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PERIODIC PROBLEM WITH QUASILINEAR DIFFERENTIAL OPERATOR AND WEAK SINGULARITY

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Periodic problem with quasilinear differential operator and weak singularity

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Abstract. We study the singular periodic boundary value problem of the form

$$\big(|u'|^{p-2}\,u'\big)' = f(t,u), \quad u(0)=u(T), \quad u'(0)=u'(T),$$

where $p\in(1,\infty)$ and $f\in Car([0,T]\times(0,\infty)$ can have a repulsive space singularity at $x=0$. On the contrary to previous results by Mawhin and Jebelean, Liu Bing and Rachůnková and Tvrdý, we need not to assume any strong force conditions. Our main existence results rely on a new antimaximum principle for periodic quasilinear periodic problem, which has an independent meaning.

Keywords. Singular problem, periodic problem, Dirichlet problem, p-Laplacian, repulsive singularity, weak singularity, lower and upper functions, antimaximum principle, quasilinear equation

Mathematics Subject Classification 2000. 34B16, 34C25, 34B15, 34B18

1. Introduction

This paper deals with singular periodic problems of the form

$$(\phi_p(u'))' = f(t,u), \tag{1.1}$$

$$u(0)=u(T), \quad u'(0)=u'(T), \tag{1.2}$$

where

$$0<T<\infty, \quad p\in(1,\infty), \quad \phi_p(y)=|y|^{p-2}\,y \quad \text{for} \quad y\in\mathbb{R} \tag{1.3}$$

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and f satisfies the Carathéodory conditions on $[0,T]\times(0,\infty)$, i.e. f has the following properties: (i) for each $x\in(0,\infty)$ the function $f(.,x)$ is measurable on $[0,T]$; (ii) for almost every $t\in[0,T]$ the function $f(t,.)$ is continuous on $(0,\infty)$; (iii) for each compact set $K\subset(0,\infty)$ the function $m_{\rm K}(t)=\sup_{x\in K}|f(t,x)|$ is Lebesgue integrable on $[0,T]$.

Second order nonlinear differential equations or systems with singularities appear naturally in the description of particles submitted to Newtonian type forces or to forces caused by compressed gases, see e.g. [12], [15] or [16]. The mathematical interest in periodic singular problems increased when the paper [22] by Lazer and Solimini appeared in 1987. Motivated by the model equation $u''=a\,u^{-\alpha}+e(t)$ with $\alpha>0, a\neq 0$ and e integrable on $[0,T]$, they investigated the existence of positive solutions to the Duffing equation $u''=g(u)+e(t)$ using topological arguments and the lower and upper functions method. The restoring force g was allowed to have an attractive singularity or a strong repulsive singularity at origin. The results by Lazer and Solimini have been generalized or extended e.g. by Habets and Sanchez [18], Mawhin [27], del Pino, Manásevich and Montero [10], Omari and Ye [29], Zhang [42] and [44], Ge and Mawhin [17], Rachůnková and Tvrdý [32] or Rachůnková, Tvrdý and Vrkoč [37]. All of these papers, when dealing with the repulsive singularity, supposed that the strong force condition is satisfied. For the case of the weak singularity, first results were delivered by Rachůnková, Tvrdý and Vrkoč in [36]. Further results were delivered later also by Bonheure and De Coster [2] and Torres [39].

Regular periodic problems with $\phi-$ or p-Laplacian on the left hand side were considered by several authors, see e.g. del Pino, Manásevich and Murúa [11] or Yan [41]. General existence principles for the regular vector problem, based on the homotopy to the averaged nonlinearity, were presented by Manásevich and Mawhin in [25] (see also Mawhin [28]).

In the well-ordered case, the lower/upper functions method was extended to periodic problems with a ϕ-Laplacian operator on the left hand side by Cabada and Pouso in [5], Jiang and Wang in [21] and Staněk in [38]. The general existence principle valid also when lower/upper functions are non-ordered was given by Rachůnková and Tvrdý in [34] and, for the case when impulses are admitted, also in [33].

The singular periodic problem for the Liénard type equation

$$\big(|u'|^{p-2}\,u'\big)'+h(u)\,u'=g(u)+e(t) \tag{1.4}$$

with g having either an attractive singularity or a strong repulsive singularity at $x=0$ was treated by Liu [24], Jebelean and Mawhin [19] and [20] and Rachůnková and Tvrdý [35].

Let us recall that a function g is said to have an *attractive* singularity at $x=0$ if

$$\liminf_{x\to 0+} g(x)=-\infty.$$

Alternatively, we say that g has a *repulsive* singularity at the origin if

$$\limsup_{x\to 0+} g(x) = +\infty \tag{1.5}$$

and g has a *strong repulsive* singularity at the origin if

$$\lim_{x\to 0+} \int_x^1 g(s)\,\mathrm{d}s = +\infty. \tag{1.6}$$

For a more detailed survey of the recent development we refer to [31, Section 5].

The main goal of this paper is a new existence result, Theorem 4.4, for problem (1.1), (1.2). As in [36, Theorem 2.5] (see also [31, Theorem 5.26]), where the classical case $p = 2$ was treated, we need not to assume that f satisfies any strong force condition. Our main tools are the lower and upper function method and a generalization of a classical antimaximum principle to the quasilinear periodic problem

$$(\phi_p(u'))' + \lambda\,\phi_p(u) = e(t), \quad u(0) = u(T), \quad u'(0) = u'(T) \tag{1.7}$$

established below in Theorem 3.2.

Our main result applies, in particular, to the Duffing type model problem

$$(\phi_p(u'))' + \mu_p\, u^{p-1} = a\,u^{-\alpha} + e(t), \quad u(0) = u(T), \quad u'(0) = u'(T), \tag{1.8}$$

where $a > 0$, $\alpha > 0$, $\mu_p = (\pi_p/T)^p$ is the least eigenvalue of a homogeneous Dirichlet problem related to (1.7) and $e \in L_1[0,T]$. In particular, we get that problem (1.8) has a positive solution if $\inf \operatorname{ess}_{t\in[0,T]} e(t) > 0$. It is worth mentioning that for $\alpha \in (0,1)$ the function $g(x) = a\,x^{-\alpha}$ does not satisfy the strong force condition (1.6).

2. Preliminaries

As usual, for an arbitrary subinterval I of $\mathbb{R}$ we denote by $C(I)$ the set of functions $x : I \to \mathbb{R}$ which are continuous on I, $C^1[0,T]$ stands for the set of functions $x \in C[0,T]$ with the first derivative continuous on $[0,T]$. Further, $L_1[0,T]$ is the set of functions $x : [0,T] \to \mathbb{R}$ which are measurable and Lebesgue integrable on $[0,T]$. $AC[0,T]$ is the set of functions absolutely continuous on $[0,T]$. For $x \in L_1[0,T]$ we put

$$\|x\|_\infty = \sup_{t\in[0,T]} \operatorname{ess} |x(t)| \quad \text{and} \quad \overline{x} = \frac{1}{T}\int_0^T x(s)\,\mathrm{d}s.$$

If $f : [0,T]\times(0,\infty) \to \mathbb{R}$ satisfies the Carathéodory conditions on $[0,T]\times(0,\infty)$ we write

$$f \in Car([0,T]\times(0,\infty)). \tag{2.1}$$

2.1 . Definition. A function $u : [0,T] \to \mathbb{R}$ is a *solution* to problem (1.1,) (1.2) if $\phi_p(u') \in AC[0,T]$, $u > 0$ on $[0,T]$, $(\phi_p(u'(t)))' = f(t,u(t))$ for a.e. $t \in [0,T]$, $u(0) = u(T)$ and $u'(0) = u'(T)$.

Notice that the requirement $\phi(u') \in AC[0,T]$ implies that $u \in C^1[0,T]$.

The singular problem (1.1), (1.2) will be also investigated through regular auxiliary problems of the form

$$(\phi_p(u'))' = \widetilde{f}(t,u), \quad u(0) = u(T), \quad u'(0) = u'(T), \tag{2.2}$$

or

$$(\phi_p(u'))' = \widetilde{f}(t,u), \quad u(a) = u(b) = 0, \tag{2.3}$$

where $\widetilde{f} \in Car([0,T] \times \mathbb{R})$ and $a, b \in \mathbb{R}$, $a < b$. As usual, by a *solution of problem* (2.2) we understand a function u such that $\phi_p(u') \in AC[0,T]$, (1.2) is true and $(\phi_p(u'(t)))' = \widetilde{f}(t,u(t))$ for a.e. $t \in [0,T]$. Analogously, u is a solution to (2.3) if $\phi_p(u') \in AC[a,b]$, $u(a) = u(b) = 0$ and $(\phi_p(u'(t)))' = \widetilde{f}(t,u(t))$ for a.e. $t \in [a,b]$.

The lower and upper functions method combined with the topological degree argument is an important tool for proofs of solvability of boundary value problems. For our purposes the following definitions of lower and upper functions associated with problems (2.2) or (2.3) are suitable.

2.2 . Definition. Let $\widetilde{f} \in Car([0,T] \times \mathbb{R})$. We say that a function $\sigma \in C[0,T]$ is a *lower function* of problem (2.2) if $\phi_p(\sigma') \in AC([0,T])$ and

$$\begin{cases} (\phi_p(\sigma'(t)))' \geq \widetilde{f}(t,\sigma(t)) & \text{for a.e. } t \in [0,T], \\ \sigma(0) = \sigma(T), \quad \sigma'(0) \geq \sigma'(T). \end{cases} \tag{2.4}$$

Analogously, $\sigma \in C[0,T]$ is a *lower function* of (2.3) if $\phi_p(\sigma') \in AC([a,b])$ and

$$\begin{cases} (\phi_p(\sigma'(t)))' \geq \widetilde{f}(t,\sigma(t)) & \text{for a.e. } t \in [a,b], \\ \sigma(a) \leq 0, \quad \sigma(b) \leq 0. \end{cases} \tag{2.5}$$

If the inequalities in (2.4) or (2.5) are reversed, then σ is called an *upper function* of (2.2) or of (2.3), respectively.

The next two assertions based on the lower and upper functions method will be useful for our purposes.

2.3 . Proposition. ([34, Theorem 3.2] or [31, Lemma 5.9]) *Assume* (1.3) *and* $\widetilde{f} \in Car([0,T] \times \mathbb{R})$. *Furthermore, let* σ_1 *and* σ_2 *be a lower and an upper function of* (2.2) *and let there be* $m \in L_1[0,T]$ *such that*

$$\widetilde{f}(t,x,y) > m(t) \ (\text{or } \widetilde{f}(t,x) < m(t)) \ \text{ for a.e. } t \in [0,T] \text{ and all } x \in \mathbb{R}.$$

Then problem (2.2) *has a solution* u *such that*

$$\min\{\sigma_1(\tau_u), \sigma_2(\tau_u)\} \le u(\tau_u) \le \max\{\sigma_1(\tau_u), \sigma_2(\tau_u)\} \quad \textit{for some } \tau_u \in [0,T].$$

2.4. Proposition. ([6, Theorem 2.1] or [40] or [30, Lemma 3.2] or [7, Theorem 3.5]) *Assume* (1.3), $\widetilde{f} \in Car([0,T]\times\mathbb{R})$ *and let* $a, b \in \mathbb{R}$, $a < b$ *be given. Furthermore, let* σ_1 *and* σ_2 *be a lower and an upper function of* (2.3) *such that* $\sigma_1 \le \sigma_2$ *on* $[a,b]$.

Then problem (2.3) *has a solution* u *such that* $\sigma_1 \le u \le \sigma_2$ *on* $[a,b]$.

3. Sign properties of quasilinear periodic problems

First, let us recall some basic known facts concerning initial value problems of the form

$$(\phi_p(u'))' + \lambda\,\phi_p(u) = 0, \tag{3.1}$$

$$u(t_0) = 0, \quad u'(t_0) = d, \tag{3.2}$$

where $p \in (1,\infty)$, $t_0 \in \mathbb{R}$, $\lambda \in \mathbb{R}$ and $d \in \mathbb{R}$. As in [8] (see also e.g. [1], [9], [13], [14], [43], [45], [26]), let us put

$$\pi_p = 2\,(p-1)^{1/p} \int_0^1 (1-s^p)^{-1/p}\,\mathrm{d}s.$$

Clearly, $\pi_2 = \pi$. Furthermore, it is known that

$$\pi_p = 2\,(p-1)^{\frac{1}{p}}\,\frac{(\pi/p)}{\sin(\pi/p)}\,.$$

(See [14, Sec. 1.1.2], but take into account that our definition differs slightly from that used in [14], where $\pi_p = 2\int_0^1 (1-s^p)^{-1/p}\,\mathrm{d}s$.) It is known (see [14, Theorem 1.1.1]) that for each $t_0 \in \mathbb{R}$, $\lambda \in \mathbb{R}$ and $d \in \mathbb{R}$ problem (3.1), (3.2) has a unique solution u on $\mathbb{R}$ which can be, by [8, sec. 3]), expressed as

$$u(t) = d\,\lambda^{-1/p}\,\sin_p(\lambda^{1/p}\,(t-t_0)) \quad \text{for } t \in \mathbb{R},$$

where the function $\sin_p : \mathbb{R} \to [-(p-1)^{1/p}, (p-1)^{1/p}]$ is defined as follows.

Let $w : [0, \pi_p/2] \to [0, (p-1)^{1/p}]$ be the inverse function to

$$x \in [0, \pi_p/2] \mapsto \int_0^x \frac{\mathrm{d}s}{(1-\frac{s^p}{p-1})^{1/p}} \in [0, (p-1)^{1/p}].$$

Further, put $\widetilde{w}(t) = w(\pi_p - t)$ for $t \in [\pi_p/2, \pi_p]$ and then $\widetilde{w}(t) = -\widetilde{w}(-t)$ for $t \in [-\pi_p, 0]$. Finally, we define $\sin_p : \mathbb{R} \to \mathbb{R}$ as the $2\,\pi_p-$ periodic extension of $\widetilde{w}$ to the whole $\mathbb{R}$. In particular, if $d = 0$, then $u \equiv 0$ on $\mathbb{R}$. Obviously, we have

$$\sin_p(t) = 0 \qquad \text{if and only if } \ t = n\,\pi_p,\ n \in \mathbb{N} \cup \{0\},$$

$$\sin_p(t) = (p-1)^{1/p} \quad \text{if and only if } \ t = (2\,n+1)\,\frac{\pi_p}{2},\ n \in \mathbb{N} \cup \{0\},$$

and

$$\sin_p(t) > 0 \qquad \text{for } \ t \in (2\,n\,\pi_p, (2\,n+1)\,\pi_p),\ n \in \mathbb{N} \cup \{0\}.$$

As a corollary, we immediately obtain that for given $a, b \in \mathbb{R}$, $a < b$, the corresponding Dirichlet problem

$$(\phi_p(u'))' + \lambda\,\phi_p(u) = 0, \quad u(a) = u(b) = 0 \tag{3.3}$$

possesses a nontrivial solution, i.e. λ is an eigenvalue for (3.3), if and only if

$$\lambda \in \left\{ \left(\frac{n\,\pi_p}{b-a} \right)^p : n \in \mathbb{N} \cup \{0\} \right\}. \tag{3.4}$$

In particular,

$$\mu_p = \left(\frac{\pi_p}{T} \right)^p. \tag{3.5}$$

is the least eigenvalue for (3.3) with $b - a = T$, wherefrom the following assertion follows.

3.1. Proposition. *Let* $p \in (1, \infty)$, $a, b \in \mathbb{R}$, $a < b$, *and let* $\lambda = \mu_p$, *where* μ_p *is given by* (3.5). *Then problem* (3.3) *has a nontrivial solution if and only if* $b - a \ge T$.

It is easy to check that the function

$$G(t,s) = \frac{T}{2\,\pi} \sin\left(\frac{\pi}{T} |t-s| \right), \quad t, s \in [0,T],$$

is the Green function for $v'' + \mu_2\,v = 0$, $v(0) = v(T)$, $v'(0) = v'(T)$ and $G(t,s)$ is nonnegative on $[0,T] \times [0,T]$. Hence, for classical linear second order periodic problems, the following *antimaximum principle* is true:

for each $h \in L_1[0,T]$ *such that* $h \ge 0$ *a.e. on* $[0,T]$, *all solutions* v *of the problem*

$$v'' + \mu_2\,v = h(t), \quad v(0) = v(T), \quad v'(0) = v'(T)$$

are nonnegative on $[0,T]$.

3.2. Theorem. *Let* $p \in (1,\infty)$ *and let* $\mu \in L_1[0,T]$ *be such that*

$$0 \le \mu \le \mu_p \quad a.e.\ on\ [0,T] \quad and \quad \overline{\mu} > 0, \tag{3.6}$$

and let $v \in C^1[0,T]$ *be such that* $\phi_p(v') \in AC[0,T]$,

$$(\phi_p(v'(t)))' + \mu(t)\,\phi_p(v(t)) \ge 0 \text{ for a.e. } t \in [0,T] \tag{3.7}$$

and

$$v(0) = v(T), \quad v'(0) = u'(T). \tag{3.8}$$

Then $v \ge 0$ *on* $[0,T]$.

Proof. Let $v \in C^1[0,T]$ be such that $\phi_p(v') \in AC[0,T]$ and (3.6) and (3.7) hold. Without any loss of generality we may assume that v is does not vanish on $[0,T]$.

Step 1. First, we show that

$$\max\{v(t) : t \in [0,T]\} > 0. \tag{3.9}$$

Assuming, on the contrary, that $v \le 0$ on $[0,T]$, we get by (3.7)

$$(\phi_p(v'(t)))' \ge -\mu(t)\,\phi_p(v(t)) \ge 0 \text{ for a.e. } t \in [0,T].$$

Therefore, v' is nondecreasing on $[0,T]$ and, taking into account (3.8), we deduce that $v' = 0$ on $[0,T]$. Consequently, $v(t) \equiv v(0) \le 0$ on $[0,T]$. Hence, (3.7) reduces to

$$-\mu(t)\,(-v(0))^{p-1} \ge 0 \text{ for a.e. } t \in [0,T].$$

However, as $\mu \ge 0$ a.e. on $[0,T]$ and $\overline{\mu} > 0$, this is possible if and only if $v(0) = 0$, i.e. $v \equiv 0$ on $[0,T]$, which contradicts our assumption that v does not vanish identically on $[0,T]$.

Step 2. Assume that $\min\{v(t) : t \in [0,T]\} < 0$. Let us extend v and μ to T-periodic functions on $\mathbb{R}$. In view of *Step* 1, there are $a, b \in \mathbb{R}$ such that $v \ge 0$ on (a,b), $v(a) = v(b) = 0$ and

$$0 < b - a < T. \tag{3.10}$$

Moreover, $v > 0$ on (a,b). Indeed, if $v(\tau) = 0$ for some $\tau \in (a,b)$, then necessarily also $v'(\tau) = 0$, i.e. $v \equiv 0$ on $\mathbb{R}$. In virtue of (3.6) and (3.7), we have

$$(\phi_p(v'(t)))' + \mu_p\,\phi_p(v(t)) \ge (\phi_p(v'(t)))' + \mu(t)\,\phi_p(v(t)) \ge 0 \text{ for a.e. } t \in [a,b]. \tag{3.11}$$

Furthermore, put

$$a_0 = a - \frac{T-b+a}{2}, \quad b_0 = a_0 + T$$

and

$$\sigma_2(t) = d\,\mu_p^{-1/p}\,\sin_p\big(\mu_p^{1/p}\,(t-a_0)\big) \quad \text{for } \ t\in\mathbb{R}$$

with $d>0$ large enough, i.e. such that $\sigma_2(t) > v(t) \ge 0$ on $[a,b]$. We have

$$(\phi_p(\sigma_2'(t)))' + \mu_p\,\phi_p(\sigma_2(t)) = 0 \text{ for a.e. } t\in[a,b]. \tag{3.12}$$

Thus, σ_2 is an upper function for (3.3). Moreover, in view of (3.11), $\sigma_1 = v$ is a lower function for (3.3). Hence, by Proposition 2.4, where we put $\widetilde{f}(t,x) = -\mu_p\,\phi_p(x)$ for t, $x\in\mathbb{R}$, there exists a nontrivial solution u to (3.3). This, due to (3.10), contradicts Proposition 3.1. □

4. Main results

First, let us recall the following a priori estimate (see [34, Lemma 2.4] or [31, Lemma 5.8]).

4.1. Lemma. *Let* $p\in(1,\infty)$ *and let* $\psi\in L_1[0,T]$. *Then*

$$\|v'\|_\infty < \phi_p^{-1}(\|\psi\|_1) \tag{4.1}$$

holds for each $v\in C^1[0,T]$ *fulfilling* $\phi_p(v')\in AC[0,T]$, $v(0)=v(T)$, $v'(0)=v'(T)$ *and* $(\phi_p(v'(t)))' > \psi(t)$ *(or* $(\phi_p(v'(t)))' < \psi(t)$*) for a.e.* $t\in[0,T]$.

Next, we prove an existence principle which relies on the comparison of the given problem (1.1), (1.2) with the related quasilinear problem fulfilling the antimaximum principle.

4.2. Theorem. *Assume* (1.3), (2.1) *and* $p\in[2,\infty)$. *Furthermore, let* $r>0$, $A\ge r$, $\mu,\beta\in L_1[0,T]$ *be such that* $\mu(t)\ge 0$ *a.e. on* $[0,T]$, $\overline{\mu}>0$,

$$\overline{\beta}\le 0 \quad \text{and} \quad f(t,x)\le\beta(t) \quad \text{for a.e. } t\in[0,T] \text{ and all } x\in[A,B] \tag{4.2}$$

and

$$f(t,x)+\mu(t)\,\phi_p(x-r)\ge 0 \quad \text{for a.e. } t\in[0,T] \text{ and all } x\in[r,B], \tag{4.3}$$

where

$$B-A\ge\frac{T}{2}\,\phi_p^{-1}(\|m\|_1),$$
$$m(t)=\max\big\{\sup\{f(t,x):\, x\in[r,A]\},\beta(t),0\big\} \quad \text{for a.e. } t\in[0,T]$$

and

$$\begin{cases} v\ge 0 \text{ on } [0,T] \text{ holds for each } v\in C^1[0,T] \text{ such that} \\ \quad \phi_p(v')\in AC[0,T], \\ \quad (\phi_p(v'(t)))'+\mu(t)\,\phi_p(v(t))\ge 0 \text{ for a.e. } t\in[0,T], \\ \quad v(0)=v(T), \quad v'(0)=v'(T). \end{cases} \tag{4.4}$$

Then problem (1.1), (1.2) *has a solution* u *such that*

$$r \le u \le B \quad \textit{on} \quad [0,T] \quad \textit{and} \quad \|u'\|_\infty < \phi_p^{-1}(\|m\|_1). \tag{4.5}$$

Proof. Part I. First, assume that $\overline{\beta} < 0$.
Step 1. Put

$$\widetilde{f}(t,x) = \begin{cases} f(t,r) - \mu(t)\,\phi_p(x-r) & \text{if } x \le r, \\ f(t,x) & \text{if } x \in [r,B], \\ f(t,B) & \text{if } x \ge B \end{cases} \tag{4.6}$$

and consider problem (2.2). We have $\widetilde{f} \in Car([0,T]\times\mathbb{R})$. Furthermore, by (4.2)–(4.6), the inequalities

$$\widetilde{f}(t,x) \le \beta(t) \quad \text{if } x \ge A \tag{4.7}$$

and

$$\widetilde{f}(t,x) + \mu(t)\,\phi_p(x-r) \ge 0 \quad \text{for all } x \in \mathbb{R} \tag{4.8}$$

are valid for a.e. $t \in [0,T]$. In particular, in view of (4.6), we have

$$\widetilde{f}(t,x) \ge h(t) := -\mu(t)\,\phi_p(B-r) \quad \text{for a.e. } t \in [0,T] \text{ and all } x \in \mathbb{R}, \tag{4.9}$$

with $h \in L_1[0,T]$.

By (4.8), $\sigma_2 \equiv r$ is an upper function of (2.2). Further, if $b = \beta - \overline{\beta}$, then $b \in L_1[0,T]$ and $\overline{b} = 0$ and it is easy to see that there is a uniquely defined $\sigma_0 \in C^1[0,T]$ such that $\phi_p(\sigma_0') \in AC[0,T]$,

$$(\phi_p(\sigma_0(t)'))' = b(t) \quad \text{for a.e. } t \in [0,T] \quad \text{and} \quad \sigma_0(0) = \sigma_0(T) = 0.$$

Now, let us choose $c^* > 0$ such that $c^* + \sigma_0 \ge A$ on $[0,T]$ and define $\sigma_1 = c^* + \sigma_0$. By (4.7) we have

$$\sigma_1(0) = \sigma_1(T) = c^*,$$
$$(\phi_p(\sigma_1'(t)))' = \beta(t) - \overline{\beta} > \beta(t) \ge \widetilde{f}(t,\sigma_1(t)) \text{ for a.e. } t \in [0,T],$$

and

$$\phi_p(\sigma_0'(T)) - \phi_p(\sigma_0'(0)) = T\,\overline{b} = 0.$$

Consequently, σ_1 is a lower function of (2.2). Therefore, by (4.9) and by Proposition 2.3, the regular problem (2.2) has a solution u such that $u(t_u) \ge r$ for some $t_u \in [0,T]$.

Step 2. We show that

$$u \ge r \text{ on } [0,T]. \tag{4.10}$$

To this aim, set $v = u - r$. By virtue of (4.8), we have

$$\big(\phi_p'(v'(t))\big)' + \mu(t)\,\phi_p(v(t)) = \widetilde{f}(t,u(t)) + \mu(t)\,\phi_p(u(t)-r) \ge 0$$

for a.e. $t \in [0,T]$. By (4.4) it follows that $v(t) \ge 0$ on $[0,T]$, i.e. (4.10) is true.

Step 3. We show that

$$u \le B \text{ on } [0,T]. \tag{4.11}$$

Indeed, by the definition of m and by (4.6) and (4.7) we have

$$\widetilde{f}(t,x) \le m(t) \quad \text{for a.e. } t \in [0,T] \text{ and all } x \ge r.$$

Hence, we can use Lemma 4.1 to get the estimate

$$\|u'\|_\infty \le \phi_p^{-1}(\|m\|_1). \tag{4.12}$$

If $u \ge A$ were valid on $[0,T]$, then taking into account the periodicity of u' and (4.7) we would get

$$0 = \int_0^T \widetilde{f}(t,u(t))\,\mathrm{d}t \le \int_0^T \beta(t)\,\mathrm{d}t = T\,\overline{\beta} < 0,$$

a contradiction. Hence,

$$\min\{u(s):\ s \in [0,T]\} < A.$$

Now, assume that

$$u^* := \max\{u(s):\ s \in [0,T]\} > A$$

and extend u to be T-periodic on $\mathbb{R}$. There are s_1, s_2 and $s^* \in \mathbb{R}$ such that

$$s_1 < s^* < s_2, \quad s_2 - s_1 < T, \quad u(s_1) = u(s_2) = A \quad \text{and} \quad u(s^*) = u^* > A.$$

In particular, due to (4.12),

$$2\,(u(s^*) - A) = \int_{s_1}^{s^*} u'(s)\,\mathrm{d}s + \int_{s_2}^{s^*} u'(s)\,\mathrm{d}s \le T\,\phi_p^{-1}(\|m\|_1),$$

wherefrom the estimate

$$u(t) - A \le \frac{T}{2}\,\phi_p^{-1}(\|m\|_1) \le B - A \text{ on } [0,T]$$

follows. Thus, (4.11) is true.

Step 4. The estimates (4.10) and (4.11) mean that $r \le u \le B$ holds on $[0,T]$. In view of (4.6), we conclude that u is a solution to (1.1), (1.2).

Part II. Now, let $\overline{\beta} = 0$. Put $n_0 = \max\{\frac{1}{r}, \frac{1}{B-A}, 3\}$. For an arbitrary $n \in \mathbb{N}$, define

$$\widetilde{f}_n(t,x) = \begin{cases} f(t,r) & \text{if } x \le r, \\ f(t,x) & \text{if } x \in [r,A], \\ f(t,x) - \mu(t)\,\phi_p\left(\frac{1}{n}\,\frac{x-A}{x-A+1}\right) & \text{if } x \in (A,B], \\ f(t,B) - \mu(t)\,\phi_p\left(\frac{1}{n}\,\frac{B-A}{B-A+1}\right) & \text{if } x \ge B. \end{cases} \tag{4.13}$$

Taking into account (4.2), we get

$$\widetilde{f}_n(t,x) = f(t,x) - \mu(t)\,\phi_p\left(\frac{1}{n}\,\frac{x-A}{x-A+1}\right) \le \beta(t) - \mu(t)\,\phi_p\left(\frac{1}{n}\,\frac{x-A}{x-A+1}\right)$$

$$\le \beta(t) - \mu(t)\,\phi_p(\frac{1}{2n^2}) \qquad \text{if } \ x \in [A + \frac{1}{n}, B]$$

and

$$\widetilde{f}_n(t,x) = f(t,B) - \mu(t)\,\phi_p\left(\frac{1}{n}\,\frac{B-A}{B-A+1}\right) \le \beta(t) - \mu(t)\,\phi_p(\frac{1}{2n^2}) \quad \text{if } \ x \ge B$$

for a.e. $t \in [0,T]$ and all $n \in \mathbb{N}$ such that $n \ge n_0$. Thus,

$$\begin{cases} \widetilde{f}_n(t,x) \le \beta_n(t) := \beta(t) - \mu(t)\,\phi_p(\frac{1}{2n^2}) \\ \qquad \text{for } \ x \ge A + \frac{1}{n}, \quad \text{for a.e. } \ t \in [0,T] \quad \text{and all } \ n \ge n_0. \end{cases} \tag{4.14}$$

Clearly,

$$\overline{\beta_n} < 0 \quad \text{and} \quad \beta_n(t) \le \beta(t) \text{ for a.e. } t \in [0,T]. \tag{4.15}$$

Furthermore, by (4.3) and (4.13), we have

$$\widetilde{f}_n(t,x) + \mu(t)\,\phi_p\left(x - (r - \frac{1}{n})\right) \ge f(t,r) \ge 0 \quad \text{if } \ x \in [r - \frac{1}{n}, r],$$

$$\widetilde{f}_n(t,x) + \mu(t)\,\phi_p\left(x - (r - \frac{1}{n})\right) = f(t,x) + \mu(t)\,\phi_p\,(x - r) \ge 0 \quad \text{if } \ x \in [r,A],$$

and, taking into account that

$$\xi^\alpha + \eta^\alpha \le (\xi + \eta)^\alpha \quad \text{holds for all } \ \xi, \eta \ge 0 \quad \text{and each } \ \alpha \ge 1,$$

$$\widetilde{f}_n(t,x) + \mu(t)\,\phi_p\left(x - (r - \frac{1}{n})\right)$$

$$= f(t,x) - \mu(t)\,\phi_p\left(\frac{1}{n}\,\frac{x-A}{x-A+1}\right) + \mu(t)\,\phi_p\left(x - r + \frac{1}{n}\right)$$

$$\geq f(t,x) + \mu(t)\,\phi_p\,(x-r) \geq 0 \quad \text{if} \ \ x \in [A,B],$$

and

$$\begin{aligned} &\widetilde{f}_n(t,x) + \mu(t)\,\phi_p\left(x - (r - \frac{1}{n})\right) \\ &\qquad = f(t,B) - \mu(t)\,\phi_p\left(\frac{1}{n}\,\frac{B-A}{B-A+1}\right) + \mu(t)\,\phi_p\left(x - (r - \frac{1}{n})\right) \\ &\qquad \geq f(t,B) + \mu(t)\,\phi_p\,(B-r) \geq 0 \quad \text{if} \ \ x \geq B. \end{aligned}$$

To summarize,

$$\widetilde{f}_n(t,x) + \mu(t)\,\phi_p\left(x - (r - \frac{1}{n})\right) \geq 0 \quad \text{for all} \ \ x \geq r - \frac{1}{n}. \tag{4.16}$$

For a.e. $t \in [0,T]$ and all $n \in \mathbb{N}$, put

$$\widetilde{m}_n(t) := \max\big\{\sup\{\widetilde{f}_n(t,x) :\, x \in [r - \frac{1}{n}, A + \frac{1}{n}]\}, b_n(t)\big\}.$$

In view of (4.13) and (4.15), we have

$$\widetilde{m}_n(t) \leq m(t) \qquad \text{for a.e. } t \in [0,T] \quad \text{and} \quad n > \frac{1}{B-A}\,.$$

This together with (4.14)-(4.16) means that, for each $n \in \mathbb{N}$ large enough, *Part* I of this proof ensures the existence of a solution u_n to the auxiliary problem

$$(\phi_p(u_n'))' = \widetilde{f}_n(t,u_n), \quad u_n(0) = u_n(T), \quad u_n'(0) = u_n'(T)$$

which satisfies the estimates

$$r - \frac{1}{n} \leq u_n(t) \leq B + \frac{1}{n} \quad \text{on} \ \ [0,T] \quad \text{and} \quad \|u_n'\|_\infty \leq \phi_p^{-1}(\|m\|_1).$$

Now, notice that

$$|\widetilde{f}_n(t,x) - \widetilde{f}(t,x)| \leq \mu(t)\,\phi_p\left(\frac{1}{n}\right) \quad \text{for a.e.} \ \ t \in [0,T], \ \text{all} \ \ x \in \mathbb{R} \text{ and all} \ \ n \in \mathbb{N},$$

where

$$\widetilde{f}(t,x) = \begin{cases} f(t,r) & \text{if} \ \ x \leq r, \\ f(t,x) & \text{if} \ \ x \in [r,B], \\ f(t,B) & \text{if} \ \ x \geq B. \end{cases}$$

Thus, in a standard way (using the Arzelá-Ascoli and the Lebesgue Dominated Convergence Theorem) we can show that the sequence $\{u_n\}_{n=1}^{\infty}$ contains a subsequence which converges in $C^1[0,T]$ to a solution u of the problem

$$(\phi_p(u'))' = \widetilde{f}(t,u), \quad u(0) = u(T), \quad u'(0) = u'(T)$$

which satisfies necessarily the estimate (4.5), i.e. solves also (1.1), (1.2). $\square$

The next supplementary assertion concerning the case $p \in (1,2)$ follows immediately from *Part* I of the previous proof.

4.3. Theorem. *Let all assumptions of Theorem* 4.2 *be satisfied, with the exceptions that* $p \in (1,2)$ *is allowed and* $\overline{\beta} < 0$ *is required in* (4.2).

Then problem (1.1), (1.2) *has a solution* u *such that* (4.5) *is true.*

Theorems 3.2, 4.2 and 4.3 yield the following new existence criterion.

4.4. Theorem. *Assume* (1.3) *and* (2.1). *Furthermore, let* $p \in (1,\infty)$, $\mu_p = (\pi_p/T)^p$ *and let* $r > 0$, $A \ge r$, $B > A$ *and* $\beta \in L_1[0,T]$ *be such that* (4.2), *with* $\overline{\beta} < 0$, *if* $p \in (1,2)$, *and* (4.3) *hold, where*

$$B - A \ge \frac{T}{2}\,\phi_p^{-1}(\|m\|_1)$$

and

$$m(t) = \max\big\{\sup\{f(t,x) : x \in [r,A]\}, \beta(t), 0\big\} \quad \textit{for a.e. } t \in [0,T].$$

Then problem (1.1), (1.2) *has a solution* u *such that* (4.5) *is true.*

In particular, for the Duffing type equation $(\phi_p(u'))' = g(u) + e(t)$ we have

4.5. Corollary. *Let* $p \in (1,\infty)$. *Suppose that* $f(t,x) = g(x) + e(t)$ *for* $x \in (0,\infty)$ *and a.e.* $t \in [0,T]$, *where* $g \in C(0,\infty)$, $e \in L_1[0,T]$, *and*

$$\overline{e} + \limsup_{x\to\infty} g(x) < 0 \tag{4.17}$$

and there is $r > 0$ *such that*

$$e(t) + g(x) + \mu_p\, x^{p-1} \ge \mu_p\, r^{p-1} \quad \textit{for a.e. } t \in [0,T] \textit{ and all } x \ge r. \tag{4.18}$$

Then problem (1.1), (1.2) *has a solution* u *such that* $u(t) \ge r$ *on* $[0,T]$.

Proof. Due to (4.17), we can find $A \ge r$ such that

$$g(x) + \overline{e} < \frac{1}{2}\left(\overline{e} + \limsup_{x\to\infty} g(x)\right) < 0 \quad \text{for } x \in [A,\infty).$$

Consequently,

$$f(t,x) = g(x) + e(t) = (g(x) + \overline{e}) + (e(t) - \overline{e}) < \frac{1}{2}\left(\overline{e} + \limsup_{x\to\infty} g(x)\right) + e(t) - \overline{e}$$

for a.e. $t \in [0,T]$ and all $x \in [A,\infty)$. Therefore (4.2) holds with

$$\beta(t) := e(t) - \overline{e} + \frac{1}{2}\left(\overline{e} + \limsup_{x\to\infty} g(x)\right)$$

and $B > A$ arbitrarily large. we have $\overline{\beta} < 0$. Furthermore, according to (4.18), by using that $(x-r)^{p-1} \geq x^{p-1} - r^{p-1}$, we deduce that f satisfies (4.3) with $B > r$ arbitrarily large.

Now, the assertion follows by Theorem 4.4. □

For the case $p \geq 2$ we have an existence result under weaker assumptions.

4.6. Corollary. *Let* $p \geq 2$. *Assume the hypothesis of the previous result, replacing condition (4.17) by the following one:*

There is $B > 0$ *such that* $\overline{e} + g(x) \leq 0$ *for all* $x \geq B$.

Then problem (1.1), (1.2) *has a solution* u *such that* $u(t) \geq r$ *on* $[0,T]$.

4.7. Example. Consider the problem

$$(\phi_p(u'))' + k\,u^{p-1}(t) = a\,u^{-\alpha} + e(t), \quad u(0) = u(T), \quad u'(0) = u'(T), \tag{4.19}$$

where $a > 0$, $\alpha > 0$, $k \in [0,\mu_p]$ and $e \in L_1[0,T]$.

It is easy to see that if $k > 0$, then the assumption (4.17) of Corollary 4.5 is satisfied for all $e \in L_1[0,T]$, while in the case $k = 0$ this condition holds whenever $\overline{e} < 0$.

Furthermore, define $h(x) = (\mu_p - k)\,x^{p-1} + a\,x^{-\alpha}$ for $x > 0$. We can verify that

$$h_* := \inf_{x\in(0,\infty)} h(x) = a\left(1 + \frac{\alpha}{p-1}\right)\left(\frac{(p-1)\,(\mu_p - k)}{\alpha\,a}\right)^{\frac{\alpha}{\alpha+p-1}}.$$

Therefore, under the assumption

$$e_* := \inf_{t\in[0,T]} \operatorname{ess} e(t) > -h_*,$$

we get for a.e. $t \in [0,T]$ and all $x > 0$

$$h(x) + e(t) \geq e_* + h_* > 0,$$

i.e. the second assumption (4.18) of Corollary 4.5 is satisfied with

$$r = \left(\frac{e_* + h_*}{\mu_p}\right)^{\frac{1}{p-1}}.$$

To summarize: by Corollary 4.5 problem (4.19) has a positive solution in the following cases:

$$k = 0, \quad \bar{e} < 0 \quad \text{and} \quad \inf_{t\in[0,T]} \operatorname{ess} e(t) > -a \left(1 + \frac{\alpha}{p-1}\right) \left(\frac{(p-1)\,\mu_p}{\alpha\, a}\right)^{\frac{\alpha}{\alpha+p-1}}$$

or

$$k \in (0, \mu_p] \quad \text{and} \quad \inf_{t\in[0,T]} \operatorname{ess} e(t) > -a \left(1 + \frac{\alpha}{p-1}\right) \left(\frac{(p-1)\,(\mu_p - k)}{\alpha\, a}\right)^{\frac{\alpha}{\alpha+p-1}}.$$

In particular, problem (1.8) has a positive solution if

$$\inf_{t\in[0,T]} \operatorname{ess} e(t) > 0.$$

For the case $p \geq 2$, from Corollary 4.6, we can verify that if

$$k = 0, \quad \bar{e} \leq 0 \quad \text{and} \quad \inf_{t\in[0,T]} \operatorname{ess} e(t) > -a \left(1 + \frac{\alpha}{p-1}\right) \left(\frac{(p-1)\,\mu_p}{\alpha\, a}\right)^{\frac{\alpha}{\alpha+p-1}},$$

then problem (4.19) has a positive solution.

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