ACADEMY OF SCIENCES
OF THE CZECH REPUBLIC

MATHEMATICAL INSTITUTE

ON THE CHARACTERISTIC INITIAL VALUE PROBLEM FOR LINEAR PARTIAL FUNCTIONAL-DIFFERENTIAL EQUATIONS OF HYPERBOLIC TYPE

Jiří Šremr

(preprint)

161
2005

ON THE CHARACTERISTIC INITIAL VALUE PROBLEM FOR LINEAR PARTIAL FUNCTIONAL–DIFFERENTIAL EQUATIONS OF HYPERBOLIC TYPE

Jiří Šremr

Abstract. Theorems on the Fredholm alternative and well–posedness of the characteristic initial value problem

$$\frac{\partial^2 u(t,x)}{\partial t\,\partial x} = \ell(u)(t,x) + q(t,x),$$
$$u(t,c) = \varphi(t) \quad \text{for} \quad t \in [a,b], \qquad u(a,x) = \psi(x) \quad \text{for} \quad x \in [c,d]$$

are established, where $\ell : C(\mathcal{D};\mathbb{R}) \to L(\mathcal{D};\mathbb{R})$ is a linear bounded operator, $q \in L(\mathcal{D};\mathbb{R})$, $\varphi : [a,b] :\to \mathbb{R}$, $\psi : [c,d] :\to \mathbb{R}$ are absolutely continuous functions such that $\varphi(a) = \psi(c)$, and $\mathcal{D} = [a,b] \times [c,d]$. Moreover, it is proved that if ℓ is a nonincreasing operator and a certain theorem on functional differential inequalities holds for the problem considered then the operator indicated is necessarily an (a,c)–Volterra one.

1. Introduction

On the rectangle $\mathcal{D} = [a,b] \times [c,d]$, we consider the characteristic initial value problem

$$\frac{\partial^2 u(t,x)}{\partial t\,\partial x} = \ell(u)(t,x) + q(t,x), \tag{1.1}$$

$$u(t,c) = \varphi(t) \quad \text{for} \quad t \in [a,b], \qquad u(a,x) = \psi(x) \quad \text{for} \quad x \in [c,d], \tag{1.2}$$

where $\ell : C(\mathcal{D};\mathbb{R}) \to L(\mathcal{D};\mathbb{R})$ is a linear bounded operator, $q \in L(\mathcal{D};\mathbb{R})$, and $\varphi : [a,b] \to \mathbb{R}$, $\psi : [c,d] \to \mathbb{R}$ are absolutely continuous functions such that $\varphi(a) = \psi(c)$. As usual, $C(\mathcal{D};\mathbb{R})$ and $L(\mathcal{D};\mathbb{R})$ denote the Banach spaces of continuous and Lebesgue integrable functions, respectively, equipped with the standard norms.

Under a solution of the problem (1.1), (1.2) is understood a function $u \in C^*(\mathcal{D};\mathbb{R})$[1] which satisfies the equation (1.1) almost everywhere on the set $\mathcal{D}$ and verifies also the condition (1.2).

The aim of the paper is to prove the Fredholm alternative and well–posedness of the problem (1.1), (1.2) (see Sections 3 and 5). Moreover, some conditions are given in Section 4 under which the problem (1.1), (1.2) has a unique solution. The results obtained are concretized for the equation with deviating arguments

$$\frac{\partial^2 u(t,x)}{\partial t\,\partial x} = p(t,x)u\Big(\tau(t,x),\mu(t,x)\Big) + q(t,x), \tag{1.1$'$}$$

where $p, q \in L(\mathcal{D};\mathbb{R})$ and $\tau : \mathcal{D} \to [a,b]$, $\mu : \mathcal{D} \to [c,d]$ are measurable functions. Finally, there is proved in Section 6 that if a certain theorem on differential inequalities

2000 Mathematics Subject Classification: 35L10, 35L15

Key words and phrases: Functional differential equation of hyperbolic type, characteristic initial value problem, nonnegative solution.

[1]For definition of the class $C^*(\mathcal{D};\mathbb{R})$, see Section 2.

holds for the problem (1.1), (1.2) with a nonincreasing operator ℓ then the operator indicated is necessarily an (a, c)–Volterra one.

We should note here that some solvability conditions and theorems on the well–posedness of the other boundary value problems for linear and nonlinear partial differential equations of hyperbolic type are given, e.g., in [3, 5, 6, 9, 10] (see also references therein).

2. Notations and Definitions

The following notation and definitions are used throughout the paper.

$\mathbb{N}$ is the set of all natural numbers.

$\mathbb{R}$ is the set of all real numbers, $\mathbb{R}_+ = [0, +\infty[$.

$\mathrm{Ent}(x)$ denotes the entire part of the number $x \in \mathbb{R}$.

$\mathcal{D} = [a, b] \times [c, d]$, where $-\infty < a < b < +\infty$ and $-\infty < c < d < +\infty$.

$C(\mathcal{D}; \mathbb{R})$ is the Banach space of continuous functions $v : \mathcal{D} \to \mathbb{R}$ equipped with the norm

$$\|v\|_C = \max\big\{|v(t, x)| : (t, x) \in \mathcal{D}\big\}.$$

$C(\mathcal{D}; \mathbb{R}_+) = \{v \in C(\mathcal{D}; \mathbb{R}) : v(t, x) \geq 0 \text{ for } (t, x) \in \mathcal{D}\}$.

$\widetilde{C}([\alpha, \beta]; \mathbb{R})$, where $-\infty < \alpha < \beta < +\infty$, is the set of absolutely continuous functions $u : [\alpha, \beta] \to \mathbb{R}$.

$C^*(\mathcal{D}; \mathbb{R})$ is the set of functions $v : \mathcal{D} \to \mathbb{R}$ admitting the representation

$$v(t, x) = v_1(t) + v_2(x) + \int_a^t \int_c^x h(s, \eta)\, d\eta\, ds \quad \text{for} \quad (t, x) \in \mathcal{D},$$

where $v_1 \in \widetilde{C}([a, b], \mathbb{R})$, $v_2 \in \widetilde{C}([c, d], \mathbb{R})$, and $h \in L(\mathcal{D}; \mathbb{R})$.

$C^2(\mathcal{D}; \mathbb{R})$ is the set of functions $v : \mathcal{D} \to \mathbb{R}$ which have continuous derivatives up to the second order.

$L(\mathcal{D}; \mathbb{R})$ is the Banach space of Lebesgue integrable functions $p : \mathcal{D} \to \mathbb{R}$ equipped with the norm

$$\|p\|_L = \iint_{\mathcal{D}} |p(t, x)|\, dt\, dx.$$

$L(\mathcal{D}; \mathbb{R}_+) = \{p \in L(\mathcal{D}; \mathbb{R}) : p(t, x) \geq 0 \text{ for almost all } (t, x) \in \mathcal{D}\}$.

$\mathcal{L}(\mathcal{D})$ is the set of linear bounded operators $\ell : C(\mathcal{D}; \mathbb{R}) \to L(\mathcal{D}; \mathbb{R})$.

$\operatorname{mes} A$ denotes the Lebesgue measure of the set $A \subset R^2$.

If X, Y are some normed spaces and $T : X \to Y$ is a linear bounded operator then $\|T\|$ denotes the norm of the operator T, i.e.,

$$\|T\| = \sup\big\{\|T(z)\|_Y : z \in X,\ \|z\|_X \leq 1\big\}.$$

Definition 2.1. An operator $\ell \in \mathcal{L}(\mathcal{D})$ is said to be nondecreasing if it maps the set $C(\mathcal{D}; \mathbb{R}_+)$ into the set $L(\mathcal{D}; \mathbb{R}_+)$. In the sequel, the set of nondecreasing operators is denoted by $\mathcal{P}(\mathcal{D})$. We say that an operator $\ell \in \mathcal{L}(\mathcal{D})$ is nonincreasing if $-\ell \in \mathcal{P}(\mathcal{D})$.

Definition 2.2. An operator $\ell \in \mathcal{L}(\mathcal{D})$ is said to be an (a, c)–Volterra operator if, for arbitrary rectangle $[a, t_0] \times [c, x_0] \subseteq \mathcal{D}$ and function $v \in C(\mathcal{D}; \mathbb{R})$ such that

$$v(t,x) = 0 \quad \text{for} \quad (t,x) \in [a, t_0] \times [c, x_0],$$

the relation

$$\ell(v)(t,x) = 0 \quad \text{for a. a.} \quad (t,x) \in [a, t_0] \times [c, x_0]$$

holds.

Analogously, we say that an operator $\Omega : L(\mathcal{D}; R) \to C(\mathcal{D}; R)$ is an (a, c)–Volterra operator if, for arbitrary rectangle $[a, t_0] \times [c, x_0] \subseteq \mathcal{D}$ and function $p \in L(\mathcal{D}; R)$ such that

$$p(t,x) = 0 \quad \text{for a. a.} \quad (t,x) \in [a, t_0] \times [c, x_0],$$

we have

$$\Omega(p)(t,x) = 0 \quad \text{for} \quad (t,x) \in [a, t_0] \times [c, x_0].$$

Remark 2.1. One can verify that $v \in C^*(\mathcal{D}; \mathbb{R})$ if and only if the following conditions are satisfied:

(a) $v(t, \cdot) \in \widetilde{C}([c, d], \mathbb{R})$ for every $t \in [a, b]$, $v(\cdot, x) \in \widetilde{C}([a, b], \mathbb{R})$ for every $x \in [c, d]$;

(b) $v_t(t, \cdot) \in \widetilde{C}([c, d], \mathbb{R})$ for almost all $t \in [a, b]$, $v_x(\cdot, x) \in \widetilde{C}([a, b], \mathbb{R})$ for almost all $x \in [c, d]$;

(c) $v_{tx} \in L(\mathcal{D}; \mathbb{R})$.

Moreover, it is clear that $C^2(\mathcal{D}; \mathbb{R}) \subset C^*(\mathcal{D}; \mathbb{R})$.

We should also note here that the set $C^*(\mathcal{D}; \mathbb{R})$ coincide with the class of absolutely continuous functions of two variables presented, e.g., in [4, 9].

In what follows, the equalities and inequalities with integrable functions are understood to hold almost everywhere.

3. Fredholm Property

The main result of this section is the following statement.

Theorem 3.1. *For the unique solvability of the problem* (1.1), (1.2) *is sufficient and necessary that the homogeneous problem*

$$\frac{\partial^2 u(t,x)}{\partial t \, \partial x} = \ell(u)(t,x), \tag{1.1$_0$}$$

$$u(t,c) = 0 \quad \textit{for} \quad t \in [a,b], \qquad u(a,x) = 0 \quad \textit{for} \quad x \in [c,d] \tag{1.2$_0$}$$

has only the trivial solution.

Definition 3.1. Let the problem (1.1_0), (1.2_0) have only the trivial solution. An operator $\Omega : L(\mathcal{D}; \mathbb{R}) \to C(\mathcal{D}; \mathbb{R})$ which assigns to everey $q \in L(\mathcal{D}; \mathbb{R})$ the solution u of the problem (1.1), (1.2_0) is reffered to as the Darboux operator of the problem (1.1_0), (1.2_0).

Remark 3.1. It follows from Theorem 3.1 that the operator Ω is well–defined. Obviously, the operator Ω is linear.

If the homogeneous problem (1.1_0), (1.2_0) has a nontrivial solution then, by virtue of Theorem 3.1, there exist functions q, φ, and ψ such that the problem (1.1), (1.2) has either no solution or infinitely many solutions. However, as it follows from the proof of Theorem 3.1, a stronger assertion can be shown in this case.

Proposition 3.1. *Let the problem* (1.1_0)*,* (1.2_0) *have a nontrivial solution. Then, for arbitrary* $\varphi \in \widetilde{C}([a,b],\mathbb{R})$ *and* $\psi \in \widetilde{C}([c,d],\mathbb{R})$ *satisfying* $\varphi(a) = \psi(c)$*, there exists a function* $q \in L(\mathcal{D};\mathbb{R})$ *such that the problem* (1.1)*,* (1.2) *has no solution.*

To prove Theorem 3.1 we need several notions and statements from functional analysis.

Definition 3.2. Let X be a Banach space, X^* be its dual space.

We say that a sequence $\{x_n\}_{n=1}^{+\infty} \subseteq X$ is weakly convergent if there exists $x \in X$ such that $f(x) = \lim\limits_{n\to+\infty} f(x_n)$ for every $f \in X^*$. The element x is said to be a weak limit of this sequence.

A set $M \subseteq X$ is reffer to be weakly relatively compact if every sequence of elements from M contains a subsequence which is weakly convergent in X.

A sequence $\{x_n\}_{n=1}^{+\infty}$ of elements fom X is said to be weakly fundamental if the sequence $\{f(x_n)\}_{n=1}^{+\infty}$ is fundamental in $\mathbb{R}$ for every $f \in X^*$.

We say that the space X is weakly complete if every weakly fundamental sequence of elements from X possesses a weak limit in X.

Definition 3.3. Let X and Y be some Banach spaces, $T : X \to Y$ be a linear bounded operator. The operator T is said to be weakly completely continuous if it maps a unit ball of X into a weakly relatively compact subset of Y.

Definition 3.4. We say that a set $M \subseteq L(\mathcal{D};\mathbb{R})$ has a property of absolutely continuous integral if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that the relation

$$\left| \iint\limits_E p(t,x)\,dt\,dx \right| < \varepsilon \quad \text{for every} \quad p \in M$$

is true whenever a measurable set $E \subseteq \mathcal{D}$ is such that $\operatorname{mes} E < \delta$.

The following three lemmata can be found in [2].

Lemma 3.1 (Theorem IV.8.6). *The space* $L(\mathcal{D};\mathbb{R})$ *is weakly complete.*

Lemma 3.2 (Theorem VI.7.6). *A linear bounded operator mapping the space* $C(\mathcal{D};\mathbb{R})$ *into a weakly complete Banach space is weakly completely continuous.*

Lemma 3.3 (Theorem IV.8.11). *If a set* $M \subseteq L(\mathcal{D};\mathbb{R})$ *is weakly relatively compact then it has a property of absolutely continuous integral.*

Now we will establish a proposition which plays a crucial role in the proof of Theorem 3.1 as well as in the proofs of statements given in Section 5.

Proposition 3.2. *Let $\ell \in \mathcal{L}(\mathcal{D})$. Then the operator $T : C(\mathcal{D};\mathbb{R}) \to C(\mathcal{D};\mathbb{R})$ defined by*

$$T(v)(t,x) = \int_a^t \int_c^x \ell(v)(s,\eta)d\eta ds \quad \text{for} \quad (t,x) \in \mathcal{D},\ v \in C(\mathcal{D};\mathbb{R}) \tag{3.1}$$

is completely continuous.

Proof. Let $M \subseteq C(\mathcal{D};\mathbb{R})$ be a bounded set. We will show that the set $T(M) = \{T(v) : v \in M\}$ is relatively compact in $C(\mathcal{D};\mathbb{R})$. According to Arzelà–Ascoli lemma, it is sufficient to show that the set $T(M)$ is bounded and equicontinuous.

Boundedness. It is clear that

$$|T(v)(t,x)| \leq \int_a^t \int_c^x |\ell(v)(s,\eta)|d\eta ds \leq \|\ell(v)\|_L \leq \|\ell\|\, \|v\|_C$$

for $(t,x) \in \mathcal{D}$ and every $v \in M$. Therefore, the set $T(M)$ is bounded in the space $C(\mathcal{D};\mathbb{R})$.

Equicontinuity. Let $\varepsilon > 0$ be arbitrary but fixed. Lemmata 3.1 and 3.2 yield that the operator ℓ is weakly completely continuous, that is, the set $\ell(M) = \{\ell(v) : v \in M\}$ is weakly relatively compact subset of $L(\mathcal{D};\mathbb{R})$. Therefore, Lemma 3.3 guarantees that there exists $\delta > 0$ such that the relation

$$\left| \iint_E \ell(v)(t,x)dtdx \right| < \frac{\varepsilon}{2} \quad \text{for} \quad v \in M \tag{3.2}$$

holds for every measurable set $E \subseteq \mathcal{D}$ satisfying $\operatorname{mes} E < \max\{b-a, d-c\}\delta$.

On the other hand, for $(t_1,x_1),(t_2,x_2) \in \mathcal{D}$ and $v \in M$, we have

$$\begin{aligned} |T(v)(t_2,x_2) - T(v)(t_1,x_1)| &= \\ &= \left| \int_a^{t_2}\int_c^{x_2} \ell(v)(s,\eta)d\eta ds - \int_a^{t_1}\int_c^{x_1} \ell(v)(s,\eta)d\eta ds \right| \leq \\ &\leq \left| \iint_{E_1} \ell(v)(s,\eta)dsd\eta \right| + \left| \iint_{E_2} \ell(v)(s,\eta)dsd\eta \right|, \end{aligned}$$

where measurable sets $E_1, E_2 \subseteq \mathcal{D}$ are such that $\operatorname{mes} E_1 \leq (d-c)|t_2 - t_1|$ and $\operatorname{mes} E_2 \leq (b-a)|x_2 - x_1|$. Hence, by virtue of (3.2), we get

$$\begin{aligned} &|T(v)(t_2,x_2) - T(v)(t_1,x_1)| < \varepsilon \\ &\qquad \text{for} \quad (t_1,x_1),(t_2,x_2) \in \mathcal{D},\ |t_2 - t_1| + |x_2 - x_1| < \delta, \text{ and } v \in M, \end{aligned}$$

i.e., the set $T(M)$ is equicontinuous in the space $C(\mathcal{D};\mathbb{R})$. □

Proof of Theorem 3.1. Let u be a solution of the problem (1.1), (1.2). It is clear that u is a solution of the equation

$$v = T(v) + f \tag{3.3}$$

in the space $C(\mathcal{D};\mathbb{R})$, where the operator T is given by (3.1) and

$$f(t,x) = -\varphi(a) + \varphi(t) + \psi(x) + \int_a^t \int_c^x q(s,\eta)d\eta ds \quad \text{for} \quad (t,x) \in \mathcal{D}. \tag{3.4}$$

Conversely, if $v \in C(\mathcal{D};\mathbb{R})$ is a solution of the equation (3.3) with f given by (3.4) then $v \in C^*(\mathcal{D};\mathbb{R})$ and v is a solution of the problem (1.1), (1.2). Hence, the problem (1.1), (1.2) and the equation (3.3) are equivalent in this sense.

Note also that u is a solution of the homogeneous problem (1.1_0), (1.2_0) if and only if u is a solution of the homogeneous equation

$$v = T(v) \tag{3.5}$$

in the space $C(\mathcal{D};\mathbb{R})$.

According to Proposition 3.2, the operator T is completely continuous. It follows from the Riesz–Schauder theory that the equation (3.3) is uniquely solvable for every $f \in C(\mathcal{D};\mathbb{R})$ if and only if the homogeneous equation (3.5) has only the trivial solution. Therefore, the assertion of theorem is true. □

Proof of Proposition 3.1. Let u_0 be a nontrivial solution of the problem (1.1_0), (1.2_0), and let $\varphi \in \widetilde{C}([a,b],\mathbb{R})$ and $\psi \in \widetilde{C}([c,d],\mathbb{R})$ be such that $\varphi(a) = \psi(c)$.

It follows from the proof of Theorem 3.1 that u_0 is also nontrivial solution of the homogeneous equation (3.5). Therefore, by the Riesz–Schauder theory, there exists $f \in C(\mathcal{D};\mathbb{R})$ such that the equation (3.3) has no solution.

Then the problem (1.1), (1.2) has no solution for $q \equiv \ell(z)$, where

$$z(t,x) = f(t,x) + \varphi(a) - \varphi(t) - \psi(x) \quad \text{for} \quad (t,x) \in \mathcal{D}.$$

Indeed, if the problem indicated has a solution u then the function $u+z$ is a solution of the equation (3.3), which is a contradiction. □

4. Existence and Uniqueness Theorems

In this section, we will establish some conditions guaranteeing the unique solvability of the problems (1.1), (1.2) and (1.1′), (1.2). We will prove, in particular, that the problem (1.1), (1.2) has a unique solution provided that the operator ℓ is an (a,c)–Volterra one. We first introduce the following notation.

Notation 4.1. Let $\ell \in \mathcal{L}(\mathcal{D})$. Define operators $\vartheta_k : C(\mathcal{D};\mathbb{R}) \to C(\mathcal{D};\mathbb{R})$, $k = 0,1,2,\dots$, by setting

$$\vartheta_0(v) = v, \quad \vartheta_k(v) = T\big(\vartheta_{k-1}(v)\big) \quad \text{for} \quad v \in C(\mathcal{D};\mathbb{R}),\ k \in \mathbb{N}, \tag{4.1}$$

where the operator T is given by (3.1).

Theorem 4.1. *Let there exist* $m \in \mathbb{N}$ *and* $\alpha \in [0,1[$ *such that the inequality*

$$\|\vartheta_m(u)\|_C \leq \alpha \|u\|_C \tag{4.2}$$

is satisfied for every solution u *of the homogeneous problem* (1.1_0), (1.2_0). *Then the problem* (1.1), (1.2) *is uniquely solvable.*

Remark 4.1. The assumption $\alpha \in [0,1[$ in the previous theorem cannot be replaced by the assumption $\alpha \in [0,1]$ (see Example 7.1).

Corollary 4.1. *Let there exist a number* $j \in \mathbb{N}$ *such that*

$$\iint\limits_{\mathcal{D}} p_j(t,x)dtdx < 1, \tag{4.3}$$

where $p_1 \equiv |p|$ *and*

$$p_{k+1}(t,x) = |p(t,x)| \int\limits_a^{\tau(t,x)} \int\limits_c^{\mu(t,x)} p_k(s,\eta)d\eta ds \quad \textit{for} \quad (t,x) \in \mathcal{D},\ k \in \mathbb{N}. \tag{4.4}$$

Then the problem $(1.1')$, (1.2) *is uniquely solvable.*

Remark 4.2. Example 7.1 shows that the strict inequality (4.3) in Corollary 4.1 cannot be replaced by the nonstrict one.

Theorem 4.2. *Let* ℓ *be an* (a,c)*–Volterra operator. Then the problem* (1.1), (1.2) *has a unique solution.*

Corollary 4.2. *Let*

$$|p(t,x)|\big(\tau(t,x) - t\big) \leq 0 \quad \textit{for} \quad (t,x) \in \mathcal{D} \tag{4.5}$$

and

$$|p(t,x)|\big(\mu(t,x) - x\big) \leq 0 \quad \textit{for} \quad (t,x) \in \mathcal{D}. \tag{4.6}$$

Then the problem $(1.1')$, (1.2) *has a unique solution.*

Proof of Theorem 4.1. According to Theorem 3.1, it is sufficient to show that the homogeneous problem (1.1_0), (1.2_0) has only the trivial solution.

Let u be a solution of the problem (1.1_0), (1.2_0). Then it is clear that

$$u(t,x) = \int\limits_a^t \int\limits_c^x \ell(u)(s,\eta)d\eta ds = T(u)(t,x) = \vartheta_1(u)(t,x) \quad \text{for} \quad (t,x) \in \mathcal{D}.$$

Using the last relation, we get

$$u(t,x) = T\big(\vartheta_1(u)\big)(t,x) = \vartheta_2(u)(t,x) \quad \text{for} \quad (t,x) \in \mathcal{D},$$

and thus, $u = \vartheta_k(u)$ for every $k \in \mathbb{N}$. Therefore, (4.2) implies

$$\|u\|_C = \|\vartheta_m(u)\|_C \leq \alpha \|u\|_C,$$

which guarantees $u \equiv 0$ because we have supposed that $\alpha \in [0,1[$. □

Proof of Corollary 4.1. Let $\ell \in \mathcal{L}(\mathcal{D})$ be defined by

$$\ell(v)(t,x) = p(t,x)v\Big(\tau(t,x), \mu(t,x)\Big) \quad \text{for} \quad (t,x) \in \mathcal{D}. \tag{4.7}$$

It is clear that

$$\big|\vartheta_k(v)(t,x)\big| \leq \int_a^t \int_c^x \big|p(s,\eta)\vartheta_{k-1}(v)\big(\tau(s,\eta), \mu(s,\eta)\big)\big| d\eta ds \leq$$

$$\leq \|v\|_C \int_a^t \int_c^x p_k(s,\eta) d\eta ds \quad \text{for} \quad (t,x) \in \mathcal{D},\ k \in \mathbb{N},\ v \in C(\mathcal{D};\mathbb{R}).$$

Therefore, the assumptions of Theorem 4.1 are satisfied for $m = j$ and

$$\alpha = \iint_{\mathcal{D}} p_j(t,x) dt dx.$$

□

To prove Theorem 4.2 we need the following lemma.

Lemma 4.1. *Let $\ell \in \mathcal{L}(\mathcal{D})$ be an (a,c)–Volterra operator. Then*

$$\lim_{k \to +\infty} \|\vartheta_k\| = 0, \tag{4.8}$$

where the operators ϑ_k are defined by (4.1).

Proof. Let $\varepsilon \in]0,1[$. According to Proposition 3.2, the operator ϑ_1 is completely continuous. Therefore, by virtue of Arzelà–Ascoli lemma, there exists $\delta > 0$ such that

$$\left| \int_a^{t_2} \int_c^{x_2} \ell(w)(s,\eta) d\eta ds - \int_a^{t_1} \int_c^{x_1} \ell(w)(s,\eta) d\eta ds \right| \leq \varepsilon \|w\|_C$$

$$\text{for} \quad (t_1,x_1), (t_2,x_2) \in \mathcal{D},\ |t_2 - t_1| + |x_2 - x_1| < \delta,\ w \in C(\mathcal{D};\mathbb{R}). \tag{4.9}$$

Let

$$n = \max\left\{ \operatorname{Ent}\left(\frac{2(b-a)}{\delta}\right), \operatorname{Ent}\left(\frac{2(d-c)}{\delta}\right) \right\},$$

$$t_i = a + i\,\frac{b-a}{n+1}, \quad x_i = c + i\,\frac{d-c}{n+1} \quad \text{for} \quad i = 0,1,\dots,n+1,$$

$$\mathcal{D}_i = [a,t_i] \times [c,x_i] \quad \text{for} \quad i = 1,2,\dots,n+1.$$

It is clear that, for any $j, r = 0,1,\dots,n$, we have

$$\big|\widetilde{t}_2 - \widetilde{t}_1\big| + |\widetilde{x}_2 - \widetilde{x}_1| < \delta \quad \text{for} \quad (\widetilde{t}_1, \widetilde{x}_1), (\widetilde{t}_2, \widetilde{x}_2) \in [t_j, t_{j+1}] \times [x_r, x_{r+1}]. \tag{4.10}$$

If $w \in C(\mathcal{D};\mathbb{R})$ then we denote

$$\|w\|_i = \|w\|_{C(\mathcal{D}_i;\mathbb{R})} \quad \text{for} \quad i = 1,2,\dots,n+1.$$

Let $v \in C(\mathcal{D};\mathbb{R})$ be arbitrary but fixed. We will show that the relation

$$\|\vartheta_k(v)\|_i \le \alpha_i(k)\varepsilon^k \|v\|_C \quad \text{for} \quad k \in \mathbb{N} \tag{4.11}$$

holds for every $i = 1,2,\dots,n+1$, where

$$\alpha_i(k) = \alpha_i k^{i-1} \quad \text{for} \quad k \in \mathbb{N},\ i = 1,2,\dots,n+1, \tag{4.12}$$

$$\alpha_1 = 1, \quad \alpha_{i+1} = i+1+i\alpha_i \quad \text{for} \quad i = 1,2,\dots,n. \tag{4.13}$$

By virtue of (4.9) and (4.10), it is easy to verify that, for any $w \in C(\mathcal{D};\mathbb{R})$, we have

$$\left| \int_a^{t_j}\int_c^{x_r} \ell(w)(s,\eta)\,d\eta ds \right| \le \min\{j,r\}\,\varepsilon\,\|w\|_C \quad \text{for} \quad j,r = 0,1,\dots,n+1. \tag{4.14}$$

Firstly, note that

$$\|\vartheta_1(v)\|_i \le i\,\varepsilon\,\|v\|_C \quad \text{for} \quad i = 1,2,\dots,n+1. \tag{4.15}$$

Indeed, according to (4.9), (4.10), and (4.14), it is obvious that

$$\begin{aligned}
&\|\vartheta_1(v)\|_i = \\
&\quad = \max\left\{ \left| \int_a^{t}\int_c^{x} \ell(v)(s,\eta)\,d\eta ds \right| : (t,x) \in \mathcal{D}_i \right\} = \left| \int_a^{t^*}\int_c^{x^*} \ell(v)(s,\eta)\,d\eta ds \right| \le \\
&\quad \le \left| \int_a^{t^*}\int_c^{x^*} \ell(v)(s,\eta)\,d\eta ds - \int_a^{t_{j_0}}\int_c^{x_{r_0}} \ell(v)(s,\eta)\,d\eta ds \right| + \left| \int_a^{t_{j_0}}\int_c^{x_{r_0}} \ell(v)(s,\eta)\,d\eta ds \right| \le \\
&\qquad \le \varepsilon\,\|v\|_C + (i-1)\,\varepsilon\,\|v\|_C = i\,\varepsilon\,\|v\|_C \quad \text{for} \quad i = 1,2,\dots,n+1,
\end{aligned}$$

where $(t^*,x^*) \in \mathcal{D}_i$ and

$$j_0 = \begin{cases} \frac{t^*-t_0}{t_1-t_0} - 1 & \text{if } \frac{t^*-t_0}{t_1-t_0} \in \mathbb{N} \\ \operatorname{Ent}\left(\frac{t^*-t_0}{t_1-t_0}\right) & \text{otherwise} \end{cases}, \quad r_0 = \begin{cases} \frac{x^*-x_0}{x_1-x_0} - 1 & \text{if } \frac{x^*-x_0}{x_1-x_0} \in \mathbb{N} \\ \operatorname{Ent}\left(\frac{x^*-x_0}{x_1-x_0}\right) & \text{otherwise} \end{cases}. \tag{4.16}$$

Further, on account of (4.9) and the fact that ℓ is an (a,c)–Volterra operator, we get

$$|\vartheta_{k+1}(v)(t,x)| = \left| \int_a^{t}\int_c^{x} \ell\big(\vartheta_k(v)\big)(s,\eta)\,d\eta ds \right| \le \varepsilon\,\|\vartheta_k(v)\|_1$$

for $(t,x) \in \mathcal{D}_1$ and $k \in \mathbb{N}$. Hence, by virtue of (4.15), we have

$$\|\vartheta_k(v)\|_1 \le \varepsilon^k\,\|v\|_C \quad \text{for} \quad k \in \mathbb{N},$$

i.e., (4.11) is true for $i = 1$.

Now suppose that the relation (4.11) holds for some $i \in \{1, 2, \dots, n\}$. We will show that the relation indicated is true also for $i + 1$. With respect to (4.9), (4.10), (4.14), and the fact that ℓ is an (a, c)–Volterra operator, we obtain

$$\|\vartheta_{k+1}(v)\|_{i+1} = \max\left\{\left|\int_a^t\int_c^x \ell\big(\vartheta_k(v)\big)(s,\eta)d\eta ds\right| : (t,x) \in \mathcal{D}_{i+1}\right\} =$$

$$= \left|\int_a^{t^*}\int_c^{x^*} \ell\big(\vartheta_k(v)\big)(s,\eta)d\eta ds\right| \leq$$

$$\leq \left|\int_a^{t^*}\int_c^{x^*} \ell\big(\vartheta_k(v)\big)(s,\eta)d\eta ds - \int_a^{t_{j_0}}\int_c^{x_{r_0}} \ell\big(\vartheta_k(v)\big)(s,\eta)d\eta ds\right| +$$

$$+ \left|\int_a^{t_{j_0}}\int_c^{x_{r_0}} \ell\big(\vartheta_k(v)\big)(s,\eta)d\eta ds\right| \leq$$

$$\leq \varepsilon\,\|\vartheta_k(v)\|_{i+1} + i\,\varepsilon\,\|\vartheta_k(v)\|_i \leq \varepsilon\,\|\vartheta_k(v)\|_{i+1} + i\,\alpha_i(k)\,\varepsilon^{k+1}\,\|v\|_C \quad \text{for} \quad k \in \mathbb{N},$$

where $(t^*, x^*) \in \mathcal{D}_{i+1}$ and j_0, r_0 are given by (4.16). Whence we get

$$\|\vartheta_{k+1}(v)\|_{i+1} \leq \varepsilon\Big(\varepsilon\,\|\vartheta_{k-1}(v)\|_{i+1} + i\,\alpha_i(k-1)\,\varepsilon^k\,\|v\|_C\Big) + \\ + i\,\alpha_i(k)\,\varepsilon^{k+1}\,\|v\|_C \quad \text{for} \quad k \in \mathbb{N}.$$

To continue this procedure, on account of (4.15), we obtain

$$\|\vartheta_{k+1}(v)\|_{i+1} \leq \\ \leq \Big(i + 1 + i\,\big(\alpha_i(1) + \dots + \alpha_i(k)\big)\Big)\,\varepsilon^{k+1}\,\|v\|_C \quad \text{for} \quad k \in \mathbb{N}. \qquad (4.17)$$

With respect to (4.12) and (4.13), it is easy to verify that

$$i + 1 + i\,\big(\alpha_i(1) + \dots + \alpha_i(k)\big) = i + 1 + i\,\alpha_i\big(1^{i-1} + \dots + k^{i-1}\big) \leq \\ \leq i + 1 + i\,\alpha_i\,k\,k^{i-1} = i + 1 + i\,\alpha_i\,k^i \leq \\ \leq (i + 1 + i\,\alpha_i)\,k^i = \alpha_{i+1}\,k^i \leq \alpha_{i+1}\,(k+1).$$

Therefore, (4.15) and (4.17) imply

$$\|\vartheta_k(v)\|_{i+1} \leq \alpha_{i+1}(k)\,\varepsilon^k\,\|v\|_C \quad \text{for} \quad k \in \mathbb{N}.$$

Thus, by induction, we have proved that the relation (4.11) is true for every $i = 1, 2, \dots, n + 1$.

Now it is already clear that, for any $k \in \mathbb{N}$, the estimate

$$\|\vartheta_k(v)\|_C = \|\vartheta_k(v)\|_{n+1} \leq \alpha_{n+1}\,k^n\,\varepsilon^k\,\|v\|_C \quad \text{for} \quad v \in C(\mathcal{D}; \mathbb{R})$$

holds. Therefore,

$$\|\vartheta_k\| \leq \alpha_{n+1}\, k^n\, \varepsilon^k \quad \text{for} \quad k \in \mathbb{N}.$$

Since we suppose that $\varepsilon \in]0,1[$, the last relation yields (4.8). □

Proof of Theorem 4.2. According to Lemma 4.1, there exists $m_0 \in \mathbb{N}$ such that $\|\vartheta_{m_0}\| < 1$. Moreover, it is clear that

$$\|\vartheta_{m_0}(v)\|_C \leq \|\vartheta_{m_0}\|\, \|v\|_C \quad \text{for} \quad v \in C(\mathcal{D};\mathbb{R})$$

because the operator ϑ_{m_0} is bounded. Therefore, the assumptions of Theorem 4.1 are satisfied for $m = m_0$ and $\alpha = \|\vartheta_{m_0}\|$. □

Proof of Corollary 4.2. The assumptions (4.5) and (4.6) guarantee that the operator ℓ given by (4.7) is an (a,c)–Volterra one. Therefore, the validity of corollary follows immediately from Theorem 4.2. □

5. Well–posedness

In this part, the well–posedness of the problems (1.1), (1.2) and (1.1′), (1.2) is investigated.

For any $k \in \mathbb{N}$, along with the problem (1.1), (1.2) we consider the perturbed problem

$$\frac{\partial^2 u(t,x)}{\partial t\, \partial x} = \ell_k(u)(t,x) + q_k(t,x), \tag{1.1$_k$}$$

$$u(t,c) = \varphi_k(t) \quad \text{for} \quad t \in [a,b], \qquad u(a,x) = \psi_k(x) \quad \text{for} \quad x \in [c,d], \tag{1.2$_k$}$$

where $\ell_k \in \mathcal{L}(\mathcal{D})$, $q_k \in L(\mathcal{D};\mathbb{R})$, and $\varphi_k \in \widetilde{C}([a,b];\mathbb{R})$, $\psi_k \in \widetilde{C}([c,d];\mathbb{R})$ are such that $\varphi_k(a) = \psi_k(c)$.

Notation 5.1. Let $\ell \in \mathcal{L}(\mathcal{D})$. Denote by $M(\ell)$ the set of all functions $y \in C^*(\mathcal{D};\mathbb{R})$ admitting the representation

$$y(t,x) = -z(a,c) + \int_a^t \int_c^x \ell(z)(s,\eta)\,d\eta ds \quad \text{for} \quad (t,x) \in \mathcal{D},$$

where $z \in C(\mathcal{D};\mathbb{R})$ and $\|z\|_C = 1$.

Theorem 5.1. *Let the problem* (1.1), (1.2) *have a unique solution* u*,*

$$\lim_{k \to +\infty} \lambda_k = 0, \tag{5.1}$$

where, for any $k \in \mathbb{N}$*,*

$$\lambda_k = \sup\left\{ \left| \int_a^t \int_c^x \Big(\ell_k(y)(s,\eta) - \ell(y)(s,\eta)\Big) d\eta ds \right| : (t,x) \in \mathcal{D},\ y \in M(\ell_k) \right\},$$

and let

$$\lim_{k\to+\infty}(1+\|\ell_k\|)\int_a^t\int_c^x\Big(\ell_k(y)(s,\eta)-\ell(y)(s,\eta)\Big)d\eta ds=0$$
$$\text{uniformly on } \mathcal{D} \text{ for every } y\in C^*(\mathcal{D};\mathbb{R}). \quad (5.2)$$

Let, moreover,

$$\lim_{k\to+\infty}(1+\|\ell_k\|)\int_a^t\int_c^x\Big(q_k(s,\eta)-q(s,\eta)\Big)d\eta ds=0 \quad \text{uniformly on } \mathcal{D} \qquad (5.3)$$

and

$$\lim_{k\to+\infty}(1+\|\ell_k\|)\|\varphi_k-\varphi\|_C=0\,, \qquad \lim_{k\to+\infty}(1+\|\ell_k\|)\|\psi_k-\psi\|_C=0. \qquad (5.4)$$

Then there exists $k_0\in\mathbb{N}$ such that, for every $k>k_0$, the problem (1.1_k), (1.2_k) *has a unique solution u_k and*

$$\lim_{k\to+\infty}\|u_k-u\|_C=0. \qquad (5.5)$$

If we suppose that the operators ℓ_k are “uniformly bounded” in the sense of the relation (5.6) then we obtain the following assertion.

Corollary 5.1. *Let the problem* (1.1), (1.2) *have a unique solution u, there exist a function $\omega\in L(\mathcal{D};\mathbb{R}_+)$ such that*

$$|\ell_k(y)(t,x)|\le\omega(t,x)\|y\|_C \quad \text{for} \quad (t,x)\in\mathcal{D},\ y\in C(\mathcal{D};\mathbb{R}),\ k\in\mathbb{N}, \qquad (5.6)$$

and let

$$\lim_{k\to+\infty}\int_a^t\int_c^x\Big(\ell_k(y)(s,\eta)-\ell(y)(s,\eta)\Big)d\eta ds=0 \quad \text{uniformly on } \mathcal{D} \qquad (5.7)$$

for every $y\in C^*(\mathcal{D};\mathbb{R})$. Let, moreover,

$$\lim_{k\to+\infty}\int_a^t\int_c^x\Big(q_k(s,\eta)-q(s,\eta)\Big)d\eta ds=0 \quad \text{uniformly on } \mathcal{D}, \qquad (5.8)$$

and

$$\lim_{k\to+\infty}\|\varphi_k-\varphi\|_C=0\,, \qquad \lim_{k\to+\infty}\|\psi_k-\psi\|_C=0. \qquad (5.9)$$

Then the conclusion of Theorem 5.1 is true.

Remark 5.1. The assumption (5.6) in the previous corollary is essential and cannot be omitted (see Example 7.2).

From Corollary 5.1, it immediately follows

Corollary 5.2. *Let the homogeneous problem* (1.1_0), (1.2_0) *have only the trivial solution. Then the Darboux operator*[2] *of the problem* (1.1_0), (1.2_0) *is continuous.*

Now, we will establish a theorem on the well–posedness of the problem $(1.1')$, (1.2). For any $k \in \mathbb{N}$, along with the equation $(1.1')$ we consider the perturbed equation

$$\frac{\partial^2 u(t,x)}{\partial t\, \partial x} = p_k(t,x)u\Big(\tau_k(t,x), \mu_k(t,x)\Big) + q_k(t,x), \tag{$1.1'_k$}$$

where $p_k, q_k \in L(\mathcal{D};\mathbb{R})$ and $\tau_k : \mathcal{D} \to [a,b]$, $\mu_k : \mathcal{D} \to [c,d]$ are measurable functions.

Theorem 5.2. *Let the problem* $(1.1')$, (1.2) *have a unique solution* u, *there exist a function* $\omega \in L(\mathcal{D};\mathbb{R}_+)$ *such that*

$$|p_k(t,x)| \le \omega(t,x) \quad \textit{for} \quad (t,x) \in \mathcal{D},\ k \in \mathbb{N}, \tag{5.10}$$

and let

$$\lim_{k\to+\infty} \int_a^t \int_c^x \Big(p_k(s,\eta) - p(s,\eta)\Big) d\eta ds = 0 \quad \textit{uniformly on } \mathcal{D}. \tag{5.11}$$

Let, moreover, the conditions (5.8) *and* (5.9) *be satisfied, and*

$$\lim_{k\to+\infty} \text{ess sup}\Big\{\big|\tau_k(t,x) - \tau(t,x)\big| : (t,x) \in \mathcal{D}\Big\} = 0, \tag{5.12}$$

$$\lim_{k\to+\infty} \text{ess sup}\Big\{\big|\mu_k(t,x) - \mu(t,x)\big| : (t,x) \in \mathcal{D}\Big\} = 0. \tag{5.13}$$

Then there exists $k_0 \in \mathbb{N}$ *such that, for every* $k > k_0$, *the problem* $(1.1'_k)$, (1.2_k) *has a unique solution* u_k *and the relation* (5.5) *is true.*

Remark 5.2. The assumption (5.10) in the previous theorem is essential and cannot be omitted (see Example 7.2).

To prove Theorem 5.1 we need the following lemma.

Lemma 5.1. *Let the problem* (1.1_0), (1.2_0) *have only the trivial solution and let the condition* (5.1) *be satisfied. Then there exist* $k_0 \in \mathbb{N}$ *and* $r_0 > 0$ *such that*

$$\|z\|_C \le r_0 \rho_k(z) \quad \textit{for} \quad k > k_0,\ z \in C^*(\mathcal{D};\mathbb{R}), \tag{5.14}$$

where

$$\rho_k(z) = |z(a,c)| + (1 + \|\ell_k\|)\, \|\Gamma_k(z)\|_C \tag{5.15}$$

and

$$\Gamma_k(z)(t,x) = z(t,c) + z(a,x) + \int_a^t \int_c^x \left(\frac{\partial^2 z(s,\eta)}{\partial s\, \partial \eta} - \ell_k(z)(s,\eta)\right) d\eta ds \tag{5.16}$$

for $(t,x) \in \mathcal{D}$.

[2]The notion of Darboux operator is introduced in Definition 3.1.

Proof. Let $T, T_k : C(\mathcal{D};\mathbb{R}) \to C(\mathcal{D};\mathbb{R})$ be operators defined by (3.1) and

$$T_k(v)(t,x) \overset{\text{def}}{=} \int_a^t \int_c^x \ell_k(v)(s,\eta)d\eta ds \quad \text{for} \quad (t,x) \in \mathcal{D},\ k \in \mathbb{N}. \tag{5.17}$$

Obviously,

$$\|T(y)\|_C \le \|\ell\|\,\|y\|_C, \quad \|T_k(y)\|_C \le \|\ell_k\|\,\|y\|_C \quad \text{for} \quad y \in C(\mathcal{D};\mathbb{R}),\ k \in \mathbb{N}.$$

Therefore, the operators T, T_k ($k \in \mathbb{N}$) are linear bounded ones and the relation

$$\|T_k\| \le \|\ell_k\| \quad \text{for} \quad k \in \mathbb{N} \tag{5.18}$$

holds. The condition (5.1) can be rewritten in the form

$$\sup\Big\{\|T_k(y) - T(y)\|_C : y \in M(\ell_k)\Big\} \to 0 \quad \text{as} \quad k \to +\infty. \tag{5.19}$$

Assume that, on the contrary, the lemma is not true. Then there exist an increasing sequence $\{k_m\}_{m=1}^{+\infty}$ of natural numbers and a sequence $\{z_m\}_{m=1}^{+\infty}$ of functions from $C^*(\mathcal{D};\mathbb{R})$ such that

$$\|z_m\|_C > m\rho_{k_m}(z_m) \quad \text{for} \quad m \in \mathbb{N}. \tag{5.20}$$

For any $m \in \mathbb{N}$ and $(t,x) \in \mathcal{D}$, we put

$$y_m(t,x) = \frac{z_m(t,x)}{\|z_m\|_C}\,, \tag{5.21}$$

$$\begin{aligned} v_m(t,x) &= y_m(t,c) + y_m(a,x) + \\ &\quad + \int_a^t \int_c^x \left(\frac{\partial^2 y_m(s,\eta)}{\partial s\,\partial\eta} - \ell_{k_m}(y_m)(s,\eta)\right) d\eta ds, \end{aligned} \tag{5.22}$$

$$y_{0m}(t,x) = y_m(t,x) - v_m(t,x), \tag{5.23}$$

$$w_m(t,x) = T_{k_m}(y_{0m})(t,x) - T(y_{0m})(t,x) + T_{k_m}(v_m)(t,x). \tag{5.24}$$

Obviously,

$$\|y_m\|_C = 1 \quad \text{for} \quad m \in \mathbb{N}, \tag{5.25}$$

$$y_{0m}(t,x) = -y_m(a,c) + T_{k_m}(y_m)(t,x) \quad \text{for} \quad (t,x) \in \mathcal{D},\ m \in \mathbb{N}, \tag{5.26}$$

and

$$y_{0m}(t,x) = -y_m(a,c) + T(y_{0m})(t,x) + w_m(t,x) \qquad \text{for} \quad (t,x) \in \mathcal{D},\ m \in \mathbb{N}. \tag{5.27}$$

On the other hand, from (5.15), (5.16), (5.18), (5.21), and (5.22), by virtue of (5.20), we get

$$\|v_m\|_C \le \frac{\rho_{k_m}(z_m)}{\|z_m\|_C(1 + \|\ell_{k_m}\|)} < \frac{1}{m(1 + \|\ell_{k_m}\|)} \quad \text{for} \quad m \in \mathbb{N}, \tag{5.28}$$

$$\|T_{k_m}(v_m)\|_C \leq \|T_{k_m}\| \, \|v_m\|_C < \frac{\|\ell_{k_m}\|}{m(1+\|\ell_{k_m}\|)} < \frac{1}{m} \quad \text{for} \quad m \in \mathbb{N}, \tag{5.29}$$

and

$$|y_m(a,c)| \leq \frac{\rho_{k_m}(z_m)}{\|z_m\|_C} < \frac{1}{m} \quad \text{for} \quad m \in \mathbb{N}. \tag{5.30}$$

The relations (5.25) and (5.26) guarantee that $y_{0m} \in M(\ell_{k_m})$ for $m \in \mathbb{N}$, and therefoere, in view of (5.19), we obtain

$$\lim_{m\to+\infty} \|T_{k_m}(y_{0m}) - T(y_{0m})\|_C = 0. \tag{5.31}$$

According to (5.29) and (5.31), it follows from (5.24) that

$$\lim_{m\to+\infty} \|w_m\|_C = 0, \tag{5.32}$$

and, by virtue of (5.25) and (5.28), the equality (5.23) implies

$$\|y_{0m}\|_C \leq \|y_m\|_C + \|v_m\|_C < 2 \quad \text{for} \quad m \in \mathbb{N}.$$

Since the sequence $\{\|y_{0m}\|_C\}_{m=1}^{+\infty}$ is bounded and the operator T is completely continuous (see Proposition 3.2), there exists a subsequence of $\{T(y_{0m})\}_{m=1}^{+\infty}$ which is convergent. Without loss of generality we can assume that the sequence $\{T(y_{0m})\}_{m=1}^{+\infty}$ is convergent, i.e., there exists $y_0 \in C(\mathcal{D};\mathbb{R})$ such that

$$\lim_{m\to+\infty} \|T(y_{0m}) - y_0\|_C = 0.$$

Then it is clear that

$$\lim_{m\to+\infty} \|y_{0m} - y_0\|_C = 0 \tag{5.33}$$

because the functions y_{0m} admit the representation (5.27) and the relations (5.30) and (5.32) are satisfied.

However, the estimate (5.28) holds for v_m and thus, the equality (5.23) yields

$$\lim_{m\to+\infty} \|y_m - y_0\|_C = 0,$$

which, together with (5.25), guarantees

$$\|y_0\|_C = 1.$$

Since the operator T is continuous and the conditions (5.30), (5.32), and (5.33) are fulfilled, the representation (5.27) of y_{0m} results in

$$y_0(t,x) = T(y_0)(t,x) \quad \text{for} \quad (t,x) \in \mathcal{D}.$$

Consequently, $y_0 \in C^*(\mathcal{D};\mathbb{R})$ and y_0 is a nontrivial solution of the problem (1.1_0), (1.2_0). But this is a contradiction because, according to our assumption, the problem indicated has no nontrivial solution. □

Proof of Theorem 5.1. Let $r_0 > 0$ and $k_0 \in \mathbb{N}$ be numbers appearing in Lemma 5.1. If, for some $k \in \mathbb{N}$, u_0 is a solution of the equation

$$\frac{\partial^2 u(t,x)}{\partial t\, \partial x} = \ell_k(u)(t,x) \tag{5.34$_k$}$$

satisfying (1.2_0) then $\rho_k(u_0) = 0$, where ρ_k is given by (5.15) and (5.16). Therefore, Lemma 5.1 guarantees that, for every $k > k_0$, the homogeneous problem (5.34_k), (1.2_0) has only the trivial solution. Hence, for every $k > k_0$, the problem (1.1_k), (1.2_k) has a unique solution u_k. We will show that the relation (5.5) is satisfied, where u is a solution of the problem (1.1), (1.2).

For any $k > k_0$, we put

$$v_k(t,x) = u_k(t,x) - u(t,x) \quad \text{for} \quad (t,x) \in \mathcal{D}.$$

Then it is clear that $v_k \in C^*(\mathcal{D};\mathbb{R})$ for $k > k_0$ and

$$\frac{\partial^2 v_k(t,x)}{\partial t\, \partial x} = \ell_k(v_k)(t,x) + \widetilde{q}_k(t,x) \quad \text{for} \quad (t,x) \in \mathcal{D},\ k > k_0, \tag{5.35}$$

$$\begin{aligned} v_k(t,c) &= \widetilde{\varphi}_k(t) \quad \text{for} \quad t \in [a,b],\ k > k_0, \\ v_k(a,x) &= \widetilde{\psi}_k(x) \quad \text{for} \quad x \in [c,d],\ k > k_0, \end{aligned} \tag{5.36}$$

where

$$\begin{gathered} \widetilde{q}_k(t,x) = \ell_k(u)(t,x) - \ell(u)(t,x) + q_k(t,x) - q(t,x) \quad \text{for} \quad (t,x) \in \mathcal{D},\ k > k_0, \\ \widetilde{\varphi}_k(t) = \varphi_k(t) - \varphi(t) \quad \text{for} \quad t \in [a,b],\ k > k_0, \\ \widetilde{\psi}_k(x) = \psi_k(x) - \psi(x) \quad \text{for} \quad x \in [c,d],\ k > k_0. \end{gathered}$$

For any $k > k_0$, we put

$$\delta_k = (1 + \|\ell_k\|) \max\left\{ \left| \widetilde{\varphi}_k(t) + \widetilde{\psi}_k(x) + \int_a^t \int_c^x \widetilde{q}_k(s,\eta)\, d\eta\, ds \right| : (t,x) \in \mathcal{D} \right\}.$$

The assumptions (5.2), (5.3), and (5.4) yield

$$\lim_{k \to +\infty} \delta_k = 0 \quad \text{and} \quad \lim_{k \to +\infty} |v_k(a,c)| = 0. \tag{5.37}$$

On the other hand, by Lemma 5.1, we get

$$\|v_k\|_C \le r_0 \rho_k(v_k) = r_0(|v_k(a,c)| + \delta_k) \quad \text{for} \quad k > k_0. \tag{5.38}$$

Therefore, (5.37) and (5.38) result in

$$\lim_{k \to +\infty} \|v_k\|_C = 0,$$

i.e., the relation (5.5) is satisfied. □

Proof of Corollary 5.1. We will show that the assumptions of Theorem 5.1 are satisfied. Indeed, the relation (5.6) yields

$$\|\ell_k\| \le \|\omega\|_L \quad \text{for} \quad k \in \mathbb{N}.$$

Therefore, it is clear that, by virtue of (5.7)–(5.9), the assumptions (5.2)–(5.4) of Theorem 5.1 are fulfilled. It remains to show that the condition (5.1) is true.

Assume that, on the contrary, the condition (5.1) does not hold. Then there exist $\varepsilon_0 > 0$, an increasing sequence $\{k_m\}_{m=1}^{+\infty}$ of natural numbers, and a sequence $\{y_m\}_{m=1}^{+\infty}$ of functions such that

$$y_m \in M(\ell_{k_m}) \quad \text{for} \quad m \in \mathbb{N} \tag{5.39}$$

and

$$\max\left\{\left|\int_a^t\int_c^x \Big(\ell_{k_m}(y_m)(s,\eta) - \ell(y_m)(s,\eta)\Big)d\eta ds\right| : (t,x) \in \mathcal{D}\right\} \ge \varepsilon_0 \quad \text{for} \quad m \in \mathbb{N}. \tag{5.40}$$

From (5.39) and Notation 5.1 we get

$$y_m(t,x) = -z_m(a,c) + \int_a^t\int_c^x \ell_{k_m}(z_m)(s,\eta)d\eta ds \quad \text{for} \quad (t,x) \in \mathcal{D},\ m \in \mathbb{N},$$

where $z_m \in C(\mathcal{D};\mathbb{R})$ and $\|z_m\|_C = 1$ for $m \in \mathbb{N}$. Since we suppose that the operators ℓ_k are uniformly bounded in the sense of condition (5.6), we obtain

$$\|y_m\|_C \le 1 + \|\omega\|_L \quad \text{for} \quad m \in \mathbb{N}.$$

Furthermore, for any $(t_1,x_1),(t_2,x_2) \in \mathcal{D}$ and $m \in \mathbb{N}$, we get

$$\begin{aligned}|y_m(t_2,x_2) - y_m(t_1,x_1)| &= \\ &= \left|\int_a^{t_2}\int_c^{x_2} \ell_{k_m}(z_m)(s,\eta)d\eta ds - \int_a^{t_1}\int_c^{x_1} \ell_{k_m}(z_m)(s,\eta)d\eta ds\right| \le \\ &\le \iint_{E_1} \omega(s,\eta)dsd\eta + \iint_{E_2} \omega(s,\eta)dsd\eta,\end{aligned}$$

where measurable sets $E_1, E_2 \subseteq \mathcal{D}$ are such that $\operatorname{mes} E_1 \le (d-c)|t_2 - t_1|$ and $\operatorname{mes} E_2 \le (b-a)|x_2 - x_1|$.

Consequently, the sequence $\{y_m\}_{m=1}^{+\infty}$ is bounded and equicontinuous in $C(\mathcal{D};\mathbb{R})$. Thus, according to Arzelà–Ascoli lemma, without loss of generality we can assume that the sequence indicated is convergent. Therefore, there exists $p_0 \in \mathbb{N}$ such that

$$\|y_m - y_{p_0}\|_C < \frac{\varepsilon_0}{2(\|\omega\|_L + \|\ell\| + 1)} \quad \text{for} \quad m \ge p_0. \tag{5.41}$$

Since $y_{p_0} \in C^*(\mathcal{D};\mathbb{R})$ and the relation (5.7) holds, there exists $p_1 \in \mathbb{N}$ such that

$$\max\left\{\left|\int\limits_a^t\int\limits_c^x \Big(\ell_k(y_{p_0})(s,\eta) - \ell(y_{p_0})(s,\eta)\Big)d\eta ds\right| : (t,x) \in \mathcal{D}\right\} < \frac{\varepsilon_0}{2} \quad \text{for} \quad k \geq p_1. \tag{5.42}$$

Now let us choose a number $M \in \mathbb{N}$ satisfying $M \geq p_0$ and $k_M \geq p_1$. Then

$$\begin{aligned}\max&\left\{\left|\int\limits_a^t\int\limits_c^x \Big(\ell_{k_M}(y_M)(s,\eta) - \ell(y_M)(s,\eta)\Big)d\eta ds\right| : (t,x) \in \mathcal{D}\right\} \leq \\ &\leq \Big(\|\omega\|_L + \|\ell\|\Big)\|y_M - y_{p_0}\|_C + \\ &\quad + \max\left\{\left|\int\limits_a^t\int\limits_c^x \Big(\ell_{k_M}(y_{p_0})(s,\eta) - \ell(y_{p_0})(s,\eta)\Big)d\eta ds\right| : (t,x) \in \mathcal{D}\right\} < \\ &< \frac{\varepsilon_0}{2}\,\frac{\|\omega\|_L + \|\ell\|}{\|\omega\|_L + \|\ell\| + 1} + \frac{\varepsilon_0}{2} < \varepsilon_0,\end{aligned}$$

which contradicts (5.40). □

To prove Theorem 5.2 we need the following lemma.

Lemma 5.2. *Let $p, p_k \in L(\mathcal{D};\mathbb{R})$ and let $\alpha, \alpha_k : \mathcal{D} \to \mathbb{R}$ be measurable and essentially bounded functions for $k \in \mathbb{N}$. Assume that the relations* (5.10) *and* (5.11) *are satisfied, and*

$$\lim_{k\to+\infty} \operatorname{ess\,sup}\Big\{|\alpha_k(t,x) - \alpha(t,x)| : (t,x) \in \mathcal{D}\Big\} = 0. \tag{5.43}$$

Then

$$\lim_{k\to+\infty}\int\limits_a^t\int\limits_c^x \Big(p_k(s,\eta)\alpha_k(s,\eta) - p(s,\eta)\alpha(s,\eta)\Big)d\eta ds = 0 \quad \textit{uniformly on } \mathcal{D}. \tag{5.44}$$

Proof. Without loss of generality we can assume that

$$|p(t,x)| \leq \omega(t,x) \quad \text{for} \quad (t,x) \in \mathcal{D}. \tag{5.45}$$

Let $\varepsilon > 0$ be arbitrary but fixed. According to (5.43), there exists $k_0 \in \mathbb{N}$ such that

$$\iint\limits_{\mathcal{D}} \omega(t,x)|\alpha_k(t,x) - \alpha(t,x)|dtdx < \frac{\varepsilon}{4} \quad \text{for} \quad k \geq k_0. \tag{5.46}$$

Since the function α is measurable and essentially bounded, there exists a function $w \in C^2(\mathcal{D};\mathbb{R})$ such that

$$\iint\limits_{\mathcal{D}} \omega(t,x)|\alpha(t,x) - w(t,x)|dtdx < \frac{\varepsilon}{4}\,. \tag{5.47}$$

For any $k \in \mathbb{N}$, we put

$$f_k(t,x) = \int_a^t \int_c^x \Big(p_k(s,\eta) - p(s,\eta)\Big) d\eta ds \quad \text{for} \quad (t,x) \in \mathcal{D}.$$

Clearly, the condition (5.11) can be rewritten in the form

$$\lim_{k \to +\infty} \|f_k\|_C = 0. \tag{5.48}$$

It can be verified by direct calculation that

$$\int_a^t \int_c^x \Big(p_k(s,\eta) - p(s,\eta)\Big) w(s,\eta) d\eta ds = f_k(t,x)w(t,x) - \\ - \int_a^t f_k(s,x)\, \frac{\partial w(s,x)}{\partial s}\, ds - \int_c^x f_k(t,\eta)\, \frac{\partial w(t,\eta)}{\partial \eta}\, d\eta + \\ + \int_a^t \int_c^x f_k(s,\eta)\, \frac{\partial^2 w(s,\eta)}{\partial s\, \partial \eta}\, d\eta ds \quad \text{for} \quad (t,x) \in \mathcal{D},\ k \in \mathbb{N}.$$

Consequently, using (5.48), we get

$$\lim_{k \to +\infty} \int_a^t \int_c^x \Big(p_k(s,\eta) - p(s,\eta)\Big) w(s,\eta) d\eta ds = 0 \quad \text{uniformly on } \mathcal{D}.$$

Hence, there exists $k_1 \geq k_0$ such that

$$\left| \int_a^t \int_c^x \Big(p_k(s,\eta) - p(s,\eta)\Big) w(s,\eta) d\eta ds \right| < \frac{\varepsilon}{4} \quad \text{for} \quad (t,x) \in \mathcal{D},\ k \geq k_1. \tag{5.49}$$

On the other hand, it is clear that, for any $(t,x) \in \mathcal{D}$ and $k \in \mathbb{N}$,

$$\int_a^t \int_c^x \Big(p_k(s,\eta)\alpha_k(s,\eta) - p(s,\eta)\alpha(s,\eta)\Big) d\eta ds = \\ = \int_a^t \int_c^x p_k(s,\eta)\Big(\alpha_k(s,\eta) - \alpha(s,\eta)\Big) d\eta ds + \\ + \int_a^t \int_c^x \Big(p_k(s,\eta) - p(s,\eta)\Big) w(s,\eta) d\eta ds + \\ + \int_a^t \int_c^x \Big(p_k(s,\eta) - p(s,\eta)\Big)\Big(\alpha(s,\eta) - w(s,\eta)\Big) d\eta ds.$$

Therefore, in view of (5.10), (5.45), (5.46), (5.47), and (5.49), we get

$$\left|\int_a^t\int_c^x \Big(p_k(s,\eta)\alpha_k(s,\eta)-p(s,\eta)\alpha(s,\eta)\Big)d\eta ds\right| \leq$$

$$\leq \iint_{\mathcal{D}} \omega(s,\eta)|\alpha_k(s,\eta)-\alpha(s,\eta)|d\eta ds+$$

$$+\left|\int_a^t\int_c^x \Big(p_k(s,\eta)-p(s,\eta)\Big)w(s,\eta)d\eta ds\right|+$$

$$+\,2\iint_{\mathcal{D}} \omega(s,\eta)|\alpha(s,\eta)-w(s,\eta)|d\eta ds<$$

$$<\frac{\varepsilon}{4}+\frac{\varepsilon}{4}+2\,\frac{\varepsilon}{4}=\varepsilon \quad \text{for} \quad (t,x)\in\mathcal{D},\ k\geq k_1\,,$$

that is, the relation (5.44) is true. □

Proof of Theorem 5.2. Let $\ell\in\mathcal{L}(\mathcal{D})$ be defined by (4.7). For any $k\in\mathbb{N}$, we put

$$\ell_k(v)(t,x)=p_k(t,x)v\Big(\tau_k(t,x),\mu_k(t,x)\Big) \quad \text{for} \quad (t,x)\in\mathcal{D}. \tag{5.50}$$

We will show that the condition (5.7) is satisfied for every $y\in C^*(\mathcal{D};\mathbb{R})$. Indeed, let $y\in C^*(\mathcal{D};\mathbb{R})$ be arbitrary but fixed. For any $k\in\mathbb{N}$, we put

$$\alpha_k(t,x)=y\big(\tau_k(t,x),\mu_k(t,x)\big),\quad \alpha(t,x)=y\big(\tau(t,x),\mu(t,x)\big) \quad \text{for} \quad (t,x)\in\mathcal{D}.$$

Then it is clear that (5.12) and (5.13) guarantee the condition (5.43). Therefore, it follows from Lemma 5.2 that the condition (5.44) holds, i.e., the condition (5.7) is true.

Consequently, the assumptions of Corollary 5.1 are satisfied. □

6. On Differential Inequalities

The main goal of this section is to prove the following statement (see Theorem 6.1): If a certain theorem on differential inequalities holds for the problem (1.1), (1.2) with a nonincreasing operator ℓ then the operator indicated is necessarily an (a,c)–Volterra one.

At first let us introduce the following definition.

Definition 6.1. We say that an operator $\ell\in\mathcal{L}(\mathcal{D})$ belongs to the set $\mathcal{S}_{ac}(\mathcal{D})$ if an arbitrary function $u\in C^*(\mathcal{D};\mathbb{R})$ satisfying

$$\frac{\partial^2 u(t,x)}{\partial t\,\partial x}\geq \ell(u)(t,x) \quad \text{for} \quad (t,x)\in\mathcal{D},$$

$$u(a,c)\geq 0,$$

$$\frac{\partial u(t,c)}{\partial t} \geq 0 \quad \text{for} \quad t \in [a,b],$$

and

$$\frac{\partial u(a,x)}{\partial x} \geq 0 \quad \text{for} \quad x \in [c,d]$$

is nonnegative on the set $\mathcal{D}$.

Remark 6.1. Obviously, if $\ell \in \mathcal{S}_{ac}(\mathcal{D})$ then the homogeneous problem (1.1_0), (1.2_0) has only the trivial solution and thus the problem (1.1), (1.2) is uniquely solvable for every q, φ, and ψ (see Theorem 3.1). Moreover, it is clear that the Darboux operator Ω[3] of the problem (1.1_0), (1.2_0) maps the set $L(\mathcal{D};\mathbb{R}_+)$ into the set $C(\mathcal{D};\mathbb{R}_+)$, i.e., Ω is a nondecreasing operator.

Remark 6.2. It is not difficult to verify that $\ell \in \mathcal{S}_{ac}(\mathcal{D})$ if and only if a certain theorem on differential inequalities is true for the problem (1.1), (1.2), i.e., whenever $u, v \in C^*(\mathcal{D};\mathbb{R})$ and $q \in L(\mathcal{D};\mathbb{R})$ are such that

$$\begin{gathered}
u_{tx}(t,x) \leq \ell(u)(t,x) + q(t,x) \quad \text{for} \quad (t,x) \in \mathcal{D}, \\
v_{tx}(t,x) \geq \ell(v)(t,x) + q(t,x) \quad \text{for} \quad (t,x) \in \mathcal{D}, \\
u(a,c) \leq v(a,c), \\
u_t(t,c) \leq v_t(t,c) \quad \text{for} \quad t \in [a,b], \\
u_x(a,x) \leq v_x(a,x) \quad \text{for} \quad x \in [c,d],
\end{gathered}$$

then

$$u(t,x) \leq v(t,x) \quad \text{for} \quad (t,x) \in \mathcal{D}.$$

One can say also that $\ell \in \mathcal{S}_{ac}(\mathcal{D})$ if and only if some kind of maximum principle holds for the probem (1.1), (1.2).

Efficient conditions gauranteeing the inclusion $\ell \in \mathcal{S}_{ac}(\mathcal{D})$ have been established in [7]. Namely, the following assertion is proved in the paper mentioned.

Proposition 6.1. *Let ℓ be a nonincreasing (a,c)–Volterra operator and let there exist a function $\gamma \in C^*_{loc}([a,b[\times[c,d[\,;\mathbb{R}_+)$[4] satisfying*

$$\frac{\partial^2 \gamma(t,x)}{\partial t\,\partial x} \leq \ell(\gamma)(t,x) \quad \textit{for} \quad (t,x) \in \mathcal{D},$$

$$\gamma(t,x) > 0 \quad \textit{for} \quad (t,x) \in [a,b[\times[c,d[\,,$$

$$\frac{\partial \gamma(t,c)}{\partial t} \leq 0 \quad \textit{for} \quad t \in [a,b[\,,$$

and

$$\frac{\partial \gamma(a,x)}{\partial x} \leq 0 \quad \textit{for} \quad x \in [c,d[\,.$$

Then the operator ℓ belongs to the set $\mathcal{S}_{ac}(\mathcal{D})$.

---

[3]The notion of the Darboux operator is given in Definition 3.1.

[4]By $C^*_{loc}([a,b[\times[c,d[\,;\mathbb{R}_+)$ we understand the set of functions $u \in C(\mathcal{D};\mathbb{R}_+)$ which satisfy $u \in C^*([a,b_0]\times[c,d_0];\mathbb{R})$ for every $b_0 \in\,]a,b[$ and $d_0 \in\,]c,d[\,$.

It follows from the next theorem that the assumption on the nonincreasing operator ℓ in Proposition 6.1 to be an (a, c)–Volterra one is necessary. Analogous results for the first and second order "ordinary" functional–differential equations are established in [1] and [8], respectively.

Theorem 6.1. *Let* $-\ell \in \mathcal{P}(\mathcal{D})$ *and* $\ell \in \mathcal{S}_{ac}(\mathcal{D})$. *Then the operator* ℓ *is an* (a, c)*–Volterra one.*

To prove this theorem we need some auxiliary assertions.

Proposition 6.2. *Let* $-\ell \in \mathcal{P}(\mathcal{D})$, $\ell \in \mathcal{S}_{ac}(\mathcal{D})$, *and* $(t_0, x_0) \in]a, b]\times]c, d]$. *Let, moreover,* u *be a solution of the problem* (1.1), (1.2), *where*

$$q(t, x) = 0 \quad \textit{for} \quad (t, x) \in [a, t_0] \times [c, x_0], \tag{6.1}$$

$$\varphi(t) = 0 \quad \textit{for} \quad t \in [a, t_0], \qquad \psi(x) = 0 \quad \textit{for} \quad x \in [c, x_0]. \tag{6.2}$$

Then

$$u(t, x) = 0 \quad \textit{for} \quad (t, x) \in [a, t_0] \times [c, x_0]. \tag{6.3}$$

Proof. Let $\mathcal{D}_0 = [a, t_0] \times [c, x_0]$. According to the inclusion $\ell \in \mathcal{S}_{ac}(\mathcal{D})$ and Remark 6.1, the problem

$$\frac{\partial^2 v(t, x)}{\partial t\, \partial x} = \ell(v)(t, x) + |q(t, x)|, \tag{6.4}$$

$$\begin{aligned} v(t, c) &= \int_a^t |\varphi'(s)|ds \quad \text{for} \quad t \in [a, b], \\ v(a, x) &= \int_c^x |\psi'(\eta)|d\eta \quad \text{for} \quad x \in [c, d] \end{aligned} \tag{6.5}$$

has a unique solution v. Moreover, by virtue of (6.2) and Remark 6.2, we get

$$v(t, x) \geq 0 \quad \text{for} \quad (t, x) \in \mathcal{D}, \tag{6.6}$$

$$v(t, x) \geq u(t, x) \quad \text{for} \quad (t, x) \in \mathcal{D}. \tag{6.7}$$

Since $-\ell \in \mathcal{P}(\mathcal{D})$, it follows from (6.4) and (6.6) that

$$v_{tx}(t, x) \leq |q(t, x)| \quad \text{for} \quad (t, x) \in \mathcal{D}.$$

Hence, on account of (6.1), (6.2), (6.5), and (6.6), we obtain

$$0 \leq v(t, x) \leq \int_a^t |\varphi'(s)|ds + \int_c^x |\psi'(\eta)|d\eta + \int_a^t \int_c^x |q(s, \eta)|d\eta ds = 0$$

for $(t, x) \in \mathcal{D}_0$, i.e.,

$$v(t, x) = 0 \quad \text{for} \quad (t, x) \in \mathcal{D}_0\,. \tag{6.8}$$

On the other hand, by virtue of (1.1), (6.4), (6.7), and the assumption $-\ell \in \mathcal{P}(\mathcal{D})$, it is obvious that

$$u_{tx}(t,x) = \ell(u-v)(t,x) + q(t,x) + \ell(v)(t,x) \geq$$
$$\geq v_{tx}(t,x) + q(t,x) - |q(t,x)| \quad \text{for} \quad (t,x) \in \mathcal{D}.$$

The last relation, (6.1), and (6.8) yield

$$u_{tx}(t,x) \geq 0 \quad \text{for} \quad (t,x) \in \mathcal{D}_0 \,.$$

However the functions φ and ψ in the initial condition (1.2) satisfy (6.2) and thus

$$u(t,x) = \varphi(t) + \psi(x) + \int_a^t \int_c^x \frac{\partial^2 u(s,\eta)}{\partial s\, \partial \eta}\, d\eta ds \geq 0 \quad \text{for} \quad (t,x) \in \mathcal{D}_0 \,. \tag{6.9}$$

Finally, (6.7)–(6.9) result in (6.3). □

It follows from the previous proposition that if the operator ℓ appearing in the equation (1.1) is nonincreasing and the theorem on functional differential inequalities holds for the problem (1.1), (1.2) then the Darboux operator Ω is necessarily an (a,c)–Volterra one. More precisely, the following assertion is true.

Corollary 6.1. *Let $-\ell \in \mathcal{P}(\mathcal{D})$ and $\ell \in \mathcal{S}_{ac}(\mathcal{D})$. Then the Darboux operator Ω of the problem* (1.1_0), (1.2_0) *is an (a,c)–Volterra one.*

We also need to be able to approximate a certain function from the set $C(\mathcal{D};\mathbb{R})$ by ones of the class $C^*(\mathcal{D};\mathbb{R})$. That is a classical question of the theory of real function but, for the sake of completness, we will show the following lemma.

Lemma 6.1. *Let $(t_0,x_0) \in]a,b] \times]c,d]$ and let $v_0 \in C(\mathcal{D};\mathbb{R})$ be a function satisfying*

$$v_0(t,x) = 0 \quad \textit{for} \quad (t,x) \in \mathcal{D}_0 \,, \tag{6.10}$$

where $\mathcal{D}_0 = [a,t_0] \times [c,x_0]$. Then there exist a sequence $\{v_n\}_{n=1}^{+\infty}$ of functions from the set $C^2(\mathcal{D};\mathbb{R})$ such that

$$v_n(t,x) = 0 \quad \textit{for} \quad (t,x) \in \mathcal{D}_0, \; n \in \mathbb{N} \tag{6.11}$$

and

$$\lim_{n \to +\infty} \|v_n - v_0\|_C = 0. \tag{6.12}$$

Proof. Let $f_n : \mathbb{R} \to \mathbb{R}$ $(n \in \mathbb{N})$ be the function defined by

$$f_n(s) = \begin{cases} 1 & \text{if} \quad s \leq 0 \\ \exp\left(\frac{n^3 s^3}{n^3 s^3 - 1}\right) & \text{if} \quad 0 < s < \frac{1}{n} \\ 0 & \text{if} \quad s \geq \frac{1}{n} \end{cases} .$$

It is clear that the functions f_n ($n \in \mathbb{N}$) are continuous together with their derivatives up to the second order and

$$0 \le f_n(s) \le 1 \quad \text{for} \quad s \in \mathbb{R},\ n \in \mathbb{N}.$$

For any $n \in \mathbb{N}$, we put

$$\chi_n(t,x) = 1 - f_n(t-t_0)f_n(x-x_0) \quad \text{for} \quad (t,x) \in \mathcal{D}.$$

Then $\chi_n \in C^2(\mathcal{D};\mathbb{R})$ for $n \in \mathbb{N}$,

$$0 \le \chi_n(t,x) \le 1 \quad \text{for} \quad (t,x) \in \mathcal{D},\ n \in \mathbb{N}, \tag{6.13}$$

$$\chi_n(t,x) = 0 \quad \text{for} \quad (t,x) \in \mathcal{D}_0,\ n \in \mathbb{N}, \tag{6.14}$$

and

$$\chi_n(t,x) = 1 \quad \text{for} \quad (t,x) \in \mathcal{D} \setminus \Big([a, t_0 + 1/n] \times [c, x_0 + 1/n]\Big),\ n \in \mathbb{N}. \tag{6.15}$$

It is well-known from functional analysis that there exists a sequence $\{w_n\}_{n=1}^{+\infty}$ of functions from the set $C^2(\mathcal{D};\mathbb{R})$ such that

$$\lim_{n \to +\infty} \|w_n - v_0\|_C = 0. \tag{6.16}$$

Put

$$v_n(t,x) = \chi_n(t,x)w_n(t,x) \quad \text{for} \quad (t,x) \in \mathcal{D},\ n \in \mathbb{N}.$$

Obviously, $v_n \in C^2(\mathcal{D};\mathbb{R})$ for $n \in \mathbb{N}$ and the relation (6.11) holds. We will show that the condition (6.12) is satisfied.

Let $\varepsilon > 0$ be arbitrary but fixed. Since the function v_0 is continuous on the rectangle $\mathcal{D}$, there exists $\delta > 0$ such that

$$|v_0(t_2,x_2) - v_0(t_1,x_1)| < \frac{\varepsilon}{2} \quad \text{for} \quad (t_1,x_1),(t_2,x_2) \in \mathcal{D},\ |t_2 - t_1| + |x_2 - x_1| < \delta. \tag{6.17}$$

According to (6.16), there exists $n_0 \in \mathbb{N}$ such that $n_0 \ge \frac{2}{\delta}$ and

$$|w_n(t,x) - v_0(t,x)| < \frac{\varepsilon}{2} \quad \text{for} \quad (t,x) \in \mathcal{D},\ n \ge n_0. \tag{6.18}$$

Therefore, in view of (6.10) and (6.17), we get

$$|v_0(t,x)| < \frac{\varepsilon}{2} \quad \text{for} \quad (t,x) \in \Big([a, t_0 + 1/n] \times [c, x_0 + 1/n]\Big) \cap \mathcal{D},\ n \ge n_0. \tag{6.19}$$

On the other hand, it is clear that the relation

$$\begin{aligned} |v_0(t,x) - v_n(t,x)| &\le \\ &\le |v_0(t,x)|\big(1 - \chi_n(t,x)\big) + |v_0(t,x) - w_n(t,x)||\chi_n(t,x)| \le \\ &\le |v_0(t,x)|\big(1 - \chi_n(t,x)\big) + |v_0(t,x) - w_n(t,x)| \end{aligned}$$

holds for $(t,x) \in \mathcal{D}$ and $n \in \mathbb{N}$. Hence, (6.15), (6.18), and (6.19) guarantee that

$$|v_0(t,x) - v_n(t,x)| < \varepsilon \quad \text{for} \quad (t,x) \in \mathcal{D},\ n \ge n_0\,,$$

i.e., the relation (6.12) holds. ☐

Now we are in position to prove Theorem 6.1.

Proof of Theorem 6.1. Assume that, on the contrary, ℓ is not (a,c)–Volterra operator. Then there exist $v_0 \in C(\mathcal{D};\mathbb{R})$ and $(t_0,x_0) \in]a,b]\times]c,d]$, $(t_0,x_0) \neq (b,d)$, such that (6.10) holds with $\mathcal{D}_0 = [a,t_0]\times[c,x_0]$ and

$$\operatorname{mes}\{(t,x)\in\mathcal{D}_0 : \ell(v_0)(t,x)\neq 0\} > 0.$$

Without loss of generality we can assume that

$$\operatorname{mes}\{(t,x)\in\mathcal{D}_0 : \ell(v_0)(t,x) < 0\} > 0. \tag{6.20}$$

At first we will show that

$$\Omega\big(\ell(|v_0|)\big)(t,x) = 0 \quad \text{for} \quad (t,x)\in\mathcal{D}_0, \tag{6.21}$$

where Ω denotes the Darboux operator of the problem (1.1_0), (1.2_0).

According to Lemma 6.1, there exists a sequence $\{v_n\}_{n=1}^{+\infty} \subset C^2(\mathcal{D};\mathbb{R})$ such that (6.11) is satisfied and

$$\lim_{n\to+\infty} \|v_n - |v_0|\,\|_C = 0. \tag{6.22}$$

Since ℓ and Ω are continuous operators (see Corollary 5.2), the relation (6.22) implies

$$\lim_{n\to+\infty} \big\|\Omega\big(\ell(v_n)\big) - \Omega\big(\ell(|v_0|)\big)\big\|_C = 0. \tag{6.23}$$

Let $z_n = \Omega\big(\ell(v_n)\big)$ for $n\in\mathbb{N}$. Then z_n is a solution of the problem

$$\frac{\partial^2 z_n(t,x)}{\partial t\,\partial x} = \ell(z_n)(t,x) + \ell(v_n)(t,x), \tag{6.24}$$

$$z_n(t,c) = 0 \quad \text{for} \quad t\in[a,b], \qquad z_n(a,x) = 0 \quad \text{for} \quad x\in[c,d]. \tag{6.25}$$

For any $n\in\mathbb{N}$, we put

$$w_n(t,x) = v_n(t,x) + z_n(t,x) \quad \text{for} \quad (t,x)\in\mathcal{D}.$$

It is clear that $w_n \in C^*(\mathcal{D};\mathbb{R})$ for $n\in\mathbb{N}$ because every function v_n belongs to the set $C^2(\mathcal{D};\mathbb{R})$. Moreover, (6.24) and (6.25) result in

$$\frac{\partial^2 w_n(t,x)}{\partial t\,\partial x} = \ell(w_n)(t,x) + \frac{\partial^2 v_n(t,x)}{\partial t\,\partial x}\,,$$

$$w_n(t,c) = v_n(t,c) \quad \text{for} \quad t\in[a,b], \qquad w_n(a,x) = v_n(a,x) \quad \text{for} \quad x\in[c,d].$$

Therefore, on account of (6.11), Proposition 6.2 implies

$$w_n(t,x) = 0 \quad \text{for} \quad (t,x)\in\mathcal{D}_0,\ n\in\mathbb{N}.$$

Hence, again by virtue of (6.11), we get

$$\Omega\big(\ell(v_n)\big)(t,x) = z_n(t,x) = -v_n(t,x) = 0 \quad \text{for} \quad (t,x)\in\mathcal{D}_0,\ n\in\mathbb{N}$$

and thus, in view of (6.23), the relation (6.21) is satisfied.

According to Remark 6.1, the problem (1.1), (1.2_0) with

$$q(t,x) = \begin{cases} -\ell(|v_0|)(t,x) & \text{if} \quad (t,x) \in \mathcal{D}_0 \\ 0 & \text{if} \quad (t,x) \in \mathcal{D} \setminus \mathcal{D}_0 \end{cases} \tag{6.26}$$

has a unique solution u. We suppose that $-\ell \in \mathcal{P}(\mathcal{D})$ and therefore

$$q(t,x) \geq 0 \quad \text{for} \quad (t,x) \in \mathcal{D} \tag{6.27}$$

and

$$\text{mes}\{(t,x) \in \mathcal{D}_0 : q(t,x) > 0\} > 0, \tag{6.28}$$

because the relation (6.20) holds. Since Ω is an (a,c)–Volterra operator (see Corollary 6.1) and $u = \Omega(q)$ it follows from (6.21) and (6.26) that

$$u(t,x) = 0 \quad \text{for} \quad (t,x) \in \mathcal{D}_0\,. \tag{6.29}$$

On the other hand, the operator ℓ belongs to the set $\mathcal{S}_{ac}(\mathcal{D})$ which garantees

$$u(t,x) \geq 0 \quad \text{for} \quad (t,x) \in \mathcal{D}, \tag{6.30}$$

because the relation (6.27) is true. By virtue of the assumption $-\ell \in \mathcal{P}(\mathcal{D})$, the equation (1.1) implies

$$u_{tx}(t,x) \leq q(t,x) \quad \text{for} \quad (t,x) \in \mathcal{D}$$

and thus, using (6.26) and (6.29), we get

$$u_{tx}(t,x) \leq 0 \quad \text{for} \quad (t,x) \in \mathcal{D}. \tag{6.31}$$

However, the function u satisfies the homogeneous initial conditions (1.2_0) and therefore the last inequality yields

$$u(t,x) \leq 0 \quad \text{for} \quad (t,x) \in \mathcal{D}.$$

Whence we get $u \equiv 0$ because the function u satisfies (6.30). Finally, the equation (1.1) implies $q \equiv 0$, which contradicts (6.28). □

7. Examples

Example 7.1. Let $p \in L(\mathcal{D}; \mathbb{R}_+)$ be such that

$$\iint_{\mathcal{D}} p(s,\eta)\,d\eta\,ds = 1$$

and let $\ell \in \mathcal{L}(\mathcal{D})$ be defined by

$$\ell(v)(t,x) = p(t,x)v(b,d) \quad \text{for} \quad (t,x) \in \mathcal{D}, \ v \in C(\mathcal{D}; \mathbb{R}).$$

Then the condition (4.2) with $\alpha = 1$ is satisfied for every $m \in \mathbb{N}$ and $v \in C(\mathcal{D};\mathbb{R})$. Moreover,

$$\iint\limits_{\mathcal{D}} p_j(s,\eta)d\eta ds = 1 \quad \text{for every} \quad j \in \mathbb{N},$$

where p_j is given by (4.4).

On the other hand, the problem (1.1_0), (1.2_0) has a nontrivial solution

$$u(t,x) = \int\limits_a^t \int\limits_c^x p(s,\eta)d\eta ds \quad \text{for} \quad (t,x) \in \mathcal{D}.$$

This example shows that the assumption $\alpha \in [0,1[$ in Theorem 4.1 cannot be replaced by the assumption $\alpha \in [0,1]$, and the strict inequality (4.3) in Corollary 4.1 cannot be replaced by the nonstrict one.

Example 7.2. Let

$$g_k(t) = k\cos k^2 t, \quad h_k(t) = -k\sin k^2 t \quad \text{for} \quad t \geq 0,\ k \in \mathbb{N}, \tag{7.1}$$

and

$$y_k(t) = -k\int\limits_0^t \exp\left(\frac{\sin k^2 t}{k} - \frac{\sin k^2 s}{k}\right)\sin k^2 s ds \quad \text{for} \quad t \geq 0,\ k \in \mathbb{N}. \tag{7.2}$$

It is not difficult to verify that, for every $k \in \mathbb{N}$,

$$y_k'(t) = g_k(t)y_k(t) + h_k(t) \quad \text{for a. a.} \quad t \geq 0, \tag{7.3}$$

$$|y_k(t)| \leq 1 + e + te^2 \quad \text{for} \quad t \geq 0, \tag{7.4}$$

and

$$\lim_{k\to+\infty} y_k(t) = \frac{t}{2} \quad \text{for} \quad t \geq 0, \tag{7.5}$$

because

$$\begin{aligned} y_k(t) = \frac{1}{k}\cos k^2 t - \frac{1}{k}\exp\left(\frac{\sin k^2 t}{k}\right) + &\\ + \frac{1}{2}\int\limits_0^t \exp\left(\frac{\sin k^2 t}{k} - \frac{\sin k^2 s}{k}\right)ds + &\\ + \frac{1}{2}\int\limits_0^t \exp\left(\frac{\sin k^2 t}{k} - \frac{\sin k^2 s}{k}\right)\cos 2k^2 s ds \quad \text{for} \quad t \geq 0.& \end{aligned}$$

Now, let $p \equiv 0$, $q \equiv 0$, $\varphi \equiv 0$, $\psi \equiv 0$, and

$$\tau(t,x) = t, \quad \mu(t,x) = x \quad \text{for} \quad (t,x) \in \mathcal{D}.$$

For any $k \in \mathbb{N}$, we put $\varphi_k \equiv 0$, $\psi_k \equiv 0$,

$$p_k(t,x) = g_k(t-a)g_k(x-c) \quad \text{for} \quad (t,x) \in \mathcal{D},$$

$$q_k(t,x) = h_k(t-a)y_k'(x-c) + y_k'(t-a)h_k(x-c) - \\ - h_k(t-a)h_k(x-c) \quad \text{for} \quad (t,x) \in \mathcal{D},$$

and

$$\tau_k(t,x) = t, \quad \mu_k(t,x) = x \quad \text{for} \quad (t,x) \in \mathcal{D}.$$

Let $\ell, \ell_k \in \mathcal{L}(\mathcal{D})$ be operators defined by (4.7) and (5.50).

According to (7.1), (7.3), and (7.4), it is clear that the assumptions of Theorem 5.2 are satisfied except of (5.10) which, in view of the proof of theorem mentioned, guarantees that the assumptions of Corollary 5.1 are fulfilled except of (5.6).

On the other hand,

$$u(t,x) = 0 \quad \text{for} \quad (t,x) \in \mathcal{D}$$

and

$$u_k(t,x) = y_k(t-a)y_k(x-c) \quad \text{for} \quad (t,x) \in \mathcal{D},\ k \in \mathbb{N}$$

are solutions of the problems (1.1′), (1.2) and ($1.1_k'$), (1.2_k), respectively, as well as of the problems (1.1), (1.2) and (1.1_k), (1.2_k), respectively. However, in view of (7.5), we get

$$\lim_{k\to+\infty} \Big(u_k(t,x) - u(t,x)\Big) = \lim_{k\to+\infty} y_k(t-a)y_k(x-c) = \\ = \frac{(t-a)(x-c)}{4} \quad \text{for} \quad (t,x) \in \mathcal{D},$$

that is, the relation (5.5) is not true.

This example shows that the assumption (5.6) in Corollary 5.1 and the assumption (5.10) in Theorem 5.2 are essential and cannot be omitted.

Acknowledgement

The research was supported by the Academy of Sciences of the Czech Republic, Institutional Research Plan No. AV0Z10190503.

References

[1] E. Bravyi, A. Lomtatidze, B. Půža, *A note on the theorem on differential inequalities,* Georgian Math. J. **7** (2000), No. 4, 627–631.

[2] N. Dunford, J. T. Schwartz, *Linear operators. I. General Theory,* Pure and Applied Mathematics, vol. 7, Interscience Publishers, New York, 1958.

[3] K. Deimling, *Absolutely continuous solutions of Cauchy problem for* $u_{xy} = f(x,y,u,u_x,u_y)$, Ann. mat. pura ed appl. **89** (1971), 381–391.

[4] O. Dzagnidze, *Some new results on the continuity and differentiability of functions of several real variables,* Proceed. of A. Razm. Math. Inst. **134** (2004), 1–138.

[5] M. Grigolia, *Some remarks on the initial problems for nonlinear hyperbolic systems,* Mem. Differential Equations Math. Phys. **34** (2005), 97–114.

[6] T. Kiguradze, *Some boundary value problems for systems of linear partial differential equations of hyperbolic type,* Mem. Differential Equations Math. Phys. **1** (1994), 1–144.

[7] A. Lomtatidze, S. Mukhigulashvili, J. Šremr, *Nonnegative solutions of the characteristic initial value problem for linear partial functional–differential equations of hyperbolic type,* Mathematical Institute, Academy of Sciences of the Czech Republic, Preprint (2005).

[8] H. Štěpánková, *A note on the theorem on differential inequalities,* E. J. Qualitative Theory of Diff. Equ. (2005), No. 7, 1–8.

[9] S. Walczak, *Absolutely continuous functions of several variables and their application to differential equations,* Bull. Pol. Acad. Sci. Math. **35** (1987), No. 11–12, 733–744.

[10] W. Walter, *Differential and integral inequalities,* Springer–Verlag, Berlin, Heidelberg, New York, 1970.

Author's Address

Mathematical Institute, Academy of Sciences of the Czech Republic, Žižkova 22, 616 62 Brno, Czech Republic

E-mail address: `sremr@ipm.cz`

ACADEMY OF SCIENCES OF THE CZECH REPUBLIC
MATHEMATICAL INSTITUTE

The preprint series was founded in 1982. Its purpose is to present manuscripts of submitted or unpublished papers and reports of members of the Mathematical Institute of the Academy of Sciences of the Czech Republic. The authors are fully responsible for the content of the preprints.

Mail address: Mathematical Institute,
Academy of Sciences of the Czech Republic
Žitná 25,
CZ-115 67 Praha 1
Czech Republic

phone: +420 222 090 711 fax: +420 222 211 638
e-mail: mathinst math.cas.cz http: //www.math.cas.cz

Latest preprints of the series:

2005

160 *A. Lomtatidze, S. Mukhigulashvili, and J. Šremr*: Nonnegative solutions of the characteristic initial value problem for linear partial functional-differential equations of hyperbolic type
159 *Andrej Rontó*: Upper bounds for the eigenvalues of compact linear operators in a preordered Banach space
158 *Martin Markl*: Cohomology operators and the Deligne conjecture

2004

157 *J. Eisner, M. Kučera, L. Recke*: A global bifurcation result for variational inequalities
156 *Irena Rachůnková, Milan Tvrdý*: Second order periodic problem with 8-Laplacian and impulses—part II
155 *Irena Rachůnková, Milan Tvrdý*: Second order periodic problem with 8-Laplacian and impulses—part I

2003

154 *J. Eisner, M. Kučera, L. Recke*: Direction and stability of bifurcation branches for variational inequalities
153 *Irena Rachůnková, Milan Tvrdý*: Periodic boundary value problems for nonlinear second order differential equations with impulses—part III
152 *Pavel Krutitskii, Dagmar Medková*: Neumann and Robin problem in a cracked domain with jump conditions on cracks

2002

151 *Irena Rachůnková, Milan Tvrdý*: Periodic boundary value problems for nonlinear second order differential equations with impulses—part II
150 *Miroslav Šilhavý*: An $O(n)$ invariant rank 1 convex function that is not polyconvex
149 *Šárka Nečasová*: Asymptotic properties of the steady fall of a body in viscous fluids
148 *Irena Rachůnková, Milan Tvrdý*: Periodic boundary value problems for nonlinear second order differential equations with impulses—part I
147 *Pavel Krutitskii, Dagmar Medková*: The harmonic Dirichlet problem for cracked domain with jump conditions on cracks
146 *Dagmar Medková*: Boundedness of the solution of the third problem for the Laplace equation
145 *George V. Jaiani, Alois Kufner*: Oscillation of cusped Euler-Bernoulli beams and Kirchhoff-love plates
144 *Emil Vitásek*: Obyčejné diferenciální rovnice (Elementární metody přibližného řešení okrajových úloh)

2001

143 *Šárka Nečasová*: Some remarks on the steady fall of body in Stokes and Oseen flow
142 *Miroslav Šilhavý*: Relaxation in a class of $SO(n)$-invariant energies related to nematic elastomers
141 *Irena Rachůnková, Milan Tvrdý, Ivo Vrkoč*: Resonance and multiplicity in periodic boundary value problems with singularity