extended Frege can count without any difficulty.

**Theorem 5.26** $\mathbf{LC}$-$EF \equiv_p \mathbf{LC}$-$SF$*, and* $\mathbf{LC}$-$EF$ *is interpretable in* $\mathbf{CPC}$-$EF$*.*

*Proof:* If $\varphi$ is a formula in variables $p_i$, $i < N$, let $F = \langle N+1, \geq\rangle$ be the chain of length $N+1$ with 0 on the top, and $\varphi^* := \varphi^F$ using the notation of Definition 5.4.

**Claim 1** *Given an* $\mathbf{LC}$-$SF$*-proof of a formula $\varphi$, we can construct in polynomial time a* $\mathbf{CPC}$-$CF$*-proof of $\varphi^*$.*

*Proof:* Exercise. □ (Claim 1)

We define the threshold circuits $T^n_k(p_0,\dots,p_{n-1})$ by induction on $n+k$:

$$\begin{aligned}
T^n_0 &:= \top,\\
T^0_{k+1} &:= \bot,\\
T^{n+1}_{k+1}(p_0,\dots,p_n) &:= T^n_{k+1}(p_0,\dots,p_{n-1}) \vee (p_n \wedge T^n_k(p_0,\dots,p_{n-1})).
\end{aligned}$$

We will omit $n$ and/or $\vec{p}$ when it can be inferred from the context.

**Claim 2** *There are poly-time constructible CF-proofs of*

*(i)* $T_k \to T_\ell$ *for* $k > \ell$,

*(ii)* $\neg T^n_{n+1}$,

*(iii)* $T^{n+1}_{k+1}(\vec{p}) \leftrightarrow T^n_{k+1}(p_0,\dots,p_{i-1},p_{i+1},\dots p_n) \vee (p_i \wedge T^n_k(p_0,\dots,p_{i-1},p_{i+1},\dots,p_n))$.

*Proof:* Easy. (Notice that classical proofs are good enough, by Theorem 3.9). □ (Claim 2)

Assume we are given a **CPC**-*CF*-proof of $\varphi^*$. We can construct an **IPC**-*CF*-proof $\pi$ of

$$\bigwedge_{\substack{i<N\\ u\le N}} (p_{i,u} \vee q_{i,u}) \to (\varphi^*)^{\mathbf{m}}$$

by Lemma 5.9. We fix any $K \le N$, and define

$$\begin{aligned}
P_{i,u} &:= T^N_{K-u}(p_0,\dots,p_{N-1}) \to p_i,\\
Q_{i,u} &:= (T_{K-u} \to p_i) \to T_{K-u+1}
\end{aligned}$$

for each $i < N$, and $u \le K$. Let $\sigma_K$ be the substitution such that $\sigma_K p_{i,u} = P_{i,u}$, and $\sigma_K q_{i,u} = Q_{i,u}$.

**Claim 3** *There are poly-time constructible* **LC**-*CF-proofs of*

*(i)* $\neg T_a \vee \bigvee_{b=a}^{N} \big((T_a \to T_b) \wedge ((T_b \to T_{b+1}) \to T_{b+1})\big)$ *for each* $a \le N+1$,

*(ii)* $P_{i,u} \vee Q_{i,u}$ *for each* $i < N$, $u \le K$,

*(iii)* $\sigma_K(\psi^u)^{\mathbf{m}} \wedge \sigma_K(\psi^u)_{\mathbf{m}} \to T_{K-u+1}$ *and* $\sigma_K(\psi^u)^{\mathbf{m}} \vee \sigma_K(\psi^u)_{\mathbf{m}}$ *for each* $u \le K$, *and each formula* $\psi$.

*Proof:* (i): By reverse induction on $a$. The base case $a = N+1$ follows from Claim 2 (ii), and the induction step from $a+1$ to $a$ follows from the instance

$$(T_a \to (T_a \to T_{a+1})) \vee ((T_a \to T_{a+1}) \to T_a)$$

of the LC axiom.

(ii): We have

$$(*) \qquad \neg T_{K-u} \vee \bigvee_{b=K-u}^{N} \big((T_{K-u} \to T_b) \wedge ((T_b \to T_{b+1}) \to T_{b+1}\big)$$

from (i) and Claim 2. For every $b \geq K-u$, we have

$$T_b^N \to T_b^{N-1}(p_0, \dots, p_{i-1}, p_{i+1}, \dots, p_{N-1}) \vee p_i$$

and

$$T_b^{N-1}(p_0, \dots, p_{i-1}, p_{i+1}, \dots) \wedge p_i \to T_{b+1}^N$$

from Claim 2 (iii), hence

$$T_b \to p_i \vee (p_i \to T_{b+1}).$$

Using an instance of the **LC**-tautology

$$(**) \qquad (\alpha \to \beta \vee \gamma) \to (\alpha \to \beta) \vee (\alpha \to \gamma),$$

we obtain

$$(T_b \to p_i) \vee (p_i \wedge T_b \to T_{b+1}).$$

Clearly

$$(T_{K-u} \to T_b) \wedge (T_b \to p_i) \to P_{i,u},$$

and

$$\begin{aligned}((T_b \to T_{b+1}) \to T_{b+1}) \wedge (p_i \wedge T_b \to T_{b+1}) \wedge (T_{K-u} \to p_i) &\to (T_b \to p_i)\\ &\to (T_b \to T_{b+1})\\ &\to T_{b+1}\\ &\to T_{K-u+1}\end{aligned}$$

using Claim 2, hence

$$((T_b \to T_{b+1}) \to T_{b+1}) \wedge (p_i \wedge T_b \to T_{b+1}) \to Q_{i,u}.$$

As $b$ was arbitrary, we obtain

$$P_{i,u} \vee Q_{i,u}$$

from $(*)$.

(iii): By straightforward induction on the complexity of $\psi$, using (ii) and

$$P_{i,u} \wedge Q_{i,u} \to T_{K-u+1}$$

for the base case. □ (Claim 3)

**Claim 4** *There are poly-time constructible* **LC**-*CF-proofs of*

$$\neg\neg T_K \wedge \neg T_{K+1} \to \big(\sigma_K(\psi^u)^{\mathbf{m}} \leftrightarrow (T_{K-u} \to \psi)\big)$$

*for every $u \leq K$, and every formula $\psi$.*

*Proof:* By induction on the complexity of $\psi$. The steps for variables, $\wedge$, $\vee$, and $\bot$ are easy, using $(**)$. We will show the step for $\to$. By definition, we have

$$\sigma_K((\psi \to \chi)^u)^{\mathbf{m}} = \bigwedge_{v \leq u} \big(\sigma_K(\psi^v)_{\mathbf{m}} \vee \sigma_K(\chi^v)^{\mathbf{m}}\big).$$

For each $v \leq u$, we have

$$\begin{aligned} \neg\neg T_K \wedge \neg T_{K+1} \wedge (T_{K-u} \to (\psi \to \chi)) \wedge \sigma_K(\psi^v)^{\mathbf{m}} &\to (T_{K-v} \to \psi) \\ &\to (T_{K-v} \to \chi) \\ &\to \sigma_K(\chi^v)^{\mathbf{m}} \end{aligned}$$

by the induction hypothesis, hence

$$\neg\neg T_K \wedge \neg T_{K+1} \wedge (T_{K-u} \to (\psi \to \chi)) \to \sigma_K(\psi^v)_{\mathbf{m}} \vee \sigma_K(\chi^v)^{\mathbf{m}}$$

by Claim 3 (iii). On the other hand, the induction hypothesis and the same Claim imply

$$\begin{aligned} \neg\neg T_K \wedge \neg T_{K+1} \wedge \bigwedge_{v \leq u} \big(\sigma_K(\psi^v)_{\mathbf{m}} \vee \sigma_K(\chi^v)^{\mathbf{m}}\big) \wedge \psi &\to \bigwedge_{v \leq u} \sigma_K(\psi^v)^{\mathbf{m}} \\ &\to \bigwedge_{v \leq u} \big(T_{K-v+1} \vee \sigma_K(\chi^v)^{\mathbf{m}}\big) \\ &\to \bigwedge_{v \leq u} \big(T_{K-v+1} \vee (T_{K-v} \to \chi)\big). \end{aligned}$$

We can prove

$$\bigwedge_{v \leq u} \big(T_{K-v+1} \vee (T_{K-v} \to \chi)\big) \to (T_{K-u} \to \chi) \vee T_{K+1}$$

by induction on $u$, thus

$$\neg\neg T_K \wedge \neg T_{K+1} \wedge \bigwedge_{v \leq u} \big(\sigma_K(\psi^v)_{\mathbf{m}} \vee \sigma_K(\chi^v)^{\mathbf{m}}\big) \to (T_{K-u} \to (\psi \to \chi)). \qquad \square \text{ (Claim 4)}$$

We substitute $\sigma_K$ in $\pi$ to get a proof of

$$\bigwedge_{i,u} (P_{i,u} \vee Q_{i,u}) \to \sigma_K(\varphi^K)^{\mathbf{m}},$$

and we apply Claims 3 and 4 to obtain

$$\neg\neg T_K \wedge \neg T_{K+1} \to (T_0 \to \varphi),$$

i.e.,

$$\neg\neg T_K \wedge \neg T_{K+1} \to \varphi.$$

As **LC** $\supseteq$ **KC**, we can prove

$$\bigvee_{K \le a} (\neg\neg T_K \wedge \neg T_{K+1}) \vee \neg\neg T_{a+1}$$

by induction on $a$. Taking $a = N$ we obtain a proof of

$$\bigvee_{K \le N} (\neg\neg T_K \wedge \neg T_{K+1}),$$

hence $\varphi$. $\square$

# 6 Separations

In this section we are going to show exponential separation of $L$-$SF$ and $L$-$EF$ for all logics of infinite branching. The proof has a purely model-theoretic part (Theorems 6.9 and 6.21) providing a description of maximal logics of infinite branching, which we have put separately in Subsection 6.1. The proof-theoretic part follows in Subsection 6.2.

## 6.1 A characterization of logics with infinite branching

**Definition 6.1** Let $\langle W, S, V\rangle$ be a transitive modal general frame, and $\langle F, R\rangle$ a finite transitive Kripke frame. A partial mapping $f$ from $W$ *onto* $F$ is called a *subreduction* of $W$ to $F$, if for every $x, y \in W$ and $u \in F$,

(i) $x\ S\ y$ and $x, y \in \mathrm{dom}(f)$ implies $f(x)\ R\ f(y)$,

(ii) if $f(x)\ R\ u$, there exists $y \in \mathrm{dom}(f)$ such that $x\ S\ y$ and $f(y) = u$,

(iii) $f^{-1}[u] \in V$.

(A total subreduction is thus a p-morphism.) A *domain* is an upper subset $d \subsetneq F$, which is not generated by a single reflexive point. A subreduction $f$ satisfies the *closed domain condition* (*CDC*) for a domain $d$, if there is no $x \in \mathrm{dom}(f){\uparrow} \smallsetminus \mathrm{dom}(f)$ such that $f(x{\uparrow}) = d$. If $D$ is a set of domains, $f$ satisfies CDC for $D$ if it satisfies CDC for every $d \in D$.

Subreductions of intuitionistic frames are defined in a similar way, except that condition (iii) is replaced with

(iii') $W \smallsetminus f^{-1}[u]{\downarrow} \in V$.

Notice that (iii') holds whenever $f^{-1}[u]$ is in the Boolean closure of $V$.

**Definition 6.2** Let $\langle F, R\rangle$ be a finite transitive Kripke frame with root $0 \in F$, and $D$ a set of domains in $F$. The *modal canonical formula* $\alpha(F, D)$ in variables $\{p_i;\ i \in F\}$ is defined as

$$\bigwedge_{i \ne j} \boxdot(p_i \vee p_j) \wedge \bigwedge_{iRj} \boxdot(\Box p_j \to p_i) \wedge \bigwedge_{i \not R j} \boxdot(p_i \vee \Box p_j) \wedge \bigwedge_{d \in D} \boxdot\Big(\bigwedge_i p_i \wedge \bigwedge_{i \notin d} \Box p_i \to \bigvee_{i \in d} \Box p_i\Big) \to p_0$$

where indices $i, j$ range over elements of $F$. We introduce the abbreviations $\alpha(F, D, \bot) = \alpha(F, D \cup \{\varnothing\})$, $\alpha(F) = \alpha(F, \varnothing)$, $\alpha^\sharp(F) = \alpha(F, F^\sharp)$, where $F^\sharp$ is the set of all non-empty domains in $F$.

If $F$ is intuitionistic, the *intuitionistic canonical formula* $\beta(F, D)$ is defined as

$$\bigwedge_{iRj}\Big(\Big(\bigwedge_{j\not R k} p_k \to p_j\Big) \to p_i\Big) \wedge \bigwedge_{d\in D}\Big(\bigwedge_{i\notin d} p_i \to \bigvee_{i\in d} p_i\Big) \to p_0.$$

The formulas $\beta(F, D, \bot)$, $\beta(F)$, and $\beta^\sharp(F)$ are defined similarly to the modal case.

**Lemma 6.3 (Zakharyaschev [7])** *A transitive modal (intuitionistic) general frame $W$ refutes $\alpha(F, D)$ ($\beta(F, D)$, resp.) if and only if there exists a subreduction of $W$ onto $F$ satisfying CDC for $D$.*

**Example 6.4** The frame formulas $\alpha^\sharp(F, \bot)$ from Definition 5.14 are (over **K4**) deductively equivalent to the canonical formulas $\alpha^\sharp(F, \bot)$. The formula $\beta$ used in the proof of Theorem 5.10 is the subframe canonical formula $\beta(F)$.

**Theorem 6.5 (Zakharyaschev [7])** *For every modal formula $\varphi$, there exists a finite sequence of canonical formulas $\alpha(F_i, D_i)$, $i < k$, such that*

$$\mathbf{K4} \oplus \varphi = \mathbf{K4} \oplus \{\alpha(F_i, D_i);\, i < k\}.$$

*For every intuitionistic formula $\varphi$, there exists a finite sequence of canonical formulas $\beta(F_i, D_i)$ such that*

$$\mathbf{IPC} + \varphi = \mathbf{IPC} + \{\beta(F_i, D_i);\, i < k\}.$$

**Example 6.6** $\mathbf{S4} = \mathbf{K4} \oplus \alpha(\bullet)$, $\mathbf{GL} = \mathbf{K4} \oplus \alpha(\circ)$, $\mathbf{K4Grz} = \mathbf{K4} \oplus \alpha(C_2)$ where $C_2$ is the 2-point cluster, $\mathbf{KC} = \mathbf{IPC} + \beta(F, \bot) = \mathbf{IPC} + \beta^\sharp(F, \bot)$ and $\mathbf{LC} = \mathbf{IPC} + \beta(F) = \mathbf{IPC} + \beta^\sharp(F)$ where $F$ is the tree of depth 2 with 2 leaves, etc.

**Lemma 6.7 (Zakharyaschev [7])** *For any $k \in \omega$,*

$$\mathbf{T}_k = \mathbf{IPC} + \beta^\sharp(F_{k+1}),$$

*where $F_{k+1}$ is the (reflexive) tree of depth 2 with $k + 1$ leaves.*

**Lemma 6.8** *Let $\langle W, R, V\rangle$ be an intuitionistic or transitive modal descriptive frame, $< = R \smallsetminus R^{-1}$, and $A \in V$. If $A \neq W$, then $W \smallsetminus A$ contains a $<$-maximal point.*

*Proof:* Assume that $W$ is a modal frame (the intuitionistic case is similar). By descriptiveness and Zorn's lemma, there exists a maximal subset $X \subseteq V$ such that $Y := \bigcap_{B\in X} \boxdot B \smallsetminus A \neq \varnothing$, where $\boxdot B := B \cap \Box B$. Pick $u \in Y$. Clearly $u \notin A$. If $u < v$, then $\{B \in V;\, v \in \boxdot B\}$ is a proper superset of $X$, hence $v \in A$. $\Box$

**Theorem 6.9** *Let $L$ be a si logic. $L$ has infinite branching if and only if $L \subseteq \mathbf{BD}_2$, or $L \subseteq \mathbf{BD}_3 + \mathbf{KC}$.*

*Proof:* Clearly $\mathbf{BD}_2$ and $\mathbf{BD}_3 + \mathbf{KC}$ have infinite branching. Let $L$ be a logic with infinite branching, and $n \geq 1$. By Lemma 6.7, there exists a descriptive $L$-frame $W$, and a subreduction $f\colon W \to F_{2n}$ with CDC for $F^{\sharp}_{2n}$. If $r$ is the root of $F_{2n}$, the set $f^{-1}[r]$ contains a maximal point $u$ by Lemma 6.8. We may assume $W = W_u$ without loss of generality, thus $f^{-1}[r] = \{u\}$. Define

$$X = \{\ell \in F_{2n} \smallsetminus \{r\};\ \exists x, y \in W\,(x < y \wedge f(x) = \ell \wedge f(y\uparrow) = \varnothing)\}.$$

Assume first $|X| \geq n$. Pick $Y \subseteq X$ of size $n$, and let $F^*_n$ be the frame $\{r\} \cup Y \cup \{*\}$, where $\ell < *$ for every $\ell \in Y$. We define a mapping $g\colon W \to F^*_n$ by

$$g(x) = \begin{cases} r & \text{if } x = u, \\ \ell & \text{if } f(x\uparrow) = \{\ell\}, \ell \in Y, \exists y \in x\uparrow f(y\uparrow) = \varnothing, \\ * & \text{otherwise.} \end{cases}$$

The definition of $Y$, and CDC of $f$ imply that $g$ is a p-morphism of $W$ onto $F^*_n$. In particular, $F^*_n \vDash L$.

If $|X| \leq n$, we pick $Y \subseteq F_{2n} \smallsetminus (X \cup \{r\})$ of size $n$, and $\ell_0 \in Y$. We define a mapping $g\colon W \to \{r\} \cup Y \simeq F_n$ as

$$g(x) = \begin{cases} r & \text{if } x = u, \\ \ell & \text{if } f(x\uparrow) = \{\ell\}, \ell \in Y, \\ \ell_0 & \text{if } f(x\uparrow) \cap Y = \varnothing. \end{cases}$$

Again, $g$ is a p-morphism of $W$ onto $F_n$, thus $F_n \vDash L$.

For every $n \geq 1$, we have $F_n \vDash L$ or $F^*_n \vDash L$. If there are infinitely many $n$ such that $F_n \vDash L$, then every finite rooted $\mathbf{BD}_2$-frame is a p-morphic image of an $L$-frame, hence it is an $L$-frame itself. As $\mathbf{BD}_2$ has the finite model property, this implies $L \subseteq \mathbf{BD}_2$. Otherwise there are infinitely many $n$ such that $F^*_n \vDash L$, hence $L \subseteq \mathbf{BD}_3 + \mathbf{KC}$ by a similar argument. □

In the rest of the present subsection, we are going to find a modal analogue of Theorem 6.9. The result and its proof will be more complicated; the main source of obstacles is that we cannot dispense with arbitrary junk by mapping it p-morphically to a singleton frame, as we did in the proof Theorem 6.9. For example, if $W$ is any generated subframe of the universal $\mathbf{K4}$-frame of rank 0, then the only p-morphic image of $W$ is itself. We start with a characterization of $\mathbf{K4BB}_k$ by canonical formulas (which incidentally shows that $\mathbf{K4BB}_k$ is finitely axiomatizable).

**Lemma 6.10** *Let $k \in \omega$, let $X_{k+1}$ be the set of all trees of depth 2 with $k+1$ leaves (where each point may be reflexive or irreflexive), and for any $F \in X_{k+1}$, let $F^{\geq 2}$ be the set of all sets of leaves of $F$ of size at least 2. Then*

$$\mathbf{K4BB}_k = \mathbf{K4} \oplus \{\alpha(F, F^{\geq 2});\ F \in X_{k+1}\}.$$

*Proof:* $\supseteq$: Let $\langle W, R\rangle$ be a finite transitive frame of branching at most $k$, and $f\colon W \to F$ a subreduction onto some $F \in X_{k+1}$. Put $< := R \smallsetminus R^{-1}$, and pick a $<$-maximal point $x \in W$ which is mapped to the root of $F$ by $f$. For each leaf $\ell \in F$, we pick $x_\ell > x$ such that $f(x_\ell) = \ell$. The points $x_\ell$ form an antichain of size $k+1$, while $W$ has branching at most $k$, hence by the pigeonhole principle there exists $y > x$ such that $y < x_\ell, x_{\ell'}$ for some $\ell \neq \ell'$. We cannot have $y \in \mathrm{dom}(f)$ by maximality of $x$, thus $y$ violates CDC for $F^{\geq 2}$.

$\subseteq$: Denote the RHS by $L$. By Theorem 6.5, it suffices to show that $L \nvdash \alpha(F, D)$ implies $\mathbf{K4BB}_k \nvdash \alpha(F, D)$ for each canonical formula $\alpha(F, D)$. Let $\langle W, R, V\rangle$ be a transitive general frame such that $W \vDash L$ and $W \nvDash \alpha(F, D)$. Let $\langle \varrho W, \varrho R\rangle$ be the reflexivization of the skeleton of $\langle W, R\rangle$, and define $\varrho V = \{\varrho(\boxdot X);\, X \in V\}$ so that $\langle \varrho W, \varrho R, \varrho V\rangle$ is a general intuitionistic frame. Similarly, let $\varrho F$ be the reflexivization of the skeleton of $F$, and $\varrho D = \{\varrho(d);\, d \in D'\}$, where

$$D' = \{d \in D;\, d \text{ is not generated by a singleton}\}.$$

We have

$$(*) \qquad \varrho W \nvDash \beta(\varrho F, \varrho D).$$

Indeed, if $f$ is a subreduction of $W$ to $F$ with CDC for $D$, then $f$ induces a mapping $\varrho f\colon \varrho W \to \varrho F$, which is easily seen to be a subreduction with CDC for $\varrho D$.

**Claim 1** $\varrho W \vDash \mathbf{T}_k$.

*Proof:* By Lemma 6.7, it suffices to show $\varrho W \vDash \beta^\sharp(F_{k+1})$. Assume for contradiction that $f\colon \varrho W \to F_{k+1}$ is a subreduction with CDC for $F^\sharp_{k+1}$, which induces a partial mapping $h$ from $W$ onto $F_{k+1}$. We will construct a $G \in X_{k+1}$ by adjusting the reflexivity of some points of $F_{k+1}$, and a subreduction $g\colon W' \to G$ from a generated subframe $W' \subseteq\cdot\, W$. Using Lemma 6.8, we pick a maximal point $u$ in $h^{-1}[r]$, and define $W' = W_u$, $g^{-1}[r] = W' \cap h^{-1}[r]{\updownarrow}$. If $u$ is irreflexive, we make $r$ irreflexive as well, otherwise we leave it reflexive. Let $\ell$ be any leaf of $F_{k+1}$. If $W' \cap (h^{-1}[\ell] \smallsetminus h^{-1}[\ell]{\downarrow}) \neq \varnothing$, we make $\ell$ irreflexive, and define $g^{-1}[\ell] = W' \cap (h^{-1}[\ell] \smallsetminus h^{-1}[\ell]{\downarrow})$, otherwise we leave $\ell$ reflexive, and put $g^{-1}[\ell] = W' \cap h^{-1}[\ell]$.

It is easy to see that $g\colon W' \to G$ is a subreduction. If $d \in G^{\geq 2}$, and $x \in W' \smallsetminus \mathrm{dom}(g)$ is such that $g(x{\uparrow}) = d$, then $h(x{\uparrow}) \supseteq d$, thus $r \in h(x{\updownarrow})$ by CDC for $F^\sharp_{k+1}$, which contradicts the definition of $g^{-1}[r]$. Hence $g$ satisfies CDC for $G^{\geq 2}$, which contradicts $W' \vDash L$. $\Box$ (Claim 1)

Claim 1 and $(*)$ imply $\mathbf{T}_k \nvdash \beta(\varrho F, \varrho D)$, hence there exists a finite $k$-ary tree $U$ such that $U \nvDash \beta(\varrho F, \varrho D)$. Assume that $|U|$ is minimal possible, and let $f\colon U \to \varrho F$ be a subreduction with CDC for $\varrho D$.

**Claim 2** *The set $\{u \in U;\, f(u{\uparrow}) = x{\uparrow}\}$ is an antichain for every $x \in \varrho F$. In particular, $f(u{\uparrow}) = x{\uparrow}$ implies $f(u) = x$.*

*Proof:* Assume for contradiction that $f(u{\uparrow}) = f(v{\uparrow}) = x{\uparrow}$ for some $u < v$. Let $U'$ be the subframe of $U$ defined by $U' = (U \smallsetminus u{\uparrow}) \cup v{\uparrow}$, and put $f' := f \restriction U'$. Then $f'(w{\uparrow}) = f(w{\uparrow})$ for every $w \in U'$, hence $f'$ is a subreduction of $U'$ to $\varrho F$ with CDC for $\varrho D$. However, $U'$ is a $k$-ary tree, and $|U'| < |U|$, which contradicts the minimality of $U$. $\Box$ (Claim 2)

We construct a modal frame $U'$ from $U$ as follows. If $u \in \mathrm{dom}(f)$, and $C$ is the cluster in $F$ such that $f(u) = \varrho(C)$, we replace $u$ with a copy of $C$. Let $f'$ be the mapping from $U'$ to $F$, induced from $f$ in the obvious way. Then $f'$ is a subreduction of $U'$ onto $F$ (Claim 2 guarantees that $f'(u) = f'(v) = x$ for no $u < v \in U'$ and irreflexive $x \in F$), and it satisfies CDC for $D'$. By Claim 2 $f'$ also satisfies CDC for all domains generated by (irreflexive) singletons, thus $f'$ satisfies CDC for $D$, and $\mathbf{K4BB}_k \nvdash \alpha(F, D)$. □

**Remark 6.11** If $k \geq 1$, the logic $\mathbf{K4} \oplus \{\alpha^\sharp(F);\, F \in X_{k+1}\}$ is strictly weaker than $\mathbf{K4BB}_k$, and in fact, it has infinite branching. However, it has the same finite frames as $\mathbf{K4BB}_k$.

By extension of the proof of Lemma 6.10, it can be shown that

$$\mathbf{K4BB}_k = \mathbf{K4} \oplus \Box\Big(\bigvee_{i\leq k} \Box\Big(\boxdot p_i \to \bigvee_{j\neq i} p_j\Big) \to \bigvee_{i\leq k} \boxdot p_i\Big) \to \bigvee_{i\leq k} \Box \bigvee_{j\neq i} p_j.$$

We skip the details as we will have no use for this explicit axiomatization.

Let us fix a transitive modal logic $L$ with infinite branching for the rest of the subsection. By Lemma 6.10, for every $k > 0$ there exists a descriptive $L$-frame $\langle W, R, V\rangle$, and a subreduction $f\colon W \to F$ with CDC for $F^{\geq 2}$ for some $F \in X_k$. Let $r$ be the root of $F$. There exists a maximal point $u \in f^{-1}[r]$ by Lemma 6.8. We may assume $W = W_u$ without loss of generality, thus $f^{-1}[r]\updownarrow$ is a cluster. As $f^{-1}[r] \in V$, we can collapse the cluster to a single point by a p-morphism, thus we may assume $f^{-1}[r] = \{u\}$. Put $A_i = f^{-1}[\ell]\updownarrow \smallsetminus \{u\}$, where $i < k$, and $\ell$ is the $i$th leaf of $F$. Moreover, let $F = W \smallsetminus \big(\{u\} \cup \bigcup_{i<k} A_i\big)$. We have the following properties (the disjointness of $A_i$'s follows from the CDC for $F^{\geq 2}$):

(i) $\langle W, R, V\rangle$ is an $L$-frame with unique root $u$,

(ii) $W$ is the disjoint union of $\{u\}$, $F$, and $A_i$, $i < k$,

(iii) $A_i \neq \varnothing$, and $A_i{\downarrow} \subseteq A_i \cup \{u\}$ for $i < k$ (hence $F$ is a generated subframe of $W$),

(iv) $\{u\}$, $F$, and $A_i$ belong to $V$.

Let us call any frame $W$ with distinguished subsets $A_i$ satisfying these conditions a *$k$-special frame*. Moreover, we call $W$ a *$\langle k, \circ\rangle$-special frame* if $u$ is reflexive, and a *$\langle k, \bullet\rangle$-special frame* if $u$ is irreflexive.

Notice that if $k' < k$, and $W$ is a $k$-special frame wrt $\{A_i;\, i < k\}$, then $W$ is a $k'$-special frame wrt $\{A_i;\, i < k'\}$. Hence if there exist $\langle k, \circ\rangle$-special frames for infinitely many $k$, then they exist for every $k$; otherwise there exist $\langle k, \bullet\rangle$-special frames for all but finitely many $k$, hence they exist for every $k$. We get:

**Lemma 6.12** *There is $\triangle \in \{\circ, \bullet\}$ such that for every $k > 0$ there exists a descriptive $\langle k, \triangle\rangle$-special frame.*

We fix an appropriate $\triangle \in \{\circ, \bullet\}$. Let $W$ be a $\langle k, \triangle\rangle$-special frame wrt $\{A_i;\, i < k\}$, and $X$ a set of variable-free formulas. We say that $W$ is *$\langle k, \triangle, X\rangle$-special*, if $x \Vdash \varphi$ for every $\varphi \in X$ and $x \in \bigcup_i A_i$. $X$ is *complete*, if $\varphi \in X$ or $\neg\varphi \in X$ for every variable-free formula $\varphi$.

**Lemma 6.13** *There exists a complete set $X$ of variable-free formulas such that for every $k > 0$ and finite $Y \subseteq X$, there exists a descriptive $\langle k, \triangle, Y\rangle$-special frame.*

*Proof:* By Zorn's lemma, there exists a maximal set $X$ of variable-free formulas which satisfies the conclusion of the lemma, we will show that $X$ is complete. Clearly $\varphi \in X$ if $X \vdash_{\mathbf{CPC}} \varphi$, it thus suffices to show that $\Box\varphi \in X$ or $\neg\Box\varphi \in X$ for every variable-free $\varphi$. Assume that $\Box\varphi \notin X$, and fix $c > 0$ and finite $Y_0 \subseteq X$ such that no descriptive $\langle c, \triangle, Y_0 \cup \{\Box\varphi\}\rangle$-special frame exists. Let $k > 0$, and $Y \subseteq X$ finite. There exists a descriptive $\langle k + c, \triangle, Y \cup Y_0\rangle$-special frame $W$ wrt $\{A_i;\, i < k + c\}$. Put $I = \{i < k + c;\, \exists x \in A_i\, x \nVdash \Box\varphi\}$. As $W$ is a $\langle k+c-|I|, \triangle, Y_0 \cup \{\Box\varphi\}\rangle$-special frame wrt $\{A_i;\, i \notin I\}$, we must have $|I| \geq k$. For each $i \in I$, put $A'_i = \{x \in A_i;\, x \nVdash \Box\varphi\}$. Then $A'_i \in V$, and $A'_i$ is a non-empty lower subset of $A_i$, thus $W$ is a $\langle |I|, \triangle, Y \cup \{\neg\Box\varphi\}\rangle$-special frame wrt $\{A'_i;\, i \in I\}$. As $k$ and $Y$ were arbitrary, we have $\neg\Box\varphi \in X$ by maximality of $X$. $\Box$

We fix a set $X$ satisfying Lemma 6.13. If $W$ is a $\langle k, \triangle, Y\rangle$-special frame, we call $W$ $\langle k, \triangle, Y\rangle$-*extraspecial,* if

$$\exists y \in F\, (x\; R\; y \wedge y \Vdash \varphi)$$

for every $x \in \bigcup_i A_i$ and every $\Diamond\varphi \in Y$. Let (XS) be the statement

(XS) *For every $k > 0$ and finite $Y \subseteq X$, there exists a descriptive $\langle k, \triangle, Y\rangle$-extraspecial frame.*

A descriptive frame $A$ is of

(i) *type* $\bullet$, if it is an irreflexive antichain (equivalently: if $A \vDash \Box\bot$),

(ii) *type* $\circ$, if all its final clusters are reflexive (equivalently: if $A \vDash \Diamond\top$),

(iii) *type* $\infty$, if it validates GL, and the rule $\Box p/p$.

Let $* \in \{\bullet, \circ, \infty\}$. A $\langle k, \triangle, Y\rangle$-extraspecial frame $W$ is called $\langle k, \triangle, Y, *\rangle$-*special* if the subframe $A_i$ of $W$ is of type $*$ for every $i < k$.

**Lemma 6.14** *If (XS) holds, there exists $* \in \{\bullet, \circ, \infty\}$ such that for every $k > 0$ and finite $Y \subseteq X$, there exists a descriptive $\langle k, \triangle, Y, *\rangle$-special frame.*

*Proof:* By an argument similar to the proof of Lemma 6.13, it suffices to show the following: for every $k > 0$ and finite $Y \subseteq X$, there exists $* \in \{\bullet, \circ, \infty\}$, and a $\langle k, \triangle, Y, *\rangle$-special frame. Fix $k$ and $Y$, and let $\langle W, R, V\rangle$ be a $\langle 3k, \triangle, Y\rangle$-extraspecial frame by (XS). Consider any $i < 3k$. If $A_i \nvDash \mathrm{GL}$, there exists a non-empty $B_i \in V$ such that $B_i \subseteq A_i$, and $\forall x \in B_i\, \exists y \in B_i\, x\; R\; y$; we put $A'_i = A_i \cap B_i{\downarrow}$, and $*_i = \circ$. If $A_i$ refutes the rule $\Box p/p$, there exists a non-empty $B_i \in V$ such that $B_i \subseteq A_i$, and $A_i \cap B_i{\downarrow} = \varnothing$; then we put $A'_i = B$, and $*_i = \bullet$. Otherwise $A_i$ is of type $\infty$, and we put $A'_i = A_i$, $*_i = \infty$.

By the construction, $A'_i$ is a nonempty lower subset of $A_i$ of type $*_i$. There exists $* \in \{\bullet, \circ, \infty\}$ such that $|\{i;\, *_i = *\}| \geq k$, hence $W$ is a $\langle k, \triangle, Y, *\rangle$-special frame wrt $\{A'_i;\, *_i = *\}$. $\Box$

**Lemma 6.15** *If (XS) fails, then for every $k > 0$ there exists a descriptive $\langle k, \triangle\rangle$-special frame such that $\bigcup_i A_i$ is an antichain of reflexive points.*

*Proof:* As above, it suffices to show that for every $k > 0$ and finite $Y \subseteq X$, there exists a $\langle k, \triangle, Y\rangle$-extraspecial frame, or a $\langle k, \triangle\rangle$-special frame which satisfies the conclusion of the lemma. There exists a descriptive $\langle 2k, \triangle, Y\rangle$-special frame by Lemma 6.13, hence there exists a descriptive $\langle k, \triangle, Y\rangle$-special frame $W$ such that either $A_i \vDash \mathbf{S5}$ for all $i < k$, or $A_i \nvDash \mathbf{S5}$ for all $i < k$. In the former case $A_i$ is an antichain of reflexive clusters by descriptiveness, and we may collapse each cluster to a single point by a p-morphism.

Assume the latter case. Let $i < k$. Notice that $\mathbf{S5} = \mathbf{K4B} \oplus \mathrm{D}$, thus $A_i$ refutes D or B. If $A_i$ refutes D (i.e., it contains an irreflexive final point), we put $A'_i = A_i \cap (A_i \smallsetminus A_i{\downarrow}){\downarrow\!\!\!\!\downarrow}$. If $x \in A'_i$, and $\Diamond\varphi \in Y$, there exists $z \in A_i \smallsetminus A_i{\downarrow}$ such that $x \le z$, and as $z \Vdash \Diamond\varphi$, there exists $y \Vdash \varphi$ such that $z\, R\, y$. Then clearly $x\, R\, y$, and $y \notin A'_i$.

If $A_i$ refutes B, then $A_i$ also refutes the rule $p \to \Diamond\Box\neg p/\neg p$, as

$$\vdash_{\mathbf{K}} \neg(\varphi \to \Box\Diamond\varphi) \to \Diamond\Box(\varphi \to \Box\Diamond\varphi).$$

Hence there exists a nonempty $B_i \in V$ such that $B_i \subseteq A_i$, and $\forall x \in B_i\, \exists y \in A_i \smallsetminus B_i{\downarrow}\; x\, R\, y$. Put $A'_i = A_i \cap B_i{\downarrow\!\!\!\!\downarrow}$. If $x \in A'_i$, and $\Diamond\varphi \in Y$, there exists $z \in A_i \smallsetminus B_i{\downarrow}$ such that $x\, R\, z$, and there exists $y$ such that $z\, R\, y$ and $y \Vdash \varphi$. Then $x\, R\, y$, and $y \notin A'_i$.

It follows that $W$ is a $\langle k, \triangle, Y\rangle$-extraspecial frame wrt $\{A'_i;\, i < k\}$. $\Box$

Let $W$ be a $\langle k, \triangle\rangle$-special frame, and $Y, Y'$ sets of variable-free formulas. We say that $W$ is a $\langle k, \triangle, Y, Y'\rangle$-*superspecial frame,* if $A = \bigcup_i A_i$ is a reflexive antichain, $\exists y \in F\, (x\, R\, y \wedge y \Vdash \varphi)$ for every $x \in A$ and $\varphi \in Y$, and $\neg\exists y \in F\, (x\, R\, y \wedge y \Vdash \varphi)$ for every $x \in A$ and $\varphi \in Y'$.

**Lemma 6.16** *If (XS) fails, there exists a set $Z$ of variable-free formulas such that for every $k > 0$, and finite $Y \subseteq Z$, $Y' \cap Z = \varnothing$, there exists a descriptive $\langle k, \triangle, Y, Y'\rangle$-superspecial frame.*

*Proof:* Using Zorn's lemma similarly to the proof of Lemma 6.13, it suffices to show the following: if $W$ is a $\langle 2k, \triangle, Y, Y'\rangle$-superspecial frame, and $\varphi$ is a variable-free formula, then $W$ is $\langle k, \triangle, Y \cup \{\varphi\}, Y'\rangle$-superspecial, or $\langle k, \triangle, Y, Y' \cup \{\varphi\}\rangle$-superspecial. For any $i < 2k$, we define $A'_i$ as $\{x \in A_i;\, \exists y \in F\, (x\, R\, y \wedge y \Vdash \varphi)\}$ or its complement in $A_i$, whichever is non-empty, and proceed in the usual way. The key point is that superspeciality wrt $Y, Y'$ is preserved, as $A_i$ is an antichain. $\Box$

Let us consider the theory $T$ in variables $r$, $p$, $p_i$ ($i \in \omega$), consisting of the following axioms (for all $i < j \in \omega$, all formulas $\varphi$, and variable-free formulas $\alpha$):

(i) $r$, $\neg p$, $\neg p_i$, $\Diamond p_i$, $\Box(p_i \to p \wedge \neg p_j)$, $\Box(\Diamond r \to r)$, $\Box(\Diamond p \to p \vee r)$, $\Box(\Diamond p_i \to p_i \vee r)$, $\Box(p_i \to \Box(p \to p_i))$, $\varphi \to \Box(r \to \varphi)$,

(ii)

$$\begin{cases} \Box\neg r & \text{if } \triangle = \bullet, \\ \Diamond r & \text{if } \triangle = \circ, \end{cases}$$

(iii)

$$\begin{cases} \Box(p \to \Diamond p) & \text{if } \neg(\mathrm{XS}) \text{ or } * = \circ, \\ \Box(\Diamond p \to r) & \text{if (XS) and } * = \bullet, \\ \left.\begin{array}{l} \Box(p \to \Box\varphi) \to \Box(p \to \varphi) \\ \Box(\Box(p \to \varphi) \to \varphi) \to \Box\varphi \end{array}\right\} & \text{if (XS) and } * = \infty, \end{cases}$$

(iv) if (XS):

$$\begin{cases} \Box(p \to \Box\alpha) & \text{if } \Box\alpha \in X, \\ \Box(\Box(\alpha \vee p) \to \neg p) & \text{if } \neg\Box\alpha \in X, \end{cases}$$

if $\neg$(XS):

$$\begin{cases} \Box(\Box(\alpha \to p) \to \neg p) & \text{if } \alpha \in Z, \\ \Box(p \to \Box(\alpha \to p)) & \text{if } \alpha \notin Z. \end{cases}$$

$T$ is $L$-consistent: it is easy to see that any finite subset of $T$ is satisfied in the root of a suitable $\langle k, \vartriangle, Y, *\rangle$-special or $\langle k, \vartriangle, Y, Y'\rangle$-superspecial frame, if we put

$$\begin{aligned} x \Vdash r &\Leftrightarrow x = u, \\ x \Vdash p_i &\Leftrightarrow x \in A_i, \\ x \Vdash p &\Leftrightarrow x \in A. \end{aligned}$$

Hence there exists a descriptive $L$-frame $\langle W, R, V\rangle$, a valuation $\Vdash\, \in V$, and a point $u \in W$ such that $u \Vdash T$. Without loss of generality $W = W_u$, and $V$ is generated by the sets

$$\begin{aligned} U &:= \{x \in W;\, x \Vdash r\}, \\ A &:= \{x \in W;\, x \Vdash p\}, \\ A_i &:= \{x \in W;\, x \Vdash p_i\}. \end{aligned}$$

We also put $F = W \smallsetminus (A \cup U)$, $A_\infty = A \smallsetminus \bigcup_i A_i$. The axioms (i), and descriptiveness of $W$ imply $U = U\downarrow = \{u\}$, $F \subseteq\cdot W$, $u \notin A$, and $\{A_i;\, i \in \omega \cup \{\infty\}\}$ are pairwise disjoint non-empty lower subsets of $A$. Axiom (ii) implies that $u$ is reflexive iff $\vartriangle = \circ$. Axioms (iii) imply that $A$ (hence also $A_i$, $i \in \omega$) is of type $*$ (or $\circ$, if $\neg$(XS)). Notice that the subframe $F$ of $W$ is 0-generated.

**Lemma 6.17** *$x\uparrow \cap F = y\uparrow \cap F$ for all $x, y \in A$.*

*Proof:* Assume for contradiction $x\ R\ z$, $y \not R\ z$ for some $z \in F$. By descriptiveness, there exists a formula $\varphi$ such that $y \Vdash \Box\varphi$, $z \nVdash \varphi$. As $F$ is 0-generated, there exists a variable-free formula $\alpha$ such that $F \Vdash \varphi \leftrightarrow \alpha$. Assume (XS), the other case is similar. As $y \Vdash \Box(\alpha \vee p) \wedge p$, we cannot have $\neg\Box\alpha \in X$ by axiom (iv), hence $\Box\alpha \in X$. But then $x \Vdash \Box\alpha$ by the same axiom, which contradicts $x\ R\ z \nVdash \alpha$. $\Box$

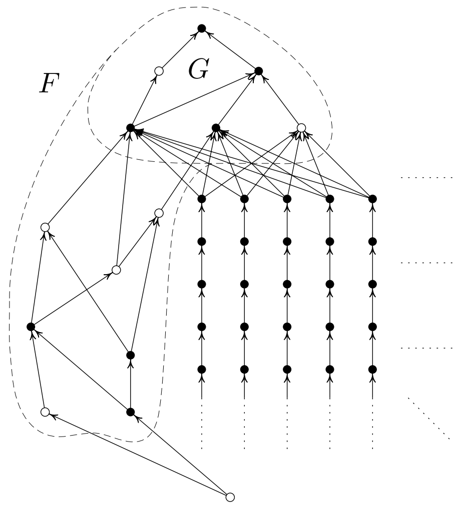


Figure 1: An example of a frame $W_{\circ,\infty,F,G}$.

Let us define

$$G := F \cap x{\uparrow}$$

for any $x \in A$. Notice that $G$ is a generated subframe of $F$. It is also closed in $F$ (i.e., an intersection of admissible subsets): if $u \in F \smallsetminus G$, there exists an admissible set $U$ such that $u \in U$, and $x \in \Box\neg U$ (hence $G \cap U = \varnothing$), as $F$ is refined.

**Definition 6.18** Let $F$ be a transitive general frame, $G$ its generated subframe, $\triangle \in \{\bullet, \circ\}$, and $* \in \{\bullet, \circ, \infty\}$. We define the frame $W_{\triangle,*,F,G}$ as follows. If $F = \varnothing$, $W_{\triangle,*,F,G}$ is a Kripke frame which consists of a root $u$, reflexive iff $\triangle = \circ$, and countably many disjoint copies of the singleton reflexive frame if $* = \circ$, the singleton irreflexive frame if $* = \bullet$, or the irreflexive descending chain of depth $\omega$ if $* = \infty$. In general, $W_{\triangle,*,F,G}$ is a union of $F$ and $W_{\triangle,*,\varnothing,\varnothing}$, where $F$ is a generated subframe of $W_{\triangle,*,F,G}$, $u$ is a root of $W_{\triangle,*,F,G}$, and $F \cap x{\uparrow} = G$ for every $x \in W_{\triangle,*,\varnothing,\varnothing} \smallsetminus \{u\}$. (See Figure 1.)

In a similar way, $W^+_{\triangle,*,F,G}$ is a union of $F$ and $W^+_{\triangle,*,\varnothing,\varnothing}$, where $W^+_{\triangle,*,\varnothing,\varnothing}$ is a general frame defined as follows. If $* \in \{\bullet, \circ\}$, the underlying Kripke frame of $W^+_{\triangle,*,\varnothing,\varnothing}$ is $W_{\triangle,*,\varnothing,\varnothing}$. We pick $x_\infty \in W^+_{\triangle,*,\varnothing,\varnothing} \smallsetminus \{u\}$, and let a subset $X$ of $W^+_{\triangle,*,\varnothing,\varnothing}$ be admissible if and only if $X$ is finite and $x_\infty \notin X$, or $X$ is cofinite and $x_\infty \in X$. If $* = \infty$, we let $W^+_{\triangle,\infty,\varnothing,\varnothing} = \langle W', R', V'\rangle$, where $W' = \{u\} \cup \{x_{\alpha,\beta};\, \alpha, \beta \in \omega + 1\}$, $u\ R'\ u$ iff $\triangle = \circ$, $u\ R'\ x_{\alpha,\beta}$, $x_{\alpha,\beta}\ R'\ x_{\gamma,\delta}$ iff $\beta = \delta$, and $\gamma < \alpha$ or $\alpha = \omega$. The ordinal topology on $\omega + 1$ induces the product topology on $W' \smallsetminus \{u\}$, and we let $V'$ consist of sets $X \subseteq W'$ such that $X \smallsetminus \{u\}$ is clopen in this topology. Notice that $W^+_{\triangle,*,F,G}$ is a descriptive frame whenever $F$ is descriptive, and $G$ is its closed generated subframe.

**Lemma 6.19** *$W$ is isomorphic to $W^+_{\triangle,*,F,G}$.*

*Proof:* Assume $* = \infty$, the other cases are similar but easier. We define a mapping $f\colon W \to W^+_{\Delta,*,F,G}$ so that $f \restriction (F \cup \{u\}) = \mathrm{id}$, and for any $x \in A_\beta$, $f(x) = x_{\alpha,\beta}$, where

$$\alpha = \begin{cases} n & \text{if } A, x \Vdash \vartheta_n := \Box^{n+1}\bot \wedge \Diamond^n\top, \\ \omega & \text{if } A, x \Vdash \Diamond^n\top \text{ for every } n \in \omega. \end{cases}$$

It is easy to see that $x \mathrel{R} y$ implies $f(x) \mathrel{R} f(y)$. Let $n, m \in \omega$. As $A_m \neq \varnothing$, and $A_m$ validates the rule $\Box p/p$, there exists $x \in A_m$ such that $A, x \Vdash \Diamond^{n+1}\top$. Moreover $A_m \vDash \mathbf{GL}$, and $\vdash_{\mathbf{GL}} \Diamond^{n+1} \to \Diamond\vartheta_n$, hence there exists an $x$ such that $f(x) = x_{n,m}$. By descriptiveness there also exists an $x$ such that $f(x) = x_{\omega,m}$, as $\{\{x \in A_m;\ A, x \Vdash \Diamond^n\top\};\ n \in \omega\} \subseteq V$ has fip. A similar compactness argument shows that there exist $x$ such that $f(x) = x_{\alpha,\omega}$ for all $\alpha$, hence $f$ is onto. If $f(x) = x_{\alpha,\beta}$, and $x_{\alpha,\beta} \mathrel{R} x_{\gamma,\delta}$, there exists a $y$ such that $x \mathrel{R} y$ and $f(y) = x_{\gamma,\delta}$: if $\gamma < \omega$, we use $\vdash_{\mathbf{GL}} \Diamond^{\gamma+1}\top \to \Diamond\vartheta_\gamma$; if $\alpha = \gamma = \omega$, we use compactness. It follows that $f$ is a p-morphism of the underlying Kripke frames.

The generators of $V$ (i.e., $A$, $A_m$, and $\{u\}$) are $f$-preimages of sets admissible in $W^+_{\Delta,*,F,G}$, hence $f$ is injective by refinedness of $W$. Conversely, admissible sets of $W^+_{\Delta,*,F,G}$ are generated by $\{u\}$, $F$, and the clopen subbasis of $(\omega+1) \times (\omega+1)$ (i.e., the sets $\{x_{\alpha,m};\ \alpha \in \omega+1\}$ and $\{x_{n,\beta};\ \beta \in \omega+1\}$ for $n, m \in \omega$), whose $f$-preimages are in $V$, hence $f$ is an isomorphism of general frames. $\Box$

**Lemma 6.20** $L(W^+_{\Delta,*,F,G}) = L(W_{\Delta,*,F,G})$.

*Proof:* Let us denote $W_{\Delta,*,F,G} = \langle W, R, V\rangle$, $W^+_{\Delta,*,F,G} = \langle W', R', V'\rangle$, and put $W^-_{\Delta,*,F,G} = \langle W, R, V''\rangle$, where $V'' = \{X \cap W;\ X \in V'\}$. It is easy to see from the definition of $V'$ that

$$W \cap X{\downarrow} = W \cap (X \cap W){\downarrow}$$

for every $X \in V'$; it follows that the mapping $f\colon V' \to V''$ defined by $f(X) = X \cap W$ is an isomorphism of modal algebras, hence $L(W^+_{\Delta,*,F,G}) = L(W^-_{\Delta,*,F,G})$. Clearly $V'' \subseteq V$, thus $L(W_{\Delta,*,F,G}) \subseteq L(W^-_{\Delta,*,F,G})$. Assume $\varphi \notin L(W_{\Delta,*,F,G})$, and fix a valuation $\Vdash \in V$ such that $x \nVdash \varphi$ for some $x \in W$. If $x \neq u$, the subframes of $W^-_{\Delta,*,F,G}$ and $W_{\Delta,*,F,G}$ generated by $x$ coincide, hence $\varphi \notin L(W^-_{\Delta,*,F,G})$. We may thus assume $u \nVdash \varphi$. Let $S$ be the (finite) set of all formulas $\psi$ such that $\Box\psi$ is a subformula of $\varphi$. Assume $* = \infty$, the other cases are similar. There exists $M < \omega$ such that for every $\psi \in S$, if $x_{n,m} \nVdash \psi$ for some $n, m < \omega$, then $x_{n,m} \nVdash \psi$ for some $n < \omega$ and $m \leq M$. Similarly, there exists $N < \omega$ such that for every $\psi \in S$ and $m \leq M$, if $x_{n,m} \nVdash \psi$ for some $n$, then $x_{n,m} \nVdash \psi$ for some $n < N$. We define a valuation $\Vdash^*$ by the following modification of $\Vdash$: for any variable $p$, $m \leq M$, and $n > N$, we put $x_{n,m} \Vdash^* p$ iff $x_{N,m} \Vdash p$; for any $m > M$ and $n$, we put $x_{n,m} \Vdash^* p$ iff $x_{n,M} \Vdash^* p$. Clearly $\Vdash^* \in V''$. A straightforward induction on complexity shows that valuation of subformulas of $\varphi$ is preserved in $u$. In particular, $u \nVdash \varphi$, thus $\varphi \notin L(W^-_{\Delta,*,F,G})$. $\Box$

The considerations above show that any logic $L$ of infinite branching is valid in a frame of the form $W_{\Delta,*,F,G}$. Conversely, it is easy to see that $L(W_{\Delta,*,F,G})$ has infinite branching, using Lemma 6.10. We have thus proved the following characterization.

**Theorem 6.21** *The following are equivalent for any transitive modal logic $L$.*

*(i) $L$ has infinite branching.*

*(ii) There exist $\triangle \in \{\bullet, \circ\}$, $* \in \{\bullet, \circ, \infty\}$, a frame $F$, and a generated subframe $G \subseteq\cdot F$, such that $W_{\triangle,*,F,G} \vDash L$.*

*Moreover, we may assume that $F$ is descriptive, 0-generated, and $G$ is closed in $F$.*

**Remark 6.22** Given a formula $\varphi$, the question whether $\mathbf{K4} \oplus \varphi$ or $\mathbf{IPC} + \varphi$ has finite branching is by definition r.e., and Theorem 6.9 immediately implies that over **IPC** it is in fact decidable. It is not hard to show from Theorem 6.21 that the property of finite branching is also decidable (more precisely *NP*-complete) over e.g. **D4**, **GL**, or $\mathbf{K4BD}_k$, and it is decidable over **K4** for $\bot$-free formulas $\varphi$. However, it is likely to be undecidable over **K4** in general.

## 6.2 Hrubeš tautologies

P. Hrubeš [10] discovered a variant of monotone interpolation for **K**-$F$, and used it to prove an unconditional exponential lower bound on the number of Frege proof lines in **K**, and other modal logics. He extended the lower bound to intuitionistic logic in [11]. We will use his results to show an exponential speed-up of *SF* over *EF* for logics of infinite branching: on the one hand, we extend the lower bounds to all logics of infinite branching, on the other hand we observe that the hard tautologies have poly-size *SF*-proofs.

**Definition 6.23** For any $k \le n$, let $\mathrm{Clique}_n^k(\vec{p}, \vec{r})$ denote the propositional formula which expresses "$\vec{r}$ encodes a clique of size $k$ in the graph on $n$ vertices defined by $\vec{p}$", and let $\mathrm{Colour}_n^k(\vec{p}, \vec{s})$ be a formula expressing "$\vec{s}$ encodes a $k$-colouring of the graph on $n$ vertices defined by $\vec{p}$". We define the *modal Hrubeš formulas*

$$\Theta_n := \mathrm{Clique}_n^{k+1}(\Box\vec{p}, \vec{r}) \to \Box\neg\,\mathrm{Colour}_n^k(\vec{p}, \vec{s}),$$

and the *intuitionistic Hrubeš formulas*

$$\Theta_n^{I,\bot} := \bigwedge_i (p_i \vee q_i) \to \neg\,\mathrm{Colour}_n^k(\vec{p}, \vec{s}) \vee \neg\,\mathrm{Clique}_n^{k+1}(\neg\vec{q}, \vec{r}),$$

where $k := \lfloor \sqrt{n} \rfloor$.

**Theorem 6.24 ([10])** *The formulas $\Theta_n$ are **K**-tautologies. If $L$ is a sublogic of **GL** or **S4**, then $\Theta_n$ requires $L$-$F$-proofs with $2^{n^{\Omega(1)}}$ lines.*

**Theorem 6.25 ([11])** *The formulas $\Theta_n^{I,\bot}$ are **IPC**-tautologies, and require **IPC**-$F$-proofs with $2^{n^{\Omega(1)}}$ lines.*

**Lemma 6.26** *There are poly-time constructible **K**-SF-proofs of $\Theta_n$, and **IPC**-SF-proofs of $\Theta_n^{I,\bot}$.*

*Proof:* There are poly-time constructible **CPC**-$F$-proofs of

$$(*)\qquad \mathrm{Clique}_n^{k+1}(\vec{p},\vec{r}) \to \neg\,\mathrm{Colour}_n^k(\vec{p},\vec{s})$$

by Buss [4]. If $\alpha(\vec{p},\vec{s})$ is any $\Box$-free formula monotone in the variables $\vec{p}$, we construct short **K**-$F$-proofs of

$$\bigwedge_i(\boxdot s_i \vee \boxdot\neg s_i) \to (\alpha(\Box\vec{p},\vec{s}) \to \Box\alpha(\vec{p},\vec{s}))$$

by straightforward induction on the complexity of $\alpha$. Taking $\alpha = \neg\,\mathrm{Colour}_n^k$, we obtain a **K**-$F$-proof of

$$\bigwedge_i(\boxdot s_i \vee \boxdot\neg s_i) \wedge \mathrm{Clique}_n^{k+1}(\Box\vec{p},\vec{r}) \to \Box\neg\,\mathrm{Colour}_n^k(\vec{p},\vec{s}),$$

using a substitution instance of $(*)$. We substitute $\top$ and $\bot$ for $s_0$ to obtain a **K**-$SF$-proof of

$$(**)\qquad \bigwedge_{i>0}(\boxdot s_i \vee \boxdot\neg s_i) \wedge \mathrm{Clique}_n^{k+1}(\Box\vec{p},\vec{r}) \to$$
$$\to \Box\neg\,\mathrm{Colour}_n^k(\vec{p},\top,s_1,s_2,\dots) \wedge \Box\neg\,\mathrm{Colour}_n^k(\vec{p},\bot,s_1,s_2,\dots).$$

As Colour is a $\Box$-free formula, there are poly-size proofs of

$$\Box(\neg\,\mathrm{Colour}_n^k(\vec{p},\top,s_1,\dots) \wedge \neg\,\mathrm{Colour}_n^k(\vec{p},\bot,s_1,\dots) \to \neg\,\mathrm{Colour}_n^k(\vec{p},s_0,s_1,\dots)),$$

which we combine with $(**)$ to get a proof of

$$\bigwedge_{i>0}(\boxdot s_i \vee \boxdot\neg s_i) \wedge \mathrm{Clique}_n^{k+1}(\Box\vec{p},\vec{r}) \to \Box\neg\,\mathrm{Colour}_n^k(\vec{p},\vec{s}).$$

We continue to eliminate the other conjuncts $\boxdot s_i \vee \boxdot\neg s_i$ in the same way.

We construct an **IPC**-$F$-proof of $(*)$ using Glivenko translation, and the intuitionistic equivalence $\neg\neg(\varphi \to \neg\psi) \leftrightarrow (\varphi \to \neg\psi)$. For any formula $\alpha(\vec{p})$, there are poly-time constructible **IPC**-$F$-proofs of

$$\bigwedge_i(p_i \vee \neg p_i) \to \alpha \vee \neg\alpha,$$

hence we obtain a proof of

$$\bigwedge_i(p_i \vee \neg p_i) \wedge \bigwedge_i(r_i \vee \neg r_i) \to \neg\,\mathrm{Colour}_n^k(\vec{p},\vec{s}) \vee \neg\,\mathrm{Clique}_n^{k+1}(\vec{p},\vec{r}).$$

We substitute $\bot$ and $\top$ for $r_0$ to get

$$\bigwedge_i(p_i \vee \neg p_i) \wedge \bigwedge_{i>0}(r_i \vee \neg r_i) \to \neg\,\mathrm{Colour}_n^k(\vec{p},\vec{s})\vee$$
$$\vee\,(\neg\,\mathrm{Clique}_n^{k+1}(\vec{p},\bot,r_1,\dots) \wedge \neg\,\mathrm{Clique}_n^{k+1}(\vec{p},\top,r_1,\dots)),$$

and use a short **CPC**-$F$-proof (hence **IPC**-$F$-proof, by Glivenko translation) of

$$\neg\,\mathrm{Clique}_n^{k+1}(\vec{p},\bot,r_1,\dots) \wedge \neg\,\mathrm{Clique}_n^{k+1}(\vec{p},\top,r_1,\dots) \to \neg\,\mathrm{Clique}_n^{k+1}(\vec{p},\vec{r})$$

to infer

$$\bigwedge_i (p_i \vee \neg p_i) \wedge \bigwedge_{i>0} (r_i \vee \neg r_i) \to \neg\,\mathrm{Colour}_n^k(\vec{p}, \vec{s}) \vee \neg\,\mathrm{Clique}_n^{k+1}(\vec{p}, \vec{r}).$$

We continue in the same way to construct an **IPC**-$SF$-proof of

$$\bigwedge_i (p_i \vee \neg p_i) \to \neg\,\mathrm{Colour}_n^k(\vec{p}, \vec{s}) \vee \neg\,\mathrm{Clique}_n^{k+1}(\vec{p}, \vec{r}).$$

Then we construct **IPC**-$SF$-proofs of

$$\bigwedge_{i<j} (p_i \vee q_i) \wedge \bigwedge_{i\geq j} (p_i \vee \neg p_i) \to \neg\,\mathrm{Colour}_n^k(\vec{p}, \vec{s}) \vee \neg\,\mathrm{Clique}_n^{k+1}(\neg q_0, \dots, \neg q_{j-1}, p_j, \dots, \vec{r})$$

by induction on $j$. The induction step from $j$ to $j+1$ is as follows. We substitute $\bot$ and $\top$ for $p_j$ to get

$$\bigwedge_{i<j} (p_i \vee q_i) \wedge \bigwedge_{i>j} (p_i \vee \neg p_i) \to \neg\,\mathrm{Colour}_n^k(\bot) \vee \neg\,\mathrm{Clique}_n^{k+1}(\bot),$$
$$\bigwedge_{i<j} (p_i \vee q_i) \wedge \bigwedge_{i>j} (p_i \vee \neg p_i) \to \neg\,\mathrm{Colour}_n^k(\top) \vee \neg\,\mathrm{Clique}_n^{k+1}(\top).$$

As $\neg\,$Colour is monotone in $p_j$, there are short proofs of

$$\neg\,\mathrm{Colour}_n^k(\bot) \to \neg\,\mathrm{Colour}_n^k(p_j),$$

hence we obtain

$$\begin{aligned}\bigwedge_{i<j} (p_i \vee q_i) \wedge \bigwedge_{i>j} (p_i \vee \neg p_i) \to \neg\,\mathrm{Colour}_n^k(p_j) \vee \\ \vee (\neg\,\mathrm{Clique}_n^{k+1}(\bot) \wedge \neg\,\mathrm{Colour}_n^k(\top)) \vee (\neg\,\mathrm{Clique}_n^{k+1}(\bot) \wedge \neg\,\mathrm{Clique}_n^{k+1}(\top)).\end{aligned}$$

We combine it with short proofs of

$$\neg\,\mathrm{Clique}_n^{k+1}(\bot) \wedge \neg\,\mathrm{Clique}_n^{k+1}(\top) \to \neg\,\mathrm{Clique}_n^{k+1}(\neg q_j),$$
$$(p_j \vee q_j) \wedge \neg\,\mathrm{Colour}_n^k(\top) \wedge \neg\,\mathrm{Clique}_n^{k+1}(\bot) \to \neg\,\mathrm{Colour}_n^k(p_j) \vee \neg\,\mathrm{Clique}_n^{k+1}(\neg q_j),$$

and obtain a proof of

$$\bigwedge_{i\leq j} (p_i \vee q_i) \wedge \bigwedge_{i>j} (p_i \vee \neg p_i) \to \neg\,\mathrm{Colour}_n^k(p_j) \vee \neg\,\mathrm{Clique}_n^{k+1}(\neg q_j). \qquad \square$$

The intuitionistic tautologies $\Theta_n^{I,\bot}$ are not quite satisfactory for our purposes, because of the following observation.

**Lemma 6.27**

*(i)* **KC**-$F$ *p-simulates* **CPC**-$F$ *wrt essentially negative formulas.*

(*ii*) *There are poly-time constructible* **KC**-*F*-*proofs of* $\Theta_n^{I,\perp}$.

*Proof:* (i): Given a classical proof of an essentially negative formula $\varphi$, we construct an intuitionistic proof of $\neg\neg\varphi$. We combine it with a poly-size **KC**-$F$-proof of $\neg\neg\varphi \to \varphi$, which we construct by induction on the complexity of $\varphi$, using the **KC**-tautology $\neg\neg(\alpha \vee \beta) \to \neg\neg\alpha \vee \neg\neg\beta$.

(ii): We use (i), and a short classical proof of $\Theta_n^{I,\perp}$ to construct a **KC**-$F$-proof of

$$\bigwedge_i (\neg\neg p_i \vee \neg\neg q_i) \to \neg\,\mathrm{Colour}_n^k(\vec{p},\vec{s}) \vee \neg\,\mathrm{Clique}_n^{k+1}(\neg\vec{q},\vec{r}),$$

which implies $\Theta_n^{I,\perp}$ by a short **IPC**-$F$-subproof using $p_i \to \neg\neg p_i$, $q_i \to \neg\neg q_i$. $\square$

The logic of the weak excluded middle **KC** certainly has infinite branching; it is arguably the most important si logic after **CPC** and **IPC**, and many other interesting si logics are its sublogics (e.g., the Kreisel–Putnam logic **KP**, the Scott logic **SL**, the Medvedev logic **ML**, logics axiomatized by the Nishimura formulas, etc.). It is thus very desirable to make sure our lower bounds apply to **KC**, which means we cannot use the original formulas $\Theta^{I,\perp}$. The problem is that there are too many negations in $\Theta^{I,\perp}$, hence the obvious solution is to devise a negation-free version of these tautologies, which we define next.

**Definition 6.28** Let $k \le n$. We define the formulas

$$\alpha_n^k(\vec{p},\vec{s},\vec{s'}) := \bigvee_{i<n}\bigwedge_{\ell<k} s'_{i,\ell} \vee \bigvee_{i,j<n}\bigvee_{\ell<k}(s_{i,\ell}\wedge s_{j,\ell}\wedge p_{i,j}),$$
$$\beta_n^k(\vec{q},\vec{r},\vec{r'}) := \bigvee_{\ell<k}\bigwedge_{i<n} r'_{i,\ell} \vee \bigvee_{i,j<n}\bigvee_{\ell<m<k}(r_{i,\ell}\wedge r_{j,m}\wedge q_{i,j}).$$

Notice that $\mathrm{Colour}_n^k(\vec{p},\vec{s}) = \neg\alpha_n^k(\vec{p},\vec{s},\neg\vec{s})$, and $\mathrm{Clique}_n^k(\vec{p},\vec{r}) = \neg\beta_n^k(\neg\vec{p},\vec{r},\neg\vec{r})$. We introduce the *negation-free* (*intuitionistic*) *Hrubeš formulas*

$$\Theta_n^I := \bigwedge_{i,j}(p_{i,j}\vee q_{i,j}) \to \Big(\bigwedge_{i,\ell}(s_{i,\ell}\vee s'_{i,\ell}) \to \alpha_n^k(\vec{p},\vec{s},\vec{s'})\Big) \vee \Big(\bigwedge_{i,\ell}(r_{i,\ell}\vee r'_{i,\ell}) \to \beta_n^{k+1}(\vec{q},\vec{r},\vec{r'})\Big),$$

where $k = \lfloor\sqrt{n}\rfloor$.

The original Hrubeš formulas $\Theta_n^{I,\perp}$ are (equivalent to) substitution instances of the $\perp$-free formulas $\Theta_n^I$, hence the lower bound of Theorem 6.25 also applies to $\Theta_n^I$.

**Lemma 6.29** *There are poly-time constructible* **IPC**-*SF*-*proofs of* $\Theta_n^I$.

*Proof:* We use a classical proof of $\alpha(\vec{p},\vec{s},\neg\vec{s}) \vee \beta(\neg\vec{p},\vec{r},\neg\vec{r})$ to construct an **IPC**-$F$-proof of

$$\bigwedge_i(p_i\vee\neg p_i)\wedge\bigwedge_i(s_i\vee\neg s_i)\wedge\bigwedge_i(r_i\vee\neg r_i) \to \alpha(\vec{p},\vec{s},\neg\vec{s})\vee\beta(\neg\vec{p},\vec{r},\neg\vec{r}).$$

We substitute $\perp$ and $\top$ for $r_0$ to obtain

$$\bigwedge_i(p_i\vee\neg p_i)\wedge\bigwedge_i(s_i\vee\neg s_i)\wedge\bigwedge_{i>0}(r_i\vee\neg r_i) \to \alpha(\vec{p},\vec{s},\neg\vec{s})\vee$$
$$\vee\,(\beta(\neg\vec{p},\top,r_1,\ldots,\perp,\neg r_1,\ldots)\wedge\beta(\neg\vec{p},\perp,r_1,\ldots,\top,\neg r_1,\ldots)).$$

As $\beta$ is monotone, we can construct short proofs of

$$\begin{aligned}\beta(\neg\vec{p},\top,r_1,\dots,\bot,\neg r_1,\dots) &\to \beta(\neg\vec{p},\top,r_1,\dots,r_0',\neg r_1,\dots)\\ &\to (r_0 \to \beta(\neg\vec{p},r_0,r_1,\dots,r_0',\neg r_1,\dots)),\\ \beta(\neg\vec{p},\bot,r_1,\dots,\top,\neg r_1,\dots) &\to \beta(\neg\vec{p},r_0,r_1,\dots,\top,\neg r_1,\dots)\\ &\to (r_0' \to \beta(\neg\vec{p},r_0,r_1,\dots,r_0',\neg r_1,\dots)),\end{aligned}$$

and we infer

$$\bigwedge_i(p_i \vee \neg p_i) \wedge \bigwedge_i(s_i \vee \neg s_i) \wedge \bigwedge_{i>0}(r_i \vee \neg r_i) \to \alpha(\vec{p},\vec{s},\neg\vec{s}) \vee ((r_0 \vee r_0') \to \beta(\neg\vec{p},\vec{r},r_0',\neg r_1,\dots)).$$

We continue in the same way to construct a proof of

$$\bigwedge_i(p_i \vee \neg p_i) \wedge \bigwedge_i(s_i \vee \neg s_i) \to \alpha(\vec{p},\vec{s},\neg\vec{s}) \vee \Big(\bigwedge_i(r_i \vee r_i') \to \beta(\neg\vec{p},\vec{r},\vec{r'})\Big).$$

We apply the same construction to $\alpha$ and $\vec{s}$, and obtain a proof of

$$\bigwedge_i(p_i \vee \neg p_i) \to \Big(\bigwedge_i(s_i \vee s_i') \to \alpha(\vec{p},\vec{s},\vec{s'})\Big) \vee \Big(\bigwedge_i(r_i \vee r_i') \to \beta(\neg\vec{p},\vec{r},\vec{r'})\Big).$$

Then we construct a proof of

$$\bigwedge_i(p_i \vee q_i) \to \Big(\bigwedge_i(s_i \vee s_i') \to \alpha(\vec{p},\vec{s},\vec{s'})\Big) \vee \Big(\bigwedge_i(r_i \vee r_i') \to \beta(\vec{q},\vec{r},\vec{r'})\Big)$$

as in Lemma 6.26. □

The next lemma shows why making the hard tautologies $\bot$-free helps.

**Lemma 6.30** *If $\Gamma$ is the set of all $\bot$-free formulas, then* $\mathbf{KC}$-*EF* $\le_{p,\Gamma}$ $\mathbf{IPC}$-*EF, and* $(\mathbf{BD}_3+\mathbf{KC})$-*EF* $\le_{p,\Gamma}$ $\mathbf{BD}_2$-*EF.*

*Proof:* Let $v$ be the classical valuation which makes all variables true. Let $\varphi^*$ be the translation which preserves variables and $\bot$, commutes with $\wedge$ and $\vee$, and

$$(\varphi \to \psi)^* = \begin{cases}\varphi^* \to \psi^* & \text{if } v(\varphi \to \psi) = 1,\\ \bot & \text{otherwise.}\end{cases}$$

Notice that $v(\varphi) = 1$, and $\varphi = \varphi^*$, for every $\varphi \in \Gamma$. If $\varphi_1,\dots,\varphi_m$ is a $\mathbf{KC}$-*CF*-proof, we construct the sequence $\varphi_1^*,\dots,\varphi_m^*$, and complete it to an $\mathbf{IPC}$-*CF*-proof. As $v$ satisfies all (classical, hence $\mathbf{KC}$) tautologies, instances of modus ponens translate to instances of modus ponens; it thus suffices to show that translations of axioms have poly-time constructible proofs.

Let $\alpha$ be an instance of the axiom

$$(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi)).$$

Then $\alpha^*$ is one of the formulas

$$(\varphi^* \to (\psi^* \to \chi^*)) \to ((\varphi^* \to \psi^*) \to (\varphi^* \to \chi^*)),$$
$$(\varphi^* \to \bot) \to ((\varphi^* \to \psi^*) \to (\varphi^* \to \chi^*)),$$
$$(\varphi^* \to (\psi^* \to \chi^*)) \to (\bot \to \bot),$$
$$(\varphi^* \to (\psi^* \to \chi^*)) \to (\bot \to (\varphi^* \to \chi^*)),$$
$$\bot \to \bot,$$

it is thus an instance of one of the tautologies

$$(p \to (q \to r)) \to ((p \to q) \to (p \to r)),$$
$$\neg p \to (q \to (p \to r)),$$
$$p \to (\bot \to q),$$
$$\bot \to \bot,$$

hence it has an **IPC**-$CF$-proof of size linear in $|\alpha|$. The other axioms of **IPC** are handled similarly.

It is easy to show by induction on $\varphi$ that $\vdash_{\mathbf{IPC}} \neg\varphi^*$ whenever $v(\varphi) = 0$, and moreover, we can construct an **IPC**-$CF$-proof of $\neg\varphi^*$ of polynomial size. Hence up to shortly provable equivalence, we have

$$(\neg\varphi)^* = \begin{cases} \bot & \text{if } v(\varphi) = 1, \\ \top & \text{if } v(\varphi) = 0. \end{cases}$$

Translations of instances of the KC-axiom $\neg\varphi \vee \neg\neg\varphi$ thus have short **IPC**-$CF$-proofs.

If $\alpha$ is an instance of the $\mathrm{BD}_3$-axiom

$$\varphi \vee (\varphi \to \psi \vee (\psi \to \chi \vee \neg\chi)),$$

then

$$\alpha^* = \varphi^* \vee (\varphi^* \to \psi^* \vee (\psi^* \to \chi^* \vee (\neg\chi)^*)),$$

as $v(\cdots \to \chi \vee \neg\chi) = 1$. Hence $\alpha^*$ follows from the instance

$$\varphi^* \vee (\varphi^* \to \psi^* \vee \neg\psi^*)$$

of $\mathrm{BD}_2$. □

**Remark 6.31** The intent of a modal analogue of Lemma 6.30 is to get rid of $F$ in $L(W_{\vartriangle,*,F,G})$-proofs of the tautologies $\Theta_n$. The actual result we obtain (Lemma 6.33) happens to be somewhat more complicated, and involves *quasi-normal* logics of the frames $W_{\bullet,*,\varnothing,\varnothing}$.

(Circuit, substitution) Frege systems for quasi-normal modal logics can be defined similarly to normal logics, by omitting the necessitation rule. I.e., a quasi-normal Frege system consists of finitely many axiom schemata, and modus ponens.

Every normal logic $L$ is also quasi-normal. If $L \supseteq \mathbf{K4}$, and $L$ is finitely axiomatizable, then $L$ is also finitely axiomatizable as a quasi-normal logic, hence it admits a quasi-normal Frege system. By (the proof of) Lemma 3.5, the normal and quasi-normal (circuit, substitution) Frege systems for $L$ are p-equivalent, as we only need the axioms of $P_{-}$ with "$k \leq 1$" due to the transitivity axiom.

**Lemma 6.32**

$$L_{qn}(W_{\bullet,\bullet,\varnothing,\varnothing}) = (\mathbf{K} \oplus \Box\Box\bot) + \Diamond\top,$$
$$L_{qn}(W_{\bullet,\circ,\varnothing,\varnothing}) = (\mathbf{K} \oplus \Box(p \leftrightarrow \Box p)) + \Diamond\top,$$
$$L_{qn}(W_{\bullet,\infty,\varnothing,\varnothing}) = (\mathbf{GL} \oplus \Box(\Box(\Box p \to q) \vee \Box(\boxdot q \to p))) + \Diamond\top + (\Box\Box p \to \Box p).$$

*Proof:* The inclusions $\supseteq$ are obvious. The logic $\mathbf{K} \oplus \Box^2\bot$ includes $\mathbf{K4BD}_2$, hence it has the finite model property; its finite rooted frames are irreflexive trees of depth at most 2. Thus assuming $\varphi \notin (\mathbf{K} \oplus \Box\Box\bot) + \Diamond\top$, there exists such a tree $W$ with root $r$, and a valuation $\Vdash$ such that $r \Vdash \Diamond\top \wedge \neg\varphi$. In particular, $r$ has at least one successor, thus there exists a p-morphism of $W_{\bullet,\bullet,\varnothing,\varnothing}$ onto $W$ which respects the roots, hence $\varphi \notin L_{qn}(W_{\bullet,\bullet,\varnothing,\varnothing})$.

The inclusion $L_{qn}(W_{\bullet,\circ,\varnothing,\varnothing}) \subseteq (\mathbf{K} \oplus \Box(p \leftrightarrow \Box p)) + \Diamond\top$ is completely analogous.

We will use some results of [13] to prove the third equation. As $\mathbf{K4} \oplus \Box\mathrm{GL} = \mathbf{GL}$, the RHS of the last equation equals the logic $A^{\Box}_{\mathbf{GL.3}}$ by [13, Thm. 4.13]. If $\varphi \notin A^{\Box}_{\mathbf{GL.3}}$, then $\varphi$ is refutable in the irreflexive root $r$ of a countable Kripke frame $W$ such that $W \smallsetminus \{r\}$ is a linearly extensible $\mathbf{GL.3}$-frame by [13, Thm. 3.5]. It is easy to see that $W$ is a p-morphic image of $W_{\bullet,\infty,\varnothing,\varnothing}$, hence $\varphi \notin L_{qn}(W_{\bullet,\infty,\varnothing,\varnothing})$. $\Box$

**Lemma 6.33** *Let $L$ be a transitive modal logic of infinite width, and $\varphi$ a formula of the form*

$$\alpha(\vec{p}, \Box\vec{p}) \to \Box\beta(\vec{p}),$$

*where $\alpha, \beta$ are $\Box$-free. If $\varphi$ has an $L$-CF-proof of size $s$, then $\varphi$ or*

$$\varphi' := \alpha(\vec{p}, \boxdot\vec{p}) \to \Box\beta(\vec{p})$$

*has a CF-proof of size $O(s)$ in $\mathbf{K} \oplus \Box^2\bot$, $\mathbf{K} \oplus \Box(p \leftrightarrow \Box p)$, or $L_{qn}(W_{\bullet,\infty,\varnothing,\varnothing})$.*

*Proof:* By Theorem 6.21, $L$ is included in the logic of a frame of the form $W_{\triangle,*,F,G}$. Let $\Vdash_0$ be the valuation in $F$ which makes all variables true. We define translations $\varphi^*$ and $\varphi^\triangle$, which preserve variables, commute with Boolean connectives, and

$$(\Box\varphi)^* = \begin{cases} \bot & \text{if } G \nVdash_0 \varphi, \\ \Box\varphi^* & \text{if } G \Vdash_0 \varphi, \end{cases}$$

$$(\Box\varphi)^\triangle = \begin{cases} \bot & \text{if } F \nVdash_0 \varphi, \\ \Box\varphi^* & \text{if } F \Vdash_0 \varphi, \text{ and } \triangle = \bullet, \\ \varphi^\triangle \wedge \Box\varphi^* & \text{if } F \Vdash_0 \varphi, \text{ and } \triangle = \circ. \end{cases}$$

Let $u$ be the root of $W_{\triangle,*,F,G}$.

**Claim 1** *If $\varphi \in L$, then $\varphi^\vartriangle \in L_{qn}(W_{\bullet,*,\varnothing,\varnothing})$. Moreover, there exists a finite sequence of formulas $\varphi_1, \dots, \varphi_k \in L_{qn}(W_{\bullet,*,\varnothing,\varnothing})$ such that for every substitution $\sigma$, $(\sigma\varphi)^\vartriangle$ is an instance of some $\varphi_i$.*

*Proof:* Let $S$ be the set of subformulas of $\varphi(\vec{p})$. For any substitution $\sigma$, we define a function $d_\sigma \colon S \to 3$ by

$$d_\sigma(\psi) = \begin{cases} 2 & \text{if } F \Vdash_0 \sigma\psi, \\ 1 & \text{if } G \Vdash_0 \sigma\psi, F \nVdash_0 \sigma\psi, \\ 0 & \text{if } G \nVdash_0 \sigma\psi. \end{cases}$$

For any $d \colon S \to 3$, and $\psi$ a subformula of $\varphi$, we define formulas $\psi^*_d(\vec{q}, \vec{r})$, $\psi^\vartriangle_d(\vec{q}, \vec{r})$ by

$$\begin{aligned} (p_i)^\vartriangle_d &= q_i, \\ (p_i)^*_d &= r_i, \\ (\psi \circ \chi)^\dagger_d &= \psi^\dagger_d \circ \chi^\dagger_d, \qquad \dagger \in \{*, \vartriangle\}, \circ \in \{\wedge, \vee, \to, \bot\}, \\ (\Box\psi)^*_d &= \begin{cases} \bot & \text{if } d(\psi) = 0, \\ \Box\psi^*_d & \text{if } d(\psi) > 0, \end{cases} \\ (\Box\psi)^\vartriangle_d &= \begin{cases} \bot & \text{if } d(\psi) < 2, \\ \Box\psi^*_d & \text{if } d(\psi) = 2, \text{ and } \vartriangle = \bullet, \\ \psi^\vartriangle_d \wedge \Box\psi^*_d & \text{if } d(\psi) = 2, \text{ and } \vartriangle = \circ. \end{cases} \end{aligned}$$

Clearly $(\sigma\varphi)^\vartriangle = \varphi^\vartriangle_{d_\sigma}((\overrightarrow{\sigma p})^\vartriangle, (\overrightarrow{\sigma p})^*)$, it thus suffices to show that $\varphi^\vartriangle_d \in L_{qn}(W_{\bullet,*,\varnothing,\varnothing})$ whenever $d = d_\sigma$ for some substitution $\sigma$. Let $\Vdash$ be a valuation of $\vec{q}$ and $\vec{r}$ in $W_{\bullet,*,\varnothing,\varnothing}$, and define a valuation $\Vdash'$ of $\vec{p}$ in $W_{\vartriangle,*,F,G}$ by

$$x \Vdash' p_i \Leftrightarrow \begin{cases} x \Vdash_0 \sigma p_i & \text{if } x \in F, \\ x \Vdash q_i & \text{if } x = u, \\ x \Vdash r_i & \text{if } x \notin F \cup \{u\}. \end{cases}$$

We have

$$W_{\vartriangle,*,F,G}, x \Vdash' \psi \Leftrightarrow F, x \Vdash_0 \sigma\psi$$

for every $x \in F$ and every $\psi$, as $F$ is a generated subframe of $W_{\vartriangle,*,F,G}$. Then a straightforward induction on the complexity of $\psi \in S$ shows

$$\begin{aligned} W_{\vartriangle,*,F,G}, x \Vdash' \psi &\Leftrightarrow W_{\bullet,*,\varnothing,\varnothing}, x \Vdash \psi^*_d, \\ W_{\vartriangle,*,F,G}, u \Vdash' \psi &\Leftrightarrow W_{\bullet,*,\varnothing,\varnothing}, u \Vdash \psi^\vartriangle_d \end{aligned}$$

for every $x \notin F \cup \{u\}$. In particular, $\varphi \in L \subseteq L(W_{\vartriangle,*,F,G})$ implies $u \Vdash \varphi^\vartriangle_d$. $\Box$ (Claim 1)

Let $\varphi_1, \dots, \varphi_m = \varphi$ be an $L$-$CF$-proof of a formula $\varphi$. We may assume the proof to be necessitation-free by Remark 6.31. Then the sequence $\varphi^\vartriangle_1, \dots, \varphi^\vartriangle_m$ can be completed to a polynomially longer (quasi-normal) $L_{qn}(W_{\bullet,*,\varnothing,\varnothing})$-$CF$-proof of $\varphi^\vartriangle$ by the claim (and Lemma 3.1). In the case $* \in \{\bullet, \circ\}$, we use the axiomatization of $L_{qn}(W_{\bullet,*,\varnothing,\varnothing})$ from Lemma 6.32, and we further eliminate the axiom $\Diamond\top$ using the feasible deduction theorem to obtain a $(\mathbf{K}\oplus\Box^2\bot)$-$CF$ or $(\mathbf{K}\oplus\Box(p\leftrightarrow\Box p))$-$CF$-proof of $\Box\bot\vee\varphi^\vartriangle$.

Assume that $\varphi = \alpha(\vec{p},\Box\vec{p})\to\Box\beta(\vec{p})$ for some propositional formulas $\alpha$, and $\beta$. Notice that $F\Vdash_0\vec{p}$ by definition. If $\vartriangle = \bullet$, the formula $\varphi^\vartriangle$ is thus one of the formulas

$$\alpha(\vec{p},\Box\vec{p})\to\Box\beta(\vec{p}),$$
$$\alpha(\vec{p},\Box\vec{p})\to\bot,$$

hence $\Box\bot\vee\varphi^\vartriangle$ implies $\varphi$. If $\vartriangle = \circ$, then $\Box\bot\vee\varphi^\vartriangle$ similarly implies $\varphi'$. $\Box$

**Remark 6.34** Both Lemmas 6.30 and 6.33 formalize in $CF$ the well-known model-theoretical argument that validity of $\bot$-free formulas is preserved under dense subframes. However, the formulas $\Theta_n$ are not actually $\bot$-free, hence we formulated Lemma 6.33 under different assumptions.

The translation $(\cdot)^\vartriangle$ in the proof of Lemma 6.33 does not make a p-simulation, as validity of variable-free formulas in $F$ or $G$ may be undecidable.

The remaining task is to prove a lower bound on $CF$-proofs of $\Theta_n$ (or $\Theta^I_n$) in the four logics $\mathbf{BD}_2$, $\mathbf{K}\oplus\Box^2\bot$, $\mathbf{K}\oplus\Box(p\leftrightarrow\Box p)$, and $L_{qn}(W_{\bullet,\infty,\varnothing,\varnothing})$. In the first three cases we could give a simple reduction to Hrubeš's theorems 6.24, 6.25. However, such a reduction seems impossible in the case of $L_{qn}(W_{\bullet,\infty,\varnothing,\varnothing})$, we thus have to include a full proof of the lower bound. We will actually give a self-contained proof of the lower bound in all four cases, because we believe that our approach using propositional valuations as in [14] (inspired by the feasible Kleene slash of [16]) is simpler and more direct than Hrubeš's proof, hence the reader may find it useful.

**Lemma 6.35** *Let $\psi$ be a formula of the form*

$$\alpha(\Box\vec{p},\vec{r})\to\Box\beta(\vec{p},\vec{s})$$

*or*

$$\alpha(\boxdot\vec{p},\vec{r})\to\Box\beta(\vec{p},\vec{s}),$$

*where $\alpha$ and $\beta$ are $\Box$-free. If $\psi$ has a $CF$-proof of size $s$ in $\mathbf{K}\oplus\Box^2\bot$, $\mathbf{K}\oplus\Box(p\leftrightarrow\Box p)$, or $L_{qn}(W_{\bullet,\infty,\varnothing,\varnothing})$, there exists a monotone circuit $C(\vec{p})$ of size $O(s^2)$ which interpolates the classical tautology*

$$\alpha(\vec{p},\vec{r})\to\beta(\vec{p},\vec{s}).$$

*Proof:* Let $\pi$ be an $L$-$CF$-proof of $\psi$. For any set $X$ of the variables $\vec{p}$, let $P(X)$ be the closure of $\pi\cup X\cup\Box X$ under modus ponens and the rule $\Box\varphi/\varphi$ if $L=\mathbf{K}\oplus\Box(p\leftrightarrow\Box p)$ or $L = L_{qn}(W_{\bullet,\infty,\varnothing,\varnothing})$, and the closure of $\pi\cup\{\Box\bot\}\cup X\cup\Box X$ under modus ponens if $L=\mathbf{K}\oplus\Box^2\bot$. We define a Boolean function $\xi(\vec{p})$ by

$$\xi(\vec{x})\Leftrightarrow\beta(\vec{p},\vec{s})\in P(\{p_i;\, x_i=1\}).$$

**Claim 1** *For any assignment $e$, if $e(\xi)=1$, then $e(\beta)=1$.*

*Proof:* Let $\sigma$ be the substitution which maps each $p_i$ satisfied by $e$ to $\top$, and put $P' = \sigma P(\{p_i;\, e(p_i) = 1\})$. Then $P'$ is included in

$$\begin{cases} \mathbf{K} \oplus \Box\bot & \text{if } L = \mathbf{K} \oplus \Box^2\bot, \\ \mathbf{K} \oplus (p \leftrightarrow \Box p) & \text{if } L = \mathbf{K} \oplus \Box(p \leftrightarrow \Box p), \\ \mathbf{GL.3} + (\Box p \to p) & \text{if } L = L_{qn}(W_{\bullet,\infty,\varnothing,\varnothing}), \end{cases}$$

which is a consistent quasi-normal logic, hence conservative over the classical logic. We have thus $\sigma\beta \in P' \subseteq \mathbf{CPC}$, which implies $e(\beta) = 1$. □ (Claim 1)

**Claim 2** *For any assignment $e$, if $e(\alpha) = 1$, then $e(\xi) = 1$.*

*Proof:* Put $P := P(\{p_i;\, e(p_i) = 1\})$, and extend $e$ to all modal formulas by

$$e(\Box\varphi) = 1 \quad \text{iff} \quad \{\varphi, \Box\varphi\} \subseteq P.$$

We have $e(\Box p_i) = e(p_i)$ ($\geq$ is obvious, and $\leq$ follows from the proof of Claim 1), hence $e(\alpha(\Box\vec{p}, \vec{r})) = e(\alpha(\boxdot\vec{p}, \vec{r})) = 1$. It thus suffices to show that $e(\varphi) = 1$ for all formulas $\varphi \in \pi$, which we prove by induction on the length of $\pi$. The induction steps for propositional axioms and rules are trivial, and the step for necessitation is immediate from $\pi \subseteq P$. The step for the K axiom follows from the closure of $P$ under modus ponens. If $\varphi$ is an axiom of the form $\Box\chi$, then either $\chi \in P$ by closure under $\Box\chi/\chi$, or $\chi = \Box\bot \in P$ by definition. If $L = L_{qn}(W_{\bullet,\infty,\varnothing,\varnothing})$, then $e(\Box^2\varphi \to \Box\varphi) = 1$ by closure under $\Box\varphi/\varphi$, $e(\Box\bot) = 0$ as $P$ is consistent by the proof of Claim 1, and $e(\Box\varphi \to \Box\Box\varphi) = 1$ by closure under modus ponens (using $\Box\varphi \to \Box\Box\varphi \in \pi \subseteq P$). □ (Claim 2)

It remains to show that $\xi$ is computable by a small monotone circuit $C$. We assume $L = \mathbf{K} \oplus \Box^2\bot$ for notational simplicity. Let $S$ be the set of subformulas of formulas in $\pi$. Notice that $|S| \leq s := |\pi|$, and $P(X) \subseteq S$ for any $X \subseteq S$. Moreover, if $U = U_0 \subseteq S$, and $U_{i+1}$ is the closure of $U_i$ under one application of modus ponens, then the closure of $U$ under modus ponens is $U_s$. We can define a monotone circuit which includes nodes $c_{\varphi,i}$ for every $\varphi \in S$, and $i \leq s$, with the meaning

$$c_{\varphi,i} = 1 \quad \text{iff} \quad \varphi \in (\pi \cup \{\Box\bot\} \cup X \cup \Box X)_i,$$

as follows: we put

$$c_{\varphi,i+1} = c_{\varphi,i} \vee \bigvee_{\psi\to\varphi\in S} (c_{\psi,i} \wedge c_{\psi\to\varphi,i})$$

for every $\varphi \in S$ and $i < s$, and

$$c_{\varphi,0} = \begin{cases} 1 & \text{if } \varphi \in \pi \cup \{\Box\bot\}, \\ p_i & \text{if } \varphi = p_i \text{ or } \varphi = \Box p_i, \\ 0 & \text{otherwise.} \end{cases}$$

It suffices to define $C(\vec{p}) = c_{\beta,s}$. □

**Lemma 6.36** *If the formula*

$$\psi = \bigwedge_i (p_i \vee q_i) \to \alpha(\vec{p}, \vec{s}) \vee \beta(\vec{q}, \vec{r})$$

*has a* $\mathbf{BD}_2$*-CF-proof of size* $s$*, then the classical tautology* $\neg\beta(\neg\vec{p}, \vec{r}) \to \alpha(\vec{p}, \vec{s})$ *has a monotone interpolant* $C(\vec{p})$ *of size* $O(s^2)$*.*

*Proof:* Let $\pi$ be a $\mathbf{BD}_2$-*CF*-proof of $\psi$. For any set $X$ of the variables $\vec{p}$, $\vec{q}$, let $P(X)$ be the closure of

$$\pi \cup \Big\{\bigwedge_i (p_i \vee q_i)\Big\} \cup \{\varphi \to \chi \vee \neg\chi;\ \varphi \vee (\varphi \to \chi \vee \neg\chi) \in \pi\} \cup X$$

under modus ponens. As in Lemma 6.35, we can define monotone circuits $C_\gamma$ of size $O(s^2)$ such that

$$e(C_\gamma(\vec{p}, \vec{q})) = 1 \quad \text{iff} \quad e\Big(\bigwedge_i (p_i \vee q_i)\Big) = 1 \wedge \gamma \in P(\{v \in \{\vec{p}, \vec{q}\};\ e(v) = 1\})$$

for $\gamma \in \{\alpha, \beta\}$, and any assignment $e$. It is easy to see that

$$(*) \qquad e(C_\gamma) = 1 \Rightarrow e(\gamma) = 1.$$

**Claim 1** *If* $e\big(\bigwedge_i (p_i \vee q_i)\big) = 1$*, then* $e(C_\alpha \vee C_\beta) = 1$*.*

*Proof:* Let $P := P(\{v \in \{\vec{p}, \vec{q}\};\ e(v) = 1\})$. We define a classical valuation $|\varphi$ of intuitionistic formulas $\varphi$ by

$$\begin{gathered} |v \text{ for every variable } v, \\ |(\varphi \wedge \chi) \quad \text{iff} \quad |\varphi \wedge |\chi, \\ |(\varphi \vee \chi) \quad \text{iff} \quad \|\varphi \vee \|\chi, \\ |(\varphi \to \chi) \quad \text{iff} \quad \|\varphi \Rightarrow |\chi, \\ \text{not } |\bot, \end{gathered}$$

where

$$\|\varphi \quad \text{iff} \quad \varphi \in P \wedge |\varphi.$$

Notice that $|\psi$ is equivalent to

$$\| \bigwedge_i (p_i \vee q_i) \Rightarrow \|\alpha \vee \|\beta,$$

which implies

$$\| \bigwedge_i (p_i \vee q_i) \Rightarrow e(C_\alpha) = 1 \vee e(C_\beta) = 1,$$

and we have

$$\| \bigwedge_i (p_i \vee q_i)$$

by the definition of $P$. It thus suffices to show that $|\varphi$ for every $\varphi \in \pi$. We proceed by induction on the length of $\pi$. The induction steps for axioms of **IPC**, and modus ponens follow from [14, L. 4.11, 4.12], as $P$ is closed under modus ponens. If $\varphi \vee (\varphi \to \chi \vee \neg\chi) \in \pi$ is an instance of the $\mathrm{BD}_2$ axiom, we have

$$\varphi \to \chi \vee \neg\chi \in P.$$

Either $\|\varphi$ and we are done, or $\neg\|\varphi$, hence $|(\varphi \to \chi \vee \neg\chi)$, thus $\|(\varphi \to \chi \vee \neg\chi)$. □ (Claim 1)

Let us define

$$C(\vec{p}) \Leftrightarrow C_\alpha(\vec{p}, \vec{\top}).$$

Clearly $C$ is a monotone circuit, and

$$C(\vec{p}) \to \alpha(\vec{p}, \vec{s})$$

is true by $(*)$. Moreover,

$$\neg\beta(\neg\vec{p}, \vec{r}) \to \neg C_\beta(\vec{p}, \neg\vec{p}) \to C_\alpha(\vec{p}, \neg\vec{p}) \to C(\vec{p})$$

by $(*)$, Claim 1, and monotonicity of $C_\alpha$. Hence $C(\vec{p})$ is an interpolant of $\neg\beta(\neg\vec{p}, \vec{r}) \to \alpha(\vec{p}, \vec{s})$. □

**Theorem 6.37** *If $L$ is a modal or si logic which has a transitive extension with infinite branching, then $L$-SF has an exponential speed-up over $L$-EF. Specifically, the Hrubeš tautologies $\Theta_n$ or $\Theta^I_n$ have poly-time constructible $L$-SF-proofs, but require $L$-EF-proofs of size $2^{n^{\Omega(1)}}$.*

*Proof:* The formulas $\Theta_n$ or $\Theta^I_n$ have poly-time $L$-*SF*-proofs by Lemmas 6.26 and 6.29. If $L$ is a modal logic, and $\Theta_n$ has an $L$-*EF*-proof of size $s$, then $\Theta_n$ or $\Theta'_n$ has a *CF*-proof of size $O(s)$ in $\mathbf{K} \oplus \Box^2\bot$, $\mathbf{K} \oplus \Box(p \leftrightarrow \Box p)$, or $L_{qn}(W_{\bullet,\infty,\varnothing,\varnothing})$ by Lemma 6.33, hence there exists a monotone circuit of size $O(s^2)$ which interpolates

$$(*) \qquad \mathrm{Clique}^{k+1}_n(\vec{p}, \vec{r}) \to \neg\,\mathrm{Colour}^k_n(\vec{p}, \vec{s})$$

by Lemma 6.35. However, any such circuit must have size $2^{\Omega(n^{1/4})}$ by [1], hence also $s \geq 2^{\Omega(n^{1/4})}$.

Assume that $L$ is si logic, and $\Theta^I_n$ has an $L$-*EF*-proof of size $s$. Then $\Theta^I_n$ has a $\mathbf{BD}_2$-*CF*-proof of size $O(s)$ by Theorem 6.9, and Lemma 6.30, as $\Theta^I_n$ is $\bot$-free. We substitute $\neg\vec{r}$ for $\vec{r'}$, and $\neg\vec{s}$ for $\vec{s'}$ to obtain a $\mathbf{BD}_2$-*CF*-proof of $\Theta^{I,\bot}_n$ of size $O(s)$. Then there exists a monotone interpolant of $(*)$ of size $O(s^2)$ by Lemma 6.36, hence $s \geq 2^{\Omega(n^{1/4})}$ as above. □

# 7 Problems

We left many questions open. First of all, we did not quite resolve the *SF* vs. *EF* problem. Our results suggest that the boundary separating logics $L$ such that $L$-*EF* $\equiv$ $L$-*SF* from the others runs somewhere between logics of finite width and logics of finite branching, but it is not clear which of these two parameters is more relevant.

**Problem 7.1** *Determine whether L-SF* $\leq$ *L-EF for* $L = \mathbf{K4BB}_k$, $\mathbf{T}_k$ $(k \geq 2)$, *or some other logic of infinite width but finite branching.*

There are however also some interesting cases of logics of finite width. For example:

**Problem 7.2** *Does L-SF* $\leq$ *L-EF hold for* $\mathbf{BW}_k$*? Does it hold for all csf logics of finite width?*

Notice that a positive solution to the latter question would settle the situation for csf logics, as all csf logics of infinite width have infinite branching, and therefore fall into the scope of Theorem 6.37.

Due to the method used, all simulations in Section 5 applied only to *coNP* logics. We believe that this is just a residue of our proof, and does not truly represent the picture. (Notice also that $\mathbf{K4BD}_k$ and many other logics of finite depth and infinite width are in *coNP*, yet separate *EF* from *SF* by 6.37.)

**Problem 7.3** *Show L-SF* $\leq$ *L-EF for some transitive logic outside of coNP* (*or better yet, for a natural class of such logics*).

Other interesting questions concern the relations of different logics. One was already stated in Section 4, we repeat it here.

**Problem 7.4** *Does L-SF simulate* (*wrt L-formulas*) $L^{wt}$*-EF or even* $L^{A}$*-EF for all modal logics L?*

Probably the most important question in this context is the relationship of modal and si logics (see [7] for the definitions):

**Problem 7.5** *Let L be a si logic, and* $L'$ *its modal companion. Does* $L'$*-EF interpret* (*in the sense of Definition 5.1*) *L-EF via Gödel translation? What about SF?*

(The translation as such is feasible. The problem is the opposite direction: whether we can extract an *L-EF*-proof of $\varphi$ from an $L'$-*EF*-proof of its translation.) Notice that the answer is affirmative if $L$ is tabular, or $L = \mathbf{LC}$: the Gödel translation commutes (up to feasibly provable equivalence) with the relevant translations to **CPC** we used in Section 5. The problem is more interesting for weaker logics, such as the basic pair $L = \mathbf{IPC}$, $L' = \mathbf{S4}$.

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