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OPERADS AND PROPS

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ABSTRACT. We review definitions and basic properties of operads, PROPs and algebras over these structures.

Dedicated to the memory of Jakub Jan Ryba (1765–1815)

Operads involve an abstraction of the family $\{Map(X^n, X)\}_{n \geq 0}$ of composable functions of several variables together with an action of permutations of variables. As such, they were originally studied as a tool in homotopy theory, specifically for iterated loop spaces and homotopy invariant structures, but the theory of operads has recently received new inspiration from homological algebra, category theory, algebraic geometry and mathematical physics. The name *operad* and the formal definition appear first in the early 1970's in J.P. May's book [86], but a year or more earlier, M. Boardman and R. Vogt [9] described the same concept under the name categories of *operators in standard form*, inspired by PROPs and PACTs of Adams and Mac Lane [67]. As pointed out in [62], also Lambek's definition of multicategory [60] (late 1960s) was almost equivalent to what is called today a colored or many-sorted operad. Another important precursor was the associahedron K that appeared in J.D. Stasheff's 1963 paper [106] on homotopy associativity of H-spaces. We do not, however, aspire to write an account on the history of operads and their applications here – we refer to the introduction of [83], to [89], [114], or to the report [105] instead.

Operads are important not in and of themselves but, like PROPs, through their representations, more commonly called *algebras* over operads or *operad algebras*. If an operad is thought of as a kind of algebraic theory, then an algebra over an operad is a model of that theory. Algebras over operads involve most of 'classical' algebras (associative, Lie, commutative associative, Poisson, &c.), loop spaces, moduli spaces of algebraic curves, vertex operator algebras, &c. *Colored* or *many-sorted* operads then describe diagrams of homomorphisms of these objects, homotopies between homomorphisms, modules, &c.

PROPs generalize operads in the sense that they admit operations with several inputs and *several* outputs. Therefore various bialgebras (associative, Lie, infinitesimal) are PROPic algebras. PROPs were also used to encode 'profiles' of structures in formal differential geometry [92, 93].

By the renaissance of operads we mean the first half of the nineties of the last century when several papers which stimulated the rebirth of interest in operads appeared [31, 34, 41, 45, 47, 49, 72]. Let us mention the most important new ideas that emerged during this period.

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First of all, operads were recognized as the underlying combinatorial structure of the moduli space of stable algebraic curves in complex geometry, and of compactifications of configuration spaces of points of affine spaces in real geometry. In mathematical physics, several very important concepts such as vertex operator algebras or various string theories were interpreted as algebras over operads. On the algebraic side, the notion of *Koszulness* of operads was introduced and studied, and the relation between resolutions of operads and deformations of their algebras was recognized. See [63] for an autochthonous account of the renaissance. Other papers that later became influential then followed in a rapid succession [30, 33, 32, 73, 75].

Let us list some most important outcomes of the renaissance of operads. The choice of the material for this incomplete catalog has been of course influenced by the author's personal expertise and inclination towards algebra, geometry and topics that are commonly called mathematical physics. We will therefore not be able to pay as much attention to other aspects of operads, such as topology, category theory and homotopy theory, as they deserve.

Complex geometry. Applications involve moduli spaces of stable complex algebraic curves of genus zero [34], enumerative geometry, Frobenius manifolds, quantum cohomology and cohomological field theory [55, 71]. The moduli space of genus zero curves exhibits an additional symmetry that leads to a generalization called *cyclic* operads [32]. *Modular* operads [33] then describe the combinatorial structure of the space of curves of arbitrary genus.

Real geometry. Compactifications of configuration spaces of points in real smooth manifolds are operads in the category of smooth manifolds with corners or modules over these operads [76]. This fact is crucial for the theory of configuration spaces with summable labels [96]. The cacti operad [117] lies behind the Chas-Sullivan product on the free loop space of a smooth manifold [13], see also [14]. Tamarkin's proof of the formality of Hochschild cochains of the algebra of functions on smooth manifolds [110] explained in [40] uses obstruction theory for operad algebras and the affirmative answer to the Deligne conjecture [17, 56].

Mathematical physics. The formality mentioned in the previous item implies the existence of the deformation quantization of Poisson manifolds [54]. We must not forget to mention the operadic interpretation of vertex operator algebras [46], string theory [49] and Connes-Kreimer's approach to renormalization [15]. Operads and multicategories are important also for Beilinson-Drinfeld's theory of chiral algebras [6].

Algebra. Operadic cohomology [1, 26, 31, 34, 83] provides a uniform treatment of all 'classical' cohomology theories, such as the Hochschild cohomology of associative algebras, Harrison cohomology of associative commutative algebras, Chevalley-Eilenberg cohomology of Lie algebras, &c. Minimal models for operads [75] offer a conceptual understanding of strong homotopy algebras, their homomorphisms and homotopy invariance [80]. Operads serve as a natural language for various types of 'multialgebras' [64, 65]. Relation between Koszulness of operads and properties of posets was studied in [27]. Also the concept of the operadic distributive law turned out to be useful [26, 74].

Model structures. It turned out [8, 31, 39, 100] that algebras over a reasonable (possibly colored) operad form a model category that generalizes the classical model structures of the categories of dg commutative associative algebras and dg Lie algebras [95, 107]. Operads, in a

reasonable monoidal model category, themselves form a model category [7, 31] such that algebras over cofibrant operads are homotopy invariant, see also [104]. Minimal operads mentioned in the previous item are particular cases of cofibrant dg-operads and the classical W -construction [9] is a functorial cofibrant replacement in the category of topological operads [116]. The above model structures are important for various constructions in the homology theory of (free or based) loop spaces [14, 43] and formulations of ‘higher’ Deligne conjecture [44].

Topology. Operads as gadgets organizing homotopy coherent structures are important in the brave new algebra approach to topological Hochschild cohomology and algebraic K -theory, see [22, 23, 90, 115], or [21] for a historical background. A description of a localized category of integral and p -adic homotopy types by E_∞ -operads was given in [69, 70]. An operadic approach to partial algebras and their completions was applied in [58] to mixed Tate motives over the rationals. See also an overview [88].

Category theory. Operads and multicategories were used as a language in which to propose a definition of weak ω -category [3, 4, 5, 61]. Operads themselves can be viewed as special kinds of algebraic theory (as can multicategories, if one allows many-sorted theories), see [85]. There are also ‘categorical’ generalizations of operads, e.g. the globular operads of [2] and T -categories of [11]. An interesting presentation of PROP-like structures in enriched monoidal categories can be found in [91].

Graph Complexes. Each cyclic operad \mathcal{P} determines a graph complex [33, 77]. As observed earlier by M. Kontsevich [52], these graph complexes are, for some specific choices of \mathcal{P} , closely related to some very interesting objects such as moduli spaces of Riemann surfaces, automorphisms of free groups or primitives in the homology of certain infinite-dimensional Lie algebras, see also [83, II.5.5]. In the same vein, complexes of directed graphs are related to PROPs [84, 111, 112, 113] and directed graphs with back-in-time edges are tied to wheeled PROPs introduced in [93].

Deformation theory and homotopy invariant structures in algebra. A concept of homotopy invariant structures in algebra parallel to the classical one in topology [9, 10] was developed in [80]. It was explained in [56, 73, 79] how cofibrant resolutions of operads or PROPs determine a cohomology theory governing deformations of related algebras. In [81], deformations were identified with solutions of the Maurer-Cartan equation of a certain strongly homotopy Lie algebra constructed in a very explicit way from a cofibrant resolution of the underlying operad or PROP.

Terminology. As we already observed, operads are abstractions of families of composable functions. Given functions $f : X^{\times n} \rightarrow X$ and $g_i : X^{\times k_i} \rightarrow X$, $1 \leq i \leq n$, one may consider the simultaneous composition

$$(I) \quad f(g_1, \dots, g_n) : X^{\times(k_1 + \dots + k_n)} \rightarrow X.$$

One may also consider, for $f : X^{\times n} \rightarrow X$, $g : X^{\times m} \rightarrow X$ and $1 \leq i \leq n$, the individual compositions

$$(II) \quad f(id, \dots, id, g, id, \dots, id) : X^{\times(m+m-1)} \rightarrow X,$$

with g at the i th place. While May’s original definition of an operad [86] was an abstraction of

type (I) compositions, there exist an alternative approach based on type (II) compositions. This second point of view was formalized in the 1963 papers by Gerstenhaber [29] and Stasheff [106]. A definition that included the symmetric group action was formulated much later in the author's paper [75] in which the two approaches were also compared.

In the presence of operadic units, these approaches are equivalent. There are, however, situations where one needs also non-unital versions, and then the two approaches lead to different structures – a non-unital structure of the second type always determines a non-unital structure of the first type, but *not vice versa!* It turns out that more common are non-unital structures of the second type; they describe, for example, the underlying combinatorial structure of the moduli space of stable complex curves.

We will therefore call the non-unital versions of the first type of operads *non-unital May's operads*, while the second version simply *non-unital operads*. We opted for this terminology, which was used already in the first version of [75], after a long hesitation, being aware that it might not be universally welcome. Note that non-unital operads are sometimes called (*Markl's*) *pseudo-operads* [75, 83].

OUTLINE OF THE PAPER:

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In the first three sections we review basic definitions of (unital and non-unital) operads and operad algebras, and give examples that illustrate these notions. The fourth section describes free operads and their relation to rooted trees. In the fifth section we explain that operads can be defined as algebras over the monad of rooted trees. In the following two sections we show that, replacing rooted trees by other types of trees, one obtains two important generalizations – cyclic and modular operads. In the last two sections, PROPs and their versions are recalled; this article is the first expository text where these structures are systematically treated.

Sections 1–3 are based on the classical book [86] by J.P. May and the author's article [75]. Sections 4–7 follow the seminal paper [34] by V. Ginzburg and M.M. Kapranov, and papers [32, 33] by E. Getzler and M.M. Kapranov. The last two sections are based on the preprint [84] of A.A. Voronov and the author, and on an e-mail message [53] from M. Kontsevich. We were also influenced by T. Leinster's concept of biased versus un-biased definitions [61]. At some places, our exposition follows the monograph [83] by S. Shnider, J.D. Stasheff and the author.

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1. OPERADS

Although operads, operad algebras and most of related structures can be defined in an arbitrary symmetric monoidal category with countable coproducts, we decided to follow the choice of [58] and formulate precise definitions only for the category $\mathbf{Mod}_{\mathbf{k}} = (\mathbf{Mod}_{\mathbf{k}}, \otimes)$ of modules over a commutative unital ring \mathbf{k} , with the monoidal structure given by the tensor product $\otimes = \otimes_{\mathbf{k}}$ over \mathbf{k} . The reason for such a decision was to give, in Section 4, a clean construction of free operads. In a general monoidal category, this construction involves the unordered \odot -product [83, Definition II.1.38] so the free operad is then a double colimit, see [83, Section II.1.9]. Our choice also allows us to write formulas involving maps in terms of elements, which is sometimes a welcome simplification. We believe that the reader can easily reformulate our definitions into other monoidal categories or consult [83, 87].

Let $\mathbf{k}[\Sigma_n]$ denote the \mathbf{k} -group ring of the symmetric group Σ_n .

Definition 1 (May’s operad). *An operad in the category of \mathbf{k} -modules is a collection $\mathcal{P} = \{\mathcal{P}(n)\}_{n \geq 0}$ of right $\mathbf{k}[\Sigma_n]$ -modules, together with \mathbf{k} -linear maps (operadic compositions)*

$$(1) \quad \gamma : \mathcal{P}(n) \otimes \mathcal{P}(k_1) \otimes \cdots \otimes \mathcal{P}(k_n) \rightarrow \mathcal{P}(k_1 + \cdots + k_n),$$

for $n \geq 1$ and $k_1, \dots, k_n \geq 0$, and a unit map $\eta : \mathbf{k} \rightarrow \mathcal{P}(1)$. These data fulfill the following axioms.

Associativity. Let $n \geq 1$ and let m_1, \dots, m_n and k_1, \dots, k_m , where $m := m_1 + \cdots + m_n$, be non-negative integers. Then the following diagram, in which $g_s := m_1 + \cdots + m_{s-1}$ and $h_s = k_{g_s+1} \cdots + k_{g_s+1}$, for $1 \leq s \leq n$, commutes.

$$\begin{array}{ccc}
 \left(\mathcal{P}(n) \otimes \bigotimes_{s=1}^n \mathcal{P}(m_s) \right) \otimes \bigotimes_{r=1}^m \mathcal{P}(k_r) & \xrightarrow{\gamma \otimes id} & \mathcal{P}(m) \otimes \bigotimes_{r=1}^m \mathcal{P}(k_r) \\
 \downarrow \text{shuffle} & & \downarrow \gamma \\
 & & \mathcal{P}(k_1 + \cdots + k_m) \\
 & & \uparrow \gamma \\
 \mathcal{P}(n) \otimes \bigotimes_{s=1}^n \left(\mathcal{P}(m_s) \otimes \bigotimes_{q=1}^{m_s} \mathcal{P}(k_{g_s+q}) \right) & \xrightarrow{id \otimes (\bigotimes_{s=1}^n \gamma)} & \mathcal{P}(n) \otimes \bigotimes_{s=1}^n \mathcal{P}(h_s)
 \end{array}$$

Equivariance. Let $n \geq 1$, let k_1, \dots, k_n be non-negative integers and $\sigma \in \Sigma_n$, $\tau_1 \in \Sigma_{k_1}, \dots, \tau_n \in \Sigma_{k_n}$ permutations. Let $\sigma(k_1, \dots, k_n) \in \Sigma_{k_1 + \cdots + k_n}$ denote the permutation that permutes n blocks

$(1, \dots, k_1), \dots, (k_{n-1} + 1, \dots, k_n)$ as σ permutes $(1, \dots, n)$ and let $\tau_1 \oplus \dots \oplus \tau_n \in \Sigma_{k_1 + \dots + k_n}$ be the block sum of permutations. Then the following diagrams commute.

$$\begin{array}{ccc}
\mathcal{P}(n) \otimes \mathcal{P}(k_1) \otimes \dots \otimes \mathcal{P}(k_n) & \xrightarrow{\sigma \otimes \sigma^{-1}} & \mathcal{P}(n) \otimes \mathcal{P}(k_{\sigma(1)}) \otimes \dots \otimes \mathcal{P}(k_{\sigma(n)}) \\
\downarrow \gamma & & \downarrow \gamma \\
\mathcal{P}(k_1 + \dots + k_n) & \xrightarrow{\sigma(k_{\sigma(1)}, \dots, k_{\sigma(n)})} & \mathcal{P}(k_{\sigma(1)} + \dots + k_{\sigma(n)}) \\
\\
\mathcal{P}(n) \otimes \mathcal{P}(k_1) \otimes \dots \otimes \mathcal{P}(k_n) & \xrightarrow{id \otimes \tau_1 \otimes \dots \otimes \tau_n} & \mathcal{P}(n) \otimes \mathcal{P}(k_1) \otimes \dots \otimes \mathcal{P}(k_n) \\
\downarrow \gamma & & \downarrow \gamma \\
\mathcal{P}(k_1 + \dots + k_n) & \xrightarrow{\tau_1 \oplus \dots \oplus \tau_n} & \mathcal{P}(k_1 + \dots + k_n)
\end{array}$$

Unitality. For each $n \geq 1$, the following diagrams commute.

$$\begin{array}{ccc}
\mathcal{P}(n) \otimes \mathbf{k}^{\otimes n} & \xrightarrow{\cong} & \mathcal{P}(n) \\
id \otimes \eta^{\otimes n} \downarrow & \nearrow \gamma & \\
\mathcal{P}(n) \otimes \mathcal{P}(1)^{\otimes n} & &
\end{array}
\qquad
\begin{array}{ccc}
\mathbf{k} \otimes \mathcal{P}(n) & \xrightarrow{\cong} & \mathcal{P}(n) \\
\eta \otimes id \downarrow & \nearrow \gamma & \\
\mathcal{P}(1) \otimes \mathcal{P}(n) & &
\end{array}$$

A straightforward modification of the above definition makes sense in any symmetric monoidal category $(\mathcal{M}, \odot, \mathbf{1})$ such as the category of differential graded modules, simplicial sets, topological spaces, &c, see [83, Definition II.1.4] or [87, Definition 1]. We then speak about *differential graded operads*, *simplicial operads*, *topological operads*, &c.

Example 2. All properties axiomatized by Definition 1 can be read from the *endomorphism operad* $\mathcal{E}nd_V = \{\mathcal{E}nd_V(n)\}_{n \geq 0}$ of a \mathbf{k} -module V . It is defined by setting $\mathcal{E}nd_V(n)$ to be the space of \mathbf{k} -linear maps $V^{\otimes n} \rightarrow V$. The operadic composition of $f \in \mathcal{E}nd_V(n)$ with $g_1 \in \mathcal{E}nd_V(k_1), \dots, g_n \in \mathcal{E}nd_V(k_n)$ is given by the usual composition of multilinear maps as

$$\gamma(f, g_1, \dots, g_n) := f(g_1 \otimes \dots \otimes g_n),$$

the symmetric group acts by

$$\gamma^\sigma(f, g_1, \dots, g_n) := f(g_{\sigma^{-1}(1)} \otimes \dots \otimes g_{\sigma^{-1}(n)}), \quad \sigma \in \Sigma_n,$$

and the unit map $\eta : \mathbf{k} \rightarrow \mathcal{E}nd_V(1)$ is given by $\eta(1) := id_V : V \rightarrow V$. The endomorphism operad can be constructed over an object of an arbitrary symmetric monoidal category with an internal hom-functor, as it was done in [83, Definition. II.1.7].

One often considers operads \mathcal{A} such that $\mathcal{A}(0) = 0$ (the trivial \mathbf{k} -module). We will indicate that \mathcal{A} is of this type by writing $\mathcal{A} = \{\mathcal{A}(n)\}_{n \geq 1}$.

Example 3. Let us denote by $\mathcal{A}ss = \{\mathcal{A}ss(n)\}_{n \geq 1}$ the operad with $\mathcal{A}ss(n) := \mathbf{k}[\Sigma_n]$, $n \geq 1$, and the operadic composition defined as follows. Let $id_n \in \Sigma_n$, $id_{k_1} \in \Sigma_{k_1}, \dots, id_{k_n} \in \Sigma_{k_n}$ be the

identity permutations. Then

$$\gamma(id_n, id_{k_1}, \dots, id_{k_n}) := id_{k_1 + \dots + k_n} \in \Sigma_{k_1 + \dots + k_n}.$$

The above formula determines $\gamma(\sigma, \tau_1, \dots, \tau_n)$ for general $\sigma \in \Sigma_n$, $\tau_1 \in \Sigma_{k_1}, \dots, \tau_n \in \Sigma_{k_n}$ by the equivariance axiom. The unit map $\eta : \mathbf{k} \rightarrow \mathcal{A}ss(1)$ is given by $\eta(1) := id_1$.

Example 4. Let us give an example of a topological operad. For $k \geq 1$, the *little k -discs operad* $\mathcal{D}_k = \{\mathcal{D}_k(n)\}_{n \geq 0}$ is defined as follows [83, Section II.4.1]. Let

$$\mathbb{D}^k := \{(x_1, \dots, x_k) \in \mathbb{R}^k; x_1^2 + \dots + x_k^2 \leq 1\}$$

be the standard closed disc in \mathbb{R}^k . A little k -disc is then a linear embedding $d : \mathbb{D}^k \hookrightarrow \mathbb{D}^k$ which is the restriction of a linear map $\mathbb{R}^k \rightarrow \mathbb{R}^k$ with parallel axes. The n -th space $\mathcal{D}_k(n)$ of the little k -disc operad is the space of all n -tuples (d_1, \dots, d_n) of little k -discs such that the images of d_1, \dots, d_n have mutually disjoint interiors. The operad structure is obvious – the symmetric group Σ_n acts on $\mathcal{D}_k(n)$ by permuting the labels of the little discs and the structure map γ is given by composition of embeddings. The unit is the identity embedding $id : \mathbb{D}^k \hookrightarrow \mathbb{D}^k$.

Example 5. The collection of normalized singular chains $C_*(\mathcal{T}) = \{C_*(\mathcal{T}(n))\}_{n \geq 0}$ of a topological operad $\mathcal{T} = \{\mathcal{T}(n)\}_{n \geq 0}$ is an operad in the category of differential graded \mathbb{Z} -modules. For a ring R , the singular homology $H_*(\mathcal{T}(n); R) = H_*(C_*(\mathcal{T}(n)) \otimes_{\mathbb{Z}} R)$ forms an operad $H_*(\mathcal{T}; R)$ in the category of graded R -modules, see [58, Section I.5] for details.

Definition 6. Let $\mathcal{P} = \{\mathcal{P}(n)\}_{n \geq 0}$ and $\mathcal{Q} = \{\mathcal{Q}(n)\}_{n \geq 0}$ be two operads. A homomorphism $f : \mathcal{P} \rightarrow \mathcal{Q}$ is a sequence $f = \{f(n) : \mathcal{P}(n) \rightarrow \mathcal{Q}(n)\}_{n \geq 0}$ of equivariant maps which commute with the operadic compositions and preserve the units.

An operad $\mathcal{R} = \{\mathcal{R}(n)\}_{n \geq 0}$ is a suboperad of \mathcal{P} if $\mathcal{R}(n)$ is, for each $n \geq 0$, a Σ_n -submodule of $\mathcal{P}(n)$ and if all structure operations of \mathcal{R} are the restrictions of those of \mathcal{P} . Finally, an ideal in the operad \mathcal{P} is the collection $\mathcal{J} = \{\mathcal{J}(n)\}_{n \geq 0}$ of Σ_n -invariant subspaces $\mathcal{J}(n) \subset \mathcal{P}(n)$ such that

$$\gamma_{\mathcal{P}}(f, g_1, \dots, g_n) \in \mathcal{J}(k_1 + \dots + k_n)$$

if either $f \in \mathcal{J}(n)$ or $g_i \in \mathcal{J}(k_i)$ for some $1 \leq i \leq n$.

Example 7. Given an operad $\mathcal{P} = \{\mathcal{P}(n)\}_{n \geq 0}$, let $\widehat{\mathcal{P}} = \{\widehat{\mathcal{P}}(n)\}_{n \geq 0}$ be the collection defined by $\widehat{\mathcal{P}}(n) := \mathcal{P}(n)$ for $n \geq 1$ and $\widehat{\mathcal{P}}(0) := 0$. Then $\widehat{\mathcal{P}}$ is a suboperad of \mathcal{P} . The correspondence $\mathcal{P} \mapsto \widehat{\mathcal{P}}$ is a full embedding of the category of operads \mathcal{P} with $\mathcal{P}(0) \cong \mathbf{k}$ into the category of operads \mathcal{A} with $\mathcal{A}(0) = 0$. Operads satisfying $\mathcal{P}(0) \cong \mathbf{k}$ have been called *unital* while operads with $\mathcal{A}(0) = 0$ *non-unital* operads. We will not use this terminology because non-unital operads will mean something different in this article, see Section 2.

An example of an operad \mathcal{A} which is not of the form $\widehat{\mathcal{P}}$ for some operad \mathcal{P} with $\mathcal{P}(0) \cong \mathbf{k}$ can be constructed as follows. Observe first that operads \mathcal{P} with the property that

$$\mathcal{P}(0) \cong \mathbf{k} \text{ and } \mathcal{P}(n) = 0 \text{ for } n \geq 2$$

are the same as augmented associative algebras. Indeed, the space $\mathcal{P}(1)$ with the operation $\circ_1 : \mathcal{P}(1) \otimes \mathcal{P}(1) \rightarrow \mathcal{P}(1)$ is clearly a unital associative algebra, augmented by the composition

$$\mathcal{P}(1) \xrightarrow{\cong} \mathcal{P}(1) \otimes \mathbf{k} \xrightarrow{\cong} \mathcal{P}(1) \otimes \mathcal{P}(0) \xrightarrow{\circ_1} \mathcal{P}(0) \cong \mathbf{k}.$$

Now take an arbitrary unital associative algebra A and define the operad $\mathcal{A} = \{\mathcal{A}(n)\}_{n \geq 1}$ by

$$\mathcal{A}(n) := \begin{cases} A, & \text{for } n = 1 \text{ and} \\ 0, & \text{for } n \neq 1, \end{cases}$$

with $\circ_1 : \mathcal{A}(1) \otimes \mathcal{A}(1) \rightarrow \mathcal{A}(1)$ the multiplication of A . It follows from the above considerations that $\mathcal{A} = \widehat{\mathcal{P}}$ for some operad \mathcal{P} with $\mathcal{P}(0) \cong \mathbf{k}$ if and only if A admits an augmentation. Therefore any unital associative algebra that does not admit an augmentation produces the desired example.

Example 8. Kernels, images, &c., of homomorphisms between operads in the category of \mathbf{k} -modules are defined componentwise. For example, if $f : \mathcal{P} \rightarrow \mathcal{Q}$ is such homomorphism, then $\text{Ker}(f) = \{\text{Ker}(f)(n)\}_{n \geq 0}$ is the collection with

$$\text{Ker}(f)(n) := \text{Ker}(f : \mathcal{P}(n) \rightarrow \mathcal{Q}(n)), \quad n \geq 0.$$

It is clear that $\text{Ker}(f)$ is an ideal in \mathcal{P} .

Also quotients are defined componentwise. If \mathcal{J} is an ideal in \mathcal{P} , then the collection $\mathcal{P}/\mathcal{J} = \{(\mathcal{P}/\mathcal{J})(n)\}_{n \geq 0}$ with $(\mathcal{P}/\mathcal{J})(n) := \mathcal{P}(n)/\mathcal{J}(n)$ for $n \geq 0$, has a natural operad structure induced by the structure of \mathcal{P} . The canonical projection $\mathcal{P} \rightarrow \mathcal{P}/\mathcal{J}$ has the expected universal property. The kernel of this projection equals \mathcal{J} .

Sometimes it suffices to consider operads without the symmetric group action. This notion is formalized by:

Definition 9 (May's non- Σ operad). *A non- Σ operad in the category of \mathbf{k} -modules is a collection $\underline{\mathcal{P}} = \{\underline{\mathcal{P}}(n)\}_{n \geq 0}$ of \mathbf{k} -modules, together with operadic compositions*

$$\underline{\gamma} : \underline{\mathcal{P}}(n) \otimes \underline{\mathcal{P}}(k_1) \otimes \cdots \otimes \underline{\mathcal{P}}(k_n) \rightarrow \underline{\mathcal{P}}(k_1 + \cdots + k_n),$$

for $n \geq 1$ and $k_1, \dots, k_n \geq 0$, and a unit map $\underline{\eta} : \mathbf{k} \rightarrow \underline{\mathcal{P}}(1)$ that fulfill the associativity and unitality axioms of Definition 1.

Each operad can be considered as a non- Σ operad by forgetting the Σ_n -actions. On the other hand, given a non- Σ operad $\underline{\mathcal{P}}$, there is an associated operad $\Sigma[\underline{\mathcal{P}}]$ with $\Sigma[\underline{\mathcal{P}}](n) := \underline{\mathcal{P}}(n) \otimes \mathbf{k}[\Sigma_n]$, $n \geq 0$, with the structure operations induced by the structure operations of $\underline{\mathcal{P}}$. Operads of this form are sometimes called *regular* operads.

Example 10. Consider the operad $\mathcal{C}om = \{\mathcal{C}om(n)\}_{n \geq 1}$ such that $\mathcal{C}om(n) := \mathbf{k}$ with the trivial Σ_n -action, $n \geq 1$, and the operadic compositions (1) given by the canonical identifications

$$\mathcal{C}om(n) \otimes \mathcal{C}om(k_1) \otimes \cdots \otimes \mathcal{C}om(k_n) \cong \mathbf{k}^{\otimes(n+1)} \xrightarrow{\cong} \mathbf{k} \cong \mathcal{C}om(k_1 + \cdots + k_n).$$

The operad $\mathcal{C}om$ is obviously not regular. Observe also that $\mathcal{C}om \cong \widehat{\mathcal{E}nd}_{\mathbf{k}}$, where $\widehat{\mathcal{E}nd}_{\mathbf{k}}$ is the endomorphism operad of the ground ring without the initial component, see Example 7 for the notation.

Let $\underline{\mathcal{A}ss}$ denote the operad $\mathcal{C}om$ considered as a non- Σ operad. Its symmetrization $\Sigma[\underline{\mathcal{A}ss}]$ then equals the operad $\mathcal{A}ss$ introduced in Example 3.

As we already observed, there is an alternative approach to operads. For the purposes of comparison, in the rest of this section and in the following section we will refer to operads viewed

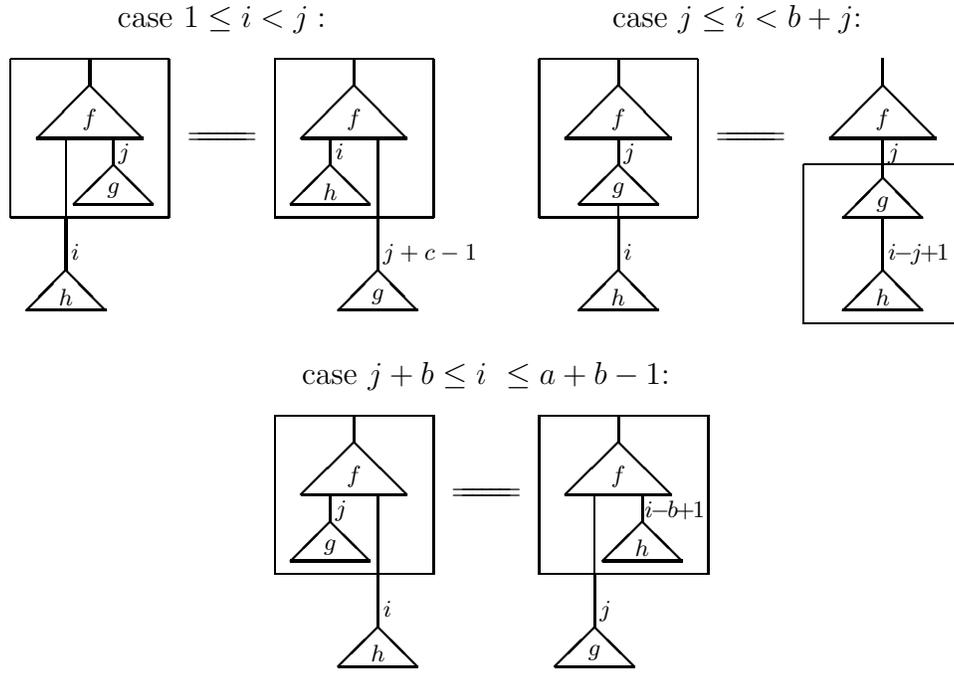


FIGURE 1. Flow charts explaining the associativity in Markl's operads.

from this alternative perspective as to Markl's operads. See also the remarks on the terminology in the introduction.

Definition 11. A Markl's operad in the category of \mathbf{k} -modules is a collection $\mathcal{S} = \{\mathcal{S}(n)\}_{n \geq 0}$ of right $\mathbf{k}[\Sigma_n]$ -modules, together with \mathbf{k} -linear maps (\circ_i -compositions)

$$\circ_i : \mathcal{S}(m) \otimes \mathcal{S}(n) \rightarrow \mathcal{S}(m + n - 1),$$

for $1 \leq i \leq m$ and $n \geq 0$. These data fulfill the following axioms.

Associativity. For each $1 \leq j \leq a$, $b, c \geq 0$, $f \in \mathcal{S}(a)$, $g \in \mathcal{S}(b)$ and $h \in \mathcal{S}(c)$,

$$(f \circ_j g) \circ_i h = \begin{cases} (f \circ_i h) \circ_{j+c-1} g, & \text{for } 1 \leq i < j, \\ f \circ_j (g \circ_{i-j+1} h), & \text{for } j \leq i < b + j, \text{ and} \\ (f \circ_{i-b+1} h) \circ_j g, & \text{for } j + b \leq i \leq a + b - 1, \end{cases}$$

see Figure 1.

Equivariance. For each $1 \leq i \leq m$, $n \geq 0$, $\tau \in \Sigma_m$ and $\sigma \in \Sigma_n$, let $\tau \circ_i \sigma \in \Sigma_{m+n-1}$ be given by inserting the permutation σ at the i th place in τ . Let $f \in \mathcal{S}(m)$ and $g \in \mathcal{S}(n)$. Then

$$(f\tau) \circ_i (g\sigma) = (f \circ_{\tau(i)} g)(\tau \circ_i \sigma).$$

Unitality. There exists $e \in \mathcal{S}(1)$ such that

$$(2) \quad f \circ_i e = e \quad \text{and} \quad e \circ_1 g = g$$

for each $1 \leq i \leq m$, $n \geq 0$, $f \in \mathcal{S}(m)$ and $g \in \mathcal{S}(n)$.

Example 12. All axioms in Definition 11 can be read from the endomorphism operad $\mathcal{E}nd_V = \{\mathcal{E}nd_V(n)\}_{n \geq 0}$ of a \mathbf{k} -module V reviewed in Example 2, with \circ_i -operations given by

$$f \circ_i g := f(id_V^{\otimes i-1} \otimes g \otimes id_V^{\otimes m-1}),$$

for $f \in \mathcal{E}nd_V(m)$, $g \in \mathcal{E}nd_V(n)$, $1 \leq i \leq m$ and $n \geq 0$.

The following proposition shows that Definition 1 describes the same objects as Definition 11.

Proposition 13. *The category of May's operads is isomorphic to the category of Markl's operads.*

Proof. Given a Markl's operad $\mathcal{S} = \{\mathcal{S}(n)\}_{n \geq 0}$ as in Definition 11, define a May's operad $\mathcal{P} = \text{May}(\mathcal{S})$ by $\mathcal{P}(n) := \mathcal{S}(n)$ for $n \geq 0$, with the γ -operations given by

$$(3) \quad \gamma(f, g_1, \dots, g_n) := (\cdots ((f \circ_n g_n) \circ_{n-1} g_{n-1}) \cdots) \circ_1 g_1$$

where $f \in \mathcal{P}(n)$, $g_i \in \mathcal{P}(k_i)$, $1 \leq i \leq n$, $k_1, \dots, k_n \geq 0$. The unit morphism $\eta : \mathbf{k} \rightarrow \mathcal{P}(1)$ is defined by $\eta(1) := e$. It is easy to verify that $\text{May}(-)$ extends to a functor from the category of Markl's operads to the category of May's operads.

On the other hand, given a May's operad \mathcal{P} , one can define a Markl's operad $\mathcal{S} = \text{Mar}(\mathcal{P})$ by $\mathcal{S}(n) := \mathcal{P}(n)$ for $n \geq 0$, with the \circ_i -operations:

$$(4) \quad f \circ_i g := \gamma(f, \underbrace{e, \dots, e}_{i-1}, g, \underbrace{e, \dots, e}_{m-i}),$$

for $f \in \mathcal{S}(m)$, $g \in \mathcal{S}(n)$, $m \geq 1$, $n \geq 0$, where $e := \eta(1) \in \mathcal{P}(1)$. It is again obvious that $\text{Mar}(-)$ extends to a functor that the functors $\text{May}(-)$ and $\text{Mar}(-)$ are mutually inverse isomorphisms between the category of Markl's operads and the category of May's operads. \square

The equivalence between May's and Markl's operads implies that an operad can be defined by specifying \circ_i -operations and a unit. This is sometimes simpler than to define the γ -operations directly, as illustrated by:

Example 14. Let Σ be a Riemann sphere, that is, a nonsingular complex projective curve of genus 0. By a puncture or a parametrized hole we mean a point p of Σ together with a holomorphic embedding of the standard closed disc $U = \{z \in \mathbb{C}; |z| \leq 1\}$ to Σ centered at the point. Thus a puncture is a holomorphic embedding $u : \tilde{U} \rightarrow \Sigma$, where $\tilde{U} \subset \mathbb{C}$ is an open neighborhood of U and $u(0) = p$. We say that two punctures $u_1 : \tilde{U}_1 \rightarrow \Sigma$ and $u_2 : \tilde{U}_2 \rightarrow \Sigma$ are disjoint, if

$$u_1(\overset{\circ}{U}) \cap u_2(\overset{\circ}{U}) = \emptyset,$$

where $\overset{\circ}{U} := \{z \in \mathbb{C}; |z| < 1\}$ is the interior of U .

Let $\widehat{\mathfrak{M}}_0(n)$ be the moduli space of Riemann spheres Σ with $n+1$ disjoint punctures $u_i : \tilde{U}_i \rightarrow \Sigma$, $0 \leq i \leq n$, modulo the action of complex projective automorphisms. The topology of $\widehat{\mathfrak{M}}_0(n)$ is a very subtle thing and we are not going to discuss this issue here; see [46]. The constructions below will be made only 'up to topology.'

Renumbering the holes u_1, \dots, u_n defines on each $\widehat{\mathfrak{M}}_0(n)$ a natural right Σ_n -action and the Σ -module $\widehat{\mathfrak{M}}_0 = \{\widehat{\mathfrak{M}}_0(n)\}_{n \geq 0}$ forms a topological operad under sewing Riemannian spheres at punctures. Let us describe this operadic structure using the \circ_i -formalism. Thus, let Σ represent an element $x \in \widehat{\mathfrak{M}}_0(m)$ and Δ represent an element $y \in \widehat{\mathfrak{M}}_0(n)$. For $1 \leq i \leq m$, let $u_i : \tilde{U}_i \rightarrow \Sigma$ be the i th puncture of Σ and let $u_0 : \tilde{U}_0 \rightarrow \Delta$ be the 0th puncture of Δ .

There certainly exists some $0 < r < 1$ such that both \tilde{U}_0 and \tilde{U}_i contain the disc $U_{1/r} := \{z \in \mathbb{C}; |z| < 1/r\}$. Let now $\Sigma_r := \Sigma \setminus u_i(U_r)$ and $\Delta_r := \Delta \setminus u_0(U_r)$. Define finally

$$\Xi := (\Sigma_r \bigsqcup \Delta_r) / \sim,$$

where the relation \sim is given by

$$\Sigma_r \ni u_i(\xi) \sim u_0(1/\xi) \in \Delta_r,$$

for $r < |\xi| < 1/r$. It is immediate to see that Ξ is a well-defined punctured Riemannian sphere, with $n + m - 1$ punctures induced in the obvious manner from those of Σ and Δ , and that the class of the punctured surface Ξ in the moduli space $\widehat{\mathfrak{M}}_0(m + n - 1)$ does not depend on the representatives Σ , Δ and on r . We define $x \circ_i y$ to be the class of Ξ .

The unit $e \in \widehat{\mathfrak{M}}_0(1)$ can be defined as follows. Let \mathbb{CP}^1 be the complex projective line with homogeneous coordinates $[z, w]$, $z, w \in \mathbb{C}$, [118, Example I.1.6]. Let $0 := [0, 1] \in \mathbb{CP}^1$ and $\infty := [1, 0] \in \mathbb{CP}^1$. Recall that we have two canonical isomorphisms $p_\infty : \mathbb{CP}^1 \setminus \infty \rightarrow \mathbb{C}$ and $p_0 : \mathbb{CP}^1 \setminus 0 \rightarrow \mathbb{C}$ given by

$$p_\infty([z, w]) := z/w \text{ and } p_0([z, w]) := w/z.$$

Then $p_\infty^{-1} : \mathbb{C} \rightarrow \mathbb{CP}^1$ (respectively $p_0^{-1} : \mathbb{C} \rightarrow \mathbb{CP}^1$) is a puncture at 0 (respectively at ∞). We define $e \in \widehat{\mathfrak{M}}_0(1)$ to be the class of $(\mathbb{CP}^1, p_0^{-1}, p_\infty^{-1})$.

It is not hard to verify that the above constructions make the collection $\widehat{\mathfrak{M}}_0 = \{\widehat{\mathfrak{M}}_0(n)\}_{n \geq 0}$ a Markl's operad. By Proposition 13, $\widehat{\mathfrak{M}}_0$ is also May's operad.

In the rest of this article, we will consider May's and Markl's operads as two versions of the same object which we will call simply a (unital) operad.

2. NON-UNITAL OPERADS

It turns out that the combinatorial structure of the moduli space of stable genus zero curves is captured by a certain non-unital version of operad. Let $\mathcal{M}_{0,n+1}$ be the moduli space of $(n + 1)$ -tuples (x_0, \dots, x_n) of distinct numbered points on the complex projective line \mathbb{CP}^1 modulo projective automorphisms, that is, transformations of the form

$$\mathbb{CP}^1 \ni [\xi_1, \xi_2] \mapsto [a\xi_1 + b\xi_2, c\xi_1 + d\xi_2] \in \mathbb{CP}^1,$$

where $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$.

The moduli space $\mathcal{M}_{0,n+1}$ has, for $n \geq 2$, a canonical compactification $\overline{\mathcal{M}}_0(n) \supset \mathcal{M}_{0,n+1}$ introduced by A. Grothendieck and F.F. Knudsen [16, 50]. The space $\overline{\mathcal{M}}_0(n)$ is the moduli space of stable $(n + 1)$ -pointed curves of genus 0:

Definition 15. *A stable $(n + 1)$ -pointed curve of genus 0 is an object*

$$(C; x_0, \dots, x_n),$$

where C is a (possibly reducible) algebraic curve with at most nodal singularities and $x_0, \dots, x_n \in C$ are distinct smooth points such that

- (i) each component of C is isomorphic to \mathbb{CP}^1 ,

- (ii) *the graph of intersections of components of C (i.e. the graph whose vertices correspond to the components of C and edges to the intersection points of the components) is a tree, and*
- (iii) *each component of C has at least three special points, where a special point means either one of the x_i , $0 \leq i \leq n$, or a singular point of C (the stability).*

It can be easily seen that a stable curve $(C; x_0, \dots, x_n)$ admits no infinitesimal automorphisms that fix marked points x_0, \dots, x_n , therefore $(C; x_0, \dots, x_n)$ is ‘stable’ in the usual sense. Observe also that $\overline{\mathcal{M}}_0(0) = \overline{\mathcal{M}}_0(1) = \emptyset$ (there are no stable curves with less than three marked points) and that $\overline{\mathcal{M}}_0(2)$ = the point corresponding to the three-pointed stable curve $(\mathbb{C}\mathbb{P}^1; \infty, 1, 0)$. The space $\mathcal{M}_{0,n+1}$ forms an open dense part of $\overline{\mathcal{M}}_0(n)$ consisting of marked curves $(C; x_0, \dots, x_n)$ such that C is isomorphic to $\mathbb{C}\mathbb{P}^1$.

Let us try to equip the collection $\overline{\mathcal{M}}_0 = \{\overline{\mathcal{M}}_0(n)\}_{n \geq 2}$ with an operad structure as in Definition 1. For $C = (C, x_1, \dots, x_n) \in \overline{\mathcal{M}}_0(n)$ and $C_i = (C_i, y_1^i, \dots, y_{k_i}^i) \in \overline{\mathcal{M}}_0(k_i)$, $1 \leq i \leq n$, let

$$(5) \quad \gamma(C, C_1, \dots, C_n) \in \overline{\mathcal{M}}_0(k_1 + \dots + k_n)$$

be the stable marked curve obtained from the disjoint union $C \sqcup C^1 \sqcup \dots \sqcup C^n$ by identifying, for each $1 \leq i \leq n$, the point $x_i \in C$ with the point $y_0^i \in C_i$, introducing a nodal singularity, and relabeling the remaining marked points accordingly. The symmetric group acts on $\overline{\mathcal{M}}_0(n)$ by

$$(C, x_0, x_1, \dots, x_n) \longmapsto (C, x_0, x_{\sigma(1)}, \dots, x_{\sigma(n)}), \quad \sigma \in \Sigma_n.$$

We have defined the γ -compositions and the symmetric group action, but there is no room for the identity, because $\overline{\mathcal{M}}_0(1)$ is empty! The above structure is, therefore, a non-unital operad in the sense of the following definition (which is formulated, as all definitions in this article, for the monoidal category of \mathbf{k} -modules).

Definition 16. *A May’s non-unital operad in the category of \mathbf{k} -modules is a collection $\mathcal{P} = \{\mathcal{P}(n)\}_{n \geq 0}$ of $\mathbf{k}[\Sigma_n]$ -modules, together with operadic compositions*

$$\gamma : \mathcal{P}(n) \otimes \mathcal{P}(k_1) \otimes \dots \otimes \mathcal{P}(k_n) \rightarrow \mathcal{P}(k_1 + \dots + k_n),$$

for $n \geq 1$ and $k_1, \dots, k_n \geq 0$, that fulfill the associativity and equivariance axioms of Definition 1.

We may as well define on the collection $\overline{\mathcal{M}}_0 = \{\overline{\mathcal{M}}_0(n)\}_{n \geq 2}$ operations

$$(6) \quad \circ_i : \overline{\mathcal{M}}_0(m) \times \overline{\mathcal{M}}_0(n) \rightarrow \overline{\mathcal{M}}_0(m + n - 1)$$

for $m, n \geq 2$, $1 \leq i \leq m$, by

$$(C^1; y_0, \dots, y_m) \times (C^2; x_0, \dots, x_n) \longmapsto (C; y_0, \dots, y_{i-1}, x_0, \dots, x_n, y_{i+1}, \dots, y_m)$$

where C is the quotient of the disjoint union $C^1 \sqcup C^2$ given by identifying x_0 with y_i at a nodal singularity, see Figure 2. The collection $\overline{\mathcal{M}}_0 = \{\overline{\mathcal{M}}_0(n)\}_{n \geq 2}$ with \circ_i -operations (6) is an example of another version of non-unital operads, recalled in:

Definition 17. *A non-unital Markl’s operad in the category of \mathbf{k} -modules is a collection $\mathcal{P} = \{\mathcal{P}(n)\}_{n \geq 0}$ of $\mathbf{k}[\Sigma_n]$ -modules, together with operadic compositions*

$$\circ_i : \mathcal{S}(m) \otimes \mathcal{S}(n) \rightarrow \mathcal{S}(m + n - 1),$$

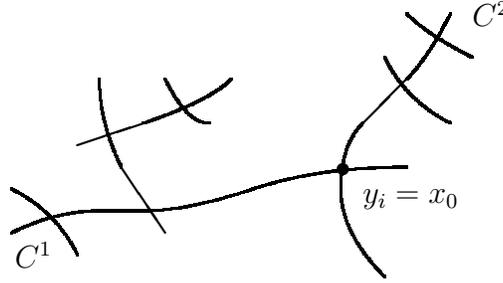


FIGURE 2. The \circ_i -compositions in $\overline{\mathcal{M}}_0 = \{\overline{\mathcal{M}}_0(n)\}_{n \geq 2}$.

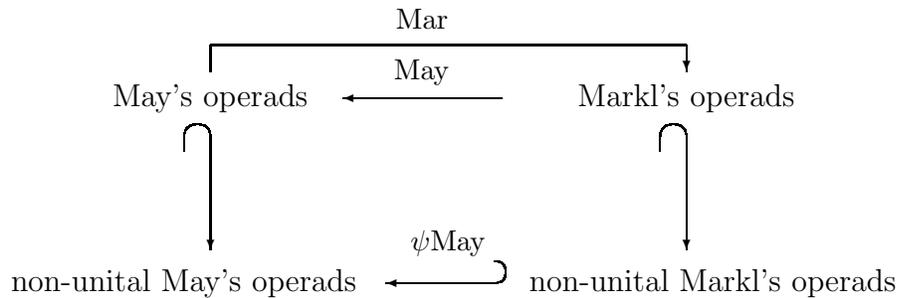
for $1 \leq i \leq m$ and $n \geq 0$, that fulfill the associativity and equivariance axioms of Definition 11.

As we saw in Proposition 13, in the presence of operadic units, May’s operads are the same as Markl’s operads. Surprisingly, the non-unital versions of these structures are *radically different* – Markl’s operads capture more information than May’s operads! This is made precise in the following:

Proposition 18. *The category of non-unital Markl’s operads is a subcategory of the category of non-unital May’s operads.*

Proof. It is easy to see that (3) defines, as in the proof of Proposition 13, a functor $\psi\text{May}(-)$ which is an embedding of the category of non-unital Markl’s operads into the category of non-unital May’s operads. \square

Observe that formula (4), inverse to (3), does not make sense without units. The relation between various versions of operads discussed so far is summarized in the following diagram of categories and their inclusions:



The following example shows that non-unital Markl’s operads form a proper sub-category of the category of non-unital May’s operads.

Example 19. We describe a non-unital May’s operad $\mathcal{V} = \{\mathcal{V}(n)\}_{n \geq 0}$ which is not of the form $\psi\text{May}(\mathcal{S})$ for some non-unital Markl’s operad \mathcal{S} . Let

$$\mathcal{V}(n) := \begin{cases} \mathbf{k}, & \text{for } n = 2 \text{ or } 4, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

The only non-trivial γ -composition is $\gamma : \mathcal{V}(2) \otimes \mathcal{V}(2) \otimes \mathcal{V}(2) \rightarrow \mathcal{V}(4)$, given as the canonical isomorphism

$$\mathcal{V}(2) \otimes \mathcal{V}(2) \otimes \mathcal{V}(2) \cong \mathbf{k}^{\otimes 3} \xrightarrow{\cong} \mathbf{k} \cong \mathcal{V}(4).$$

Suppose that $\mathcal{V} = \text{May}(\mathcal{S})$ for some non-unital Markl's operad \mathcal{S} . Then, according to (3), for $f, g_1, g_2 \in \mathcal{V}(2)$,

$$\gamma(f, g_1, g_2) = (f \circ_2 g_2) \circ_1 g_1.$$

Since $(f \circ_2 g) \in \mathcal{V}(3) = 0$, this would imply that γ is trivial, which is not true.

Proposition 21 below shows that Markl's rather than May's non-unital operads are true non-unital versions of operads. We will need the following definition in which $\mathcal{K} = \{\mathcal{K}(n)\}_{n \geq 1}$ is the trivial (unital) operad with $\mathcal{K}(1) := \mathbf{k}$ and $\mathcal{K}(n) = 0$, for $n \neq 1$.

Definition 20. *An augmentation of an operad \mathcal{P} in the category of \mathbf{k} -modules is a homomorphism $\epsilon : \mathcal{P} \rightarrow \mathcal{K}$. Operads with an augmentation are called augmented operads. The kernel*

$$\overline{\mathcal{P}} := \text{Ker}(\epsilon : \mathcal{P} \rightarrow \mathcal{K})$$

is called the augmentation ideal.

The following proposition was proved in [75].

Proposition 21. *The correspondence $\mathcal{P} \mapsto \overline{\mathcal{P}}$ is an isomorphism between the category of augmented operads and the category of Markl's non-unital operads.*

Proof. The \circ_i -operations of \mathcal{P} obviously restrict to $\overline{\mathcal{P}}$, making it a non-unital Markl's operad. It is simple to describe a functorial inverse $\mathcal{S} \mapsto \widetilde{\mathcal{S}}$ of the correspondence $\mathcal{P} \mapsto \overline{\mathcal{P}}$. For a Markl's non-unital operad \mathcal{S} , denote by $\widetilde{\mathcal{S}}$ the collection

$$(7) \quad \widetilde{\mathcal{S}}(n) := \begin{cases} \mathcal{S}(n), & \text{for } n \neq 1, \text{ and} \\ \mathcal{S}(1) \oplus \mathbf{k}, & \text{for } n = 1. \end{cases}$$

The \circ_i -operations of $\widetilde{\mathcal{S}}$ are uniquely determined by requiring that they extend the \circ_i -operations of \mathcal{S} and satisfy (2), with the unit $e := 0 \oplus 1_{\mathbf{k}} \in \mathcal{S}(1) \oplus \mathbf{k} = \widetilde{\mathcal{S}}(1)$. Informally, $\widetilde{\mathcal{S}}$ is obtained from the Markl's non-unital operad \mathcal{S} by adjoining a unit. \square

Observe that if \mathcal{S} were a May's, not Markl's, non-unital operad, the construction of $\widetilde{\mathcal{S}}$ described in the above proof would not make sense, because we would not know how to define

$$\gamma(f, \underbrace{e, \dots, e}_{i-1}, g, \underbrace{e, \dots, e}_{m-i})$$

for $f \in \mathcal{S}(m)$, $g \in \mathcal{S}(n)$, $m \geq 2$, $n \geq 0$, $1 \leq i \leq m$. Proposition 21 should be compared to the obvious statement that the category of augmented unital associative algebras is isomorphic to the category of (non-unital) associative algebras. In the following proposition, \mathbf{Oper} denotes the category of \mathbf{k} -linear operads and $\psi\mathbf{Oper}$ the category of \mathbf{k} -linear Markl's non-unital operads.

Proposition 22. *Let \mathcal{P} be an augmented operad and \mathcal{Q} an arbitrary operad in the category of \mathbf{k} -modules. Then there exists a natural isomorphism*

$$(8) \quad \text{Mor}_{\mathbf{Oper}}(\mathcal{P}, \mathcal{Q}) \cong \text{Mor}_{\psi\mathbf{Oper}}(\overline{\mathcal{P}}, \psi\text{May}(\mathcal{Q})).$$

The proof is simple and we leave it to the reader. Combining (8) with the isomorphism of Proposition 21 one obtains a natural isomorphism

$$(9) \quad \text{Mor}_{\mathbf{Oper}}(\tilde{\mathcal{S}}, \mathcal{Q}) \cong \text{Mor}_{\psi\mathbf{Oper}}(\mathcal{S}, \psi\text{May}(\mathcal{Q}))$$

which holds for each Markl's non-unital operad \mathcal{S} and operad \mathcal{Q} . Isomorphism (9) means that $\tilde{\cdot} : \psi\mathbf{Oper} \rightarrow \mathbf{Oper}$ and $\psi\text{May} : \mathbf{Oper} \rightarrow \psi\mathbf{Oper}$ are adjoint functors. This adjunction will be used in the construction of free operads in Section 4.

In the rest of this article, non-unital Markl's operads will be called simply non-unital operads. This will not lead to confusion, since all non-unital operads referred to in the rest of this article will be Markl's.

3. OPERAD ALGEBRAS

As we already remarked, operads are important through their representations called operad algebras or simply algebras.

Definition 23. *Let V be a \mathbf{k} -module and $\mathcal{E}nd_V$ the endomorphism operad of V recalled in Example 2. A \mathcal{P} -algebra is a homomorphism of operads $\rho : \mathcal{P} \rightarrow \mathcal{E}nd_V$.*

The above definition admits an obvious generalization into an arbitrary symmetric monoidal category with an internal hom-functor. The last assumption is necessary for the existence of the 'internal' endomorphism operad, see [83, Definition II.1.20]. Definition 23 can be however unwrapped into the form given in [58, Definition 2.1] that makes sense in an arbitrary symmetric monoidal category without the internal hom-functor assumption:

Proposition 24. *Let \mathcal{P} be an operad. A \mathcal{P} -algebra is the same as a \mathbf{k} -module V together with maps*

$$(10) \quad \alpha : \mathcal{P}(n) \otimes V^{\otimes n} \rightarrow V, \quad n \geq 0,$$

that satisfy the following axioms.

Associativity. For each $n \geq 1$ and non-negative integers k_1, \dots, k_n , the following diagram commutes.

$$\begin{array}{ccc} \left(\mathcal{P}(n) \otimes \bigotimes_{s=1}^n \mathcal{P}(k_s) \right) \otimes \bigotimes_{s=1}^n V^{\otimes k_s} & \xrightarrow{\gamma \otimes id} & \mathcal{P}(k_1 + \dots + k_n) \otimes V^{\otimes(k_1 + \dots + k_n)} \\ \downarrow \text{shuffle} & & \downarrow \alpha \\ \mathcal{P}(n) \otimes \bigotimes_{s=1}^n \left(\mathcal{P}(k_s) \otimes V^{\otimes k_s} \right) & \xrightarrow{id \otimes \left(\bigotimes_{s=1}^n \alpha \right)} & \mathcal{P}(n) \otimes V^{\otimes n} \\ & & \uparrow \alpha \\ & & V \end{array}$$

Equivariance. For each $n \geq 1$ and $\sigma \in \Sigma_n$, the following diagram commutes.

$$\begin{array}{ccc}
\mathcal{P}(n) \otimes V^{\otimes n} & \xrightarrow{\sigma \otimes \sigma^{-1}} & \mathcal{P}(n) \otimes V^{\otimes n} \\
\searrow \alpha & & \swarrow \alpha \\
& V &
\end{array}$$

Unitality. For each $n \geq 1$, the following diagram commutes.

$$\begin{array}{ccc}
\mathbf{k} \otimes V & \xrightarrow{\cong} & V \\
\eta \otimes id \downarrow & \nearrow \alpha & \\
\mathcal{P}(1) \otimes V & &
\end{array}$$

We leave as an exercise to formulate a version of Proposition 24 that would use \circ_i -operations instead of γ -operations.

Example 25. In this example we verify, using Proposition 24, that algebras over the operad $\mathcal{Com} = \{\mathcal{Com}(n)\}_{n \geq 1}$ recalled in Example 10 are ordinary commutative associative algebras. To simplify the exposition, let us agree that v 's with various subscripts denote elements of V . Since $\mathcal{Com}(n) = \mathbf{k}$ for $n \geq 1$, the structure map (10) determines, for each $n \geq 1$, a linear map $\mu_n : V^{\otimes n} \rightarrow V$ by

$$\mu_n(v_1, \dots, v_n) := \alpha(1_n, v_1, \dots, v_n),$$

where 1_n denotes in this example the unit $1_n \in \mathbf{k} = \mathcal{Com}(n)$. The associativity of Proposition 24 says that

$$(11) \quad \mu_n(\mu_{k_1}(v_1, \dots, v_{k_1}), \dots, \mu_{k_n}(v_{k_1+\dots+k_{n-1}+1}, \dots, v_{k_1+\dots+k_n})) = \mu_{k_1+\dots+k_n}(v_1, \dots, v_{k_1+\dots+k_n}),$$

for each $n, k_1, \dots, k_n \geq 1$. The equivariance of Proposition 24 means that each μ_n is fully symmetric

$$(12) \quad \mu_n(v_1, \dots, v_n) = \mu_n(v_{\sigma(1)}, \dots, v_{\sigma(n)}), \quad \sigma \in \Sigma_n,$$

and the unitality implies that μ_1 is the identity map,

$$(13) \quad \mu_1(v) = v.$$

The above structure can be identified with a commutative associative multiplication on V . Indeed, the bilinear map $\cdot := \mu_2 : V \otimes V \rightarrow V$ is clearly associative:

$$(14) \quad (v_1 \cdot v_2) \cdot v_3 = v_1 \cdot (v_2 \cdot v_3)$$

and commutative:

$$(15) \quad v_1 \cdot v_2 = v_2 \cdot v_1.$$

On the other hand, $\mu_1(v) := v$ and

$$\mu_n(v_1, \dots, v_n) := (\dots(v_1 \cdot v_2) \dots v_{n-1}) \cdot v_n \quad \text{for } n \geq 2$$

defines multilinear maps $\{\mu_n : V^{\otimes n} \rightarrow V\}$ satisfying (11)–(13). It is equally easy to verify that algebras over the operad \mathcal{Ass} introduced in Example 3 are ordinary associative algebras.

Following Leinster [61], one could say that (11)–(13) is an *unbiased* definition of associative commutative algebras, while (14)–(15) is a definition of the same object *biased* towards bilinear operations. Operads therefore provide unbiased definitions of algebras.

Example 26. Let us denote by $UCom$ the endomorphism operad $\mathcal{E}nd_{\mathbf{k}}$ of the ground ring \mathbf{k} . It is easy to verify that $UCom$ -algebras are *unital* commutative associative algebras. We leave it to the reader to describe the operad $UAss$ governing unital associative operads.

Algebras over a non- Σ operad $\underline{\mathcal{P}}$ are defined as algebras, in the sense of Definition 23, over the symmetrization $\Sigma[\underline{\mathcal{P}}]$ of $\underline{\mathcal{P}}$. Algebras over non-unital operads discussed in Section 2 are defined by appropriate obvious modifications of Definition 23.

Example 27. Let Y be a topological space with a base point $*$ and \mathbb{S}^k the k -dimensional sphere, $k \geq 1$. The k -fold loop space $\Omega^k Y$ is the space of all continuous maps $\mathbb{S}^k \rightarrow Y$ that send the south pole of \mathbb{S}^k to the base point of Y . Equivalently, $\Omega^k Y$ is the space of all continuous maps $\lambda : (\mathbb{D}^k, \mathbb{S}^{k-1}) \rightarrow (Y, *)$ from the standard closed k -dimensional disc \mathbb{D}^k to Y that map the boundary \mathbb{S}^{k-1} of \mathbb{D}^k to the base point of Y . Let us show, following Boardman and Vogt [10], that $\Omega^k Y$ is a natural topological algebra over the little k -discs operad $\mathcal{D}_k = \{\mathcal{D}_k(n)\}_{n \geq 0}$ recalled in Example 4.

The action $\alpha : \mathcal{D}_k(n) \times (\Omega^k Y)^{\times n} \rightarrow \Omega^k Y$ is, for $n \geq 0$, defined as follows. Given $\lambda_i : (\mathbb{D}^k, \mathbb{S}^{k-1}) \rightarrow (Y, *) \in \Omega^k Y$, $1 \leq i \leq n$, and little k -discs $d = (d_1, \dots, d_n) \in \mathcal{D}_k(n)$ as in Example 4, then

$$\alpha(d, \lambda_1, \dots, \lambda_n) : (\mathbb{D}^k, \mathbb{S}^{k-1}) \rightarrow (Y, *) \in \Omega^k Y$$

is the map defined to be $\lambda_i : \mathbb{D}^k \rightarrow Y$ (suitably rescaled) on the image of d_i , and to be $*$ on the complement of the images of the maps d_i , $1 \leq i \leq n$.

Therefore each k -fold loop space is a \mathcal{D}_k -space. The following classical theorem is a certain form of the inverse statement.

Theorem 28. (Boardman-Vogt [10], May [86]) *A path-connected \mathcal{D}_k -algebra X has the weak homotopy type of a k -fold loop space.*

The connectedness assumption in the above theorem can be weakened by assuming that the \mathcal{D}_k -action makes the set $\pi_0(X)$ of path components of X a group.

Example 29. The non-unital operad $\overline{\mathcal{M}}_0$ of stable pointed curves of genus 0 (also called the *configuration (non-unital) operad*) recalled on page 12 is a non-unital operad in the category of smooth complex projective varieties. It therefore makes sense, as explained in Example 5, to consider its homology operad $H_*(\overline{\mathcal{M}}_0, \mathbf{k}) = \{H_*(\overline{\mathcal{M}}_0(n), \mathbf{k})\}_{n \geq 2}$.

An algebra over this non-unital operad is called a (tree level) *cohomological conformal field theory* or a *hyper-commutative algebra* [55]. It consist of a family $\{(-, \dots, -) : V^{\otimes n} \rightarrow V\}_{n \geq 2}$ of linear operations which are totally symmetric, that is

$$(v_{\sigma(1)}, \dots, v_{\sigma(n)}) = (v_1, \dots, v_n),$$

for each permutation $\sigma \in \Sigma_n$. Moreover, we require the following form of associativity:

$$(16) \quad \sum_{(S,T)} ((u, v, x_i; i \in S), w, x_j; j \in T) = \sum_{(S,T)} (u, (v, w, x_i; i \in S), x_j; j \in T),$$

where $u, v, w, x_1, \dots, x_n \in V$ and (S, T) runs over disjoint decompositions $S \sqcup T = \{1, \dots, n\}$. For $n = 0$, (16) means the (usual) associativity of the bilinear operation $(-, -)$, i.e. $((u, v), w) = (u, (v, w))$. For $n = 1$ we get

$$((u, v), w, x) + ((u, v, x), w) = (u, (v, w, x)) + (u, (v, w), x).$$

Example 30. In this example, \mathbf{k} is a field of characteristic 0. The non-unital operad $\overline{\mathcal{M}}_0(\mathbb{R}) = \{\overline{\mathcal{M}}_0(\mathbb{R})(n)\}_{n \geq 2}$ of real points in the configuration operad $\overline{\mathcal{M}}_0$ is called the *mosaic non-unital operad* [19]. Algebras over the homology $H_*(\overline{\mathcal{M}}_0(\mathbb{R}), \mathbf{k}) = \{H_*(\overline{\mathcal{M}}_0(\mathbb{R})(n), \mathbf{k})\}_{n \geq 2}$ of this operad were recently identified [25] with *2-Gerstenhaber algebras*, which are structures (V, μ, τ) consisting of a commutative associative product $\mu : V \otimes V \rightarrow V$ and an anti-symmetric degree +1 ternary operation $\tau : V \otimes V \otimes V \rightarrow V$ which satisfies the generalized Jacobi identity

$$\sum_{\sigma} \text{sgn}(\sigma) \cdot \tau(\tau(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}), x_{\sigma(4)}, x_{\sigma(5)}) = 0,$$

where the summation runs over all (3, 2)-unshuffles $\sigma(1) < \sigma(2) < \sigma(3)$, $\sigma(4) < \sigma(5)$. Moreover, the ternary operation τ is tied to the multiplication μ by the distributive law

$$\tau(\mu(s, t), u, v) = \mu(\tau(s, u, v), t) + (-1)^{(1+|u|+|v|)|s|} \cdot \mu(s, \tau(t, u, v)), \quad s, t, u, v \in V,$$

saying that the assignment $s \mapsto \tau(s, u, v)$ is a degree $(1 + |u| + |v|)$ -derivation of the associative commutative algebra (V, μ) , for each $u, v \in V$.

4. FREE OPERADS AND TREES

The purpose of this section is three-fold. First, we want to study free operads because each operad is a quotient of a free one. The second reason why we are interested in free operads is that their construction involves trees. Indeed, it turns out that rooted trees provide ‘pasting schemes’ for operads and that, replacing trees by other types of graphs, one can introduce several important generalizations of operads, such as cyclic operads, modular operads, and PROPs. The last reason is that the free operad functor defines a monad which provides an unbiased definition of operads as algebras over this monad. Everything in this section is written for \mathbf{k} -linear operads, but the constructions can be generalized into an arbitrary symmetric monoidal category with countable coproducts $(\mathcal{M}, \odot, \mathbf{1})$ whose monoidal product \odot is distributive over coproducts, see [83, Section II.1.9].

Recall that a Σ -module is a collection $E = \{E(n)\}_{n \geq 0}$ in which each $E(n)$ is a right $\mathbf{k}[\Sigma_n]$ -module. There is an obvious forgetful functor $\square : \mathbf{Oper} \rightarrow \Sigma\text{-mod}$ from the category \mathbf{Oper} of \mathbf{k} -linear operads to the category $\Sigma\text{-mod}$ of Σ -modules.

Definition 31. *The free operad functor is a left adjoint [38, § II.7] $\Gamma : \Sigma\text{-mod} \rightarrow \mathbf{Oper}$ to the forgetful functor $\square : \mathbf{Oper} \rightarrow \Sigma\text{-mod}$. This means that there exists a functorial isomorphism*

$$\text{Mor}_{\mathbf{Oper}}(\Gamma(E), \mathcal{P}) \cong \text{Mor}_{\Sigma\text{-mod}}(E, \square(\mathcal{P}))$$

for an arbitrary Σ -module E and operad \mathcal{P} . The operad $\Gamma(E)$ is the free operad generated by the Σ -module E . Similarly, the free non-unital operad functor is a left adjoint $\Psi : \Sigma\text{-mod} \rightarrow \psi\text{Oper}$ of the obvious forgetful functor $\square_\psi : \psi\text{Oper} \rightarrow \Sigma\text{-mod}$, that is

$$\text{Mor}_{\psi\text{Oper}}(\Psi(E), \mathcal{S}) \cong \text{Mor}_{\Sigma\text{-mod}}(E, \square_\psi(\mathcal{S})),$$

where E is a Σ -module and \mathcal{S} a non-unital operad. The non-unital operad $\Psi(E)$ is the free non-unital operad generated by the Σ -module E .

Let $\tilde{} : \psi\text{Oper} \rightarrow \text{Oper}$ be the functor of ‘adjoining the unit’ considered in the proof of Proposition 21 on page 14. Functorial isomorphism (9) implies that one may take

$$(17) \quad \Gamma := \tilde{\Psi},$$

which means that the free operad $\Gamma(E)$ can be obtained from the free non-unital operad $\Psi(E)$ by formally adjoining the unit.

Let us indicate how to construct the free non-unital operad $\Psi(E)$, a precise description will be given later in this section. The free non-unital operad $\Psi(E)$ must be built up from all formal \circ_i -compositions of elements of E modulo the axioms listed in Definition 11. For instance, given $f \in E(2)$, $g \in E(3)$, $h \in E(2)$ and $l \in E(0)$, the component $\Psi(E)(5)$ must contain the following five compositions

$$(18) \quad \begin{aligned} & (f \circ_1 (g \circ_2 l)) \circ_3 h, \quad (f \circ_2 h) \circ_1 (g \circ_2 l), \quad ((f \circ_2 h) \circ_1 g) \circ_2 l, \\ & ((f \circ_1 g) \circ_2 l) \circ_3 h \quad \text{and} \quad ((f \circ_1 g) \circ_4 h) \circ_2 l. \end{aligned}$$

The elements in (18) can be depicted by the ‘flow diagrams’ of Figure 3 on page 20. Nodes of these diagrams are decorated by elements f, g, h and l of E in such a way that an element of $E(n)$ decorates a node with n input lines, $n \geq 0$. Thin ‘amoebas’ indicate the nesting which specifies the order in which the \circ_i -operations are performed. The associativity of Definition 11 however says that the result of the composition does not depend on the order, therefore the amoebas can be erased and the common value of the compositions represented by



Let us look more closely how diagram (19) determines an element of the (still hypothetical) free non-unital operad $\Psi(E)$. The crucial fact is that the underlying graph of (19) is a planar rooted tree. Recall that a *tree* is a finite connected simply connected graph without loops and multiple edges. For a tree T we denote, as usual, by $\text{Vert}(T)$ the set of vertices and $\text{Edg}(T)$ the set of edges of T . The number of edges adjacent to a vertex $v \in \text{Vert}(T)$ is called the *valence* of v and denoted $\text{val}(v)$. We assume that one is given a subset

$$\text{ext}(T) \subset \{v \in \text{Vert}(T); \text{val}(v) = 1\}$$

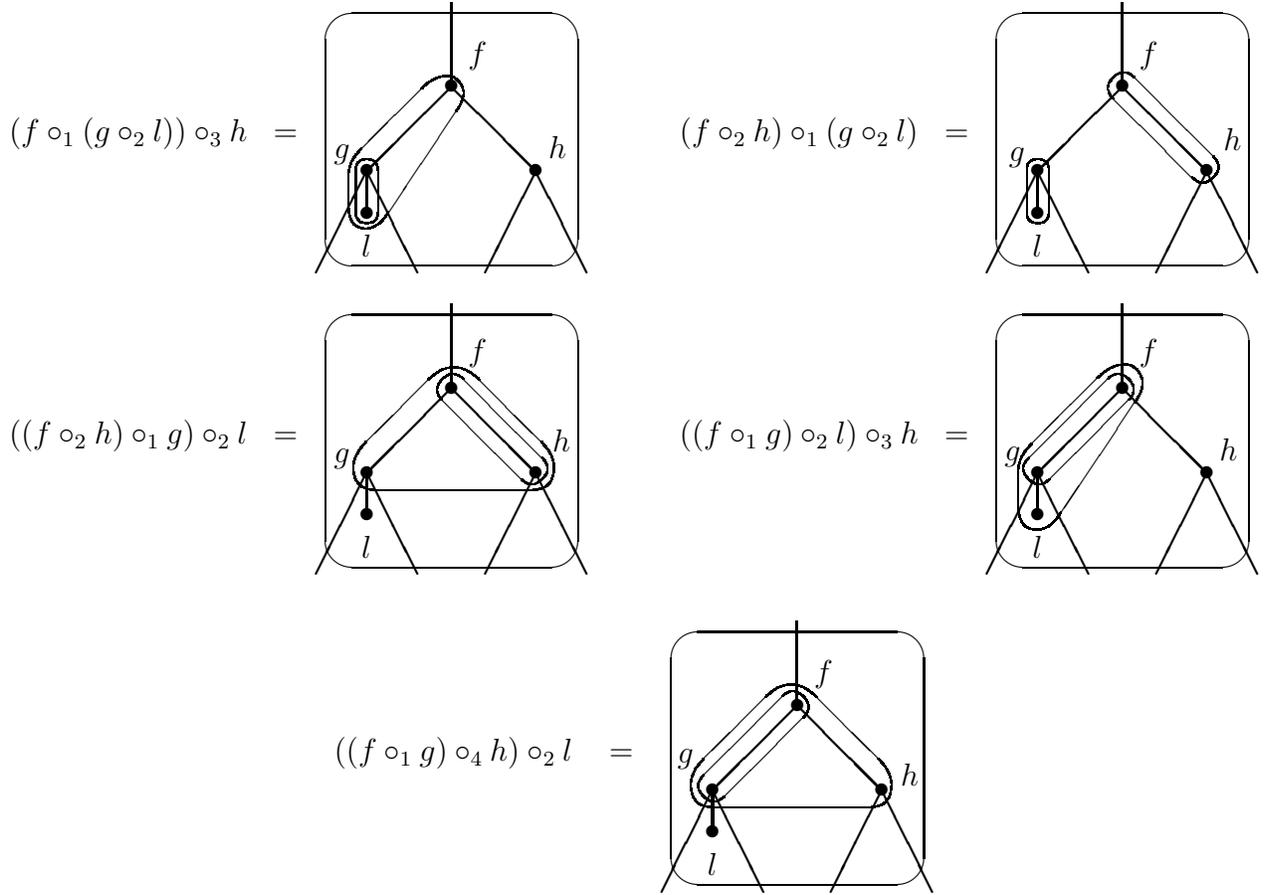


FIGURE 3. Flow diagrams in non-unital operads.

of *external* vertices, the remaining vertices are *internal*. Let us denote

$$\text{vert}(T) := \text{Vert}(T) \setminus \text{ext}(T)$$

the set of all internal vertices. Henceforth, we will assume that our trees have at least one internal vertex. This excludes at this stage the *exceptional tree* consisting of two external vertices connected by an edge.

Edges adjacent to external vertices are the *legs* of T . A tree is *rooted* if one of its legs, called the *root*, is marked and all other edges are oriented, pointing to the root. The legs different from the root are the *leaves* of T . For example, the tree in (19) has 4 internal vertices decorated f , g , h and l , and 4 leaves. Finally, the *planarity* means that an embeddings of T into the plane is specified. In our pictures, the root will always be placed on the top. By a vertex we will always mean an internal one.

The planarity and a choice of the root of the underlying tree of (19) specifies a total order of the set $\text{in}(v)$ of input edges of each vertex $v \in \text{vert}(T)$ as well as a total order of the set $\text{Leaf}(T)$

of the leaves of T , by numbering from the left to the right:

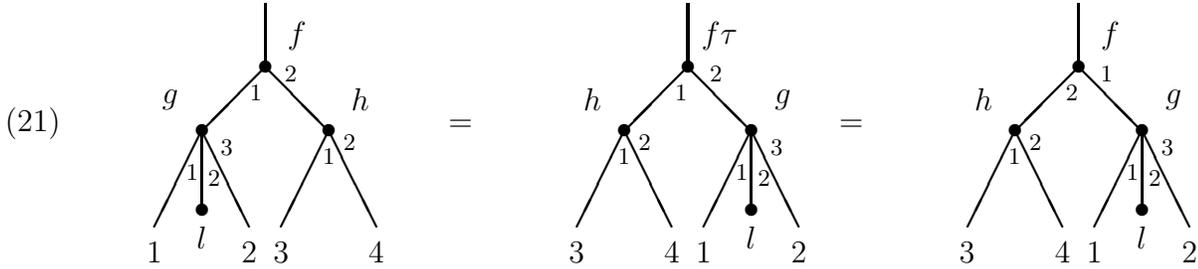


This tells us that l should be inserted into the second input of g , g into the first input of f and h into the second input of f . Using ‘abstract variables’ v_1, v_2, v_3 and v_4 , the element represented by (20) can also be written as the ‘composition’ $f(g(v_1, l, v_2), h(v_3, v_4))$.

Now we need to take into account also the symmetric group action. If τ is the generator of Σ_2 , then the obvious equality

$$f(g(v_1, l, v_2), h(v_3, v_4)) = f\tau(h(v_3, v_4), g(v_1, l, v_2))$$

of ‘abstract compositions’ coming from the equivariance of Definition 11 translates into the following equality of flow diagrams:



Relation (21) shows that the equivariance of Definition 11 violates the linear orders induced by the planar embedding of T . This leads us to the conclusion that the flow diagrams describing elements of free non-unital operads are (abstract, non-planar) rooted, leaf-labeled decorated trees.

Let us describe, after these motivations, a precise construction of $\Psi(E)$. The first subtlety one needs to understand is how to decorate vertices of non-planar trees. To this end, we need to explain how each Σ -module $E = \{E(n)\}_{n \geq 0}$ naturally extends into a functor (denoted again E) from the category \mathbf{Set}_f of finite sets and their bijections to the category of \mathbf{k} -modules. If X and Y are finite sets, denote by

(22)

$$Bij(Y, X) := \{\vartheta : X \xrightarrow{\cong} Y\}$$

the set of all isomorphisms between X and Y (notice the unexpected direction of the arrow!). It is clear that $Bij(Y, X)$ is a natural left Aut_Y - right Aut_X -bimodule, where $Aut_X := Bij(X, X)$ and $Aut_Y := Bij(Y, Y)$ are the sets of automorphisms with group structure given by composition. For a finite set $S \in \mathbf{Set}_f$ of cardinality n and a Σ -module $E = \{E(n)\}_{n \geq 0}$ define $E(S)$ to be

(23)

$$E(S) := E(n) \times_{\Sigma_n} Bij([n], S)$$

where, as usual, $[n] := \{1, \dots, n\}$ and, of course, $\Sigma_n = Aut_{[n]}$.

Let us recall that a *(leaf-) labeled rooted n -tree* is a rooted tree T together with a specified bijection $\ell : \text{Leaf}(T) \xrightarrow{\sim} [n]$. Let \mathbf{Tree}_n be the category of labeled rooted n -trees and their bijections. For $T \in \mathbf{Tree}_n$ define

$$(24) \quad E(T) := \bigotimes_{v \in \text{vert}(T)} E(\text{in}(v))$$

where $\text{in}(v)$ is, as before, the set of all input edges of a vertex $v \in \text{vert}(T)$. It is easy to verify that $E \mapsto E(T)$ defines a functor from the category \mathbf{Tree}_n to the category of \mathbf{k} -modules.

Recall that the colimit of a covariant functor $F : \mathcal{D} \rightarrow \mathbf{Mod}_{\mathbf{k}}$ is the quotient

$$\text{colim}_{x \in \mathcal{D}} F(x) = \bigoplus_{x \in \mathcal{D}} F(x) / \sim,$$

where \sim is the equivalence generated by

$$F(y) \ni a \sim F(f)(a) \in F(z),$$

for each $a \in F(y)$, $y, z \in \mathcal{D}$ and $f \in \text{Mor}_{\mathcal{D}}(y, z)$. Define finally

$$(25) \quad \Psi(E)(n) := \text{colim}_{T \in \mathbf{Tree}_n} E(T), \quad n \geq 0.$$

The following theorem was proved in [83, II.1.9].

Theorem 32. *There exists a natural non-unital operad structure on the Σ -module*

$$\Psi(E) = \{\Psi(E)(n)\}_{n \geq 0},$$

with the \circ_i -operations given by the grafting of trees and the symmetric group re-labeling the leaves, such that $\Psi(E)$ is the free non-unital operad generated by the Σ -module E .

One could simplify (25) by introducing $\mathcal{T}ree(n)$ as the set of *isomorphism* classes of n -trees from \mathbf{Tree}_n and defining $\Psi(E)$ by the formula

$$(26) \quad \Psi(E)(n) = \bigoplus_{[T] \in \mathcal{T}ree(n)} E(T), \quad n \geq 0,$$

which does not involve the colimit. The drawback of (26) is that it assumes a choice of a representative $[T]$ of each isomorphism class in $\mathcal{T}ree(n)$, while (25) is functorial and admits simple generalizations to other types of operads and PROPs. See [83, Section II.1.9] for other representations of the free non-unital operad functor.

Having constructed the free non-unital operad $\Psi(E)$, we may use (17) to define the free operad $\Gamma(E)$. This is obviously equivalent to enlarging, in (25) for $n = 1$, the category \mathbf{Tree}_n by the *exceptional rooted tree* $\begin{array}{c} | \\ \hline \end{array}$ with one leg and no internal vertex. If we denote this enlarged category of trees and their isomorphisms (which however differs from \mathbf{Tree}_n only at $n = 1$) by \mathbf{UTree}_n , we may represent the free operad as

$$(27) \quad \Gamma(E)(n) := \text{colim}_{T \in \mathbf{UTree}_n} E(T), \quad n \geq 0.$$

If E is a Σ -module such that $E(0) = E(1) = 0$, then (26) reduces to a summation over *reduced trees*, that is trees whose all vertices have at least two input edges. By simple combinatorics, the

number of isomorphism classes of reduced trees in \mathbf{Tree}_n is finite for each $n \geq 0$. This implies the following proposition that says that operads are relatively small objects.

Proposition 33. *Let $E = \{E(n)\}_{n \geq 0}$ be a Σ -module such that*

$$E(0) = E(1) = 0$$

and that $E(n)$ are finite-dimensional for $n \geq 2$. Then the spaces $\Psi(E)(n)$ and $\Gamma(E)(n)$ are finite-dimensional for each $n \geq 0$.

We close this section by showing how the free operad functor can be used to define operads. It follows from general principles that any operad \mathcal{P} is a quotient $\mathcal{P} = \Gamma(E)/(R)$, where E and R are Σ -modules and (R) is the operadic ideal (see Definition 6) generated by R in $\Gamma(E)$.

Example 34. The commutative associative operad $\mathcal{C}om$ recalled in Example 10 is generated by the Σ -module

$$E_{\mathcal{C}om}(n) := \begin{cases} \mathbf{k} \cdot \mu, & \text{if } n = 2 \text{ and} \\ 0, & \text{if } n \neq 2. \end{cases}$$

where $\mathbf{k} \cdot \mu$ is the trivial representation of Σ_2 . The ideal of relations is generated by

$$R_{\mathcal{C}om} := \text{Span}_{\mathbf{k}}\{\mu(\mu \otimes id) - \mu(id \otimes \mu)\} \subset \Gamma(E_{\mathcal{C}om})(3),$$

where $\mu(\mu \otimes id) - \mu(id \otimes \mu)$ is the obvious shorthand for $\gamma(\mu, \mu, e) - \gamma(\mu, e, \mu)$, with e the unit of $\Gamma(E_{\mathcal{C}om})$.

Similarly, the operad $\mathcal{A}ss$ for associative algebras reviewed in Example 3 is generated by the Σ -module $E_{\mathcal{A}ss}$ such that

$$E_{\mathcal{A}ss}(n) := \begin{cases} \mathbf{k}[\Sigma_2], & \text{if } n = 2 \text{ and} \\ 0, & \text{if } n \neq 2. \end{cases}$$

The ideal of relations is generated by the $\mathbf{k}[\Sigma_3]$ -closure $R_{\mathcal{A}ss}$ of the associativity

$$(28) \quad \alpha(\alpha \otimes id) - \alpha(id \otimes \alpha) \in \Gamma(E_{\mathcal{A}ss})(3),$$

where α is a generator of the regular representation $E_{\mathcal{A}ss}(2) = \mathbf{k}[\Sigma_2]$.

Example 35. The operad $\mathcal{L}ie$ governing Lie algebras is the quotient $\mathcal{L}ie := \Gamma(E_{\mathcal{L}ie})/(R_{\mathcal{L}ie})$, where $E_{\mathcal{L}ie}$ is the Σ -module

$$E_{\mathcal{L}ie}(n) := \begin{cases} \mathbf{k} \cdot \beta & \text{if } n = 2 \text{ and} \\ 0 & \text{if } n \neq 2, \end{cases}$$

with $\mathbf{k} \cdot \beta$ is the signum representation of Σ_2 . The ideal of relations $(R_{\mathcal{L}ie})$ is generated by the Jacobi identity:

$$(29) \quad \beta(\beta \otimes id) + \beta(\beta \otimes id)c + \beta(\beta \otimes id)c^2 = 0,$$

in which $c \in \Sigma_3$ is the cyclic permutation $(1, 2, 3) \mapsto (2, 3, 1)$.

Example 36. We show how to describe the presentations of the operads $\mathcal{A}ss$ and $\mathcal{L}ie$ given in Examples 34 and 35 in a simple graphical language. The generator α of $E_{\mathcal{A}ss}$ is an operation with two inputs and one output, so we depict it as \blacktriangle . The associativity (28) then reads as

$$\blacktriangle = \blacktriangle,$$

therefore $\mathcal{A}ss = \Gamma(\blacktriangleright)/(\blacktriangleright = \blacktriangleleft)$. Also the operad for $\mathcal{L}ie$ algebras is generated by one bilinear operation \blacktriangleright , but this time the operation is anti-symmetric

$$\begin{array}{c} \blacktriangleright \\ 1 \quad 2 \end{array} = - \begin{array}{c} \blacktriangleright \\ 2 \quad 1 \end{array}.$$

The Jacobi identity (29) reads

$$\begin{array}{c} \blacktriangleright \\ 1 \quad 2 \quad 3 \end{array} + \begin{array}{c} \blacktriangleright \\ 2 \quad 3 \quad 1 \end{array} + \begin{array}{c} \blacktriangleright \\ 3 \quad 1 \quad 2 \end{array} = 0.$$

The kind of description used in the above examples is ‘tautological’ in the sense that it just says that the operad \mathcal{P} governing a certain type of algebras is generated by operations of these algebras, with an appropriate symmetry, modulo the axioms satisfied by these operations. It does not say directly anything about the properties of the individual spaces $\mathcal{P}(n)$, $n \geq 0$. Describing these individual components may be a very nontrivial task, see for example the formula for the Σ_n -modules $\mathcal{L}ie(n)$ given in [83, page 50]. Operads in Examples 34 and 35 are quadratic in the sense of the following:

Definition 37. *An operad \mathcal{P} is quadratic if it has a presentation $\mathcal{P} = \Gamma(E)/(R)$, where $E = \mathcal{P}(2)$ and $R \subset \Gamma(E)(3)$.*

Quadratic operads form a very important class of operads. Each quadratic operad \mathcal{P} has a *quadratic dual* $\mathcal{P}^!$ [34], [83, Definition II.3.37] which is a quadratic operad defined, roughly speaking, by dualizing the generators of \mathcal{P} and replacing the relations of \mathcal{P} by their annihilator in the dual space. For example, $\mathcal{A}ss^! = \mathcal{A}ss$, $\mathcal{C}om^! = \mathcal{L}ie$ and $\mathcal{L}ie^! = \mathcal{C}om$. A quadratic operad \mathcal{P} is *Koszul* if it has the homotopy type of the bar construction of its quadratic dual [34], [83, Definition II.3.40]. For quadratic Koszul operads, there is a deep understanding of the derived category of the corresponding algebras. Operads $\mathcal{A}ss$, $\mathcal{C}om$ and $\mathcal{L}ie$ above, as well as most quadratic operads one encounters in everyday life, are Koszul.

5. UNBIASED DEFINITIONS

In this section, we review the definition of a triple (monad) and give, in Theorem 40, a description of unital and non-unital operads in terms of algebras over a triple. The relevant triples come from the endofunctors Ψ and Γ recalled in Section 4. Let $End(\mathcal{C})$ be the strict symmetric monoidal category of endofunctors on a category \mathcal{C} where multiplication is the composition of functors.

Definition 38. *A triple (also called a monad) T on a category \mathcal{C} is an associative and unital monoid (T, μ, ν) in $End(\mathcal{C})$. The multiplication $\mu : TT \rightarrow T$ and unit morphism $\nu : id \rightarrow T$ satisfy the axioms given by commutativity of the diagrams in Figure 4.*

Triples arise naturally from pairs of adjoint functors. Given an adjoint pair [38, II.7]

$$\begin{array}{ccc} & G & \\ A & \xrightarrow{\quad} & B \\ & F & \end{array},$$

with associated functorial isomorphism

$$Mor_{\mathbf{A}}(F(X), Y) \cong Mor_{\mathbf{B}}(X, G(Y)), \quad X \in \mathbf{B}, \quad Y \in \mathbf{A},$$

$$\begin{array}{ccc}
 \begin{array}{ccc}
 TTT & \xrightarrow{T\mu} & TT \\
 \mu T \downarrow & & \downarrow \mu \\
 TT & \xrightarrow{\mu} & T
 \end{array} &
 \begin{array}{ccc}
 T & \xrightarrow{Tv} & TT \\
 id \swarrow & & \searrow \mu \\
 & T &
 \end{array} &
 \begin{array}{ccc}
 T & \xrightarrow{vT} & TT \\
 id \swarrow & & \searrow \mu \\
 & T &
 \end{array}
 \end{array}$$

FIGURE 4. Associativity and unit axioms for a triple.

$$\begin{array}{ccc}
 \begin{array}{ccc}
 T(T(A)) & \xrightarrow{T(\alpha)} & T(A) \\
 \mu \downarrow & & \downarrow \alpha \\
 T(A) & \xrightarrow{\alpha} & A
 \end{array} &
 \begin{array}{ccc}
 A & \xrightarrow{v_A} & T(A) \\
 id \swarrow & & \searrow \alpha \\
 & A &
 \end{array}
 \end{array}$$

 FIGURE 5. T -algebra structure.

there is a triple in \mathbf{B} defined by $T := GF$. The unit of the adjunction $id \rightarrow GF$ defines the unit v of the triple and the counit of the adjunction $FG \rightarrow id$ induces a natural transformation $GF GF \rightarrow GF$ which defines the multiplication μ . In fact, it is a theorem of Eilenberg and Moore [20] that all triples arise in this way from adjoint pairs. This is exactly the situation with the free operad and free non-unital operad functors that were described in Section 4. We will show how operads and non-unital operads can actually be defined using the concept of an algebra over a triple:

Definition 39. A T -algebra or algebra over the triple T is an object A of \mathcal{C} together with a structure morphism $\alpha : T(A) \rightarrow A$ satisfying

$$\alpha(T(\alpha)) = \alpha(\mu_A) \text{ and } \alpha v_A = id_A,$$

see Figure 5.

The category of T -algebras in \mathcal{C} will be denoted $\mathbf{Alg}_T(\mathcal{C})$. Since the free non-unital operad functor Ψ and the free operad functor Γ described in Section 4 are left adjoints to $\square_\psi : \psi\mathbf{Oper} \rightarrow \Sigma\text{-mod}$ and $\square : \mathbf{Oper} \rightarrow \Sigma\text{-mod}$, respectively, the functors $\square_\psi\Psi$ (denoted simply Ψ) and $\square\Gamma$ (denoted Γ) define triples on $\Sigma\text{-mod}$.

Theorem 40. A Σ -module \mathcal{S} is a Ψ -algebra if and only if it is a non-unital operad and it is a Γ -algebra if and only if it is an operad. In shorthand:

$$\mathbf{Alg}_\Psi(\Sigma\text{-mod}) \cong \psi\mathbf{Oper} \text{ and } \mathbf{Alg}_\Gamma(\Sigma\text{-mod}) \cong \mathbf{Oper}.$$

Proof. We outline first the proof of the implication in the direction from algebra to non-unital operad. Let \mathcal{S} be a Ψ -algebra. The restriction of the structure morphism $\alpha : \Psi(\mathcal{S}) \rightarrow \mathcal{S}$ to the components of $\Psi(\mathcal{S})$ supported on trees with one internal edge defines the non-unital operad composition maps \circ_i , as indicated by:

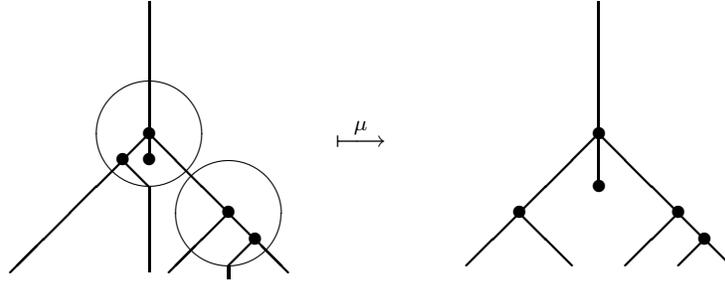
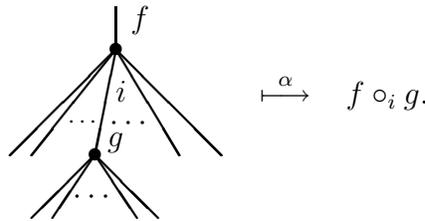


FIGURE 6. Bracketed trees. The left picture shows an element of $\Psi\Psi(E)(5)$ while the right picture shows the same element interpreted, after erasing the braces indicated by thin cycles, as an element of $\Psi(E)(5)$. For simplicity, we did not show the decoration of vertices by elements of E .



In the opposite direction, for a non-unital operad \mathcal{S} , the Ψ -algebra structure $\alpha : \Psi(\mathcal{S}) \rightarrow \mathcal{S}$ is the contraction along the edges of underlying trees, using the \circ_i -operations. The proof that Γ -algebras are operads is similar. \square

Let us change our perspective and consider formula (25) as defining an endofunctor $\Psi : \Sigma\text{-mod} \rightarrow \Sigma\text{-mod}$, ignoring that we already know that it represents free non-unital operads. We are going to construct maps

$$\mu : \Psi\Psi \rightarrow \Psi \text{ and } \nu : id \rightarrow \Psi$$

making Ψ a triple on the category $\Sigma\text{-mod}$. Let us start with the triple multiplication μ . It follows from (25) that, for each Σ -module E ,

$$\Psi\Psi(E)(n) := \operatorname{colim}_{T \in \operatorname{Tree}_n} \Psi(E)(T), \quad n \geq 0.$$

The elements in the right hand side are represented by rooted trees T with vertices decorated by elements of $\Psi(E)$, while elements of $\Psi(E)$ are represented by rooted trees with vertices decorated by E . We may therefore imagine elements of $\Psi\Psi(E)$ as ‘bracketed’ rooted trees, in the sense indicated in Figure 6. The triple multiplication $\mu_E : \Psi\Psi(E) \rightarrow \Psi(E)$ then simply erases the braces. The triple unit $\nu_E : E \rightarrow \Psi(E)$ identifies elements of E with decorated corollas:

$$E(n) \ni e \quad \longleftrightarrow \quad \begin{array}{c} \text{---} e \text{---} \\ | \\ \bullet \\ / \quad \backslash \\ \dots \end{array} \in \Psi(E)(n), \quad n \geq 0.$$

$\underbrace{\hspace{2cm}}_{n \text{ inputs}}$

It is not difficult to verify that the above constructions indeed make Ψ a triple, compare [83, § II.1.12]. Now we can define non-unital operads as algebras over the triple (Ψ, μ, ν) . The

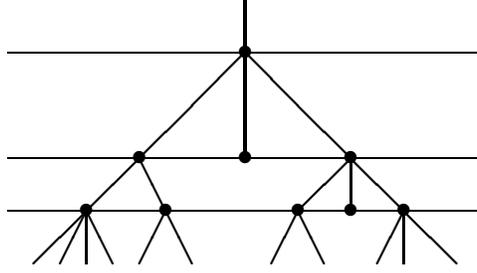


FIGURE 7. A May's tree.

advantage of this approach is that, by replacing \mathbf{Tree}_n in (25) by another category of trees or graphs, one may obtain triples defining other types of operads and their generalizations.

We have already seen in (27) that enlarging \mathbf{Tree}_n into \mathbf{UTree}_n by adding the exceptional tree, one gets the triple Γ describing (unital) operads. It is not difficult to see that non-unital May's operads are related to the category \mathbf{MTree}_n of *May's trees* which are, by definition, rooted trees whose vertices can be arranged into levels as in Figure 7. Non-unital May's operads are then algebras over the triple $\mathbf{M} : \Sigma\text{-mod} \rightarrow \Sigma\text{-mod}$ defined by

$$\mathbf{M}(E)(n) := \operatorname{colim}_{T \in \mathbf{MTree}_n} E(T), \quad n \geq 0.$$

These observations are summarized in the first three lines of the table in Figure 14 on page 45.

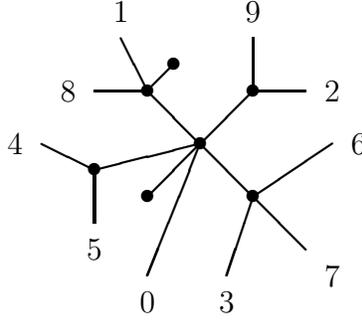
6. CYCLIC OPERADS

In the following two sections we use the approach developed in Section 5 to introduce cyclic and modular operads. We recalled, in Example 14, the operad $\widehat{\mathfrak{M}}_0 = \{\widehat{\mathfrak{M}}_0(n)\}_{n \geq 0}$ of Riemann spheres with parametrized labeled holes. Each $\widehat{\mathfrak{M}}_0(n)$ was a right Σ_n -space, with the operadic right Σ_n -action permuting the labels $1, \dots, n$ of the holes u_1, \dots, u_n . But each $\widehat{\mathfrak{M}}_0(n)$ obviously admits a higher type of symmetry which interchanges labels $0, \dots, n$ of *all* holes, including the label of the 'output' hole u_0 . Another example admitting a similar higher symmetry is the configuration (non-unital) operad $\overline{\mathfrak{M}}_0 = \{\overline{\mathfrak{M}}_0(n)\}_{n \geq 2}$.

These examples indicate that, for some operads, there is no clear distinction between 'inputs' and the 'output.' Cyclic operads, introduced by E. Getzler and M.M. Kapranov in [32], formalize this phenomenon. They are, roughly speaking, operads with an extra symmetry that interchanges the output with one of the inputs. Let us recall some notions necessary to give a precise definition.

We remind the reader that in this section, as well as everywhere in this article, main definitions are formulated over the underlying category of \mathbf{k} -modules, where \mathbf{k} is a commutative associative unital ring. However, for some constructions, we will require \mathbf{k} to be a *field*; we will indicate this as usual by speaking about *vector spaces* instead of \mathbf{k} -modules.

Let Σ_n^+ be the permutation group of the set $\{0, \dots, n\}$. The group Σ_n^+ is, of course, non-canonically isomorphic to the symmetric group Σ_{n+1} . We identify Σ_n with the subgroup of Σ_n^+ consisting of permutations $\sigma \in \Sigma_n^+$ such that $\sigma(0) = 0$. If $\tau_n \in \Sigma_n^+$ denotes the cycle $(0, \dots, n)$, that is, the permutation with $\tau_n(0) = 1, \tau_n(1) = 2, \dots, \tau_n(n) = 0$, then τ_n and Σ_n generate Σ_n^+ .

FIGURE 8. A cyclic labeled tree from \mathbf{Tree}_9^+ .

Recall that a *cyclic Σ -module* or a Σ^+ -*module* is a sequence $W = \{W(n)\}_{n \geq 0}$ such that each $W(n)$ is a (right) $\mathbf{k}[\Sigma_n^+]$ -module. Let Σ^+ -**mod** denote the category of cyclic Σ -modules. As (ordinary) operads were Σ -modules with an additional structure, cyclic operads are Σ^+ -modules with an additional structure.

We will also need the following ‘cyclic’ analog of (23): if X is a set with $n + 1$ elements and $W \in \Sigma^+$ -**mod**, then

$$(30) \quad W((X)) := W(n) \times_{\Sigma_n^+} \mathit{Bij}([n]^+, X),$$

where $[n]^+ := \{0, \dots, n\}$, $n \geq 0$. Double brackets in $W((X))$ remind us that the n th piece of the cyclic Σ -module $W = \{W(n)\}_{n \geq 0}$ is applied on a set with $n + 1$ elements, using the extended Σ_n^+ -symmetry. Therefore

$$W(\{\{0, \dots, n\}\}) \cong W(n) \quad \text{while} \quad W(\{0, \dots, n\}) \cong W(n + 1), \quad n \geq 0.$$

Pasting schemes for cyclic operads are *cyclic (leg-) labeled n -trees*, by which we mean unrooted trees as on page 19, with legs labeled by the set $\{0, \dots, n\}$. An example of such a tree is given in Figure 8. Since we do not assume a choice of the root, the edges of a cyclic tree C are not directed and it does not make sense to speak about inputs and the output of a vertex $v \in \mathit{vert}(C)$. Let \mathbf{Tree}_n^+ be the category of cyclic labeled n -trees and their bijections.

For a cyclic Σ -module W and a cyclic labeled tree T we have the following cyclic version of the product (24)

$$W((T)) := \bigotimes_{v \in \mathit{vert}(T)} W(\mathit{edge}(v)).$$

The conceptual difference between (24) and the above formula is that instead of the set $\mathit{in}(v)$ of incoming edges of a vertex v of a rooted tree, here we use the set $\mathit{edge}(v)$ of *all* edges incident with v . Let, finally, $\Psi_+ : \Sigma^+$ -**mod** \rightarrow Σ^+ -**mod** be the functor

$$(31) \quad \Psi_+(W)(n) := \mathop{\mathrm{colim}}_{T \in \mathbf{Tree}_n^+} W((T)), \quad n \geq 0,$$

equipped with the triple structure of ‘forgetting the braces’ similar to that reviewed on page 26. We will use also the ‘extended’ triple $\Gamma_+ : \Sigma^+$ -**mod** \rightarrow Σ^+ -**mod**,

$$\Gamma_+(W)(n) := \mathop{\mathrm{colim}}_{T \in \mathbf{UTree}_n^+} W((T)), \quad n \geq 0,$$

where \mathbf{UTree}_n^+ is the obvious extension of the category \mathbf{Tree}_n^+ by the exceptional tree \lfloor .

Definition 41. A cyclic (resp. non-unital cyclic) operad is an algebra over the triple Γ_+ (resp. the triple Ψ_+) introduced above.

In the following proposition, which slightly improves [32, Theorem 2.2], $\tau_n \in \Sigma_n^+$ denotes the cycle $(0, \dots, n)$.

Proposition 42. A non-unital cyclic operad is the same as a non-unital operad $\mathcal{C} = \{\mathcal{C}(n)\}_{n \geq 0}$ (Definition 11) such that the right Σ_n -action on $\mathcal{C}(n)$ extends, for each $n \geq 0$, to an action of Σ_n^+ with the property that for $p \in \mathcal{C}(m)$ and $q \in \mathcal{C}(n)$, $1 \leq i \leq m$, $n \geq 0$, the composition maps satisfy

$$(p \circ_i q)\tau_{m+n-1} = \begin{cases} (q\tau_n) \circ_n (p\tau_m), & \text{if } i = 1, \text{ and} \\ (p\tau_m) \circ_{i-1} q, & \text{for } 2 \leq i \leq m. \end{cases}$$

The above structure is a (unital) cyclic operad if moreover there exists a Σ_1^+ -invariant operadic unit $e \in \mathcal{C}(1)$.

Proposition 42 gives a biased definition of cyclic operads whose obvious modification (see [83, Definition II.5.2]) makes sense in an arbitrary symmetric monoidal category. We can therefore speak about topological cyclic operads, differential graded cyclic operads, simplicial cyclic operads &c. Observe that there are no *non-unital cyclic May's operads* because it does not make sense to speak about levels in trees without a choice of the root.

Example 43. Let V be a finite dimensional vector space and $B : V \otimes V \rightarrow \mathbf{k}$ a nondegenerate symmetric bilinear form. The form B induces the identification

$$\mathit{Lin}(V^{\otimes n}, V) \ni f \longmapsto \widehat{B}(f) := B(-, f(-)) \in \mathit{Lin}(V^{\otimes(n+1)}, \mathbf{k})$$

of the spaces of linear maps. The standard right Σ_n^+ -action

$$\widehat{B}(f)\sigma(v_0, \dots, v_n) = \widehat{B}(f)(v_{\sigma^{-1}(0)}, \dots, v_{\sigma^{-1}(n)}), \quad \sigma \in \Sigma_n^+, \quad v_0, \dots, v_n \in V,$$

defines, via this identification, a right Σ_n^+ -action on $\mathit{Lin}(V^{\otimes n}, V)$, that is, on the n th piece of the endomorphism operad $\mathcal{E}nd_V = \{\mathcal{E}nd_V(n)\}_{n \geq 0}$ recalled in Example 2. It is easy to show that, with the above action, $\mathcal{E}nd_V$ is a cyclic operad in the monoidal category of vector spaces, called the *cyclic endomorphism operad* of the pair $V = (V, B)$. The biased definition of cyclic operads given in Proposition 42 can be read off from this example.

Example 44. We saw in Example 7 that a unital operad $\mathcal{A} = \{\mathcal{A}(n)\}_{n \geq 0}$ such that $\mathcal{A}(n) = 0$ for $n \neq 1$ is the same as a unital associative algebra. Similarly, it can be easily shown that a cyclic operad $\mathcal{C} = \{\mathcal{C}(n)\}_{n \geq 0}$ satisfying $\mathcal{C}(n) = 0$ for $n \neq 1$ is the same as a unital associative algebra A with a linear involutive antiautomorphism, by which we mean a \mathbf{k} -linear map $*$: $A \rightarrow A$ such that

$$(ab)^* = b^*a^*, \quad (a^*)^* = a \quad \text{and} \quad 1^* = 1,$$

for arbitrary $a, b \in A$.

Let $\mathcal{P} = \Gamma(E)/(R)$ be a quadratic operad as in Definition 37. The action of Σ_2 on E extends to an action of Σ_2^+ , via the sign representation $\text{sgn} : \Sigma_2^+ \rightarrow \{\pm 1\} = \Sigma_2$. It can be easily verified that this action induces a cyclic operad structure on the free operad $\Gamma(E)$. In particular, $\Gamma(E)(3)$ is a right Σ_3^+ -module.

Definition 45. *We say that the operad \mathcal{P} is a cyclic quadratic operad if, in the above presentation, R is a Σ_3^+ -invariant subspace of $\Gamma(E)(3)$.*

If the condition of the above definition is satisfied, \mathcal{P} has a natural induced cyclic operad structure.

Example 46. By [32, Proposition 3.6], all quadratic operads generated by a one-dimensional space are cyclic quadratic, therefore the operads $\mathcal{L}ie$ and $\mathcal{C}om$ are cyclic quadratic. Also the operads $\mathcal{A}ss$ and the operad $\mathcal{P}oiss$ for Poisson algebras are cyclic quadratic [32, Proposition 3.11]. A surprisingly simple operad which is cyclic and quadratic, but not cyclic quadratic, is constructed in [82, Remark 15].

The operad $\widehat{\mathcal{M}}_0$ of Riemann spheres with labeled punctures reviewed in Example 14 is a topological cyclic operad. The configuration operad $\overline{\mathcal{M}}_0$ recalled on page 12 is a non-unital topological cyclic operad. Important examples of non-cyclic operads are the operad *pre- $\mathcal{L}ie$* for pre-Lie algebras [82, Section 3] and the operad *$\mathcal{L}eib$* for Leibniz algebras [32, § 3.15].

Let \mathcal{C} be an operad, $\alpha : \mathcal{C}(n) \otimes V^{\otimes n} \rightarrow V$, $n \geq 0$, a \mathcal{C} -algebra with the underlying vector space V as in Proposition 24 and $B : V \otimes V \rightarrow U$ a bilinear form on V with values in a vector space U . We can form a map

$$(32) \quad \tilde{B}(\alpha) : \mathcal{C}(n) \otimes V^{\otimes(n+1)} \rightarrow U, \quad n \geq 0,$$

by the formula

$$\tilde{B}(\alpha)(c \otimes v_0 \otimes \cdots \otimes v_n) := B(v_0, \alpha(c \otimes v_1 \otimes \cdots \otimes v_n)), \quad c \in \mathcal{C}(n), \quad v_0, \dots, v_n \in V.$$

Suppose now that the operad \mathcal{C} is cyclic, in particular, that each $\mathcal{C}(n)$ is a right Σ_n^+ -module. We say that the bilinear form $B : V \otimes V \rightarrow U$ is *invariant* [32, Definition 4.1], if the maps $\tilde{B}(\alpha)$ in (32) are, for each $n \geq 0$, invariant under the diagonal action of Σ_n^+ on $\mathcal{C}(n) \otimes V^{\otimes(n+1)}$. We leave as an exercise to verify that the invariance of $\tilde{B}(\alpha)$ for $n = 1$ together with the existence of the operadic unit implies that B is symmetric,

$$B(v_0, v_1) = B(v_1, v_0), \quad v_0, v_1 \in V.$$

Definition 47. *A cyclic algebra over a cyclic operad \mathcal{C} is a \mathcal{C} -algebra structure on a vector space V together with a nondegenerate invariant bilinear form $B : V \otimes V \rightarrow \mathbf{k}$.*

By [83, Proposition II.5.14], a cyclic algebra is the same as a cyclic operad homomorphism $\mathcal{C} \rightarrow \mathcal{E}nd_V$, where $\mathcal{E}nd_V$ is the cyclic endomorphism operad of the pair (V, B) recalled in Example 43.

Example 48. A cyclic algebra over the cyclic operad $\mathcal{C}om$ is a *Frobenius algebra*, that is, a structure consisting of a commutative associative multiplication $\cdot : V \otimes V \rightarrow V$ as in Example 25

together with a non-degenerate symmetric bilinear form $B : V \otimes V \rightarrow \mathbf{k}$, invariant in the sense that

$$B(a \cdot b, c) = B(a, b \cdot c), \quad \text{for all } a, b, c \in V.$$

Similarly, a cyclic Lie algebra is given by a Lie bracket $[-, -] : V \otimes V \rightarrow V$ and a non-degenerate symmetric bilinear form $B : V \otimes V \rightarrow \mathbf{k}$ satisfying

$$B([a, b], c) = B(a, [b, c]), \quad \text{for } a, b, c \in V.$$

For algebras over cyclic operads, one may introduce cyclic cohomology that generalizes the classical cyclic cohomology of associative algebras [12, 66, 109] as the non-abelian derived functor of the universal bilinear form [32], [83, Proposition II.5.26]. Let us close this section by mentioning two examples of operads with other types of higher symmetries. The symmetry required for *anticyclic operads* differs from the symmetry of cyclic operads by the sign [83, Definition II.5.20]. *Dihedral operads* exhibit a symmetry governed by the dihedral groups [82, Definition 16].

7. MODULAR OPERADS

Let us consider again the Σ^+ -module $\widehat{\mathfrak{M}}_0 = \{\widehat{\mathfrak{M}}_0(n)\}_{n \geq 0}$ of Riemann spheres with punctures. We saw that the operation $M, N \mapsto M \circ_i N$ of sewing the 0th hole of the surface N to the i th hole of the surface M defined on $\widehat{\mathfrak{M}}_0$ a cyclic operad structure. One may generalize this operation by defining, for $M \in \widehat{\mathfrak{M}}_0(m)$, $N \in \widehat{\mathfrak{M}}_0(n)$, $0 \leq i \leq m$, $0 \leq j \leq n$, the element $M_i \circ_j N \in \widehat{\mathfrak{M}}_0(m+n-1)$ by sewing the j th hole of M to the i th hole of N . Under this notation, $\circ_i = {}_i \circ_0$. In the same manner, one may consider a single surface $M \in \widehat{\mathfrak{M}}_0(n)$, choose labels i, j , $0 \leq i \neq j \leq n$, and sew the i th hole of M along the j th hole of the *same* surface. The result is a new surface $\xi_{\{i,j\}}(M)$, with $n-2$ holes and genus 1.

This leads us to the system $\widehat{\mathfrak{M}} = \{\widehat{\mathfrak{M}}(g, n)\}_{g \geq 0, n \geq -1}$, where $\widehat{\mathfrak{M}}(g, n)$ denotes now the moduli space of genus g Riemann surfaces with $n+1$ holes. Observe that we include $\widehat{\mathfrak{M}}(g, n)$ also for $n = -1$; $\widehat{\mathfrak{M}}(g, -1)$ is the moduli space of Riemann surfaces of genus g . The operations ${}_i \circ_j$ and $\xi_{\{i,j\}}$ act on $\widehat{\mathfrak{M}}$. Clearly, for $M \in \widehat{\mathfrak{M}}(g, m)$ and $N \in \widehat{\mathfrak{M}}(h, n)$, $0 \leq i \leq m$, $0 \leq j \leq n$ and $g, h \geq 0$,

$$(33) \quad M_i \circ_j N \in \widehat{\mathfrak{M}}(g+h, m+n-1)$$

and, for $m \geq 1$ and $g \geq 0$,

$$(34) \quad \xi_{\{i,j\}}(M) \in \widehat{\mathfrak{M}}(g+1, m-2).$$

A particular case of (33) is the non-operadic composition

$$(35) \quad {}_{0 \circ 0} : \widehat{\mathfrak{M}}(g, 0) \times \widehat{\mathfrak{M}}(h, 0) \rightarrow \widehat{\mathfrak{M}}(g+h, -1), \quad g, h \geq 0.$$

Modular operads are abstractions of the above structure satisfying a certain additional stability condition. The following definitions, taken from [33], are made for the category of \mathbf{k} -modules, but they can be easily generalized to an arbitrary symmetric monoidal category with finite colimits, whose monoidal product \odot is distributive over colimits. Let us introduce the underlying category for modular operads.

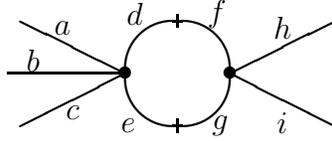


FIGURE 9. The sputnik.

A *modular Σ -module* is a sequence $\mathcal{E} = \{\mathcal{E}(g, n)\}_{g \geq 0, n \geq -1}$ of \mathbf{k} -modules such that each $\mathcal{E}(g, n)$ has a right $\mathbf{k}[\Sigma_n^+]$ -action. We say that \mathcal{E} is *stable* if

$$(36) \quad \mathcal{E}(g, n) = 0 \quad \text{for } 2g + n - 1 \leq 0$$

and denote \mathbf{MMod} the category of stable modular Σ -modules.

Stability (36) says that $\mathcal{E}(g, n)$ is trivial for $(g, n) = (0, -1), (1, -1), (0, 0)$ and $(0, 1)$. We will sometimes express the stability of \mathcal{E} by writing $\mathcal{E} = \{\mathcal{E}(g, n)\}_{(g, n) \in \mathfrak{S}}$, where

$$\mathfrak{S} := \{(g, n) \mid g \geq 0, n \geq -1 \text{ and } 2g + n - 1 > 0\}.$$

Recall that a genus g Riemann surface with k marked points is stable if it does not admit infinitesimal automorphisms. This happens if and only if $2(g - 1) + k > 0$, that is, excluded is the torus with no marked points and the sphere with less than three marked points. Thus the stability property of modular Σ -modules is analogous to the stability of Riemann surfaces.

Now we introduce graphs that serve as pasting schemes for modular operads. The naive notion of a graph as we have used it up to this point is not subtle enough; we need to replace it by a more sophisticated:

Definition 49. A graph Γ is a finite set $\text{Flag}(\Gamma)$ (whose elements are called flags or half-edges) together with an involution σ and a partition λ . The vertices $\text{vert}(\Gamma)$ of a graph Γ are the blocks of the partition λ , we assume also that the number of these blocks is finite. The edges $\text{Edg}(\Gamma)$ are pairs of flags forming a two-cycle of σ . The legs $\text{Leg}(\Gamma)$ are the fixed points of σ .

We also denote by $\text{edge}(v)$ the flags belonging to the block v or, in common speech, half-edges adjacent to the vertex v . We say that graphs Γ_1 and Γ_2 are *isomorphic* if there exists a set isomorphism $\varphi : \text{Flag}(\Gamma_1) \rightarrow \text{Flag}(\Gamma_2)$ that preserves the partitions and commutes with the involutions. We may associate to a graph Γ a finite one-dimensional cell complex $|\Gamma|$, obtained by taking one copy of $[0, \frac{1}{2}]$ for each flag, a point for each block of the partition, and imposing the following equivalence relation: The points $0 \in [0, \frac{1}{2}]$ are identified for all flags in a block of the partition λ with the point corresponding to the block, and the points $\frac{1}{2} \in [0, \frac{1}{2}]$ are identified for pairs of flags exchanged by the involution σ .

We call $|\Gamma|$ the *geometric realization* of Γ . Observe that empty blocks of the partition generate isolated vertices in the geometric realization. We will usually make no distinction between the graph and its geometric realization. As an example (taken from [33]), consider the graph with $\{a, b, \dots, i\}$ as the set of flags, the involution $\sigma = (df)(eg)$ and the partition $\{a, b, c, d, e\} \cup \{f, g, h, i\}$. The geometric realization of this graph is the ‘sputnik’ in Figure 9.

Let us introduce labeled versions of the above notions. A (*vertex-*) *labeled graph* is a connected graph Γ together with a map g (the *genus map*) from $\text{vert}(\Gamma)$ to the set $\{0, 1, 2, \dots\}$. Labeled

graphs Γ_1 and Γ_2 are isomorphic if there exists an isomorphism preserving the labels of the vertices. The *genus* $g(\Gamma)$ of a labeled graph Γ is defined by

$$(37) \quad g(\Gamma) := b_1(\Gamma) + \sum_{v \in \text{vert}(\Gamma)} g(v),$$

where $b_1(\Gamma) := \dim H_1(|\Gamma|)$ is the first Betti number of the graph $|\Gamma|$, i.e. the number of independent circuits of Γ . A graph Γ is *stable* if

$$2(g(v) - 1) + |\text{edge}(v)| > 0,$$

at each vertex $v \in \text{vert}(\Gamma)$.

For $g \geq 0$ and $n \geq -1$, let $\mathbf{MGr}(g, n)$ be the groupoid whose objects are pairs (Γ, ℓ) consisting of a stable (vertex-) labeled graph Γ of genus g and an isomorphism $\ell : \text{Leg}(\Gamma) \rightarrow \{0, \dots, n\}$ labeling the legs of Γ by elements of $\{0, \dots, n\}$. Morphisms of $\mathbf{MGr}(g, S)$ are isomorphisms of vertex-labeled graphs preserving the labeling of the legs. The stability implies, via an elementary combinatorial topology that, for each fixed $g \geq 0$ and $n \geq -1$, there is only a finite number of isomorphism classes of stable graphs $\Gamma \in \mathbf{MGr}(g, n)$, see [33, Lemma 2.16].

We will also need the following obvious generalization of (30): if $\mathcal{E} = \{\mathcal{E}(g, n)\}_{g \geq 0, n \geq -1}$ is a modular Σ -module and X a set with $n + 1$ elements, then

$$(38) \quad \mathcal{E}((g, X)) := \mathcal{E}(g, n) \times_{\Sigma_n^+} \text{Bij}([n]^+, X), \quad g \geq 0, \quad n \geq -1.$$

For a modular Σ -module $\mathcal{E} = \{\mathcal{E}(g, n)\}_{g \geq 0, n \geq -1}$ and a labeled graph Γ , let $\mathcal{E}((\Gamma))$ be the product

$$(39) \quad \mathcal{E}((\Gamma)) := \bigotimes_{v \in \text{vert}(\Gamma)} \mathcal{E}((g(v), \text{edge}(v))).$$

Evidently, the correspondence $\Gamma \mapsto \mathcal{E}((\Gamma))$ defines a functor from the category $\mathbf{MGr}(g, n)$ to the category of \mathbf{k} -modules and their isomorphisms. We may thus define an endofunctor \mathbb{M} on the category \mathbf{MMod} of stable modular Σ -modules by the formula

$$\mathbb{M}\mathcal{E}(g, n) := \text{colim}_{\Gamma \in \mathbf{MGr}(g, n)} \mathcal{E}((\Gamma)), \quad g \geq 0, \quad n \geq -1.$$

Choosing a representative for each isomorphism class in $\mathbf{MGr}(g, n)$, one obtains the identification

$$(40) \quad \mathbb{M}\mathcal{E}(g, n) \cong \bigoplus_{[\Gamma] \in \{\mathbf{MGr}(g, n)\}} \mathcal{E}((\Gamma))_{\text{Aut}(\Gamma)}, \quad g \geq 0, \quad n \geq -1,$$

where $\{\mathbf{MGr}(g, n)\}$ is the set of isomorphism classes of objects of the groupoid $\mathbf{MGr}(g, n)$ and the subscript $\text{Aut}(\Gamma)$ denotes the space of coinvariants. Stability (36) implies that the summation in the right-hand side of (40) is finite. Formula (40) generalizes (26) which does not contain coinvariants because there are no nontrivial automorphisms of leaf-labeled trees. On the other hand, stable labeled graphs with nontrivial automorphisms are abundant, an example can be easily constructed from the graph in Figure 9. The functor \mathbb{M} carries a triple structure of ‘erasing the braces’ similar to the one used on pages 26 and 28.

Definition 50. A modular operad is an algebra over the triple $\mathbb{M} : \mathbf{MMod} \rightarrow \mathbf{MMod}$.

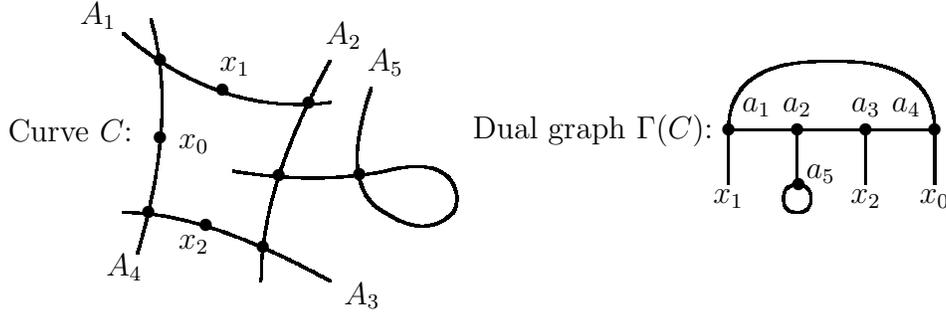


FIGURE 10. A stable curve and its dual graph. The curve C on the left has five components A_i , $1 \leq i \leq 5$, and three marked points x_0 , x_1 and x_2 . The dual graph $\Gamma(C)$ on the right has five vertices a_i , $1 \leq i \leq 5$, corresponding to the components of the curve and three legs labeled by the marked points.

Therefore a modular operad is a stable modular Σ -module $\mathcal{A} = \{\mathcal{A}(g, n)\}_{(g, n) \in \mathfrak{S}}$ equipped with operations that determine coherent contractions along stable modular graphs. Observe that the stability condition is built firmly into the very definition. Very crucially, modular operads *do not have units*, because such a unit ought to be an element of the space $\mathcal{A}(0, 1)$ which is empty, by (36).

One can easily introduce un-stable modular operads and their unital versions, but the main motivating example reviewed below is stable. We will consider an extension of the Grothendieck-Knudsen configuration operad $\overline{\mathcal{M}}_0 = \{\overline{\mathcal{M}}_0(n)\}_{n \geq 2}$ consisting of moduli spaces of stable curves of arbitrary genera in the sense of the following generalization of Definition 15:

Definition 51. A stable $(n + 1)$ -pointed curve, $n \geq 0$, is a connected complex projective curve C with at most nodal singularities, together with a ‘marking’ given by a choice $x_0, \dots, x_n \in C$ of smooth points. The stability means, as usual, that there are no infinitesimal automorphisms of C fixing the marked points and double points.

The stability in Definition 51 is equivalent to saying that each smooth component of C isomorphic to the complex projective space $\mathbb{C}P^1$ has at least three special points and that each smooth component isomorphic to the torus has at least one special point, where by a special point we mean either a double point or a node.

The *dual graph* $\Gamma = \Gamma(C)$ of a stable $(n + 1)$ -pointed curve $C = (C, x_0, \dots, x_n)$ is a labeled graph whose vertices are the components of C , edges are the nodes and its legs are the points $\{x_i\}_{0 \leq i \leq n}$. An edge e_y corresponding to a nodal point y joins the vertices corresponding to the components intersecting at y . The vertex v_K corresponding to a branch K is labeled by the genus of the normalization of K . See [37, page 23] for the normalization and recall that a curve is normal if and only if it is nonsingular. The construction of $\Gamma(C)$ from a curve C is visualized in Figure 10.

Let us denote by $\overline{\mathcal{M}}_{g, n+1}$ the coarse moduli space [37, page 347] of stable $(n + 1)$ -pointed curves C such that the dual graph $\Gamma(C)$ has genus g , in the sense of (37). The genus of $\Gamma(C)$ in fact equals the arithmetic genus of the curve C , thus $\overline{\mathcal{M}}_{g, n+1}$ is the coarse moduli space of stable curves of arithmetic genus g with $n + 1$ marked points. By a result of P. Deligne, F.F. Knudsen and D. Mumford [18, 51, 50], $\overline{\mathcal{M}}_{g, n+1}$ is a projective variety.

Observe that, for a curve $C \in \overline{\mathcal{M}}_{0,n+1}$, the graph $\Gamma(C)$ must necessarily be a tree and all components of C must be smooth of genus 0, therefore $\overline{\mathcal{M}}_{0,n+1}$ coincides with the moduli space $\overline{\mathcal{M}}_0(n)$ of genus 0 stable curves with $n + 1$ marked points that we discussed in Section 2. Dual graphs of curves $C \in \overline{\mathcal{M}}_{g,n+1}$ are stable labeled graphs belonging to $\mathbf{MGr}(g, n + 1)$.

The symmetric group Σ_n^+ acts on $\overline{\mathcal{M}}_{g,n+1}$ by renumbering the marked points, therefore

$$\overline{\mathcal{M}} := \{\overline{\mathcal{M}}(g, n)\}_{g \geq 0, n \geq -1},$$

with $\overline{\mathcal{M}}(g, n) := \overline{\mathcal{M}}_{g,n+1}$, is a modular Σ -module in the category of projective varieties. Since there are no stable curves of genus g with $n + 1$ punctures if $2g + n - 1 \leq 0$, $\overline{\mathcal{M}}$ is a *stable* modular Σ -module. Let us define the contraction along a stable graph $\Gamma \in \mathbf{MGr}(g, n)$

$$(41) \quad \alpha_\Gamma : \overline{\mathcal{M}}(\Gamma) = \prod_{v \in \text{vert}(\Gamma)} \overline{\mathcal{M}}(g(v), \text{edge}(v)) \rightarrow \overline{\mathcal{M}}(g, n)$$

by gluing the marked points of curves from $\overline{\mathcal{M}}(g(v), \text{edge}(v))$, $v \in \text{vert}(\Gamma)$, according to the graph Γ . To be more precise, let

$$\prod_{v \in \text{vert}(\Gamma)} C_v, \quad \text{where } C_v \in \overline{\mathcal{M}}(g(v), \text{edge}(v)),$$

be an element of $\overline{\mathcal{M}}(\Gamma)$. Let e be an edge of the graph Γ connecting vertices v_1 and v_2 , $e = \{y_{v_1}^e, y_{v_2}^e\}$, where $y_{v_i}^e$ is a marked point of the component C_{v_i} , $i = 1, 2$, which is also the name of the corresponding flag of the graph Γ . The curve $\alpha_\Gamma(C)$ is then obtained by the identifications $y_{v_1}^e = y_{v_2}^e$, introducing a nodal singularity, for all $e \in \text{Edg}(\Gamma)$. The procedure is the same as that described for the tree level in Section 2. As proved in [33, § 6.2], the contraction maps (41) define on the stable modular Σ -module of coarse moduli spaces $\overline{\mathcal{M}} = \{\overline{\mathcal{M}}(g, n)\}_{(g,n) \in \mathfrak{S}}$ a modular operad structure in the category of complex projective varieties.

Let us look more closely at the structure of the modular triple \mathbb{M} . Given a (stable or unstable) modular Σ -module \mathcal{E} , there is, for each $g \geq 0$ and $n \geq -1$, a natural decomposition

$$\mathbb{M}(\mathcal{E})(g, n) = \mathbb{M}_0(\mathcal{E})(g, n) \oplus \mathbb{M}_1(\mathcal{E})(g, n) \oplus \mathbb{M}_2(\mathcal{E})(g, n) \oplus \cdots,$$

with $\mathbb{M}_k(\mathcal{E})(g, n)$ the subspace obtained by summing over graphs Γ with $\dim H_1(|\Gamma|) = k$, $k \geq 0$. In particular, $\mathbb{M}_0(\mathcal{E})(g, n)$ is a summation over simply connected graphs. It is not difficult to see that $\mathbb{M}_0(\mathcal{E})$ is a *subtriple* of $\mathbb{M}(\mathcal{E})$. This shows that modular operads are \mathbb{M}_0 -algebras with some additional operations (the ‘contractions’) that raise the genus and generate the higher components \mathbb{M}_k , $k \geq 1$, of the modular triple \mathbb{M} .

There seems to be a belief expressed in the proof of [33, Lemma 3.4] and also in [33, Theorem 3.7] that, in the stable case, the triple \mathbb{M}_0 is equivalent to the non-unital cyclic operad triple Ψ_+ , but it is not so. The triple \mathbb{M}_0 is *much bigger*, for example, if $a \in \mathcal{E}(1, 0)$, then $\mathbb{M}_0(\mathcal{E})(2, -1)$ contains a non-operadic element



which can be also written, using (35), as $a_0 \circ_0 a$. The corresponding part $\Psi_+(\mathcal{E})(-1)$ of the cyclic triple is empty. In the Grothendieck-Knudsen modular operad $\overline{\mathcal{M}}$, an element of the above type is realized by two tori meeting at a nodal point.

On the other hand, the triple \mathbb{M}_0 restricted to the subcategory of stable modular Σ -modules \mathcal{E} such that $\mathcal{E}(g, n) = 0$ for $g > 0$ indeed coincides with the non-unital cyclic operad triple Ψ_+ , as was in fact proved in [33, page 81]. Therefore, given a modular operad $\mathcal{A} = \{\mathcal{A}(g, n)\}_{(g,n) \in \mathfrak{S}}$, there is an induced non-unital cyclic operad structure on the cyclic collection $\mathcal{A}^b := \{\mathcal{A}(0, n)\}_{n \geq 2}$. We will call \mathcal{A}^b the *associated cyclic operad*. For example, the cyclic operad associated to the Grothendieck-Knudsen modular operad $\overline{\mathcal{M}}$ equals its genus zero part $\overline{\mathcal{M}}_0$.

A *biased* definition of modular operads can be found in [83, Definition II.5.35]. It is formulated in terms of operations

$$\{\circ_j : \mathcal{A}(g, m) \otimes \mathcal{A}(h, n) \rightarrow \mathcal{A}(g+h, m+n); 0 \leq i \leq m, 0 \leq j \leq n, g, h \geq 0\}$$

together with contractions

$$\{\xi_{\{i,j\}} : \mathcal{A}(g, m) \rightarrow \mathcal{A}(g+1, m-2); m \geq 1, g \geq 0\}$$

that generalize (33) and (34).

Example 52. Let $V = (V, B)$ be a vector space with a symmetric inner product $B : V \otimes V \rightarrow \mathbf{k}$. Denote, for each $g \geq 0$ and $n \geq -1$,

$$\mathcal{E}nd_V(g, n) := V^{\otimes(n+1)}.$$

It is clear from definition (39) that, for any labeled graph $\Gamma \in \mathbf{MGr}(g, n)$, $\mathcal{E}nd_V((\Gamma)) = V^{\otimes \text{Flag}(\Gamma)}$.

Let $B^{\otimes \text{Edg}(\Gamma)} : V^{\otimes \text{Flag}(\Gamma)} \rightarrow V^{\otimes \text{Leg}(\Gamma)}$ be the multilinear form which contracts the factors of $V^{\otimes \text{Flag}(\Gamma)}$ corresponding to the flags which are paired up as edges of Γ . Then we define $\alpha_\Gamma : \mathcal{E}nd_V((\Gamma)) \rightarrow \mathcal{E}nd_V(g, n)$ to be the map

$$\alpha_\Gamma : \mathcal{E}nd_V((\Gamma)) = V^{\otimes \text{Flag}(\Gamma)} \xrightarrow{B^{\otimes \text{Edg}(\Gamma)}} V^{\otimes \text{Leg}(\Gamma)} \xrightarrow{V^{\otimes \ell}} V^{\otimes(n+1)} = \mathcal{E}nd_V(g, n),$$

where $\ell : \text{Leg}(\Gamma) \rightarrow \{0, \dots, n\}$ is the labeling of the legs of Γ . It is easy to show that the compositions $\{\alpha_\Gamma; \Gamma \in \mathbf{MGr}(g, n)\}$ define on $\mathcal{E}nd_V$ the structure of an un-stable unital modular operad, see [33, § 2.25].

An *algebra* over a modular operad \mathcal{A} is a vector space V with an inner product B , together with a morphism $\rho : \mathcal{A} \rightarrow \mathcal{E}nd_V$ of modular operads. Several important structures are algebras over modular operads. For example, an algebra over the homology $H_*(\overline{\mathcal{M}})$ of the Grothendieck-Knudsen modular operad is the same as a cohomological field theory in the sense of [55]. Other physically relevant algebras over modular operads can be found in [33, 78, 83]. Relations between modular operads, chord diagrams and Vassiliev invariants are studied in [42].

8. PROPS

Operads are devices invented to describe structures consisting of operations with several inputs and *one* output. There are, however, important structures with operations having several inputs and *several* outputs. Let us recall the most prominent one:

Example 53. A (associative) *bialgebra* is a \mathbf{k} -module V with a *multiplication* $\mu : V \otimes V \rightarrow V$ and a *comultiplication* (also called a *diagonal*) $\Delta : V \rightarrow V \otimes V$. The multiplication is associative:

$$\mu(\mu \otimes id_V) = \mu(id_V \otimes \mu),$$

the comultiplication is coassociative:

$$(\Delta \otimes id_V)\Delta = (id_V \otimes \Delta)\Delta$$

and the usual compatibility between μ and Δ is assumed:

$$(42) \quad \Delta(u \cdot v) = \Delta(u) \cdot \Delta(v) \quad \text{for } u, v \in V,$$

where $u \cdot v := \mu(u, v)$ and the dot \cdot in the right hand side denotes the multiplication induced on $V \otimes V$ by μ . Loosely speaking, bialgebras are Hopf algebras without unit, counit and antipode.

PROPs (an abbreviation of **product** and **permutation category**) describe structures as in Example 53. Although PROPs are more general than operads, they appeared much sooner, in a 1965 Mac Lane's paper [68]. This might be explained by the fact that the definition of PROPs is more compact than that of operads – compare Definition 54 below with Definition 1 in Section 1. PROPs then entered the ‘renaissance of operads’ in 1996 via [73].

Definition 54 uses the notion of a symmetric strict monoidal category which we consider so basic and commonly known that we will not recall it, standard citations are [68, 67], see also [83, § II.1.1]. An example is the category $\mathbf{Mod}_{\mathbf{k}}$ of \mathbf{k} -modules, with the monoidal product \odot given by the tensor product $\otimes = \otimes_{\mathbf{k}}$, the symmetry $S_{U,V} : U \otimes V \rightarrow V \otimes U$ defined as $S_{U,V}(u, v) := v \otimes u$ for $u \in U$ and $v \in V$, and the unit $\mathbf{1}$ the ground ring \mathbf{k} .

Definition 54. *A (\mathbf{k} -linear) PROP (called a theory in [73]) is a symmetric strict monoidal category $\mathbf{P} = (\mathbf{P}, \odot, S, \mathbf{1})$ enriched over $\mathbf{Mod}_{\mathbf{k}}$ such that*

- (i) *the objects are indexed by (or identified with) the set $\mathbb{N} = \{0, 1, 2, \dots\}$ of natural numbers, and*
- (ii) *the product satisfies $m \odot n = m + n$, for any $m, n \in \mathbb{N} = \text{Ob}(\mathbf{P})$ (hence the unit $\mathbf{1}$ equals 0).*

Recall that the $\mathbf{Mod}_{\mathbf{k}}$ -enrichment in the above definition means that each hom-set $Mor_{\mathbf{P}}(m, n)$ is a \mathbf{k} -module and the operations of the monoidal category \mathbf{P} (the composition \circ , the product \odot and the symmetry S) are compatible with this \mathbf{k} -linear structure.

For a PROP \mathbf{P} denote $\mathbf{P}(m, n) := Mor_{\mathbf{P}}(m, n)$. The symmetry S induces, via the canonical identifications $m \cong \mathbf{1}^{\odot m}$ and $n \cong \mathbf{1}^{\odot n}$, on each $\mathbf{P}(m, n)$ a structure of (Σ_m, Σ_n) -bimodule (left Σ_m -right Σ_n -module such that the left action commutes with the right one). Therefore a PROP is a collection $\mathbf{P} = \{\mathbf{P}(m, n)\}_{m, n \geq 0}$ of (Σ_m, Σ_n) -bimodules, together with two types of compositions, *horizontal*

$$\otimes : \mathbf{P}(m_1, n_1) \otimes \cdots \otimes \mathbf{P}(m_s, n_s) \rightarrow \mathbf{P}(m_1 + \cdots + m_s, n_1 + \cdots + n_s),$$

induced, for all $m_1, \dots, m_s, n_1, \dots, n_s \geq 0$, by the monoidal product \odot of \mathbf{P} , and *vertical*

$$\circ : \mathbf{P}(m, n) \otimes \mathbf{P}(n, k) \rightarrow \mathbf{P}(m, k),$$

given, for all $m, n, k \geq 0$, by the categorial composition. The monoidal unit is an element $e := \mathbf{1} \in \mathbf{P}(1, 1)$. In Definition 54, $\mathbf{Mod}_{\mathbf{k}}$ can be replaced by an arbitrary symmetric strict monoidal category.

Let $\mathbf{P} = \{\mathbf{P}(m, n)\}_{m, n \geq 0}$ and $\mathbf{Q} = \{\mathbf{Q}(m, n)\}_{m, n \geq 0}$ be two PROPs. A *homomorphism* $f : \mathbf{P} \rightarrow \mathbf{Q}$ is a sequence $f = \{f(m, n) : \mathbf{P}(m, n) \rightarrow \mathbf{Q}(m, n)\}_{m, n \geq 0}$ of bi-equivariant maps which commute with both the vertical and horizontal compositions. An *ideal* in a PROP \mathbf{P} is a system

$\mathbb{I} = \{\mathbb{I}(m, n)\}_{m, n \geq 0}$ of left Σ_m - right Σ_n -invariant subspaces $\mathbb{I}(m, n) \subset \mathbb{P}(m, n)$ which is closed, in the obvious sense, under both the vertical and horizontal compositions. Kernels, images, &c., of homomorphisms between PROPs, as well as quotients of PROPs by PROPic ideals, are defined componentwise, see [73, 111, 112, 113] for details.

Example 55. The *endomorphism* PROP of a \mathbf{k} -module V is the system

$$\mathcal{E}nd_V = \{\mathcal{E}nd_V(m, n)\}_{m, n \geq 0}$$

with $\mathcal{E}nd_V(m, n)$ the space of linear maps $Lin(V^{\otimes n}, V^{\otimes m})$ with n ‘inputs’ and m ‘outputs,’ $e \in \mathcal{E}nd_V(1, 1)$ the identity map, horizontal composition given by the tensor product of linear maps, and vertical composition by the ordinary composition of linear maps.

Also algebras over PROPs can be introduced in a very concise way:

Definition 56. A \mathbb{P} -algebra is a strict symmetric monoidal functor $\lambda : \mathbb{P} \rightarrow \mathbf{Mod}_{\mathbf{k}}$ of enriched monoidal categories. The value $\lambda(1)$ is the underlying space of the algebra ρ .

It is easy to see that a \mathbb{P} -algebra is the same as a PROP homomorphism $\rho : \mathbb{P} \rightarrow \mathcal{E}nd_V$. As in Proposition 24, a \mathbb{P} -algebra is determined by a system

$$\alpha : \mathbb{P}(m, n) \otimes V^{\otimes n} \rightarrow V^{\otimes m}, \quad m, n, \geq 0,$$

of linear maps satisfying appropriate axioms.

As before, the first step in formulating an unbiased definition of PROPs is to specify their underlying category. A Σ -bimodule is a system $E = \{E(m, n)\}_{m, n \geq 0}$ such that each $E(m, n)$ is a left $\mathbf{k}[\Sigma_m]$ - right $\mathbf{k}[\Sigma_n]$ -bimodule. Let $\Sigma\text{-bimod}$ denote the category of Σ -bimodules. For $E \in \Sigma\text{-bimod}$ and finite sets Y, X with m resp. n elements put

$$E(Y, X) := \text{Bij}(Y, [m]) \times_{\Sigma_m} E(m, n) \times_{\Sigma_n} \text{Bij}([n], X), \quad m, n \geq 0,$$

where $\text{Bij}(-, -)$ is the same as in (22). Pasting schemes for PROPs are *directed* (m, n) -graphs, by which we mean finite, not necessary connected, graphs in the sense of Definition 49 such that

- (i) each edge is equipped with a direction
- (ii) there are no directed cycles and
- (iii) the set of legs is divided into the set of inputs labeled by $\{1, \dots, n\}$ and the set of outputs labeled by $\{1, \dots, m\}$.

An example of a directed graph is given in Figure 11. We denote by $\mathbf{Gr}(m, n)$ the category of directed (m, n) -graphs and their isomorphisms. The direction of edges determines at each vertex $v \in \text{vert}(G)$ of a directed graph G a disjoint decomposition

$$\text{edge}(v) = \text{in}(v) \sqcup \text{out}(v)$$

of the set of edges adjacent to v into the set $\text{in}(v)$ of incoming edges and the set $\text{out}(v)$ of outgoing edges. The pair $(\#(\text{out}(v)), \#(\text{in}(v))) \in \mathbb{N} \times \mathbb{N}$ is called the *biarity* of v . To incorporate the unit, we need to extend the category $\mathbf{Gr}(m, n)$, for $m = n$, into the category $\mathbf{UGr}(m, n)$ by allowing the exceptional graph

$$\uparrow \uparrow \uparrow \cdots \uparrow \in \mathbf{UGr}(n, n), \quad n \geq 1,$$

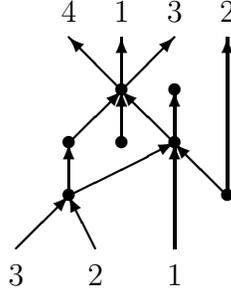


FIGURE 11. A directed graph from $\mathbf{Gr}(4, 3)$.

with n inputs, n outputs and no vertices. For a graph $G \in \mathbf{UGr}(m, n)$ and a Σ -bimodule E , let

$$E(G) := \bigotimes_{v \in \text{vert}(G)} E(\text{out}(v), \text{in}(v)).$$

and

$$(43) \quad \Gamma_{\mathbf{P}}(E)(m, n) := \text{colim}_{G \in \mathbf{UGr}(m, n)} E(G), \quad m, n \geq 0.$$

The Σ -bimodule $\Gamma_{\mathbf{P}}(E)$ is a PROP, with the vertical composition given by the disjoint union of graphs, the horizontal composition by grafting the legs, and the unit the exceptional graph $\uparrow \in \Gamma_{\mathbf{P}}(E)(1, 1)$. The following proposition follows from [84] and [111, 112, 113]:

Proposition 57. *The PROP $\Gamma_{\mathbf{P}}(E)$ is the free PROP generated by the Σ -bimodule E .*

As in the previous sections, (43) defines a triple $\Gamma_{\mathbf{P}} : \Sigma\text{-bimod} \rightarrow \Sigma\text{-bimod}$ with the triple multiplication of erasing the braces. According to general principles [20], Proposition 57 is almost equivalent to

Proposition 58. *PROPs are algebras over the triple $\Gamma_{\mathbf{P}}$.*

One may obviously consider *non-unital PROPs* defined as algebras over the triple

$$\Psi_{\mathbf{P}}(E)(m, n) := \text{colim}_{G \in \mathbf{Gr}(m, n)} E(G), \quad m, n \geq 0,$$

and develop a theory parallel to the theory of non-unital operads reviewed in Section 2.

Example 59. We will use the graphical language explained in Example 36. Let $\Gamma(\blacktriangleright, \blacktriangleright)$ be the free PROP generated by one operation \blacktriangleright of biarity $(1, 2)$ and one operation \blacktriangleright of biarity $(2, 1)$. As we noticed already in [72, 73], the PROP \mathbf{B} describing bialgebras equals

$$\mathbf{B} = \Gamma(\blacktriangleright, \blacktriangleright) / \mathfrak{l}_{\mathbf{B}},$$

where $\mathfrak{l}_{\mathbf{B}}$ is the PROPic ideal generated by

$$(44) \quad \blacktriangleright - \blacktriangleright, \quad \blacktriangleright - \blacktriangleright \quad \text{and} \quad \blacktriangleright - \blacktriangleright.$$

In the above display we denoted

$$\begin{aligned} \blacktriangleright &:= \blacktriangleright \circ (\blacktriangleright \otimes e), \quad \blacktriangleleft := \blacktriangleright \circ (e \otimes \blacktriangleright), \quad \blacktriangleright := (\blacktriangleright \otimes e) \circ \blacktriangleright, \quad \blacktriangleleft := (e \otimes \blacktriangleright) \circ \blacktriangleright, \\ \blacktriangleright &:= \blacktriangleright \circ \blacktriangleright \quad \text{and} \quad \blacktriangleright \circ \blacktriangleright := (\blacktriangleright \otimes \blacktriangleright) \circ \kappa \circ (\blacktriangleright \otimes \blacktriangleright), \end{aligned}$$

where $\kappa \in \Sigma_4$ is the permutation

$$(45) \quad \kappa := \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix} = \begin{array}{c} \bullet \bullet \bullet \bullet \\ | \times | \\ \bullet \bullet \bullet \bullet \end{array}.$$

The above description of \mathbf{B} is ‘tautological,’ but B. Enriquez and P. Etingof found in [24, Proposition 6.2] the following basis of the \mathbf{k} -linear space $\mathbf{B}(m, n)$ for arbitrary $m, n \geq 1$. Let $\blacktriangleright \in \mathbf{B}(1, 2)$ be the equivalence class, in $\mathbf{B} = \Gamma(\blacktriangleright, \blacktriangleright)/\mathbf{I}_{\mathbf{B}}$, of the generator $\blacktriangleright \in \Gamma(\blacktriangleright, \blacktriangleright)(1, 2)$ (we use the same symbol both for a generator and its equivalence class). Define $\blacktriangleright^{[1]} := e \in \mathbf{B}(1, 1)$ and, for $a \geq 2$, let

$$\blacktriangleright^{[a]} := \blacktriangleright \circ (\blacktriangleright \otimes e) \circ (\blacktriangleright \otimes e^{\otimes 2}) \circ \cdots \circ (\blacktriangleright \otimes e^{\otimes (a-2)}) \in \mathbf{B}(1, a).$$

Let $\blacktriangleright_{[b]} \in \mathbf{B}(b, 1)$ has the obvious similar meaning. The elements

$$(46) \quad (\blacktriangleright^{[a_1]} \otimes \cdots \otimes \blacktriangleright^{[a_m]}) \circ \sigma \circ (\blacktriangleright_{[b^1]} \otimes \cdots \otimes \blacktriangleright_{[b^n]}),$$

where $\sigma \in \Sigma_N$ for some $N \geq 1$, and $a_1 + \cdots + a_m = b^1 + \cdots + b^n = N$, form a \mathbf{k} -linear basis of $\mathbf{B}(m, n)$. This result can also be found in [57]. See also [59, 94] for the bialgebra PROP viewed from a different perspective.

Example 60. Each operad \mathcal{P} generates a unique PROP \mathbf{P} such that $\mathbf{P}(1, n) = \mathcal{P}(n)$ for each $n \geq 0$. The components of such a PROP are given by

$$\mathbf{P}(m, n) = \bigoplus_{r_1 + \cdots + r_k = n} [\mathcal{P}(1, r_1) \otimes \cdots \otimes \mathcal{P}(1, r_k)] \times_{\Sigma_{r_1} \times \cdots \times \Sigma_{r_k}} \Sigma_n,$$

for each $m, n \geq 0$. The (topological) PROPs considered in [10] are all of this type. On the other hand, Example 59 shows that not each PROP is of this form. A PROP \mathbf{P} is generated by an operad if and only if it has a presentation $\mathbf{P} = \Gamma_{\mathbf{P}}(E)/(R)$, where E is a Σ -bimodule such that $E(m, n) = 0$ for $m \neq 1$ and R is generated by elements in $\Gamma_{\mathbf{P}}(E)(1, n)$, $n \geq 0$.

9. PROPERADS, DIOPERADS AND $\frac{1}{2}$ PROPS

As we saw in Proposition 33, under some mild assumptions, the components of free operads are finite-dimensional. In contrast, PROPs are huge objects. For example, the component $\Gamma_{\mathbf{P}}(\blacktriangleright, \blacktriangleright)(m, n)$ of the free PROP $\Gamma_{\mathbf{P}}(\blacktriangleright, \blacktriangleright)$ used in the definition of the bialgebra PROP \mathbf{B} in Example 59 is infinite-dimensional for each $m, n \geq 1$, and also the components of the bialgebra PROP \mathbf{B} itself are infinite-dimensional, as follows from the fact that the Enriquez-Etingof basis (46) of $\mathbf{B}(m, n)$ has, for $m, n \geq 1$, infinitely many elements.

To handle this combinatorial explosion of PROPs combined with lack of suitable filtrations, smaller versions of PROPs were invented. Let us begin with the simplest modification which we use as an example which explains the general scheme of modifying PROPs. Denote $\mathbf{UGr}_c(m, n)$

the full subcategory of $\mathbf{UGr}(m, n)$ consisting of *connected* graphs and consider the triple defined by

$$(47) \quad \Gamma_c(E)(m, n) := \operatorname{colim}_{G \in \mathbf{UGr}_c(m, n)} E(G), \quad m, n \geq 0,$$

for $E \in \Sigma\text{-bimod}$. The following notion was introduced by B. Vallette [111, 112, 113].

Definition 61. Properads are algebras over the triple $\Gamma_c : \Sigma\text{-bimod} \rightarrow \Sigma\text{-bimod}$.

A properad is therefore a Σ -bimodule with operations that determine coherent contractions along connected graphs. A biased definition of properads is given in [111, 112, 113]. Since Γ_c is a subtriple of Γ_P , each PROP is automatically also a properad. Therefore one may speak about the *endomorphism properad* $\mathcal{E}nd_V$ and define *algebras* over a properad P as properad homomorphisms $\rho : P \rightarrow \mathcal{E}nd_V$. Algebras over other versions of PROPs recalled below can be defined in a similar way.

Example 62. Associative bialgebras reviewed in Example 59 are algebras over the properad B defined (tautologically) as the quotient of the free properad $\Gamma_c(\blacktriangleright, \blacktriangleright)$ by the properadic ideal generated by the elements listed in (44). We leave as an exercise to describe the sub-basis of (46) that span $B(m, n)$, $m, n \geq 1$.

The following slightly artificial structure exists over PROPs but not over properads. It consists of a ‘multiplication’ $\mu = \blacktriangleright : V \otimes V \rightarrow V$, a ‘comultiplication’ $\Delta = \blacktriangleright : V \rightarrow V \otimes V$ and a linear map $f = \blacktriangleright : V \rightarrow V$ satisfying $\Delta \circ \mu = f \otimes f$ or, diagrammatically

$$\blacktriangleright = \blacktriangleright \blacktriangleright.$$

This structure cannot be a properad algebra because the graph on the right hand side of the above display is not connected.

Properads are still huge objects. The first really small version of PROPs were dioperads introduced in 2003 by W.L. Gan [28]. As a motivation for his definition, consider the following:

Example 63. A *Lie bialgebra* is a vector space V with a Lie algebra structure $[-, -] = \blacktriangleright : V \otimes V \rightarrow V$ and a Lie diagonal $\delta = \blacktriangleright : V \rightarrow V \otimes V$. We assume that $[-, -]$ and δ are related by

$$(48) \quad \delta[a, b] = \sum ([a_{(1)}, b] \otimes a_{(2)} + [a, b_{(1)}] \otimes b_{(2)} + a_{(1)} \otimes [a_{(2)}, b] + b_{(1)} \otimes [a, b_{(2)}])$$

for any $a, b \in V$, with the Sweedler notation $\delta a = \sum a_{(1)} \otimes a_{(2)}$ and $\delta b = \sum b_{(1)} \otimes b_{(2)}$.

Lie bialgebras are governed by the PROP $\text{LieB} = \Gamma(\blacktriangleright, \blacktriangleright) / \mathfrak{I}_{\text{LieB}}$, where \blacktriangleright and \blacktriangleright are now *antisymmetric* and $\mathfrak{I}_{\text{LieB}}$ denotes the ideal generated by

$$(49) \quad \begin{array}{c} \diagup \\ 1 \ 2 \ 3 \end{array} + \begin{array}{c} \diagup \\ 2 \ 3 \ 1 \end{array} + \begin{array}{c} \diagup \\ 3 \ 1 \ 2 \end{array}, \quad \begin{array}{c} 1 \ 2 \ 3 \\ \diagdown \\ \diagup \end{array} + \begin{array}{c} 2 \ 3 \ 1 \\ \diagdown \\ \diagup \end{array} + \begin{array}{c} 3 \ 1 \ 2 \\ \diagdown \\ \diagup \end{array} \quad \text{and} \quad \begin{array}{c} 1 \ 2 \\ \diagdown \\ \diagup \\ 1 \ 2 \end{array} - \begin{array}{c} 1 \ 2 \\ \diagdown \\ \diagup \\ 1 \ 2 \end{array} - \begin{array}{c} 1 \ 2 \\ \diagdown \\ \diagup \\ 1 \ 2 \end{array} + \begin{array}{c} 2 \ 1 \\ \diagdown \\ \diagup \\ 1 \ 2 \end{array} + \begin{array}{c} 2 \ 1 \\ \diagdown \\ \diagup \\ 1 \ 2 \end{array},$$

with labels indicating the corresponding permutations of the inputs and outputs.

We observe that all graphs in (49) are not only connected as demanded for properads, but also simply-connected. This suggests considering the full subcategory $\mathbf{UGr}_D(m, n)$ of $\mathbf{UGr}(m, n)$

consisting of *connected simply-connected* graphs and the related triple

$$(50) \quad \Gamma_{\mathbb{D}}(E)(m, n) := \operatorname{colim}_{G \in \mathbf{UGr}_{\mathbb{D}}(m, n)} E(G), \quad m, n \geq 0.$$

Definition 64. Dioperads are algebras over the triple $\Gamma_{\mathbb{D}} : \Sigma\text{-bimod} \rightarrow \Sigma\text{-bimod}$.

A biased definition of dioperads can be found in [28]. As observed by T. Leinster, dioperads are more or less equivalent to polycategories, in the sense of [108], with one object. Lie bialgebras reviewed in Example 63 are algebras over a dioperad. Another important class of dioperad algebras is recalled in:

Example 65. An *infinitesimal bialgebra* [48] (called in [26, Example 11.7] a *mock bialgebra*) is a vector space V with an associative multiplication $\cdot : V \otimes V \rightarrow V$ and a coassociative comultiplication $\Delta : V \rightarrow V \otimes V$ such that

$$\Delta(a \cdot b) = \sum (a_{(1)} \otimes a_{(2)} \cdot b + a \cdot b_{(1)} \otimes b_{(2)})$$

for any $a, b \in V$. It is easy to see that the axioms of infinitesimal bialgebras are encoded by the following simply connected graphs:

$$\begin{array}{c} \text{A} \\ \text{---} \\ \text{B} \end{array} - \begin{array}{c} \text{A} \\ \text{---} \\ \text{B} \end{array}, \quad \begin{array}{c} \text{Y} \\ \text{---} \\ \text{Y} \end{array} - \begin{array}{c} \text{Y} \\ \text{---} \\ \text{Y} \end{array} \quad \text{and} \quad \begin{array}{c} \text{X} \\ \text{---} \\ \text{Y} \end{array} - \begin{array}{c} \text{Y} \\ \text{---} \\ \text{Y} \end{array} - \begin{array}{c} \text{Y} \\ \text{---} \\ \text{Y} \end{array}.$$

Observe that associative bialgebras recalled in Example 53 cannot be defined over dioperads, because the rightmost graph in (44) is not simply connected. The following proposition, which should be compared to Proposition 33, shows that dioperads are of the same size as operads.

Proposition 66. Let $E = \{E(m, n)\}_{m, n \geq 0}$ be a Σ -bimodule such that

$$(51) \quad E(m, n) = 0 \text{ for } m + n \leq 2$$

and that $E(m, n)$ is finite-dimensional for all remaining m, n . Then the components $\Gamma_{\mathbb{D}}(E)(m, n)$ of the free dioperad $\Gamma_{\mathbb{D}}(E)$ are finite-dimensional, for all $m, n \geq 0$.

The proof, similar to the proof of Proposition 33, is based on the observation that the assumption (51) reduces the colimit (50) to a summation over reduced trees (trees whose all vertices have at least three adjacent edges).

An important problem arising in connection with deformation quantization is to find a reasonably small, explicit cofibrant resolution of the bialgebra PROP \mathbf{B} . Here by a resolution we mean a differential graded PROP \mathbf{R} together with a homomorphism $\beta : \mathbf{R} \rightarrow \mathbf{B}$ inducing a homology isomorphism. Cofibrant in this context means that \mathbf{R} is of the form $(\Gamma_{\mathbb{P}}(E), \partial)$, where the generating Σ -bimodule E decomposes as $E = \bigoplus_{n \geq 0} E_n$ and the differential decreases the filtration, that is

$$\partial(E_n) \subset \Gamma_{\mathbb{P}}(E)_{<n}, \quad \text{for each } n \geq 0,$$

where $\Gamma_{\mathbb{P}}(E)_{<n}$ denotes the sub-PROP of $\Gamma(E)$ generated by $\bigoplus_{j < n} E_j$. This notion is an PROPic analog of the Koszul-Sullivan algebra in rational homotopy theory [36]. Several papers devoted to finding \mathbf{R} appeared recently [57, 99, 97, 98, 101, 103, 102]. The approach of [79] is based on the observation that \mathbf{B} is a deformation, in the sense explained below, of the PROP describing structures recalled in the following:

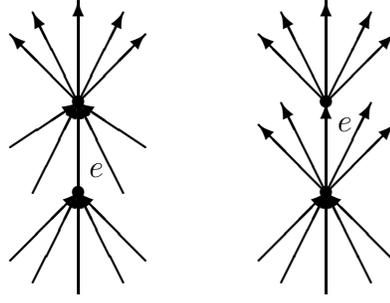


FIGURE 12. Edges allowed in a $\frac{1}{2}$ graph.

Definition 67. A half-bialgebra or simply a $\frac{1}{2}$ bialgebra is a vector space V with an associative multiplication $\mu : V \otimes V \rightarrow V$ and a coassociative comultiplication $\Delta : V \rightarrow V \otimes V$ that satisfy

$$(52) \quad \Delta(u \cdot v) = 0, \quad \text{for each } u, v \in V.$$

We chose this strange name because (52) is indeed one half of the compatibility relation (42) of associative bialgebras. $\frac{1}{2}$ bialgebras are algebras over the PROP

$$\frac{1}{2}\mathbf{B} := \Gamma(\blacktriangleright) / (\blacktriangleright = \blacktriangleright, \blacktriangleright = \blacktriangleright, \blacktriangleright = 0).$$

Now define, for a formal variable t , \mathbf{B}_t to be the quotient of the free PROP $\Gamma(\blacktriangleright, \blacktriangleright)$ by the ideal generated by

$$\blacktriangleright = \blacktriangleright, \quad \blacktriangleright = \blacktriangleright, \quad \blacktriangleright = t \cdot \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array}.$$

Thus \mathbf{B}_t is a one-parametric family of PROPs with the property that $\mathbf{B}_0 = \frac{1}{2}\mathbf{B}$. At a generic t , \mathbf{B}_t is isomorphic to the bialgebra PROP \mathbf{B} . In other words, the PROP for bialgebras is a deformation of the PROP for $\frac{1}{2}$ bialgebras. According to general principles of homological perturbation theory [35], one may try to construct the resolution \mathbf{R} as a perturbation of a cofibrant resolution $\frac{1}{2}\mathbf{R}$ of the PROP $\frac{1}{2}\mathbf{B}$. Since $\frac{1}{2}\mathbf{B}$ is simpler than \mathbf{B} , one may expect that resolving $\frac{1}{2}\mathbf{B}$ would be a simpler task than resolving \mathbf{B} .

For instance, one may realize that $\frac{1}{2}$ bialgebras are algebras over a dioperad $\frac{1}{2}\mathbf{B}$, use [28] to construct a resolution $\frac{1}{2}\mathbf{R}$ of the dioperad $\frac{1}{2}\mathbf{B}$, and then take $\frac{1}{2}\mathbf{R}$ to be the PROP generated by $\frac{1}{2}\mathbf{R}$. More precisely, one denotes

$$(53) \quad F_1 : \mathbf{diOp} \rightarrow \mathbf{PROP}$$

the left adjoint to the forgetful functor $\mathbf{PROP} \xrightarrow{\square_1} \mathbf{diOp}$ and defines $\frac{1}{2}\mathbf{R} := F_1(\frac{1}{2}\mathbf{R})$.

The problem is that we do not know whether the functor F_1 is exact, so it is not clear if $\frac{1}{2}\mathbf{R}$ constructed in this way is really a resolution of $\frac{1}{2}\mathbf{B}$. To get around this subtlety, M. Kontsevich observed that $\frac{1}{2}$ bialgebras live over a version of PROPs which is smaller than dioperads. It can be defined as follows.

Let an (m, n) - $\frac{1}{2}$ graph be a connected simply-connected directed (m, n) -graph whose each edge e has the following property: either e is the unique outgoing edge of its initial vertex or e is the unique incoming edge of its terminal vertex, see Figure 12. An example of an (m, n) - $\frac{1}{2}$ graph is given in Figure 13. Let $\mathbf{Gr}_{\frac{1}{2}}(m, n)$ be the category of (m, n) - $\frac{1}{2}$ graphs and their isomorphisms. Define a triple $\Gamma_{\frac{1}{2}} : \Sigma\text{-bimod} \rightarrow \Sigma\text{-bimod}$ by

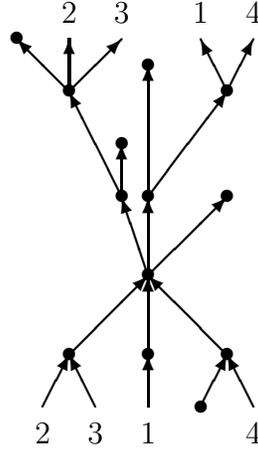


FIGURE 13. A graph from $\text{Gr}_{\frac{1}{2}}(4, 4)$.

$$(54) \quad \Gamma_{\frac{1}{2}}(E)(m, n) := \operatorname{colim}_{G \in \text{Gr}_{\frac{1}{2}}(m, n)} E(G), \quad m, n \geq 0.$$

Definition 68. A $\frac{1}{2}$ PROP (called a meager PROP in [53]) is an algebra over the triple $\Gamma_{\frac{1}{2}} : \Sigma\text{-bimod} \rightarrow \Sigma\text{-bimod}$.

A biased definition of $\frac{1}{2}$ PROPs can be found in [53, 79, 84]. We followed the original convention of [53] that $\frac{1}{2}$ PROPs do not have units; the unital version of $\frac{1}{2}$ PROPs can be defined in an obvious way, compare also the remarks in [79].

Example 69. $\frac{1}{2}$ bialgebras are algebras over a $\frac{1}{2}$ PROP which we denote $\frac{1}{2}\mathbf{b}$. Another example of structures that can be defined over $\frac{1}{2}$ PROPs are *Lie $\frac{1}{2}$ bialgebras* consisting of a Lie algebra bracket $[-, -] : V \otimes V \rightarrow V$ and a Lie diagonal $\delta : V \rightarrow V \otimes V$ satisfying one-half of (48):

$$\delta[a, b] = 0.$$

Let us denote by

$$F : \frac{1}{2}\text{PROP} \rightarrow \text{PROP}$$

the left adjoint to the forgetful functor $\text{PROP} \xrightarrow{\square} \frac{1}{2}\text{PROP}$ from the category of PROPs to the category of $\frac{1}{2}$ PROPs. M. Kontsevich observed that, in contrast to $F_1 : \text{diOp} \rightarrow \text{PROP}$ in (53), F is a *polynomial* functor, which immediately implies the following important theorem [53, 84].

Theorem 70. *The functor $F : \frac{1}{2}\text{PROP} \rightarrow \text{PROP}$ is exact.*

Now one may take a resolution $\frac{1}{2}\mathbf{r}$ of the $\frac{1}{2}$ PROP $\frac{1}{2}\mathbf{b}$ and put $\frac{1}{2}\mathbf{R} := F(\frac{1}{2}\mathbf{r})$. Theorem 70 guarantees that $\frac{1}{2}\mathbf{R}$ defined in this way is indeed a resolution of the PROP $\frac{1}{2}\mathbf{B}$. Let us mention that there are also other structures invented to study resolutions of the PROP \mathbf{B} , as $\frac{2}{3}$ PROPs of Shoikhet [101], matrons of Saneblidze and Umble [99], or special PROPs considered in [79].

Pasting schemes	corresponding structures
rooted trees	non-unital operads
May's trees	non-unital May's operads
extended rooted trees	operads
cyclic trees	non-unital cyclic operads
extended cyclic trees	cyclic operads
stable labeled graphs	modular operads
extended directed graphs	PROPs
extended connected directed graphs	properads
extended connected 1-connected dir. graphs	dioperads
$\frac{1}{2}$ graphs	$\frac{1}{2}$ PROPs

FIGURE 14. Pasting schemes and the structures they define.

The constructions reviewed in this section can be organized into the following chain of inclusions of full subcategories:

$$\text{Oper} \subset \frac{1}{2}\text{PROP} \subset \text{diOp} \subset \text{Proper} \subset \text{PROP}.$$

The general scheme behind all these constructions is the following. We start by choosing a sub-groupoid $\mathbf{SGr} = \bigsqcup_{m,n \geq 0} \mathbf{SGr}(m, n)$ of $\mathbf{Gr} := \bigsqcup_{m,n \geq 0} \mathbf{Gr}(m, n)$ (or a subgroupoid of $\mathbf{UGr} := \bigsqcup_{m,n \geq 0} \mathbf{UGr}(m, n)$ if we want units). Then we define a functor $\Gamma_{\mathbf{S}} : \Sigma\text{-bimod} \rightarrow \Sigma\text{-bimod}$ by

$$\Gamma_{\mathbf{S}}(E)(m, n) := \text{colim}_{G \in \mathbf{SGr}(m, n)} E(G), \quad m, n \geq 0.$$

It is easy to see that $\Gamma_{\mathbf{S}}$ is a subtriple of the PROP triple $\Gamma_{\mathbf{P}}$ if and only if the following two conditions are satisfied:

- (i) the groupoid \mathbf{SGr} is *hereditary* in the sense that, given a graph from \mathbf{SGr} with vertices decorated by graphs from \mathbf{SGr} , then the graph obtained by ‘forgetting the braces’ again belongs to \mathbf{SGr} , and
- (ii) \mathbf{SGr} contains all directed corollas.

Hereditary (i) is necessary for $\Gamma_{\mathbf{S}}$ to be closed under the triple multiplication of $\Gamma_{\mathbf{P}}$ while (ii) guarantees that $\Gamma_{\mathbf{S}}$ has an unit. Plainly, all the three choices used above – \mathbf{UGr}_c , \mathbf{UGr}_D and $\mathbf{Gr}_{\frac{1}{2}}$ – satisfy the above assumptions. Let us mention that one may modify the definition of PROPs also by *enlarging* the category $\mathbf{Gr}(m, n)$, as was done for *wheeled PROPs* in [93]. Pasting schemes and the corresponding structures reviewed in this article are listed in Figure 14.

REFERENCES

[1] D. Balavoine. Homology and cohomology with coefficients, of an algebra over a quadratic operad. *J. Pure Appl. Algebra*, 132:221–258, 1998.
 [2] M. A. Batanin. Monoidal globular categories as a natural environment for the theory of weak n -categories. *Adv. in Math.*, 136(1):39–103, 1998.

- [3] M.A. Batanin. Homotopy coherent category theory and A_∞ -structures in monoidal categories. *J. Pure Appl. Algebra*, 123(1-3):67–103, 1998.
- [4] M.A. Batanin. The Eckmann-Hilton argument, higher operads and E_n -spaces. Preprint `math.CT/0207281`, July 2002.
- [5] M.A. Batanin. The combinatorics of iterated loop spaces. Preprint `math.CT/0301221`, January 2003.
- [6] A. Beilinson and V. Drinfel'd. *Chiral Algebras*, volume 51 of *Colloquium Publications*. American Mathematical Society, 2004.
- [7] C. Berger and I. Moerdijk. Axiomatic homotopy theory for operads. *Comment. Math. Helv.*, 78(4):805–831, 2003.
- [8] C. Berger and I. Moerdijk. Resolution of coloured operads and rectification of homotopy algebras. Preprint `math.AT/0512576`, December 2005.
- [9] J.M. Boardman and R.M. Vogt. Homotopy-everything H -spaces. *Bull. Amer. Math. Soc.*, 74(6):1117–1122, 1968.
- [10] J.M. Boardman and R.M. Vogt. *Homotopy Invariant Algebraic Structures on Topological Spaces*. Springer-Verlag, 1973.
- [11] A. Burroni. T -catégories (catégories dans un triple). *Cahiers Topologie Géom. Différentielle*, 12:215–321, 1971.
- [12] P. Cartier. Homologie cycliques: rapport sur des travaux récents de Connes, Karoubi, Loday, Quillen... Séminaire Bourbaki 621, 1983–84.
- [13] M. Chas and D. Sullivan. String topology. Preprint `math.GT/9911159`, November 1999.
- [14] F.R. Cohen and A.A. Voronov. Notes on string topology. Preprint `math.GT/0503625`, March 2005.
- [15] A. Connes and D. Kreimer. Renormalization in quantum field theory and the Riemann-Hilbert problem I: The Hopf algebra structure of graphs and the main theorem. *Comm. Math. Phys.*, 210(1):249–273, 2000.
- [16] P. Deligne. Résumé des premiers exposés de A. Grothendieck. In *Groupes de Monodromie en Géométrie Algébrique*, volume 288, pages 1–24. Springer-Verlag, Berlin, 1972.
- [17] P. Deligne. A letter to Stasheff, Gerstenhaber, May, Schechtman and Drinfel'd. Unpublished, 1993.
- [18] P. Deligne and D. Mumford. *The Irreducibility of the Space of Curves of Given Genus*, volume 36 of *Inst. Hautes Études Sci. Publ. Math.* I.H.E.S., 1969.
- [19] S. L. Devadoss. Tessellations of moduli spaces and the mosaic operad. In J.P. Meyer, J. Morava, and W.S. Wilson, editors, *Homotopy Invariant Algebraic Structures*, volume 239 of *Contemporary Math.*, pages 91–114, 1999.
- [20] S. Eilenberg and J.C. Moore. Adjoint functors and triples. *Illinois J. Math.*, 9:381–389, 1965.
- [21] A.D. Elmendorf. The development of structured ring spectra. In A. Baker and B. Richter, editors, *Structured ring spectra*, volume 315 of *London Math. Soc. Lecture Note Series*. London Math. Soc., 2004.
- [22] A.D. Elmendorf, I. Kříž, M.A. Mandell, and J.P. May. Modern foundations of stable homotopy theory. In I.M. James, editor, *Handbook of Algebraic Topology*, pages 217–257. North Holland, 1995.
- [23] A.D. Elmendorf, I. Kříž, M.A. Mandell, and J.P. May. *Rings, Modules, and Algebras in Stable Homotopy Theory*, volume 47 of *Mathematical Surveys and Monographs*. Amer. Math. Soc., 1997.
- [24] B. Enriquez and P. Etingof. On the invertibility of quantization functors. Preprint `math.QA/0306212`, June 2003.
- [25] P. Etingof, A. Henriques, J. Kamnitzer, and E. Rains. The cohomology ring of the real locus of the moduli space of stable curves of genus 0 with marked points. Preprint `math.AT/0507514`, August 2005.
- [26] T.F. Fox and M. Markl. Distributive laws, bialgebras, and cohomology. In J.-L. Loday, J.D. Stasheff, and A.A. Voronov, editors, *Operads: Proceedings of Renaissance Conferences*, volume 202 of *Contemporary Math.*, pages 167–205. Amer. Math. Soc., 1997.
- [27] B. Fresse. Koszul duality of operads and homology of partition posets. Preprint `math.AT/0301365`, January 2003.
- [28] W.L. Gan. Koszul duality for dioperads. *Math. Res. Lett.*, 10(1):109–124, 2003.
- [29] M. Gerstenhaber. The cohomology structure of an associative ring. *Ann. of Math.*, 78(2):267–288, 1963.
- [30] E. Getzler. Operads and moduli spaces of genus 0 Riemann surfaces. In R. Dijkgraaf, C. Faber, and G. van der Geer, editors, *The Moduli Space of Curves*, volume 129 of *Progress in Mathematics*, pages 99–230. Birkhäuser, 1995.
- [31] E. Getzler and J.D.S. Jones. Operads, homotopy algebra, and iterated integrals for double loop spaces. Preprint `hep-th/9403055`, March 1994.

- [32] E. Getzler and M.M. Kapranov. Cyclic operads and cyclic homology. In S.-T. Yau, editor, *Geometry, Topology and Physics for Raoul Bott*, volume 4 of *Conf. Proc. Lect. Notes. Geom. Topol.*, pages 167–201. International Press, 1995.
- [33] E. Getzler and M.M. Kapranov. Modular operads. *Compositio Math.*, 110(1):65–126, 1998.
- [34] V. Ginzburg and M.M. Kapranov. Koszul duality for operads. *Duke Math. J.*, 76(1):203–272, 1994.
- [35] V.K.A.M. Gugenheim and J.D. Stasheff. On perturbations and A_∞ -structures. *Bull. Soc. Math. Belgique*, 38:237–246, 1986.
- [36] S. Halperin. *Lectures on Minimal Models*, volume 261 of *Memoirs Soc. Math. France, Nouv. Ser. 9-10*. Soc. Mat. France, 1983.
- [37] R. Hartshorne. *Algebraic Geometry*, volume 52 of *Graduate Texts in Mathematics*. Springer-Verlag, 1977.
- [38] P.J. Hilton and U. Stambach. *A Course in Homological Algebra*, volume 4 of *Graduate texts in Mathematics*. Springer-Verlag, 1971.
- [39] V. Hinich. Homological algebra of homotopy algebras. *Comm. Algebra*, 10(25):3291–3323, 1997.
- [40] V. Hinich. Tamarkin’s proof of Kontsevich formality theorem. *Forum Mathematicum*, 15:591–614, 2003.
- [41] V. Hinich and V.V. Schechtman. Homotopy Lie algebras. *Advances in Soviet Mathem.*, 16(2):1–28, 1993.
- [42] V. Hinich and A. Vaintrob. Cyclic operads and algebra of chord diagrams. *Selecta Math. (N.S.)*, 8(2):237–282, 2002.
- [43] P. Hu. Higher string topology on general spaces. Preprint [math.AT/0401081](#), January 2004.
- [44] P. Hu, I. Kříž, and A.A. Voronov. On Kontsevich’s Hochschild cohomology conjecture. Preprint [math.AT/0309369](#), September 2003.
- [45] Y.-Z. Huang. Operadic formulation of topological vertex algebras and Gerstenhaber or Batalin-Vilkovisky algebras. *Comm. Math. Phys.*, 164:145–161, 1994.
- [46] Y.-Z. Huang. *Two-dimensional conformal geometry and vertex operator algebras*. Birkhäuser Boston Inc., Boston, MA, 1997.
- [47] Y.-Z. Huang and J. Lepowsky. Vertex operator algebras and operads. In *The Gel’fand Mathematics Seminars, 1990–1992*, pages 145–161. Birkhäuser, Boston, 1993.
- [48] S. A. Joni and G.-C. Rota. Coalgebras and bialgebras in combinatorics. *Stud. Appl. Math.*, 61(2):93–139, 1979.
- [49] T. Kimura, J.D. Stasheff, and A.A. Voronov. On operad structures of moduli spaces and string theory. *Comm. Math. Phys.*, 171:1–25, 1995.
- [50] F.F. Knudsen. The projectivity of the moduli space of stable curves, II: The stacks $M_{g,n}$. *Math. Scand.*, 52:161–199, 1983.
- [51] F.F. Knudsen and D. Mumford. The projectivity of the moduli space of stable curves I: Preliminaries on “det” and “Div”. *Math. Scand.*, 39:19–55, 1976.
- [52] M. Kontsevich. Formal (non)commutative symplectic geometry. In *The Gel’fand mathematics seminars 1990–1992*. Birkhäuser, 1993.
- [53] M. Kontsevich. An e-mail message to M. Markl, November 2002.
- [54] M. Kontsevich. Deformation quantization of Poisson manifolds. *Lett. Math. Phys.*, 66(3):157–216, December 2003.
- [55] M. Kontsevich and Y. Manin. Gromov-Witten classes, quantum cohomology, and enumerative geometry. *Comm. Math. Phys.*, 164:525–562, 1994.
- [56] M. Kontsevich and Y. Soibelman. Deformations of algebras over operads and Deligne’s conjecture. Preprint [math.QA/0001151](#), January 2000.
- [57] M. Kontsevich and Y. Soibelman. Deformation theory of bialgebras, Hopf algebras and tensor categories. Preprint, March 2002.
- [58] I. Kříž and J.P. May. *Operads, Algebras, Modules and Motives*, volume 233 of *Astérisque*. Société Mathématique de France, 1995.
- [59] S. Lack. Composing PROPS. *Theory Appl. Categ.*, 13:No. 9, 147–163 (electronic), 2004.
- [60] J. Lambek. Deductive systems and categories II. In P. Hilton, editor, *Category Theory, Homology Theory and their Applications I*, volume 86 of *Springer Lecture Notes in Mathematics*, pages 76–122, 1969.
- [61] T. Leinster. *Higher operads, higher categories*, volume 298 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2004.
- [62] T. Leinster. Operads in higher-dimensional category theory. *Theory Appl. Categ.*, 12:73–194, 2004.
- [63] J.-L. Loday. La renaissance des opérades. *Séminaire Bourbaki*, 792:47–74. 47ème année 1994-95.

- [64] J.-L. Loday. Algèbres ayant deux opérations associatives (digèbres). *C. R. Acad. Sci. Paris Sér. I Math.*, 321(2):141–146, 1995.
- [65] J.-L. Loday, A. Frabetti, F. Chapoton, and F. Goichot. *Dialgebras and Related Operads*, volume 1763 of *Lecture Notes in Mathematics*. Springer-Verlag, 2001.
- [66] J.-L. Loday and D. Quillen. Cyclic homology and the Lie algebra homology of matrices. *Comment. Math. Helv.*, 59:565–591, 1984.
- [67] S. Mac Lane. Natural associativity and commutativity. *Rice Univ. Stud.*, 49(1):28–46, 1963.
- [68] S. Mac Lane. Categorical algebra. *Bull. Amer. Math. Soc.*, 71:40–106, 1965.
- [69] M.A. Mandell. E_∞ algebras and p -adic homotopy theory. *Topology*, 40(1):43–94, 2001.
- [70] M.A. Mandell. Cochains and homotopy type. Preprint [math.AT/0311016](#), November 2003.
- [71] Y. I. Manin. *Frobenius Manifolds, Quantum Cohomology, and Moduli Spaces*, volume 47 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 1999.
- [72] M. Markl. Deformations and the coherence. *Proc. of the Winter School ‘Geometry and Physics,’ Zdr̃kov, Bohemia, January 1993, Supplemento ai Rend. Circ. Matem. Palermo, Serie II*, 37:121–151, 1994.
- [73] M. Markl. Cotangent cohomology of a category and deformations. *J. Pure Appl. Algebra*, 113:195–218, 1996.
- [74] M. Markl. Distributive laws and Koszulness. *Ann. Inst. Fourier (Grenoble)*, 46(4):307–323, 1996.
- [75] M. Markl. Models for operads. *Comm. Algebra*, 24(4):1471–1500, 1996.
- [76] M. Markl. A compactification of the real configuration space as an operadic completion. *J. Algebra*, 215:185–204, 1999.
- [77] M. Markl. Cyclic operads and homology of graph complexes. In *Proceedings of the 18th Winter School “Geometry and physics”, Srñ, Czech Republic, January 10-17, 1998*, volume 59 of *Supplem. ai Rend. Circ. Matem. Palermo, Ser. II*, pages 161–170, 1999.
- [78] M. Markl. Loop homotopy algebras in closed string field theory. *Comm. Math. Phys.*, 221:367–384, 2001.
- [79] M. Markl. A resolution (minimal model) of the PROP for bialgebras. Preprint [math.AT/0209007](#), September 2002.
- [80] M. Markl. Homotopy algebras are homotopy algebras. *Forum Mathematicum*, 16(1):129–160, January 2004. Preprint available as [math.AT/9907138](#).
- [81] M. Markl. Intrinsic brackets and the L_∞ -deformation theory of bialgebras. Preprint [math.AT/0411456](#), November 2004.
- [82] M. Markl and E. Remm. Algebras with one operation including Poisson and other Lie-admissible algebras. Preprint [math.AT/0412206](#), to appear in *J. of Algebra*.
- [83] M. Markl, S. Shnider, and J. D. Stasheff. *Operads in Algebra, Topology and Physics*, volume 96 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, Rhode Island, 2002.
- [84] M. Markl and A.A. Voronov. PROPped up graph cohomology. Preprint [math.QA/0307081](#), July 2003.
- [85] L. Mauri. Algebraic theories in monoidal categories. Preprint, available from the author’s web-site <http://math.rutgers.edu/~mauri/>, 2005.
- [86] J.P. May. *The Geometry of Iterated Loop Spaces*, volume 271 of *Lecture Notes in Mathematics*. Springer-Verlag, New York, 1972.
- [87] J.P. May. Definitions: Operads, algebras and modules. In J.L. Loday, J.D Stasheff, and A.A. Voronov, editors, *Operads: Proceedings of Renaissance Conferences*, volume 202 of *Contemporary Math.*, pages 1–7. Am. Math. Soc., 1997.
- [88] J.P. May. Operadic tensor products and smash products. In J.L. Loday, J.D Stasheff, and A.A. Voronov, editors, *Operads: Proceedings of Renaissance Conferences*, volume 202 of *Contemporary Math.*, pages 287–303. Am. Math. Soc., 1997.
- [89] J.P. May. Operads, algebras, and modules. In J.L. Loday, J.D Stasheff, and A.A. Voronov, editors, *Operads: Proceedings of Renaissance Conferences*, volume 202 of *Contemporary Math.*, pages 15–31. Am. Math. Soc., 1997.
- [90] J.P. May. Brave new worlds in stable homotopy theory. In M. Mahowald, editor, *Proceedings of a Conference on Homotopy Theory, Evanston, IL, USA*, volume 220 of *Contemporary Math.*, pages 193–212. American Mathematical Society, 1998.
- [91] J.P. May. Caterads and algebras over caterads. Preprint, available from the author’s home page, December 2004.
- [92] S.A. Merkulov. PROP profile of Poisson geometry. Preprint [math.DG/0401034](#), January 2003.
- [93] S.A. Merkulov. PROP profile of deformation quantization. Preprint [math.QA/0412257](#), December 2004.

- [94] T. Pirashvili. On the PROP corresponding to bialgebras. *Cah. Topol. Géom. Différ. Catég.*, 43(3):221–239, 2002.
- [95] D. Quillen. Rational homotopy theory. *Ann. of Math.*, 90:205–295, 1969.
- [96] P. Salvatore. Configuration spaces with summable labels. In J. Aguadé, editor, *Cohomological methods in homotopy theory. Proceedings of the Barcelona conference on algebraic topology (BCAT), Bellaterra, Spain, June 4-10, 1998*, volume 196 of *Prog. Math.*, pages 375–395, Basel, 2001. Birkhäuser Publ.
- [97] S. Sanedlidze and R. Umble. The biderivative and A_∞ -bialgebras. Preprint [math.AT/0406270](#), April 2004.
- [98] S. Sanedlidze and R. Umble. The biderivative, matrons and A_∞ -bialgebras. Preprint, July 2004.
- [99] S. Sanedlidze and R. Umble. Matrons, A_∞ -bialgebras and the polytopes KK. Preprint [math.AT/0508017](#), August 2005.
- [100] R. Schwänzl and R. M. Vogt. The categories of A_∞ - and E_∞ -monoids and ring spaces as closed simplicial and topological model categories. *Arch. Math. (Basel)*, 56(4):405–411, 1991.
- [101] B. Shoikhet. A concept of $\frac{2}{3}$ PROP and deformation theory of (co)associative coalgebras. Preprint [math.QA/0311337](#), November 2003.
- [102] B. Shoikhet. The CROCs, non-commutative deformations, and (co)associative bialgebras. Preprint [math.QA/0306143](#), June 2003.
- [103] B. Shoikhet. An explicit deformation theory of (co)associative bialgebras. Preprint [math.QA/0310320](#), October 2003.
- [104] M. Spitzweck. Operads, algebras and modules in general model categories. Preprint [math.AT/0101102](#), January 2001.
- [105] J.D. Stasheff. Math. Reviews reports MR0420609 (54#8623a) and MR0420610 (54#8623b) on [10] and [86]. Available from MathSciNet.
- [106] J.D. Stasheff. Homotopy associativity of H-spaces I,II. *Trans. Amer. Math. Soc.*, 108:275–312, 1963.
- [107] D. Sullivan. Infinitesimal computations in topology. *Inst. Hautes Études Sci. Publ. Math.*, 47:269–331, 1977.
- [108] M.E. Szabo. Polycategories. *Comm. Algebra*, 3(8):663–689, 1975.
- [109] D. Tamarkin and B. Tsygan. Cyclic formality and index theorems. *Lett. Math. Phys.*, 56:85–97, 2001.
- [110] D.E. Tamarkin. Another proof of M. Kontsevich formality theorem. Preprint [math.QA/9803025](#), March 1998.
- [111] B. Vallette. *Dualité de Koszul des PROPs*. PhD thesis, Université Louis Pasteur, 2003.
- [112] B. Vallette. A Koszul duality for props. Preprint [math.AT/0411542](#), November 2004.
- [113] B. Vallette. Koszul duality for PROPs. *Comptes Rendus Math.*, 338(12):909–914, June 2004.
- [114] R.M. Vogt. My time as Mike Boardman’s student and our work on infinite loop spaces. Preprint, 1998.
- [115] R.M. Vogt. Introduction to algebra over “brave new rings”. *Rend. Circ. Mat. Palermo (2) Suppl.*, (59):49–82, 1999. The 18th Winter School “Geometry and Physics” (Srńí, 1998).
- [116] R.M. Vogt. Cofibrant operads and universal E -infinity operads. *Topology Appl.*, 133(1):69–87, September 2003.
- [117] A.A. Voronov. Notes on universal algebra. preprint [math.QA/0111009](#), November 2001.
- [118] R.O. Wells. *Differential Analysis on Complex Manifolds*, volume 65 of *Graduate Texts in Mathematics*. Springer-Verlag, 1980.

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