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THE ORNSTEIN UHLENBECK BRIDGE AND APPLICATIONS TO MARKOV SEMIGROUPS

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Abstract. For an arbitrary Hilbert space-valued Ornstein-Uhlenbeck process we construct the Ornstein-Uhlenbeck Bridge connecting a given starting point x and an endpoint y provided y belongs to a certain linear subspace of full measure. We derive also a stochastic evolution equation satisfied by the OU Bridge and study its basic properties. The OU Bridge is then used to investigate the Markov transition semigroup defined by a stochastic evolution equation with additive noise. We provide an explicit formula for the transition density and study its regularity. These results are applied to show some basic properties of the transition semigroup. Given the Strong Feller property and the existence of invariant measure we show that all L^p functions are transformed into continuous functions thus generalising the Strong Feller property. We also show that transition operators are q-summing for some $q > p > 1$, in particular of Hilbert-Schmidt type.

1. Introduction

Let (Z_t^x) be an Ornstein-Uhlenbeck process on a separable Hilbert space H. By this we mean that (Z_t^x) is a solution to a linear stochastic evolution equation

$$\begin{cases} dZ_t^x = AZ_t^x dt + \sqrt{Q} dW_t, \\ Z_0^x = x \in H. \end{cases} \tag{1.1}$$

In the above equation (W_t) is a standard cylindrical Wiener process defined on a certain stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P})$ and $Q = Q^* \geqslant 0$ is a bounded operator on H. We assume that the operator $(A, \mathrm{dom}(A))$ is a generator of a C_0-semigroup (S_t) on H. Under the assumptions given below the solution to (1.1) is defined by the formula

$$Z_t^x = S_t x + \int_0^t S_{t-s} \sqrt{Q} dW_s. \tag{1.2}$$

The aim of this paper is to study the basic properties of the Ornstein-Uhlenbeck Bridge (sometimes called a Pinned Ornstein-Uhlenbeck process) $\left(\hat{Z}_t^{x,y}\right)$ associated to the Ornstein-Uhlenbeck process (Z_t^x) and its applications. Let us recall informally, that this process is defined via the formula

$$\mathbb{P}\left(Z_t^x \in B \middle| Z_T^x = y\right) = \mathbb{P}\left(\hat{Z}_t^{x,y} \in B\right), \quad t < T,$$

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where $x, y \in H$ and $B \subset H$ is a Borel set, so it is the Ornstein-Uhlenbeck process "conditioned to go from x at time $t = 0$ to y at time $t = T$" (a rigorous definition is given in Section 2, cf. Def. 2.15). The importance of various types of bridge processes in the theory of finite dimensional diffusions is well recognised, see for example [22]. In infinite dimensional framework this concept was developed in [19] in order to study regularity of transition semigroup of certain linear and nonlinear diffusions on Hilbert space. In [17] and [18] an Ornstein-Uhlenbeck Bridge is introduced in order to obtain lower estimates on the transition kernel of some semilinear stochastic evolution equations. Those estimates provide a powerful tool to study exponential ergodicity and V-uniform ergodicity for such equations and, in particular, the rate of convergence to invariant measure, providing explicit estimates on the constants in the definition of exponential ergodicity, as has been shown in our previous paper [12].

In the present paper the OU Bridge is studied under much more general conditions and in more detail. In particular, unlike in [12] we do not assume that the OU process is strongly Feller, which is a rather strong requirement in infinite dimensions (the strong Feller property is assumed only in in Section 4 devoted to applications to transition densities of semilinear equations, where it is a natural condition). We provide also further applications of the OU Bridge to the analysis of transition densities and the regularity of associated Markov semigroups. Regularity of strongly Feller transition semigroups was studied by different methods in [9] (see also references therein). We use methods completely different from [9] and obtain stronger results but for bounded drifts only while the aforementioned paper allows linearly growing drifts. Closely related results for semigroups that are not strongly Feller may be found in [4]. For the regularity of strongly Feller semigroups associated to the OU process we refer to [6].

Let us describe the contents of this paper. In Section 2 we provide, for the reader's convenience, some relevant facts about linear measurable mappings and conditional distributions of Hilbert space valued Gaussian random vectors. Then we give a definition of the OU Bridge and some basic results on OU processes and OU Bridges. Some of the technical results from [12] that are needed in the sequel are stated without proof and others (Lemma 2.8, Proposition 2.11 and Lemma 3.3) are reproved under more general conditions. In Section 3, a stochastic equation for the OU Bridge is derived. A new Brownian Motion adapted to the filtration of the Ornstein Uhlenbeck Bridge is obtained and then it is shown that the Bridge process is a unique mild (and weak) solution of a linear nonhomogenous stochastic evolution equation with singular coefficients. Section 4 is devoted to applications of the previous results to semilinear stochastic equations; at first continuity of Markov transition densities (with respect to the Gaussian invariant measure ν that is an invariant measure with respect to the OU process) is proved (Theorem 4.5 and Remark 4.10). Note that (for a fixed initial value) the continuity of densities in infinite dimensional case is a rather strong requirement (so is, in a sense, continuity of the mappings $y \to \mathbb{E}[Z_t^x | Z_T^x = y]$, etc.) The difficulties lie in the form of conditioned processes and transition densities (typically, (2.27) and (4.13)) which involve inverses of injective Hilbert-Schmidt operators. These are in infinite dimensions always unbounded and only densely defined (cf. Example 4.12 for an illustration of this fact). Furthermore, in Section 4 the Markov semigroup is shown to map the space $L^p(H,\nu)$, $p>1$, into the space of continuous functions on H (Theorem 4.6) and is also shown to be Hilbert-Schmidt on $L^2(H,\nu)$ and q-summing (in particular, compact) as a mapping $L^p(H,\nu)\to L^q(H,\nu)$ even if $q>p$ provided the gap between q and p is not too large (Theorem 4.7). At the end of the section the results are illustrated in the case of one-dimensional semilinear stochastic parabolic equation (Example 4.11) in which case the conditions imposed in the paper are verified or specified. In Example 4.12 it is shown that even in simple (in fact, linear) infinite dimensional cases densities may be irregular and conditions for regularity are specified.

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2. Preliminaries on OU Processes and Bridges

In this section we collect, for the reader convenience, some properties of infinite-dimensional OU processes and Gaussian random variables which will be useful in the paper. We also define the OU Bridge and recall some known results that will be useful in the sequel.

2.1. **Measurable Linear Mappings.** Let H be a real separable Hilbert space and let $\mu=N(0,C)$ be a centered Gaussian measure on H with the covariance operator C such that $\overline{\operatorname{im}(C)}=H$. The space $H_C=\operatorname{im}\left(C^{1/2}\right)$ endowed with the norm $|x|_C=\left|C^{-1/2}x\right|$ can be identified as the Reproducing Kernel Hilbert Space of the measure μ. In the sequel we will denote by $\{e_n: n\geqslant 1\}$ the eigenbasis of C and by $\{c_n: n\geqslant 1\}$ the corresponding set of eigenvalues:

$$Ce_n=c_ne_n,\quad n\geqslant 1.$$

For any $h\in H$ we define

$$\phi_n(x)=\sum_{k=1}^{n}\frac{1}{\sqrt{c_k}}\langle h,e_k\rangle\langle x,e_k\rangle,\quad x\in H.$$

The following two lemmas are well known (see e.g. [12]):

Lemma 2.1. *The sequence (ϕ_n) converges in $L^2(H,\mu)$ to a limit ϕ and*

$$\int_H|\phi(x)|^2\,\mu(dx)=|h|^2.$$

Moreover, there exists a measurable linear space $\mathscr{M}_h\subset H$, such that $\mu(\mathscr{M}_h)=1$, ϕ is linear on $\mathscr{M}_h$ and

$$\phi(x)=\lim_{n\to\infty}\phi_n(x),\quad x\in\mathscr{M}_h. \tag{2.1}$$

We will use the notation $\phi(x)=\left\langle h,C^{-1/2}x\right\rangle$.

Let H_1 be another real separable Hilbert space and let $T:H\to H_1$ be a bounded operator. The Hilbert-Schmidt norm of T will be denoted by $\|T\|_{HS}$. Let

$$\tilde{T}_nx=\sum_{k=1}^{n}\frac{1}{\sqrt{c_k}}\langle x,e_k\rangle Te_k,\quad x\in H.$$

Lemma 2.2. *Let $T : H \to H_1$ be a Hilbert-Schmidt operator. Then the sequence $\left(\tilde{T}_n\right)$ converges in $L^2(H, \mu; H_1)$ to a limit $\tilde{T}$ and*

$$\int_H \left|\tilde{T}(x)\right|_{H_1}^2 \mu(dx) = \|T\|_{HS}^2 .$$

Moreover, there exists a measurable linear space $\mathscr{M}_T \subset H$, such that $\mu(\mathscr{M}_T) = 1$, $\tilde{T}$ is linear on $\mathscr{M}_T$ and

$$\tilde{T}(x) = \lim_{n\to\infty} \tilde{T}_n x, \quad x \in \mathscr{M}_T. \tag{2.2}$$

We will use the notation $TC^{-1/2}x := \tilde{T}(x)$.

The above procedure is specified in the following Lemma (the proof of which may be found in [12]) to operator-valued functions:

Lemma 2.3. *Let $K(t,s) : H \to H$ be an operator-valued, strongly measurable function, such that for each $a \in (0,T)$*

$$\int_0^a \int_0^a \|K(t,s)\|_{HS}^2 \, dsdt < \infty. \tag{2.3}$$

Then the following holds.
(a) There exists a Borel set $B \subset [0,T]^2$ of full Lebesgue measure such that the measurable linear mapping $K(t,s)C^{-1/2}$ is well defined for all $(s,t) \in B$.
(b) There exists a measurable mapping $f : [0,T)^2 \times H \to H$ and a measurable linear space $\mathscr{M} \subset H$ of full measure such that $f(t,s,y) = K(t,s)C^{-1/2}y$ for $y \in \mathscr{M}$ and for each $a < T$

$$\int_0^a |f(t,s,y)| \, ds < \infty$$

for almost all $t \in [0,T]$. We will use the notation $K(t,s)C^{-1/2}y := f(t,s,y)$.

2.2. **Conditional Distributions.** Let H_1 and H_2 be two real separable Hilbert spaces and let $(X,Y) \in H_1 \times H_2$ be a Gaussian vector with mean values

$$m_X = \mathbb{E}X, \quad \text{and} \quad m_Y = \mathbb{E}Y.$$

The covariance operator of X is determined by the equation

$$\mathbb{E}\langle X - m_X, h\rangle \langle X - m_X, k\rangle = \langle C_X h, k\rangle, \quad h,k \in H_1, \tag{2.4}$$

and a similar condition determines the covariance C_Y of Y. The covariance operator $C_{XY} : H_1 \to H_2$ is defined by the condition

$$\langle C_{XY} h, k\rangle = \mathbb{E}\langle X - m_X, h\rangle \langle Y - m_Y, k\rangle, \quad h \in H_1, k \in H_2,$$

and then $C_{XY}^* = C_{YX}$. For a linear closable operator G on H the closure of G will be denoted by $\overline{G}$. The next theorem is well known, see for example [16]

Theorem 2.4. *Assume that C_X is injective. Then the following holds.*
(a) We have
$$\operatorname{im}(C_{YX}) \subset \operatorname{im}\left(C_X^{1/2}\right), \tag{2.5}$$
the operator $T = C_X^{-1/2} C_{YX}$ is of Hilbert-Schmidt type on H and $T^* = \overline{C_{XY} C_X^{-1/2}}$.
(b) We have
$$\mathbb{E}(Y|X) = m_Y + T^* C_X^{-1/2}(X - m_X), \quad \mathbb{P}^X - a.s.$$
(c) The conditional distribution of Y given X is Gaussian $N\left(\mathbb{E}(Y|X), C_{Y|X}\right)$, where
$$C_{Y|X} = C_Y - T^*T.$$
Moreover, the random variables $T^* C_X^{-1/2} X$ and $\left(Y - T^* C_X^{-1/2} X\right)$ are independent.

2.3. **Some Properties of the Ornstein-Uhlenbeck Process.** The following hypothesis is a standing assumption for the rest of the paper.

Hypothesis 2.5. *For every $t > 0$*
$$\int_0^t \left\|S_s Q^{1/2}\right\|_{HS}^2 ds < \infty, \tag{2.6}$$
and
$$\overline{\operatorname{im}(Q_t)} = H, \tag{2.7}$$
where, in view of (2.6)
$$Q_t = \int_0^t S_s Q S_s^* ds. \tag{2.8}$$
is a well defined trace class operator.

It is well known that if Hypothesis 2.5 holds then the process (1.2) is a well defined H-valued, Gaussian and Markov process, see [8].

Let μ denote the probability law of the process $\{Z_t^0 : t \in [0,1]\}$ that is concentrated on $L^2(0,T;H)$ and let $\mathscr{L} : L^2(0,T;H) \to C(0,T;H)$ be defined by the formula
$$\mathscr{L}u(t) = \int_0^t S_{t-s} Q^{1/2} u(s) ds. \tag{2.9}$$
Note that, cf. [8], $\operatorname{im}(\mathscr{L}) = RKHS(\mu)$ (the Reproducing Kernel Hilbert Space of the measure μ). We will use the notation μ_t^x for the Gaussian measure $N(S_t x, Q_t)$ and μ_t for μ_t^0. By the properties of Gaussian distribution μ_t^x is the probability distribution of a random variable Z_t^x and we set $Z_t = Z_t^0$. In the rest of this subsection we give several statements on properties of the family of covariance operators $\{Q_t : t \leqslant T\}$ that will be useful later.

The definition of Q_t given in 2.8 yields immediately a simple identity that will be frequently used:
$$Q_T = Q_t + S_t Q_{T-t} S_t^*, \quad t \leqslant T. \tag{2.10}$$

Lemma 2.6. *We have*

$$\operatorname{im}\left(Q_t^{1/2}\right) \subset \operatorname{im}\left(Q_T^{1/2}\right), \quad t \leqslant T,$$

hence the operator $U_t = Q_T^{-1/2} Q_t^{1/2}$ *is bounded on* H *for every* $t \leqslant T$ *and* $\|U_t\| \leqslant 1$. *Moreover,* $U_t^* = \overline{Q_t^{1/2} Q_T^{-1/2}}$, *the closure of the operator* $Q_t^{1/2} Q_T^{-1/2}$ *defined on the domain* $\operatorname{im}\left(Q_T^{1/2}\right)$.

Proof. From the definition of the covariance operators Q_t it follows that $|Q_t x|^2 \leqslant |Q_T x|^2$ for each $x \in H$ and $0 \leqslant t \leqslant T$ and the conclusion easily follows. □

Lemma 2.7. *(a) The operator* $V_t = Q_T^{-1/2} S_{T-t} Q_t^{1/2}$ *is well defined and bounded on* H *and*

$$\|V_t\| \leqslant 1, \quad t \in (0, T). \tag{2.11}$$

Moreover,

$$\lim_{t \to T} V_t^* x = \lim_{t \to T} V_t x = x, \quad x \in H. \tag{2.12}$$

(b) For any $t \in [0, T]$

$$Q_{T-t} = Q_T^{1/2} \left(I - V_t V_t^*\right) Q_T^{1/2}. \tag{2.13}$$

Proof. The inequality (2.11) has been proved in [20], the convergence (2.12) in [12]. Part (b) follows immediately from (2.10). □

Under a slightly stronger condition we show that the inequality (2.11) is sharp, more precisely, we have

Lemma 2.8. *The following conditions are equivalent:*
(a) For any $t \in (0, T]$

$$\operatorname{im}\left(Q_t^{1/2}\right) = \operatorname{im}\left(Q_T^{1/2}\right). \tag{2.14}$$

(b) $\operatorname{im}(U_t)$ *is dense in* H *for each* $t \in (0, T)$.
(c) We have

$$\|V_t\| < 1, \quad t \in (0, T). \tag{2.15}$$

Proof. Obviously (a) implies (b).
To prove that (b) implies (c) note first that (2.10) yields

$$\left|Q_{T-t}^{1/2} x\right|^2 = \left|Q_T^{1/2} x\right|^2 - \left|V_t^* Q_T^{1/2} x\right|^2,$$

hence putting $y = Q_T^{1/2} x$ we obtain

$$\left|Q_{T-t}^{1/2} Q_T^{-1/2} y\right|^2 = |y|^2 - |V_t^* y|^2.$$

Assume that $\|V_t^*\| = 1$ for a certain $t \in (0,T)$. Since $\mathrm{im}\left(Q_T^{1/2}\right)$ is dense in H, there exists a sequence $y_n \in \mathrm{im}\left(Q_T^{1/2}\right)$, such that $|y_n| = 1$ and $|V_t^* y_n| \to 1$. Therefore,

$$\lim_{n\to\infty}\left|Q_{T-t}^{1/2}Q_T^{-1/2}y_n\right|^2 = \lim_{n\to\infty}\left(1-|V_t^*y_n|^2\right) = 0. \tag{2.16}$$

Let y_{n_k} be a subsequence converging weakly to $y \in H$. Since

$$\mathrm{im}\left(Q_{T-t}^{1/2}\right) \subset \mathrm{im}\left(Q_T^{1/2}\right), \quad t \leqslant T,$$

and

$$\left(Q_T^{-1/2}Q_{T-t}^{1/2}\right)^* = \overline{Q_{T-t}^{1/2}Q_T^{-1/2}},$$

we find that

$$Q_{T-t}^{1/2}Q_T^{-1/2}y_{n_k} \to \overline{Q_{T-t}^{1/2}Q_T^{-1/2}}y, \quad \text{weakly},$$

and by (2.16) we obtain $\overline{Q_{T-t}^{1/2}Q_T^{-1/2}}y = 0$ and since $|V_t^*y| = 1$ we obtain $y \neq 0$. It follows that the range of the operator $Q_T^{-1/2}Q_{T-t}^{1/2}$ is not dense in H, which shows that (b) implies (c).

Finally, assume that (c) holds. Then (2.13) and Proposition B1 in [8] yield

$$\mathrm{im}\left(Q_{T-t}^{1/2}\right) = \mathrm{im}\left(Q_T^{1/2}\left(I - V_tV_t^*\right)^{1/2}\right).$$

Since $\|V_t\| < 1$, the operator $I - V_tV_t^* : H \to H$ is an isomorphism, hence

$$\mathrm{im}\left(Q_{T-t}^{1/2}\right) = \mathrm{im}\left(Q_T^{1/2}\right), \quad t < T,$$

and (a) follows. □

Remark 2.9. Necessary and sufficient conditions for (2.14) to hold are not known but it was proved to be satisfied in the following cases.

(a) If

$$\mathrm{im}\left(S_t\right) \subset \mathrm{im}\left(Q_t^{1/2}\right), \quad t > 0,$$

then (2.14) holds. It is known that the above condition is equivalent to the strong Feller property of the OU transition semigroup $R_t\phi(x) = \mathbb{E}\phi\left(Z_t^x\right)$, see [8] for details.

(b) Assume that the process (Z_t^x) admits a nondegenerate invariant measure ν and $\mathrm{im}(Q)$ is dense in H. Let $H_Q = \mathrm{im}\left(Q^{1/2}\right)$ be endowed with the norm $|x|_Q = \left|Q^{-1/2}x\right|$. Assume that H_Q is invariant for the semigroup (S_t) and its restriction to H_Q is a C_0-semigroup in H_Q. Then (2.14) holds, see [11]. These assumptions are satisfied for any process (Z_t^x) with the transition semigroup analytic in $L^2(H,\nu)$, in particular they are satisfied for any reversible OU process.

We define the operator $B : Q_T^{1/2}(H) \to L^2(0,T;H)$,

$$Bx(t) = Q^{1/2}S_{T-t}^*Q_T^{-1/2}x, \quad t \in [0,T], \quad x \in Q_T^{1/2}(H).$$

The following simple Lemma has been proved in [12]:

Lemma 2.10. *(a) The operator B with the domain* $\mathrm{dom}(B) = Q_T^{1/2}(H)$ *extends to a bounded operator (still denoted by B) $B : H \to L^2(0,T;H)$. Moreover,*

$$|Bx|_{L^2(0,T;H)} = |x|_H, \quad x \in H.$$

(b) Setting

$$H \ni x \to \mathscr{K}x(t) = K_t x \in L^2(0,T;H), \tag{2.17}$$

where

$$K_t = Q_t^{1/2} V_t^*, \tag{2.18}$$

we have $\mathscr{K} = \mathscr{L}B$. In particular the operator $\mathscr{K} : H \to C(0,T;H)$ is bounded.

2.4. **Fundamentals on OU Bridge.** In the present subsection we give the definition and some basic properties of the OU Bridge.
Since $V_t^* = \overline{Q_t^{1/2} S_{T-t}^* Q_T^{-1/2}}$ is bounded, the operator K_t is of Hilbert-Schmidt type on H for each $t \in [0,T)$. Also, $\mathscr{K} : H \to L^2(0,T;H)$ is Hilbert-Schmidt.

Note that if K_t is defined by (2.18) then, in view of Lemma 2.2, the measurable function $K_t Q_T^{-1/2}$ is well defined for each $t \in [0,T]$. We will start from the definition of the process $(\hat{Z}_t)$,

$$\hat{Z}_t = Z_t - K_t Q_T^{-1/2} Z_T, \quad t \in [0,1), \quad \text{and} \quad \hat{Z}_1 = 0.$$

Proposition 2.11. *(a) An H-valued Gaussian process $\left(\hat{Z}_t\right)$ is independent of Z_T.*
(b) The covariance operator $\hat{Q}_t$ of $\hat{Z}_t$ is given by

$$\hat{Q}_t = Q_t^{1/2} \left(I - V_t^* V_t\right) Q_t^{1/2}. \tag{2.19}$$

(c) The process $\left(\hat{Z}_t\right)$ is mean-square continuous on $[0,T]$.
(d) If, moreover, one of the equivalent conditions (a)-(c) of Lemma 2.8 holds then

$$\mathrm{im}\left(\hat{Q}_t^{1/2}\right) = \mathrm{im}\left(Q_t^{1/2}\right), \quad t \in (0,T). \tag{2.20}$$

Proof. Theorem 2.4 yields immediately (a) since $\hat{Z}_t = Z_t - \mathbb{E}\left(Z_t | Z_T\right)$. Invoking (c) of Theorem 2.4 with $C_X = Q_T$, $C_Y = Q_t$ and $T^* = K_t$ and (2.18) we obtain

$$\hat{Q}_t = Q_t - K_t K_t^* = Q_t^{1/2} \left(I - V_t^* V_t\right) Q_t^{1/2}, \quad t < T.$$

Using (2.11) we find easily that

$$\lim_{t \to 0} \mathrm{tr}\left(\hat{Q}_t\right) = 0. \tag{2.21}$$

To prove that

$$\lim_{t \to T} \mathrm{tr}\left(\hat{Q}_t\right) = 0, \tag{2.22}$$

we note first that

$$\mathrm{tr}\left(\hat{Q}_t\right) = \mathrm{tr}\left(\left(I - V_t^* V_t\right)\left(Q_t - Q_T\right)\right) + \mathrm{tr}\left(\left(I - V_t^* V_t\right) Q_T\right).$$

Next, it is easy to see that

$$0 \leqslant \lim_{t\to T} \operatorname{tr}\left(\left(I - V_t^* V_t\right)\left(Q_T - Q_t\right)\right) \leqslant \lim_{t\to T} \operatorname{tr}\left(Q_T - Q_t\right) = 0. \tag{2.23}$$

Finally,

$$\begin{aligned}\operatorname{tr}\left(\left(I - V_t^* V_t\right) Q_T\right) &= \operatorname{tr}\left(Q_T\right) - \operatorname{tr}\left(V_t Q_T V_t^*\right) \\ &= \operatorname{tr}\left(Q_T\right) - \sum_{k=1}^{\infty} \left| Q_T^{1/2} V_t^* e_k \right|^2,\end{aligned}$$

where $\{e_k : k \geqslant 1\}$ is a CONS in H. Therefore,

$$\lim_{t\to T} \operatorname{tr}\left(\left(I - V_t^* V_t\right) Q_T\right) = 0 \tag{2.24}$$

by Lemma 2.7 and the Dominated Convergence Theorem. Combining (2.23) and (2.24) we obtain (2.22) and, consequently, (c). Part (d) follows immediately from Lemma 2.8 and (2.19). □

Proposition 2.12. *The conditional distribution of the process* (Z_t^x) *in the space* $H_2 = L^2(0,T;H)$ *given* Z_T^x *is* $N(\lambda, \overline{Q})$*, where*

$$\lambda(t) = S_t x + K_t Q_T^{-1/2} Z_T, \tag{2.25}$$

$$\overline{Q} = \tilde{Q} - \mathscr{K}\mathscr{K}^*, \tag{2.26}$$

where $\tilde{Q}$ *is the covariance operator of the process* (Z_t^x) *in* H_2, $\tilde{Q} : H \to H_2$,

$$[\tilde{Q}y](t) = \int_0^t R(t,s) y(s) ds, \quad y \in H_2,$$

and

$$R(t,s)z = \int_0^s S_{t-r} Q S_{s-r}^* z dr, \quad z \in H, \quad 0 \leqslant s \leqslant t \leqslant T,$$

and $\mathscr{K} : H \to H_1$ *is defined in (2.17).*

Proof. We use Theorem 2.4 with $H_1 = H$, $H_2 = L^2(0,T;H)$, $X = Z_t^x$, $Y = (Z_t^x)$, $C_X = Q_T$, and $C_Y = \tilde{Q}$. By the definition of the covariance C_{XY},

$$\langle C_{XY} k, h\rangle_{L^2(0,T;H)} = \mathbf{E}\,\langle Z_T^x, k\rangle\,\langle Z^x, h\rangle_{L^2(0,T,H)}, \quad k \in H_1,\, h \in H_2,$$

it is easy to compute $[C_{XY}k](t) = Q_t S_{T-t}^* k$, $t \in [0,T]$. Hence we have $T^* = \overline{C_{XY} C_X^{-1/2}} = \mathscr{K}$ and $T : H_2 \to H_1$, $Ty = \mathscr{K}^* y = \int_0^T K_t^* y(t) dt$. By Theorem 2.4 we have that

$$\overline{Q} = C_Y - T^* T = \tilde{Q} - \mathscr{K}\mathscr{K}^*,$$

and

$$\lambda(t) = \mathbb{E}(Z_t^x | Z_T^x) = \mathbb{E}(S_t x + Z_t | Z_T^x) = \mathbb{E}(S_t x + \hat{Z}_t + K_t Q_t^{-1/2} Z_T | Z_T^x)$$

which yields $\lambda(t) = S_t x + K_t Q_T^{-1/2} Z_T$, because $\hat{Z}_t$ and Z_T^x are stochastically independent, hence (2.25) and (2.26) hold true.

□

Recall that μ_T denotes the probability law of Z_T on H.

Proposition 2.13. *There exists a Borel subspace $\mathscr{M} \subset H$ such that $\mu_T(\mathscr{M}) = 1$ and for all $x \in H$ and $y \in S_T x + \mathscr{M}$ the H-valued Gaussian process*

$$\hat{Z}_t^{x,y} = Z_t^x - \mathscr{K} Q_T^{-1/2} \left(Z_T^x - y\right), \tag{2.27}$$

is well defined with paths in $L^2(0,T;H)$ and

$$\hat{Z}_t^{x,y} = S_t x - \mathscr{K} Q_T^{-1/2} \left(S_T x - y\right) + \hat{Z}_t, \quad \mathbb{P} - a.s. \tag{2.28}$$

Proof. By Lemma 2.2 we can choose a measurable linear space $\mathscr{M}$ such that $\mathscr{K} Q_T^{-1/2}$ is linear on $\mathscr{M}$ and $\mu_T(\mathscr{M}) = 1$. Therefore, $\mathscr{K} Q_T^{-1/2}\left(Z_T^x - y\right)$ is well defined for any $y \in S_T x + \mathscr{M}$ and (2.28) holds. □

Theorem 2.14. *Let $\Phi : L^2(0,T;H) \to \mathbb{R}$ be a Borel mapping such that*

$$\mathbb{E}\left|\Phi\left(Z^x\right)\right| < \infty.$$

Then

$$\mathbb{E}\left(\Phi\left(Z^x\right) \middle| Z_T^x = y\right) = \mathbb{E}\Phi\left(\hat{Z}^{x,y}\right), \quad \mu_T^x - a.e. \tag{2.29}$$

where the left-hand side of (2.29) is defined as a function $g_\Phi = g_\Phi(y) \in L^1(H, \mu_T^x)$ such that $\mathbb{E}(\Phi(Z^x)|Z_T^x) = g_\Phi(Z_T^x)$ $\mathbb{P}$-a.s.

Proof. We have to show that

$$\mathbb{E}(\Phi(Z^x)|Z_T^x) = \mathbb{E}(\Phi(\hat{Z}^{x,y}))|_{Z_T^x=y} \quad \mathbb{P} - a.s.$$

By Proposition 2.12 we have

$$\mathbb{E}(\Phi(Z^x)|Z_T^x) = \int_H \Phi(z) N(\lambda, \overline{Q})(dz) \quad \mathbb{P} - a.s., \tag{2.30}$$

where λ and $\overline{Q}$ are defined by (2.25) and (2.26), respectively. On the other hand, the covariance operator $\hat{Q}$ of the process $\hat{Z}_t^{x,y}$ in H_2 is by (2.28) the same as the one of $\hat{Z}_t$. Since $Z_t = \hat{Z}_t + K_t Q_T^{-1/2} Z_T$ and the summands on the right-hand side are independent random variables, we obtain $\tilde{Q} = \hat{Q} + \mathscr{K}\mathscr{K}^*$, that is, $\hat{Q} = \overline{Q}$. Also, we have

$$\mathbb{E}\hat{Z}_t^{x,y} = S_t x - \mathscr{K} Q_T^{-1/2}(S_T x - y),$$

and therefore

$$\mathbb{E}(\Phi(\hat{Z}^{x,y}))|_{Z_T^x=y} = \int_H \Phi(z) N(S_t x - K_t Q_T^{-1/2}(S_T x - y), \overline{Q})(dz)|_{Z_T^x=y} = \int_H \Phi(z) N(\lambda, \overline{Q})(dz)$$

$\mathbb{P} - a.s.$, which together with (2.30) concludes the proof.

□

Definition 2.15. *Given x, $y \in H$ and an H-valued OU process (Z_t^x), a process $(\hat{Z}_t^{x,y})$ satisfying (2.29) is called an Ornstein-Uhlenbeck Bridge (connecting points x at time $t = 0$ and y at time $t = T$). The probability law of the process $(\hat{Z}_t^{x,y})$ in the space $L^2(0,T;H)$ will be denoted by $\hat{\mu}^{x,y}$.*

Thus we have shown that the OU Bridge may be written in the form (2.27) or (2.28) and its probability law $\hat{\mu}^{x,y}$ is $N(\gamma, \overline{Q})$ where $\gamma(t) = \mathbb{E}[\lambda(t)|Z_T^x = y] = S_t x - K_t Q_T^{-1/2}(S_T x - y)$ $\mu_T^x - a.e.$

The following Theorem has been proved in [12] :

Theorem 2.16. *Let $\mathscr{E}$ be a Banach space such that $\mu(\mathscr{E}) = 1$. Then $\hat{\mu}^{0,y}(\mathscr{E}) = 1$ for $y \in \mathscr{M}$.*

3. SDE associated to the OU Bridge

The main purpose of this Section is to show that the OU Bridge $(\hat{Z}_t^{x,y})$ solves an affine non-autonomous stochastic forward equation with the intial datum x, where the drift contains y as a parameter. As the formulae for the coefficients of this equation are rather cumbersome, we at first outline the main idea. The OU Bridge will be shown to satisfy the equation of the form

$$d\hat{Z}_t^{x,y} = (A + f_1(t)\hat{Z}_t^{x,y})dt + f_2(t)ydt + Q^{1/2}d\zeta_t, \quad t \in (0,T), \tag{3.1}$$

with the initial condition $\hat{Z}_0^{x,y} = x$, where (ζ_t) is again a cylindrical Wiener process, $f_1(t) = -Q^{1/2}F_t^* Q_{T-t}^{-1/2} S_{T-s}$, $f_2(t) = Q^{1/2}F_t^* Q_{T-t}^{-1/2}$ and F_t is defined in (3.6). In particular, the equation (3.1) corresponds to the well-known equation for the finite-dimensional Brownian Bridge (see Example 4.12), however in infinite-dimensional case we cannot expect it to possess a strong solution. We consider two concepts of solutions: mild and weak. First we show that the OU Bridge solves (3.1) in the mild sense, i.e.

$$\hat{Z}_t^{x,y} = S_t x + \int_0^t S_{t-r} f_1(r)\hat{Z}_r^{x,y}dr + \int_0^t S_{t-r}f_2(r)ydr + \int_0^t S_{t-r}Q^{1/2}d\zeta_r, \quad t \in [0,T), \tag{3.2}$$

holds (Theorem 3.8). Then it is shown that $(\hat{Z}_t^{x,y})$ is also a weak solution to (3.1), that is,

$$\left\langle \hat{Z}_t^{x,y}, h \right\rangle = \langle x, h\rangle + \int_0^t \left\langle \hat{Z}_s^{x,y}, A^* h \right\rangle ds - \int_0^t \left\langle f_1(s)\hat{Z}_s^{x,y}, h \right\rangle ds + \int_0^t \langle f_2(s)y, h\rangle\, ds + \langle \zeta_t, Q^{1/2}h\rangle \tag{3.3}$$

for $h \in \mathrm{dom(A^*)}$ (Corollary 3.9).

In the sequel we will need the following

Hypothesis 3.1. *For any $t > 0$*

$$\mathrm{im}\left(S_t Q^{1/2}\right) \subset \mathrm{im}\left(Q_t^{1/2}\right). \tag{3.4}$$

Remark 3.2. Condition (3.4) is satisfied in some important cases.

(a) If the process (Z_t^x) is strong Feller then $\mathrm{im}\,(S_t) \subset \mathrm{im}\left(Q_t^{1/2}\right)$ and therefore (3.4) holds.

(b) Let $H_Q = Q^{1/2}(H)$ be endowed with the norm $|x|_Q = \left|Q^{-1/2}x\right|$, where Q is assumed to be nondegenerate. Assume that $S_t H_Q \subset H_Q$ for all $t \geqslant 0$ and (S_t) restricted to H_Q is a

C_0-semigroup. It was proved in [11] that in this case $S_t(H) \subset Q_t^{1/2}(H)$ for all $t > 0$ and there exists $c > 0$ such that

$$\left\| Q_t^{-1/2} S_t Q^{1/2} \right\| \leqslant \frac{c}{\sqrt{t}}, \quad t > 0.$$

Assume additionally that the process (Z_t^x) admits a Gaussian invariant measure ν. Then, cf. [11], (S_t) is a C_0-semigroup on H_Q if the transition semigroup of the process (Z_t^x) is analytic on $L^2(H,\nu)$, in particular this holds for a symmetric Ornstein-Uhlenbeck process. Explicit conditions for the analyticity and symmetry of the transition semigroup of the process (Z_t^x) in $L^2(H,\nu)$ may be found in [11] and [7].

Lemma 3.3. *Assume that Hypothesis 3.1 holds. Then the function*

$$t \to \left| Q_t^{-1/2} S_t Q^{1/2} h \right|,$$

is nonincreasing on $(0,\infty)$ *for each* $h \in H$.

Proof. By Lemma 2.7 we have

$$\left\| Q_{t+s}^{-1/2} S_t Q_s^{1/2} \right\| \leqslant 1. \tag{3.5}$$

By assumption the operator $Q_{t+s}^{-1/2} S_{t+s} Q^{1/2}$ is well defined and bounded and $S_s Q^{1/2} h \in \operatorname{im}\left(Q_s^{1/2}\right)$. Therefore, by (3.5)

$$\begin{aligned} \left| Q_{t+s}^{-1/2} S_{s+t} Q^{1/2} h \right| &= \left| Q_{t+s}^{-1/2} S_t Q_s^{1/2} Q_s^{-1/2} S_s Q^{1/2} h \right| \\ &\leqslant \left| Q_s^{-1/2} S_s Q^{1/2} h \right|, \end{aligned}$$

and (b) follows. ☐

Let

$$Y_u = \int_u^T S_{T-s} Q^{1/2} dW_s, \quad u \leqslant T.$$

Since the operator-valued function $t \to Q_t$ is continuous in the weak operator topology and all the operators Q_t are compact for $t > 0$, there exists a measurable choice of eigenvectors $\{e_k(t) : k \geqslant 1\}$ and eigenvalues $\{\lambda_k(t) : k \geqslant 1\}$. For each $n \geqslant 1$ we define a process

$$X_u^n = \sum_{k=1}^n \frac{1}{\sqrt{\lambda_k(T-u)}} \left\langle Y_u, e_k(T-u) \right\rangle F_u^* e_k(T-u),$$

where

$$F_u = Q_{T-u}^{-1/2} S_{T-u} Q^{1/2}. \tag{3.6}$$

Lemma 3.4. *There exists a measurable stochastic process* (X_u) *defined on* $[0,T)$ *such that for each* $a < T$

$$\lim_{n\to\infty} \mathbb{E} \int_0^a \left| X_u^n - X_u \right|^2 du = 0. \tag{3.7}$$

and for each $h \in H$ and $a < T$ the series

$$\langle X_u, h\rangle = \sum_{k=1}^{\infty} \frac{1}{\sqrt{\lambda_k(T-u)}} \langle Y_u, e_k(T-u)\rangle \langle e_k(T-u), F_u h\rangle \tag{3.8}$$

converges in $L^2(0,a)$ in mean square. Moreover, if $0 \leqslant u \leqslant v < T$ then for all $h, k \in H$

$$\mathbb{E}\langle X_u, h\rangle \langle X_v, k\rangle = \left\langle F_u h, Q_{T-u}^{-1/2} Q_{T-v}^{1/2} F_v k \right\rangle, \tag{3.9}$$

where the operator $Q_{T-u}^{-1/2} Q_{T-v}^{1/2}$ is bounded.

Proof. For $u \leqslant v \leqslant T$

$$\mathbb{E}\langle Y_u, h\rangle \langle Y_v, k\rangle = \langle Q_{T-v} h, k\rangle, \quad h, k \in H. \tag{3.10}$$

Therefore

$$\mathbb{E}\langle X_u^n - X_u^m, h\rangle^2 = \sum_{j=m+1}^{n} \frac{1}{\lambda_k(T-u)} \mathbb{E}\langle Y_u, e_k(T-u)\rangle^2 \langle e_k(T-u), F_u h\rangle^2$$

$$= \sum_{j=m+1}^{n} \langle e_k(T-u), F_u h\rangle^2 \underset{n,m\to\infty}{\longrightarrow} 0, \tag{3.11}$$

hence the process

$$\langle X_u, h\rangle = \sum_{k=1}^{\infty} \frac{1}{\sqrt{\lambda_k(T-u)}} \langle Y_u, e_k(T-u)\rangle \langle e_k(T-u), F_u h\rangle = \left\langle Q_{T-u}^{-1/2} Y_u, F_u h \right\rangle$$

is well defined for each $h \in H$ and $u < T$. For u, v such that $0 < u \leqslant v < T$ we have

$$\operatorname{im}\left(Q_{T-v}^{1/2}\right) \subset \operatorname{im}\left(Q_{T-u}^{1/2}\right). \tag{3.12}$$

Let P_n is an orthogonal projection on $\operatorname{lin}\{e_k(T-v) : k \leqslant n\}$ and $F_u^n = P_n F_u$. Then $Q_{T-u}^{-1/2} F_u^n$ is bounded on H. Let

$$X_u^n = \left(Q_{T-u}^{-1/2} F_u^n\right)^* Y_u.$$

By (3.10)

$$\mathbb{E}\langle X_u^n, h\rangle \langle X_v^n k\rangle = \left\langle Q_{T-v} Q_{T-u}^{-1/2} F_u^n h, Q_{T-v}^{-1/2} F_v^n k \right\rangle$$

$$= \left\langle F_u^n h, Q_{T-u}^{-1/2} Q_{T-v}^{1/2} F_v^n k \right\rangle.$$

By (3.12) the operator $Q_{T-u}^{-1/2} Q_{T-v}^{1/2}$ is bounded and therefore

$$\mathbb{E}\left\langle Q_{T-u}^{-1/2} Y_u, F_u h \right\rangle \left\langle Q_{T-v}^{-1/2} Y_v, F_v k \right\rangle = \lim_{n\to\infty} \mathbb{E}\langle X_u^n, h\rangle \langle X_v^n, k\rangle$$

$$= \left\langle F_u h, Q_{T-u}^{-1/2} Q_{T-v}^{1/2} F_v k \right\rangle.$$

It follows from (3.9) that

$$\mathbb{E}\langle X_u, h\rangle^2 = |F_u h|^2,$$

and by Lemma 3.3 we obtain for $u \leqslant a$

$$\mathbb{E}\left\langle X_u^n, h\right\rangle^2 \leqslant \mathbb{E}\left\langle X_u, h\right\rangle^2 \leqslant |h|^2 \left\|F_{T-a}\right\|^2 .$$

Then (3.11) and the Dominated Convergence Theorem yield

$$\lim_{n,m\to\infty}\int_0^a \sup_{|h|\leqslant 1}\mathbb{E}\left\langle X_u^n - X_u^m, h\right\rangle^2 du = 0.$$

As a consequence we find that (3.7) holds for any $a \in (0,T)$. □

By Lemma 3.3 a cylindrical process

$$I_t = \int_0^t F_u^* Q_{T-u}^{-1/2} Y_u du$$

is well defined, that is for any $h \in H$ the real-valued process

$$\langle I_t, h\rangle = \int_0^t \left\langle Q_{T-u}^{-1/2} Y_u, F_u h\right\rangle du$$

is well defined for all $t < T$.

Lemma 3.5. *The cylindrical process*

$$\zeta_t = W_t - \int_0^t F_u^* Q_{T-u}^{-1/2} Y_u du, \quad t \leqslant T,$$

is a standard cylindrical Wiener process on H.

The proof of this Lemma is omitted; it is a word by word repetition of the proof of Lemma 4.7 in [12] if we use Lemmas 3.3 and 3.4 above.

Theorem 3.6. *For all $t < T$*

$$\mathbb{E}\int_0^t \left|S_{t-s} Q^{1/2} F_s^* Q_{T-s}^{-1/2} S_{T-s}\hat{Z}_s\right|^2 ds < \infty, \tag{3.13}$$

and

$$\hat{Z}_t = -\int_0^t S_{t-s} Q^{1/2} F_s^* Q_{T-s}^{-1/2} S_{T-s}\hat{Z}_s ds + \int_0^t S_{t-s} Q^{1/2} d\zeta_s, \quad \mathbb{P} - a.s. \tag{3.14}$$

Proof. We will show first that the operator $Q_{T-s}^{-1/2} S_{T-s}\hat{Q}_s S_{T-s}^* Q_{T-s}^{-1/2}$ is bounded. Let $h, k \in H$. Then by Proposition 2.11 and (2.10) we obtain

$$\begin{aligned}
\left\langle S_{T-s}\hat{Q}_s S_{T-s}^* h, k\right\rangle &= \left\langle S_{T-s} Q_s S_{T-s}^* h, k\right\rangle - \left\langle S_{T-s} Q_s^{1/2} V_s^* V_s Q_s^{1/2} S_{T-s}^* h, k\right\rangle \\
&= \left\langle \left(Q_T - Q_{T-s}\right) h, k\right\rangle - \left\langle Q_T^{-1/2} S_{T-s} Q_s S_{T-s}^* h, Q_T^{-1/2} S_{T-s} Q_s S_{T-s}^* k\right\rangle \\
&= \left\langle \left(Q_T - Q_{T-s}\right) h, k\right\rangle - \left\langle Q_T^{-1/2}\left(Q_T - Q_{T-s}\right) h, Q_T^{-1/2}\left(Q_T - Q_{T-s}\right) k\right\rangle \\
&= \left\langle \left(Q_T - Q_{T-s}\right) h, k\right\rangle - \left\langle \left(Q_T - Q_{T-s}\right) Q_T^{-1}\left(Q_T - Q_{T-s}\right) h, k\right\rangle \\
&= \left\langle \left(Q_T - Q_{T-s}\right) h, k\right\rangle - \left\langle \left(Q_T - Q_{T-s}\right)\left(I - Q_T^{-1} Q_{T-s}\right) h, k\right\rangle
\end{aligned}$$

$$= \left\langle \left(Q_{T-s} - Q_{T-s} Q_T^{-1} Q_{T-s} \right) h, k \right\rangle = \left\langle Q_{T-s}^{1/2} \left(I - Q_{T-s}^{1/2} Q_T^{-1} Q_{T-s}^{1/2} \right) Q_{T-s}^{1/2} h, k \right\rangle .$$

Since the operator $Q_{T-s}^{1/2} Q_T^{-1} Q_{T-s}^{1/2}$ is bounded for $s < T$ we find that the operator

$$T_s = Q_{T-s}^{-1/2} S_{T-s} \hat{Q}_s S_{T-s}^* Q_{T-s}^{-1/2} = I - Q_{T-s}^{1/2} Q_T^{-1} Q_{T-s}^{1/2} \tag{3.15}$$

is bounded as well . Therefore, for $s \leqslant T - \epsilon$ Lemma 3.3 and (3.15) yield

$$\mathbb{E} \left| S_{t-s} Q^{1/2} F_s^* \left(Q_{T-s}^{-1/2} S_{T-s} \hat{Z}_s \right) \right|^2$$

$$= \left\| S_{t-s} Q^{1/2} F_s^* \left(Q_{T-s}^{-1/2} S_{T-s} \hat{Q}_s^{1/2} \right) \right\|_{HS}^2 \leqslant \left\| S_{t-s} Q^{1/2} \right\|_{HS}^2 \left\| F_s \right\|^2 \left\| T_s^{1/2} \right\|$$

$$\leqslant \left\| S_{t-s} Q^{1/2} \right\|_{HS}^2 \left\| F_{T-\epsilon} \right\|^2 ,$$

which completes the proof of (3.13). As a byproduct of the argument given above we proved also that the process $Q_{T-s}^{-1/2} S_{T-s} \hat{Z}_s$ is well defined for all $s \leqslant T$. Now, we are ready to prove (3.14). By Lemma 3.5 we have

$$\hat{Z}_t = Z_t - K_t Q_T^{-1/2} Z_T$$

$$= \int_0^t S_{t-s} Q^{1/2} d\zeta_s + \int_0^t S_{t-s} Q^{1/2} F_s^* Q_{1-s}^{-1/2} Y_s ds - K_t Q_T^{-1/2} Z_T,$$

and since

$$Y_s = Z_T - S_{T-s} Z_s = Z_T - S_{T-s} K_s Q_T^{-1/2} Z_T - S_{T-s} \hat{Z}_s,$$

we find that

$$\hat{Z}_t = \int_0^t S_{t-s} Q^{1/2} d\zeta_s - \int_0^t S_{t-s} Q^{1/2} F_s^* Q_{T-s}^{-1/2} S_{T-s} \hat{Z}_s ds$$

$$+ \int_0^t S_{t-s} Q^{1/2} F_s^* Q_{T-s}^{-1/2} \left(Z_T - S_{T-s} K_s Q_T^{-1/2} Z_T \right) ds - K_t Q_T^{-1/2} Z_T.$$

It remains to show that

$$\int_0^t S_{t-s} Q^{1/2} F_s^* Q_{T-s}^{-1/2} \left(Z_T - S_{T-s} K_s Q_T^{-1/2} Z_T \right) ds - K_t Q_T^{-1/2} Z_T = 0. \tag{3.16}$$

To this end note first that

$$K_t Q_T^{-1/2} Z_T = \left(\int_0^t S_{t-s} Q^{1/2} F_s^* ds \right) Q_T^{-1/2} Z_T, \tag{3.17}$$

and

$$S_{T-t} K_t Q_T^{-1/2} Z_T = \left(\int_0^t S_{T-s} Q^{1/2} F_s^* ds \right) Q_T^{-1/2} Z_T$$

$$= \left(Q_T - Q_{T-t} \right) Q_T^{-1} Z_T = Z_T - Q_{T-t} Q_T^{-1} Z_T,$$

and thereby

$$Z_T - S_{T-t} K_t Q_T^{-1/2} Z_T = Q_{T-t} Q_T^{-1} Z_T. \tag{3.18}$$

Finally, (3.18) and the definition of F_s^* give

$$\int_0^t S_{t-s}Q^{1/2}F_s^*Q_{T-s}^{-1/2}\left(Z_T - S_{T-s}K_sQ_T^{-1/2}Z_T\right)ds$$

$$= \int_0^t S_{t-s}Q^{1/2}F_s^*Q_{T-s}^{-1/2}Q_{T-s}Q_T^{-1}Z_Tds = \left(\int_0^t S_{t-s}Q^{1/2}F_s^*ds\right)Q_T^{-1/2}Z_T,$$

and (3.16) follows from (3.17). □

We will consider now the general case of the bridge $\left(\hat{Z}_t^{x,y}\right)$ connecting points $x \in H$ and y. We will impose the stronger condition (2.14) which is now formulated as a separate hypothesis:

Hypothesis 3.7. *For any $t \in (0,T]$*

$$\operatorname{im}\left(Q_t^{1/2}\right) = \operatorname{im}\left(Q_T^{1/2}\right).$$

For $y \in H_1 := \operatorname{im}(Q_T^{1/2})$ we define

$$Ny(t) = \int_0^t S_{t-s}Q^{1/2}F_s^*Q_{T-s}^{-1/2}yds, \quad t \leqslant T-\epsilon.$$

Theorem 3.8. *Assume that Hypotheses 3.1 and 3.7 hold. Then the following holds.*
(a) The operator $N : H_1 \to L^2(0, T-\epsilon; H)$ is Hilbert-Schmidt.
(b) For any $x \in H$ and $y \in \mathscr{M}$

$$\hat{Z}_t^{x,y} = S_tx - \int_0^t S_{t-s}Q^{1/2}F_s^*Q_{T-s}^{-1/2}S_{T-s}\hat{Z}_s^{x,y}ds + \int_0^t S_{t-s}Q^{1/2}d\zeta_s$$
$$+ \int_0^t S_{t-s}Q^{1/2}F_s^*Q_{T-s}^{-1/2}yds. \tag{3.19}$$

Proof. Recall that by Lemma 2.10 (b) we have $\mathscr{K} = \mathscr{L}B$, hence for $z \in \mathscr{M}$

$$K_tQ_T^{-1/2}z = \int_0^t S_{t-s}Q^{1/2}B_sQ_T^{-1/2}zds. \tag{3.20}$$

Next, for $s \leqslant T-\epsilon$

$$\sup_{s\leqslant T-\epsilon}\left\|Q_{T-s}^{-1/2}Q_T^{1/2}\right\| = \left\|Q_\epsilon^{-1/2}Q_T^{1/2}\right\| < \infty,$$

and invoking Lemma 3.3 we find that

$$\left\|NQ_T^{1/2}\right\|_{HS}^2 \leqslant \int_0^{T-\epsilon}\left\|\int_0^t S_{t-s}Q^{1/2}F_s^*Q_{T-s}^{-1/2}Q_T^{1/2}ds\right\|_{HS}^2 dt$$
$$\leqslant \left(\int_0^T \left\|S_sQ^{1/2}\right\|_{HS}^2 ds\right)\left(\int_0^{T-\epsilon}\left\|F_s^*Q_{T-s}^{-1/2}Q_T^{1/2}\right\|^2 ds\right)$$
$$\leqslant \|F_{T-\epsilon}\|^2\left\|Q_\epsilon^{-1/2}Q_T^{1/2}\right\|^2\left(\int_0^T\left\|S_sQ^{1/2}\right\|_{HS}^2 ds\right) < \infty.$$

Therefore, the measurable function

$$y \to \int_0^t S_{t-s}Q^{1/2}F_s^*Q_{T-s}^{-1/2}yds,$$

is well defined. We are ready now for the proof of (3.19). Let $x, y \in \operatorname{im}\left(Q_T^{1/2}\right)$. Then Hypothesis 3.7 yields $S_Tx \in \operatorname{im}\left(Q_T^{1/2}\right)$, hence $y \in \mathscr{M}$. By (2.28) we have

$$\hat{Z}_t^{x,y} = \hat{Z}_t + S_tx - K_tQ_T^{-1/2}\left(S_Tx - y\right),$$

and Theorem 3.6 yields

$$\begin{aligned}
\hat{Z}_t^{x,y} &= S_tx - K_tQ_T^{-1/2}S_Tx + K_tQ_T^{-1/2}y \\
&- \int_0^t S_{t-s}Q^{1/2}F_s^*Q_{T-s}^{-1/2}S_{T-s}\hat{Z}_sds + \int_0^t S_{t-s}Q^{1/2}d\zeta_s \\
&= S_tx - K_tQ_T^{-1/2}S_Tx + K_tQ_T^{-1/2}y \\
&- \int_0^t S_{t-s}Q^{1/2}F_s^*Q_{T-s}^{-1/2}S_{T-s}\left(\hat{Z}_s^{x,y} - S_sx + K_sQ_T^{-1/2}S_Tx - K_sQ_T^{-1/2}y\right)ds + \int_0^t S_{t-s}Q^{1/2}d\zeta_s \\
&= -K_tQ_T^{-1/2}S_Tx + \int_0^t S_{t-s}Q^{1/2}F_s^*Q_{T-s}^{-1/2}S_{T-s}\left(S_s - K_sQ_T^{-1/2}S_T\right)xds \\
&+K_tQ_T^{-1/2}y + \int_0^t S_{t-s}Q^{1/2}F_s^*Q_{T-s}^{-1/2}S_{T-s}K_sQ_T^{-1/2}yds \\
&+S_tx - \int_0^t S_{t-s}Q^{1/2}F_s^*Q_{T-s}^{-1/2}S_{T-s}\hat{Z}_s^{x,y}ds + \int_0^t S_{t-s}Q^{1/2}d\zeta_s \\
&=: H_tx + G_ty + S_tx - \int_0^t S_{t-s}Q^{1/2}F_s^*Q_{T-s}^{-1/2}S_{T-s}\hat{Z}_s^{x,y}ds + \int_0^t S_{t-s}Q^{1/2}d\zeta_s. \qquad (3.21)
\end{aligned}$$

We will show first that

$$G_ty = \int_0^t S_{t-s}Q^{1/2}F_s^*Q_{T-s}^{-1/2}yds. \qquad (3.22)$$

For $y \in \operatorname{im}\left(Q_T^{1/2}\right)$

$$\begin{aligned}
S_{T-t}K_ty = \int_0^t S_{t-s}QS_{T-s}^*Q_T^{-1/2}yds &= \int_0^t S_{T-s}QS_{T-s}^*Q_T^{-1/2}yds \\
&= \left(Q_T - Q_{T-t}\right)Q_T^{-1/2}y, \qquad (3.23)
\end{aligned}$$

and therefore

$$\begin{aligned}
F_s^*Q_{T-s}^{-1/2}S_{T-s}K_sy &= F_s^*Q_{T-s}^{-1/2}Q_T^{1/2}y - F_s^*Q_{T-s}^{1/2}Q_T^{-1/2}y \\
&= F_s^*Q_{T-s}^{-1/2}Q_T^{1/2}y - Q^{1/2}S_{T-s}^*Q_T^{-1/2}y.
\end{aligned}$$

Hence, taking Lemma 2.10 (b) into account we find that

$$G_ty = K_tQ_T^{-1/2}y + \int_0^t S_{t-s}Q^{1/2}F_s^*Q_{T-s}^{-1/2}S_{T-s}K_sQ_T^{-1/2}yds$$

$$= K_t Q_T^{-1/2} y + \int_0^t S_{t-s} Q^{1/2} F_s^* Q_{T-s}^{-1/2} y ds - K_t Q_T^{-1/2} y,$$

and (3.22) follows. Next, we claim that for $x \in \operatorname{im}\left(Q_T^{1/2}\right)$

$$H_t x = 0. \tag{3.24}$$

Indeed, using (3.23) we obtain

$$H_t x = -K_t Q_T^{-1/2} S_T x + \int_0^t S_{t-s} Q^{1/2} F_s^* Q_{T-s}^{-1/2} S_{T-s} \left(S_s - K_s Q_T^{-1/2} S_T \right) x ds$$

$$= -K_t Q_T^{-1/2} x + \int_0^t S_{t-s} Q^{1/2} F_s^* Q_{T-s}^{-1/2} S_T x ds - \int_0^t S_{t-s} Q^{1/2} F_s^* Q_{T-s}^{-1/2} S_{T-s} K_s Q_T^{-1/2} S_T x ds$$

$$= -K_t Q_T^{-1/2} x + \int_0^t S_{t-s} Q^{1/2} F_s^* Q_{T-s}^{-1/2} S_T x ds$$

$$- \int_0^t S_{t-s} Q^{1/2} F_s^* Q_{T-s}^{-1/2} \left(Q_T - Q_{T-t} \right) Q_T^{-1/2} S_T x ds = 0,$$

which yields (3.24) for $x \in \operatorname{im}\left(Q_T^{1/2}\right)$ and therefore for all $x \in H$. Finally, combining (3.21), (3.22) and (3.24) we obtain (3.19). □

Corollary 3.9. *Assume Hypotheses 3.1 and 3.7. Then for each $t < T$, and $h \in \operatorname{dom}(A^*)$ and all $x \in H$ and $y \in \mathscr{M}$*

$$\left\langle \hat{Z}_t^{x,y}, h \right\rangle = \langle x, h \rangle + \int_0^t \left\langle \hat{Z}_s^{x,y}, A^* h \right\rangle ds - \int_0^t \left\langle F_s^* Q_{T-s}^{-1/2} S_{T-s} \hat{Z}_s^{x,y}, Q^{1/2} h \right\rangle ds$$

$$+ \int_0^t \left\langle F_s^* Q_{T-s}^{-1/2} y, Q^{1/2} h \right\rangle ds + \left\langle \zeta_t, Q^{1/2} h \right\rangle .$$

Proof. On any interval $[0, T_0]$ with $T_0 < T$ and for any $y \in \mathscr{M}$ the functions

$$s \to Q^{1/2} F_s^* Q_{T-s}^{-1/2} S_{T-s} \hat{Z}_s^{x,y} \quad \text{and} \quad s \to Q^{1/2} F_s^* Q_{T-s}^{-1/2} y$$

are $\mathbb{P}$-a.s. Bochner integrable by Theorem 3.8 and therefore standard results about the equivalence of weak and strong solutions of deterministic and stochastic evolution equations can be applied to prove the corollary, see for example [1] for deterministic and [3], [21] for stochastic versions. □

4. Applications to Semilinear Equations

In this Section, transition densities and Markov semigroups defined by semilinear stochastic equations are studied using the OU Bridge. Throughout the Section we assume (beside (2.5)) that the OU process (Z_t^x) is strongly Feller, that is, the condition

$$\operatorname{im}(S_t) \subset \operatorname{im}(Q_t^{1/2}), \quad t \in (0, T), \tag{4.1}$$

is satisfied. Note that (4.1) trivially implies the preceding Hypotheses 3.1 and 3.7 (or (2.14)). Let $(\mathscr{P}, \|.\|_{var})$ denote the space of probability measures on the Borel sets of H

endowed with the metric of total variation. We start from a simple proposition where some continuity properties of the OU Bridge are given.

Proposition 4.1. *(a) For each $t \in (0,T), y \in \mathscr{M}$, where $\mathscr{M}$ has been defined in Proposition 2.13, the mappings*

$$x \mapsto \hat{Z}_t^{x,y}(\omega), \quad H \to H, \tag{4.2}$$

$$x \mapsto \hat{Z}^{x,y}(\omega), \quad H \to L^2(0,T;H), \tag{4.3}$$

are continuous for $\mathbb{P}$-almost all $\omega \in \Omega$, and the mapping

$$x \mapsto \hat{\mu}_t^{x,y}, \quad H \to (\mathscr{P}, \|.\|_{var}), \tag{4.4}$$

is continuous.

(b) If, moreover, for each $t \in (0,T)$ we have $\overline{K_t Q_T^{-1/2}} \in \mathscr{L}(\hat{H}, H)$, where $\hat{H}$ is a separable Banach space continuously embedded into H, then the mapping $y \mapsto \hat{Z}_t^{x,y}(\omega)$ is $\hat{H} \to H$ $\mathbb{P}$-a.s. continuous. Similarly, if

$$\overline{\mathscr{K} Q_T^{-1/2}} \in \mathscr{L}(\hat{H}, L^2(0,T;H)) \tag{4.5}$$

then $\mathscr{M} \supset \hat{H}$ and the mapping $y \mapsto \hat{Z}^{x,y}(\omega)$ is $\mathbb{P}$-a.s. $\hat{H} \to L^2(0,T;H))$ continuous.

Proof. (a) By (4.1) we have that $S_T x \in \operatorname{im}(Q_T^{1/2})$ for each $x \in H$ and hence $S_T x \in \mathscr{M}$ by construction of $\mathscr{M}$, hence $y \in \mathscr{M}$. Furthermore, (4.1) implies that the mappings $\mathscr{K} Q_T^{-1/2} S_T$ and $K_t Q_T^{-1/2} S_T$, $t \in (0,T]$, are in $\mathscr{L}(H, L^2(0,T;H))$ and $\mathscr{L}(H)$, respectively, and (4.2) and (4.3) follow by (2.28).

To show (4.4) we recall Proposition 2.11 and Lemma 2.8 , by which we have $\operatorname{im}(\hat{Q}_t^{1/2}) = \operatorname{im}(Q_t^{1/2})$. Hence the measures $(\hat{\mu}_T^{x,y}), x \in H$, are equivalent and

$$\psi^y(t,x,z) = \frac{d\mu_t^{x,y}}{d\mu_t^{0,y}}(z) = \exp\left(-\frac{1}{2}\left|Q_t^{-1/2} S_t x\right|^2 + \frac{1}{2}\left|Q_T^{-1/2} S_T x\right|^2 + \left\langle Q_t^{-1/2} z, Q_t^{-1/2} S_t x\right\rangle\right). \tag{4.6}$$

Indeed, by the Cameron-Martin formula we have

$$\psi^y(t,x,z) = \exp\left(-\frac{1}{2}\left|\hat{Q}_t^{-1/2} m\right|^2 + \left\langle \hat{Q}_t^{-1/2} z, \hat{Q}_t^{-1/2} m\right\rangle\right),$$

where $m = Q_t^{1/2}\left(I - V_t^* V_t\right) Q_t^{-1/2} S_t x$. Then using (2.19) we get (4.6) and the assertion easily follows.

The proof of part (b) is completely analogous. □

Remark 4.2. (a) The equivalent form of the density (4.6) is

$$\psi^y(t,x,z) = \exp\left(-\frac{1}{2}\left|\left(I - V_t^* V_t\right)^{1/2} Q_t^{-1/2} S_t x\right|^2 + \left\langle Q_t^{-1/2} z, Q_t^{-1/2} S_t x\right\rangle\right).$$

(b) Note that the OU Bridge $(\hat{Z}_t^{x,y})$ satisfies the SDE (3.19) which defines an (inhomogeneous) Markov process on the interval $(0,T)$. By (4.4) this process is strongly Feller.

Now consider a stochastic semilinear evolution equation of the form

$$dX_t = AX_t dt + F(X_t)dt + \sqrt{Q}dW_t, \quad X_0 = x \in H \tag{4.7}$$

where A, W_t and Q are as before and $F : H \to H$ is a nonlinear continuous mapping. Suppose that $\mathrm{im}(F) \subset \mathrm{im}(Q^{1/2})$ and set $G := Q^{-1/2}F$.

Hypothesis 4.3. *The mapping $G : H \to H$ is bounded and continuous.*

Now we formulate technical assumptions on the linear part of the equation. For simplicity of presentation, it is stated in the form that is verifiable in examples and includes all assumptions made previously in the paper.

Hypothesis 4.4. *Assume either*
(i) $\mathrm{dim}H < \infty$ *or*
(ii) There exist $\alpha \in (0,1)$ *and* $\beta < \frac{1+\alpha}{2}$ *such that*

$$\int_0^{T_0} t^{-\alpha}\|S_t Q^{1/2}\|_{HS}^2 dt < \infty \quad \textit{and}$$

$$\|Q_t^{-1/2}S_t\| \leqslant \frac{c}{t^\beta}, \quad t \in (0, T_0),$$

for some $c > 0$ *and* $T_0 > 0$.

Conditions from (ii) are often used in the theory of stochastic equations and have been widely studied (cf.[8] or [12], see also the Example below). Note that Hypothesis 4.4 (ii) implies all previous assumptions made in the paper on the linear part of the equation (4.7) (i.e., all except for Hypothesis 4.3).
It is well known (see e.g. [21]) that under Hypotheses 4.3 and 4.4 equation (4.7) defines an H-valued Markov process as a solution to the integral equation

$$X_t = S_t x + \int_0^t S_{t-r}F(X_r)dr + \int_0^t S_{t-r}\sqrt{Q}d\widetilde{W}_r, \quad t \geqslant 0, \tag{4.8}$$

where $\widetilde{W}_t$ is a standard cylindrical Wiener process on H defined on a suitable probability space.
Finally, we assume that the OU process defined by the linear equation (1.1) has an invariant measure ν that will be used as a reference measure. This is equivalent to the condition

$$\sup_{t>0} tr(Q_t) < \infty. \tag{4.9}$$

If (4.9) holds then ν is a centered Gaussian measure with the covariance operator

$$Q_\infty = \int_0^\infty S_t Q S_t^* dt.$$

Moreover, it has been shown in [5] that $S_t Q_\infty^{1/2}(H) \subset Q_\infty^{1/2}(H)$ and the family of operators

$$S_0(t) = Q_\infty^{-1/2} S_t Q_\infty^{1/2}, \quad t \geqslant 0,$$

defines a C_0-semigroup of contractions on H. Moreover, if part (ii) of Hypothesis 4.4 holds then $\|S_0(t)\| < 1$ for all $t > 0$.

Denote by (P_t) the transition Markov semigroup defined by the equation (4.7) and set

$$P(t,x,\Gamma) = P_t 1_\Gamma(x),\ x \in H,\ t > 0$$

and Γ Borel sets in H, and

$$d(t,x,y) = \frac{P(t,x,dy)}{\nu(dy)}.$$

It is standard to see that the density d exists, because Girsanov Theorem may be used to show the equivalence of measures $P(t,x,dy) \sim \mu_t^x$, and $\mu_t^x \sim \nu$ by (4.1) (see e.g. [12]).

Theorem 4.5. *Let Hypotheses 4.3, 4.4 and (4.9) be satisfied and let $T > 0$ be fixed. Then for ν-almost all $y \in H$ the mapping $x \mapsto d(T,x,y)$ is continuous on H.*

Theorem 4.6. *Let Hypotheses 4.3, 4.4 and (4.9) be satisfied. Then for $p > 1$, $T > 0$, we have*

$$P_T(L^p(H,\nu)) \subset \mathscr{C}(H),$$

that is, the semigroup (P_t) maps the space $L^p(H,\nu)$ into the space of continuous functions on H.

For $p, q > 1$ we introduce the notation

$$\|P_t\|_{p,q} = \left(\int_H \left(\int_H d^{p'}(t,x,y)\nu(dy)\right)^{q/p'} \nu(dx)\right)^{1/q},$$

where $p' = \frac{p}{p-1}$. Note that $\|P_t\|_{2,2}$ is a Hilbert-Schmidt norm of P_t. Moreover, if $\|P_t\|_{p,q} < \infty$ then the operator $P_t : L^p(H,\nu) \to L^q(H,\nu)$ is compact. Under assumptions more general than ours necessary and sufficient conditions were given in [4] for boundedness of the operator $P_t : L^p(H,\nu) \to L^q(H,\nu)$. In the theorem below we use different arguments based on the formula for transition densities to show that a stronger property holds: $\|P_t\|_{p,q} < \infty$.

Theorem 4.7. *Let Hypotheses 4.3, 4.4 and (4.9) be satisfied. Then for any fixed $T > 0$ and $p, q > 1$ satisfying*

$$q < 1 + \frac{p-1}{\|S_0(T)\|^2}$$

we have $\|P_T\|_{p,q} < \infty$. In particular, the operator $P_T : L^p(H,\nu) \to L^q(H,\nu)$ is q-summing.

Corollary 4.8. *If*

$$q < 1 + \frac{1}{\|S_0(T)\|^2}$$

then $P_T : L^2(H,\nu) \to L^q(H,\nu)$ is γ-radonifying. In particular, $P_T : L^2(H,\nu) \to L^2(H,\nu)$ is Hilbert-Schmidt.

By the above mentioned equivalence of probabilities we may write

$$d(T,x,y) = \frac{P(T,x,dy)}{\mu_T^x(dy)} \cdot \frac{\mu_T^x(dy)}{\mu_T^0(dy)} \cdot \frac{\mu_T^0(dy)}{\nu(dy)} \tag{4.10}$$

$$=: h(T,x,y) \cdot g(T,x,y) \cdot k(T,y), \tag{4.11}$$

where k does not depend on x, g is given by the Cameron-Martin formula

$$g(T,x,y)=\exp\{\left\langle x,\overline{S_T^*Q_T^{-1/2}}Q_T^{-1/2}y\right\rangle-\frac{1}{2}|Q_T^{-1/2}S_Tx|^2\} \tag{4.12}$$

for ν-almost all $y\in H$, and h may be expressed by means of the OU Bridge $(\hat{Z}_t^{x,y})$,

$$h(T,x,y)=\mathbb{E}\exp\{\rho(\hat{Z}^{x,y})-\int_0^T\left\langle G(\hat{Z}_s^{x,y}),B_1(s)\hat{Z}_s+B_2(s)x-B_3(s)y\right\rangle ds\} \tag{4.13}$$

(cf.[12], Theorem 5.2), where

$$\rho(\hat{Z}^{x,y})=\int_0^T\left\langle G(\hat{Z}_s^{x,y}),d\zeta_s\right\rangle-\frac{1}{2}\int_0^T|G(\hat{Z}_s^{x,y})|^2ds$$

and (ζ_t) is a standard cylindrical Wiener process defined in Lemma 3.5,

$$B_1(s)=(Q_{T-s}^{-1/2}S_{T-s}Q^{1/2})^*Q_{T-s}^{-1/2}S_{T-s},$$

$$B_2(s)=(Q_T^{-1/2}S_{T-s}Q^{1/2})^*Q_T^{-1/2}S_T,$$

$$B_3(s)y=(Q_T^{-1/2}S_{T-s}Q^{1/2})^*Q_T^{-1/2}y,\quad y\in\operatorname{im}\left(Q_T^{1/2}\right).$$

From Lemma 2.10 it follows that

$$\int_0^T|B_2(s)x|^2ds=|Q_T^{-1/2}S_Tx|^2,\quad x\in H, \tag{4.14}$$

and by [12], Proposition 4.9, we have that

$$\mathbb{E}\int_0^T|B_1(s)\hat{Z}_t|ds<\infty \tag{4.15}$$

and

$$\int_0^T|B_3(s)y|ds<\infty \tag{4.16}$$

for ν- almost all $y\in\mathscr{M}$ (with no loss of generality we may assume that (4.16) holds for all $y\in\mathscr{M}$, $\nu(\mathscr{M})=1$). The proofs of Theorems 4.5, 4.6 and 4.7 are based on the following technical lemma:

Lemma 4.9. *Given $T>0$ and $q\in[0,\infty)$, there exists a constant $k_q>0$ such that*

$$\begin{aligned}h_q(T,x,y):&=\mathbb{E}\exp\{q(\rho(\hat{Z}^{x,y})-\int_0^T\left\langle G(\hat{Z}_s^{x,y}),B_1(s)\hat{Z}_s+B_2(s)x-B_3(s)y\right\rangle ds)\}\\&\leqslant k_q\exp\{k_q(|x|+\int_0^T|B_3(s)y|ds)\}\end{aligned} \tag{4.17}$$

for all $x\in H$ and $y\in\mathscr{M}$, in particular,

$$h(t,x,y)\leqslant k_1\exp\{k_1(|x|+\int_0^T|B_3(s)y|ds)\}.$$

Proof. By the Cauchy inequality we have

$$h_q(T,x,y) \leqslant (\mathbb{E}\exp\{2q\rho(\hat{Z}^{x,y}\})^{1/2} \tag{4.18}$$
$$\times(\mathbb{E}\exp\{2q(\int_0^T |\left\langle G(\hat{Z}_s^{x,y}), B_1(s)\hat{Z}_s + B_2(s)x - B_3(s)y\right\rangle |ds)\})^{1/2}$$

and since the process $s \mapsto G(\hat{Z}_s^{x,y})$ is bounded the first expectation on the right-hand side of (4.18) is bounded (uniformly w.r.t. x and y). By (4.14) and (4.16) we thus have

$$h_q(T,x,y) \leqslant C_q(\mathbb{E}\exp\{C_q \int_0^T (|B_1(s)\hat{Z}_s| + |B_2(s)x| + |B_3(s)y|)ds\})^{1/2} \tag{4.19}$$
$$\leqslant \tilde{C}_q \exp\{\tilde{C}_q(|Q_t^{-1/2}S_T x| + \int_0^T |B_3(s)y|ds)\}(\mathbb{E}\exp\{\tilde{C}_q \int_0^T |B_1(s)\hat{Z}_s|ds\})^{1/2}$$

for some C_q, $\tilde{C}_q$, and (4.17) follows by (4.15) and the Fernique inequality. □

Proof of Theorem 4.5. Without loss of generality (dropping, if necessary, a set of ν-measure zero) we may suppose that $g(T,x,y)$ and $k(T,y)$ are defined for all $y \in \mathscr{M}$. By (4.12) we have that the mapping $x \mapsto g(T,x,y)k(T,y)$ is continuous, so we only have to prove continuity of the mapping $x \mapsto h(T,x,y)$, $y \in \mathscr{M}$, $T > 0$. Let $x_n \to x_0$ in H. First we show (possibly, for a subsequence) that

$$\lim_{n\to\infty} \exp\{\rho(\hat{Z}^{x_n,y}) - \int_0^T \left\langle G(\hat{Z}_s^{x_n,y}), B_1(s)\hat{Z}_s + B_2(s)x_n - B_3(s)y\right\rangle ds\} \tag{4.20}$$
$$= \exp\{\rho(\hat{Z}^{x_0,y}) - \int_0^T \left\langle G(\hat{Z}_s^{x_0,y}), B_1(s)\hat{Z}_s + B_2(s)x_0 - B_3(s)y\right\rangle ds\}$$

$\mathbb{P}$-a.s. We have

$$\int_0^T \left|\left\langle G(\hat{Z}_s^{x_n,y}), B_1(s)\hat{Z}_s + B_2(s)x_n - B_3(s)y\right\rangle - \int_0^T \left\langle G(\hat{Z}_s^{x_0,y}), B_1(s)\hat{Z}_s + B_2(s)x_0 - B_3(s)y\right\rangle\right| ds$$
$$\leqslant \int_0^T |G(\hat{Z}_s^{x_n,y}) - G(\hat{Z}_s^{x_0,y})|(|B_1(s)\hat{Z}_s| + |B_2(s)x_0| + |B_3(s)y|)ds$$
$$+ \int_0^T \left|G(\hat{Z}_s^{x_0,y})| \cdot |B_2(s)(x_n - x_0)\right| ds, \tag{4.21}$$

which tends to zero by continuity and boundedness of G, (4.14) and Dominated Convergence Theorem. Also, we have

$$\mathbb{E}|\rho(\hat{Z}^{x_n,y}) - \rho(\hat{Z}^{x_0,y})| \leqslant C\left(\left(\mathbb{E}\int_0^T |G(\hat{Z}_s^{x_n,y}) - G(\hat{Z}_s^{x_0,y})|^2 ds\right)^{1/2}\right.$$
$$\left. + \mathbb{E}\int_0^T |G(\hat{Z}_s^{x_n,y}) - G(\hat{Z}_s^{x_0,y})|^2 ds\right),$$

which again tends to zero by Dominated Convergence Theorem, so there is a subsequence converging $\mathbb{P}$-a.s. Taking into account (4.21) we obtain (4.20). By (4.17) (used, for instance, with $q=2$) the random variables on the left-hand side of (4.20) are integrable uniformly in n, hence the convergence in (4.20) holds also in the space $L^1(\Omega)$ and, consequently, we obtain $h(T,x_n,y)\to h(T,x_0,y)$. Since we may choose a subsequence with this property from an arbitrary sequence $x_n\to x_0$, the convergence takes place for the whole sequence.

Proof of Theorem 4.6. Let $T>0$, $\phi\in L^p(H,\nu)$ and $x_n\to x_0$ in H. Then

$$\begin{aligned}|P_T\phi(x_n)-P_T\phi(x_0)| &\leqslant \int_H |\phi(y)||d(T,x_n)-d(T,x_0,y)|\nu(dy)\\ &\leqslant (\int_H |\phi|^p d\nu)^{1/p}(\int_H |d(T,x_n,y)-d(T,x_0,y)|^{p'}\nu(dy))^{1/p'},\end{aligned}$$

so by Theorem 4.5 it suffices to show that

$$\int_H (d(T,x_n,y))^q\nu(dy)<c_q,\quad q\in(1,\infty), \tag{4.22}$$

where c_q does not depend on n. The same property (uniform boundedness in arbitrary $L^q(H,\nu)$) has been shown for Gaussian densities $g(T,x_n,\cdot)$ and $k(T,\cdot)$ in [6], so we only have to show (4.22) where $d(T,x_n,y)$ is replaced by $h(T,x_n,y)$. However, by Lemma 4.9 and Hölder inequality we have

$$\int_H (h(T,x_n,y))^q\nu(dy)\leqslant\int_H h_q(T,x_n,y)\nu(dy) \tag{4.23}$$

$$\leqslant k_q\exp\{k_q|x_n|\}\int_H \exp\{\int_0^T |B_3(s)y|ds\}\nu(dy)<c_q$$

where c_q does not depend on n, since the sequence x_n is obviously bounded and

$$\int_H \exp\{\int_0^T |B_3(s)y|ds\}\nu(dy)<\infty$$

by (4.16), (4.1) and the Fernique inequality.

Proof of Theorem 4.7. We can rewrite (4.10) in the form

$$d(T,x,Y)=h(T,x,y)H(t,x,y),$$

where

$$H(T,x,y)=\frac{\mu_T^x(dy)}{\nu(dy)}.$$

Invoking the Hölder inequality we obtain

$$\begin{aligned}\|P_T\phi\|^q_{L^q(H,\nu)} &= \int_H\left(\int_H hH\phi\nu(dy)\right)^q\nu(dx)\\ &\leqslant\int_H\left(\left(\int_H h^{p'}H^{p'}\nu(dy)\right)^{1/p'}\left(\int_H|\phi|^p\nu(dy)\right)^{1/p}\right)^q\nu(dx) \qquad (4.24)\\ &= \|\phi\|_p^q\int_H\left(\int_H h^{p'}H^{p'}\nu(dy)\right)^{q/p'}\nu(dx).\end{aligned}$$

It remains to show that

$$K = \int_H \left(\int_H h^{p'} H^{p'} \nu(dy) \right)^{q/p'} \nu(dx) < \infty. \tag{4.25}$$

Indeed, using successively the Hölder equality we obtain for any $r > 1$

$$\begin{aligned} K &\leqslant \int_H \left(\int_H h^{p'r'} \nu(dy) \right)^{q/p'r'} \left(\int_H H^{p'r} \nu(dy) \right)^{q/p'r} \nu(dx) \\ &\leqslant \left(\int_H \left(\int_H h^{p'r'} \nu(dy) \right)^{q/p'} \nu(dx) \right)^{1/r'} \left(\int_H \left(\int_H H^{p'r} \nu(dy) \right)^{q/p'} \nu(dx) \right)^{1/r}. \end{aligned} \tag{4.26}$$

It was shown in [6] that

$$\int_H \left(\int_H H^{a'} \nu(dy) \right)^{b/a'} \nu(dx) < \infty, \tag{4.27}$$

for any $a, b \geqslant 1$, such that

$$b \leqslant 1 + \frac{a-1}{\|S_0(T)\|^2}. \tag{4.28}$$

Putting

$$a = \frac{p'r}{p'r-1} \quad \text{and} \quad b = qr,$$

we find that there exists $r > 1$ such that (4.28) holds. Therefore, for such an r

$$\int_H \left(H^{p'r} \nu(dy) \right)^{q/p'} \nu(dx) = \int_H \left(H^{a'} \nu(dy) \right)^{b/a'} \nu(dx) < \infty. \tag{4.29}$$

Next, we need to show that

$$\int_H \left(\int_H h^{p'r'} \nu(dy) \right)^{q/p'} \nu(dx) < \infty. \tag{4.30}$$

To prove (4.30) we note that if $\frac{q}{p'} \geqslant 1$ then

$$\int_H \left(\int_H h^{p'r'} \nu(dy) \right)^{q/p'} \nu(dx) \leqslant \int_H \int_H h^{r'q} \nu(dy) \nu(dx)$$

However, using Lemma 4.9 for $\tilde{q} = r'q$ we have

$$\begin{aligned} \int_H \int_H (h(T,x,y))^{\tilde{q}} \nu(dx)\nu(dy) &\leqslant \int_H \int_H h_{\tilde{q}}(T,x,y) \nu(dx)\nu(dy) \\ &\leqslant \int_H \int_H k_{\tilde{q}} \exp\{k_{\tilde{q}}(|x| + \int_0^T |B_3(s)y|ds)\} \nu(dx)\nu(dy) \\ &\leqslant k_{\tilde{q}} \int_H \exp\{k_{\tilde{q}}|x|\} \nu(dx) \int_H \exp\{k_{\tilde{q}} \int_0^T |B_3(s)y|ds)\} \nu(dy) \end{aligned}$$

$$= k_{\tilde q}\mathbb{E}e^{k_{\tilde q}|\tilde Z|}\cdot \mathbb{E}\exp\{k_{\tilde q}\int_0^T |B_3(s)\tilde Z|ds\}$$

where $\tilde Z$ is an arbitrary random variable with probability distribution ν. By (4.16), (4.1) and the Fernique inequality we conclude that (4.30) holds true. The proof of (4.30) for the case when $\frac{q}{p'} < 1$ is even simpler and is omitted. The fact that for $p = 2$ the operator P_T is γ-radonifying, hence Hilbert-Schmidt for $p = q = 2$ now follows from the representation of γ-radonifying operators, see [2].

Remark 4.10. There is a natural question whether the transition density is regular (continuous) "in y", that is, whether the mapping $y \mapsto d(T, x, y)$ is continuous, at least on a certain subspace $\hat H \subset H$) of full measure. In the Gaussian case the formulas for the density may be used to conclude that if

$$\overline{S_T^* Q_T^{-1}} \in \mathscr{L}(\hat H, H) \tag{4.31}$$

then $y \to g(T, x, y)$ is continuous on $\hat H$ for all $T > 0$ and $x \in H$ (cf. the Cameron-Martin formula (4.12)). A similar well-known formula for $k(T, y)$ (see e.g. [6]) yields $\hat H \to H$ continuity of the mapping $y \mapsto k(T, y)$ provided

$$C(T) := \overline{Q_\infty^{-1/2}(I - S_0(T)S_0^*(T))^{-1}S_0(T)S_0^*(T)Q_\infty^{-1/2}} \in \mathscr{L}(\hat H, H) \tag{4.32}$$

where $S_0(T) = Q_\infty^{-1/2} S_T Q_\infty^{1/2}$. Following the proof of Theorem 4.5 we can easily see that the remaining factor, the function $h(T, x, y)$ is continuous in $y \in \hat H$ if the mapping $y \to \hat Z_t^{x,y}$ is $\hat H \to H$ a.s. continuous (which by Proposition 4.1 (b) happens if $\overline{K_t Q_T^{-1/2}} \in \mathscr{L}(\hat H, H)$, $t < T$) and

$$B_3 \in \mathscr{L}(\hat H, L^1(0, T; H)). \tag{4.33}$$

In fact, a more careful analysis of the situation shows that if (4.31)-(4.33) is satisfied, we already have the joint continuity of the mapping $(x, y) \to p(T, x, y)$ on $H \times \hat H$.

We are able to verify these additional conditions in some important cases (supposing that the standing assumptions of this Section (4.3), (4.4) and (4.9) are satisfied).

(a) All conditions (4.31)-(4.33) are satisfied if $\dim H < \infty$.

(b) In the commutative case the conditions (4.31) and (4.32) are satisfied with $\hat H = H$ by the strong Feller property. However, condition (4.33) is not satisfied with $\hat H = H$ even in simple infinite - dimensional situations (cf. Example 4.12 below).

(c) Assume also that the generator A has bounded imaginary powers and (for simplicity) $Q = I$. Under these assumptions the OU semigroup (R_t) is analytic in $L^2(H, \nu)$ and moreover its generator L is variational, see [11] for details and for more general results. In particular these conditions are satisfied if A is a variational operator in a bounded domain with Dirichlet boundary conditions (for instance). Then it follows from [10] that the $S_0(t) = Q_\infty^{-1/2} S_t Q_\infty^{1/2}$ defines a C_0-semigroup of contractions in the domain of the operator $Q_\infty^{-1/2}$ endowed with the norm $|x|_0 = \left|Q_\infty^{-1/2}x\right|$. Therefore, for $h \in \operatorname{im}\left(Q_\infty^{1/2}\right)$ we obtain

$S_0(t)h \in \operatorname{im}\left(Q_\infty^{1/2}\right)$ and

$$|S_0(t)h|_0 = \left|Q_\infty^{-1} S_t Q_\infty Q_\infty^{-1/2} h\right| \leqslant \left|Q_\infty^{-1/2} h\right|,$$

or equivalently $\|Q_\infty^{-1} S_t Q_\infty\| \leqslant 1$ and $V_t = Q_\infty^{-1} S_t Q_\infty$ is a C_0-semigroup in H. Hence our first condition is satisfied with $\hat{H} = H$. Note that in this case results in [5] yield the existence of a dual OU process Z^* such that $R_t^* \phi(y) = \mathbb{E}\phi\left(Z(t,y)^*\right)$ and

$$\begin{cases} dZ^*(t,y) = BZ^*(t,y)dt + dW_t, \\ Z(0,y) = y, \end{cases}$$

where $B = \overline{Q_\infty A^* Q_\infty^{-1}}$ is a generator of the C_0-semigroup V_t. Note also that the existence of the process Z^* follows from the general theory of nonsymmetric Dirichlet forms, see [15]. In this case we could construct a dual bridge $Z^{*y,x}$ from $Z^{x,y}$ by time reversal.

Example 4.11. Consider the semilinear stochastic heat equation

$$\frac{\partial u}{\partial t}(t,\xi) = \frac{\partial^2 u}{\partial \xi^2}(t,\xi) + f(u(t,\xi)) + \eta(t,\xi), \quad (t,\xi) \in \mathbb{R}_+ \times (0,1), \tag{4.34}$$

with an initial condition and Dirichlet boundary conditions

$$u(0,\xi) = x(\xi), \quad u(t,0) = u(t,1) = 0, \quad t \geqslant 0,\ \xi \in (0,1) \tag{4.35}$$

where $f : \mathbb{R} \to \mathbb{R}$ is bounded and continuous and η denoted formally a space-dependent white noise. As well known (see e.g. [8] for fundamentals on the theory of stochastic evolution equations) the system (4.34) - (4.35) may be understood as an equation of the form (4.7) in the space $H = L^2(0,1)$ where $A = \frac{\partial^2}{\partial \xi^2}$, $\operatorname{dom}(A) = H_0^1(0,1) \cap H^2(0,1)$, $F : H \to H$, $F(y)(\xi) := f(y(\xi))$, $y \in H$, $\xi \in (0,1)$, and $\sqrt{Q}$ is a bounded operator on $H = L^2(0,1)$. We assume that the operator Q is boundedly invertible on H, (i.e., the noise is nondegenerate). Then Hypothesis 4.3 is obviously satisfied and Hypothesis 4.4 (ii) is satisfied with $\beta = \frac{1}{2}$ and arbitrary $\alpha \in (0, \frac{1}{2})$ (cf.[12], Example 9.2 and references therein).

Thus the conclusions of Theorems 4.5, 4.7 and 4.6 hold true in the present example.

Example 4.12. Let (e_n) denote an ONB of a Hilbert space H and assume that the operators A and Q are given by sequences if their eigenvalues $(-\alpha_n)$, (λ_n),

$$Ae_n = -\alpha_n e_n, \quad 0 < \alpha_n \to \infty,$$

and

$$Qe_n = \lambda_n e_n, \quad 0 < \lambda_n \leqslant \sup \lambda_n < \infty,$$

(note that in the previous example the operator A satisfies this condition with $\alpha_n \sim n^2$). In this "diagonal case" all Hypotheses made in the paper may be expressed and verified in terms of the sequences $(-\alpha_n)$, (λ_n). More specifically,

$$\sum \frac{\lambda_n}{\alpha_n} < \infty \tag{4.36}$$

is equivalent to Hypothesis 2.5 ; in that case all results of Section 2 on the OU Bridge hold true (obviously, (4.36) is also necessary and sufficient for the OU process to be well defined in H).

Furthermore, it is easy to compute that

$$Q_t e_n = \frac{\lambda_n}{\alpha_n}(1 - e^{-2\alpha_n t})e_n, \tag{4.37}$$

and so Hypotheses 3.1 and 3.7 are always satisfied. Therefore (under condition (4.36)) the differential equations for the OU Bridge has the mild and weak solutions described in Theorem 3.8 and Corollary 3.9, respectively.

This equation splits into a sequence of independent one-dimensional equations for particular coordinates $\hat{z}_n^{x,y}(t) := \left\langle \hat{Z}_t^{x,y}, e_n \right\rangle$. We obtain

$$d\hat{z}_n^{x,y}(t) = [-\alpha_n \hat{z}_n^{x,y}(t) - 2\alpha_n e^{-\alpha_n(T-t)}(1 - e^{-2\alpha_n(T-t)})^{-1}(e^{-\alpha_n(T-t)}\hat{z}_n^{x,y}(t) - y_n)]dt + \sqrt{\lambda_n}d\zeta_n(t)$$

for $t \in (0, T)$ with the initial condition

$$\hat{z}_n^{x,y}(0) = x_n,$$

where $x_n = \langle x, e_n \rangle$, $y_n = \langle y, e_n \rangle$ and $\zeta_n(t) = \langle \zeta_t, e_n \rangle$. The mild and weak formulas from Theorem 3.8 and Corollary 3.9 may be easily expressed as well.
Note that if dimH $< \infty$, the condition (4.36) is automatically satisfied. In this case the above equation has obviously a strong solution. Here we need not have to assume that the eigenvalues α_n are all negative, only $\alpha_n \neq 0$. If $\alpha_n = 0$ for some n the corresponding equation takes the form

$$d\hat{z}_n^{x,y}(t) = \frac{y_n - \hat{z}_n^{x,y}(t)}{T - t}dt + \sqrt{\lambda_n}d\zeta_n(t), \quad t \in (0, T),$$

which is a well-known equation for a one-dimensional Brownian Bridge.
In Section 4, where the semilinear equations are considered, our standing assumption was (4.1) (the strong Feller property for the OU process), which in the present example is equivalent to

$$\sup_n \frac{\alpha_n}{\lambda_n} e^{-2\alpha_n t} < C_t, \quad t > 0, \tag{4.38}$$

where $C_t < \infty$ (intuitively, the noise term is "sufficiently nondegenerate"). The condition (4.9) (existence of the invariant measure for the OU process) is automatically satisfied and the conditions of Hypothesis 4.4 have been often studied in the past and may be easily formulated in terms of sequences (α_n) and (λ_n) (cf. Section 3 in [13]). For instance, if $\sum (1/\alpha_n)^{1-\epsilon}$ holds for some $\epsilon > 0$, $\lambda_n > c > 0$ and the nonlinear term F is bounded and continuous, the conclusions of Theorems 4.5- 4.7 hold true (in particular, the transition densities are "continuous in x").
The continuity of transition density "in the variable y" may be veriified by means of Remark 4.10 . It is easy to compute eigenvalue expansions of all operators that appear there. We

have

$$K_t Q_T^{-1} e_n = \frac{1 - e^{-2\alpha_n t}}{1 - e^{-2\alpha_n T}} e^{-\alpha_n (T-t)} e_n, \tag{4.39}$$

$$Q_T^{-1} S_T^* e_n = 2e^{-\alpha_n T} \frac{\alpha_n}{\lambda_n} (1 - e^{-\alpha_n T})^{-1} e_n, \tag{4.40}$$

$$C(T) e_n = 2e^{-2\alpha_n T} \frac{\alpha_n}{\lambda_n} (1 - e^{-2\alpha_n T})^{-1} e_n, \tag{4.41}$$

$$B_3(s) e_n = 2e^{-\alpha_n (T-s)} \frac{\alpha_n}{\sqrt{\lambda_n}} (1 - e^{-\alpha_n T})^{-1} e_n. \tag{4.42}$$

As an illustrative example consider the case when the "nonlinear term" F is, in fact, a constant element of H, $F = \sum F_n < F, e_n >$. The solution to the equation (4.7) has the form

$$X_t = S_t x + a(t) + Z_t, \quad t \geqslant 0,$$

where $a(t) := \int_0^t S_{t-s} F ds$. In order to satisfy Hypothesis 4.3 we assume that $F \in \mathrm{im}(\mathrm{Q}^{1/2})$ for a given $T > 0$, which is equivalent to

$$\sum \frac{F_n^2}{\lambda_n} < \infty. \tag{4.43}$$

The regularity "in x" of the density may be then obtained as a particular case of the preceding part of the Example. However, since the solution is Gaussian, we may conclude directly by the Cameron-Martin formula that the mapping $x \to d(T, x, y)$ is continuous (in fact, smooth), for fixed y from a set of measure one, if and only if the strong Feller property (the condition (4.38)) is satisfied and $a(T) \in \mathrm{im}(\mathrm{Q}_\mathrm{T}^{1/2})$ holds. The latter condition is equivalent to $(-A)^{-1/2} F \in \mathrm{im}(\mathrm{Q}^{1/2})$ or equivalently,

$$\sum \frac{F_n^2}{\lambda_n \alpha_n} < \infty \tag{4.44}$$

(this is obviously a weaker condition that (4.43), which in this case is not needed). Now, let us check the regularity in the variable y for a fixed $x \in H$. Assume that the OU process is strongly Feller ((i.e., 4.38) is satisfied). It is easy see that the mapping $y \to k(T, y)$) is continuous. The continuity $y \to g(T, x, y)$ is equivalent to the inclusion $Q_T^{-1} S_T^* \in \mathscr{L}(H)$, which in terms of the eigenvalues is expressed as

$$\sup_n \frac{\alpha_n}{\lambda_n} e^{-\alpha_n T} < C_T. \tag{4.45}$$

This would be, for a fixed $T > 0$, a stronger demand than the strong Feller property (4.38) but if we require (4.38) and (4.45) for each $T > 0$, they are equivalent. By Gaussianity, the remaining factor h may be again expressed by the Cameron-Martin formula

$$h(T, x, y) = \frac{N(S_T x + a(T), Q_T)(dy)}{N(S_t x, Q_T)(dy)} = \exp(< Q_T^{-1/2} a(T), Q_T^{-1/2} y > - \frac{1}{2} |Q_T^{-1/2} a(T)|^2). \tag{4.46}$$

Now it is easy to see that the mapping $y \to h(T,x,y)$ is continuous (and in fact, smooth) for each $x \in H$ if and only if $a(T) \in \mathrm{im}(\mathrm{Q_T})$, which turns out to be the same as $F \in \mathrm{im}(\mathrm{Q})$, or equivalently,

$$\sum \frac{F_n^2}{\lambda_n^2} < \infty. \tag{4.47}$$

Obviously, (4.47) is stronger than (4.43), which shows that for the continuity "in x" the Hypothesis 4.3, which makes the Girsanov theorem applicable, is in general unnecessary. For continuity "in y" in our example, even stronger condition (4.47) is necessary. However, our formulation of the problem is not "symmetric in x and y": While x is the initial value that is supposed to be arbitrary, y is just a variable in the densities and we obtain continuity $x \to d(T,x,y)$ only for y from a set of measure one.

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