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Milan Tvrdý

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THE NORMAL FORM AND THE STABILITY OF SOLUTIONS OF A SYSTEM OF DIFFERENTIAL EQUATIONS IN THE COMPLEX DOMAIN

MILAN TVRDÝ, Praha

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1. INTRODUCTION

The stability of the trivial solution $\boldsymbol{x} \equiv \boldsymbol{O}$ of the system (1,1) of differential equations described below will be investigated in this paper in some critical cases.

We shall consider the system of differential equations

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{X}(\boldsymbol{x}) \tag{1,1}$$

with the following properties $(\boldsymbol{P})$:

a) $\boldsymbol{A}$ is a $n \times n$-matrix whose elements $a_{i,j}$ $(i, j = 1, 2, \ldots, n)$ are complex constants.

b) $\boldsymbol{x}$ is a column n-vector whose components x_j $(j = 1, 2, \ldots, n)$ are complex-valued functions of a real variable t.

c) $\boldsymbol{X}$ is a column n-vector whose components X_j $(j = 1, 2, \ldots, n)$ are complex-valued functions of n complex variables $x_1, x_2, \ldots, x_n$ holomorphic for $|x_j| < H$; $j = 1, 2 \ldots, n$ $(H > 0)$ and such that their developments into power series in $x_1, x_2, \ldots, x_n$ begin with terms of the second or higher order.

We shall follow the papers of O. VEJVODA ([1]) and O. GÖTZ ([2]), who are to my knowledge the only ones who have so far dealt with the problem in question. The former treated two critical cases, particularly that of one zero root and that of one purely imaginary root of the characteristic equation of the matrix $\boldsymbol{A}$. The latter investigated the case of the n-tuple zero root of the characteristic equation of the matrix $\boldsymbol{A}$. In this paper the critical case of an l-tuple $(l \leqq n)$ zero root and the case of several purely imaginary roots (under some additional assumptions) are dealt with.

The proper stability investigations are preceded by the treatment of the normal form of the system (1,1). A general theorem (Theorem 3,4) supplementing the well-known results of H. POINCARE ([3]), H. DULAC ([4]), K. L. SIEGEL ([8], [9]) A. D. BRJUNO ([5]–[7]) and E. PESCHL and L. REICH ([15]–[17]) is proved using the method of majorant series*).

I should like to express my gratitude to Prof. OTTO VEJVODA whose advices contributed considerably to the improvement of this paper.

2. PRELIMINARIES

2,1. Assumptions and notations. The following assumptions, notations and conventions will be kept throughout the paper.

A complex number z being given, Re z, Im z, $\bar{z}$ and $|z|$ have their usual meaning. The space of all complex numbers (with the usual norm $|\ldots|$) will be denoted by $\boldsymbol{K}$. The space of all column n-vectors $\boldsymbol{x}$ whose components are elements of $\boldsymbol{K}$ will be denoted by $\boldsymbol{K}_n$. The components of given vector $\boldsymbol{x} \in \boldsymbol{K}_n$ will be denoted by $x_1, x_2, \ldots$ $\ldots, x_n$. Given $\boldsymbol{x} \in \boldsymbol{K}_n$, $\|\boldsymbol{x}\|$ denotes the norm of $\boldsymbol{x}$ (in $\boldsymbol{K}_n$) defined by $\|\boldsymbol{x}\| = (\sum_{j=1}^{n} |x_j|^2)^{1/2}$.

Speaking about the system (1,1) we shall always suppose that this system fulfils the conditions ($\boldsymbol{P}$). Let H be the positive constant from the conditions ($\boldsymbol{P}$). Then the set of all $\boldsymbol{x} \in \boldsymbol{K}_n$ such that $\|\boldsymbol{x}\| < H$ will be denoted by $\boldsymbol{\Omega}$. (Hence $\boldsymbol{\Omega} = \mathrm{E}[\boldsymbol{x} \in \boldsymbol{K}_n : \|\boldsymbol{x}\| < H]$.) If some complex-valued n-vector function $\boldsymbol{Z}(\boldsymbol{z})$ of n complex variables $z_1, z_2, \ldots, z_n$ satisfies condition c) in ($\boldsymbol{P}$) we shall write $\boldsymbol{Z}(\boldsymbol{z}) = [\boldsymbol{z}]_2$ and we shall say that $\boldsymbol{Z}(\boldsymbol{z})$ is of the type $[\boldsymbol{z}]_2$. The matrix $\boldsymbol{A}$ in (1,1) will be called the matrix of the linear terms of the system (1,1).

Given $n \times n$-matrix $\boldsymbol{A}$, the elements of $\boldsymbol{A}$ will be denoted by $a_{i,j}$ $(i, j = 1, 2, \ldots, n)$. The unit $n \times n$-matrix will be denoted by $\boldsymbol{I}$. $\boldsymbol{A}$ being the matrix of the linear terms of the system (1,1), the characteristic equation of the matrix $\boldsymbol{A}$:

$$\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = 0 \tag{2,1}$$

will be also called the characteristic equation of the system (1,1). In this paper each root of the characteristic equation (2,1) of the system (1,1) will be counted so many times as corresponds to its algebraical multiplicity. These roots will usually be denoted by $\lambda_1, \lambda_2, \ldots, \lambda_n$, the enumeration of them being performed by the following rule: Let $\mu_1, \mu_2, \ldots, \mu_r$ $(r \leqq n)$ be all mutually distinct roots of (2,1) (arbitrarily enumerated) with algebraic multiplicity $n_1, n_2, \ldots, n_r$, respectively. Then $\lambda_{n_1+n_2+\ldots+n_{j-1}+k} = \mu_j$ $(k = 1, 2, \ldots, n_j)$ holds for each $j = 1, 2, \ldots, r$ $(n_0 = 0)$.

For any nonnegative integer k the space of all (row) n-vectors $\boldsymbol{p}$ whose components

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During this article was printed I got acquinted also with hitherto unpublished papers ([18], [19]) of E. Peschl and L. Reich.

$p_1, p_2, \ldots, p_n$ are nonnegative integers such that $p_1 + p_2 + \ldots + p_n \geqq k$ holds will be denoted by $\mathcal{M}_k(\mathcal{M}_0 = \mathcal{M})$. Given $\boldsymbol{p}, \boldsymbol{q} \in \mathcal{M}$, the expressions $\boldsymbol{p} + \boldsymbol{q}$ or $\boldsymbol{p} - \boldsymbol{q}$ have the obvious sense. Given $\boldsymbol{x} \in \boldsymbol{K}_n$ and $\boldsymbol{p} \in \mathcal{M}$, $\boldsymbol{x}^{\boldsymbol{p}}$ will be written briefly instead of $x_1^{p_1} x_2^{p_2} \ldots x_n^{p_n}$. Given $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \boldsymbol{K}_n$, $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ will be written briefly instead of $\lambda_1\mu_1 + \lambda_2\mu_2 + \ldots + \lambda_n\mu_n$. Given a natural number k less than $n + 1$ and $\boldsymbol{\lambda} \in \boldsymbol{K}_n$, the space of all (row) n-vectors $\boldsymbol{p} \in \mathcal{M}$ for which $(\boldsymbol{\lambda}, \boldsymbol{p}) = \lambda_k$ holds will be denoted by $\mathcal{N}_k(\boldsymbol{\lambda})$. Given natural number k less than $n + 1$, the element $\boldsymbol{q}$ of $\mathcal{M}$ with $q_k = 1$ and $q_j = 0$ for $j \neq k, j = 1, 2, \ldots, n$ will be denoted by $\mathbf{e}_k$.

Finally, $\boldsymbol{G}$ being a subset of $\boldsymbol{K}_n$, $\overline{\boldsymbol{G}}$ denotes the closure of $\boldsymbol{G}$ in the topology given by the norm $\|\ldots\|$ defined above.

2,2. Some remarks concerning the system (1,1). Let us remember some properties of the system (1,1) (which satisfies conditions $(\boldsymbol{P})$).

a) $\boldsymbol{x} \equiv \boldsymbol{O}$ is the solution of the system (1,1) on the interval $(-\infty, \infty)$.

b) $\boldsymbol{x}(t)$ being the solution of the system (1,1) on the interval (a, b) and t_0 being an arbitrary real number, $\boldsymbol{x}(t + t_0)$ is a solution of the system (1,1) on the interval $(a - t_0, b - t_0)$.

c) By a standard theorem to any $\xi \in \boldsymbol{\Omega}$ there exists on $\langle 0, \infty)$ the maximal interval $I_\xi = \langle 0, T_\xi)$ $(T_\xi > 0)$ on which the solution $\boldsymbol{x}(t, \xi)$ with $\boldsymbol{x}(0, \xi) = \xi$ exists, $\boldsymbol{x}(t, \xi)$ belonging to $\boldsymbol{\Omega}$ for every $t \in I_\xi$.

d) We shall say (in accordance with the terminology of O. Götz in [2]) that the given system (1,1) has a domain of boundedness for $t \geqq 0$ if there is a domain $\boldsymbol{B} \subset \boldsymbol{K}_n$ with the following properties:

(i) $\boldsymbol{O} \in \boldsymbol{B} \subset \boldsymbol{\Omega}$.

(ii) The solution $\boldsymbol{x}(t, \xi)$ (with $\boldsymbol{x}(0, \xi) = \xi$) is defined on the whole interval $\langle 0, \infty)$ for any $\xi \in \boldsymbol{B}$.

(iii) There is a compact set $\boldsymbol{C}$ such that $\boldsymbol{B} \subset \boldsymbol{C} \subset \boldsymbol{\Omega}$ holds, $\boldsymbol{x}(t, \xi)$ belonging to $\boldsymbol{C}$ for every $\xi \in \boldsymbol{B}$ and for every $t \geqq 0$.

(Such domain $\boldsymbol{B}$ will be called the domain of boundedness of the system (1,1) for $t \geqq 0$).

e) Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the roots of the characteristic equation of the system (1,1) ordered according to the rule mentioned in sec. 2,1. It is well known that there always exists such a regular $n \times n$-matrix $\mathbf{S}$ of complex constants that the substitution $\boldsymbol{x} = \mathbf{S}\boldsymbol{y}$ transforms the system (1,1) into the system

$$\dot{\mathbf{y}} = \mathbf{B}\mathbf{y} + \mathbf{Y}(\mathbf{y}), \tag{2,2}$$

which satisfies again the conditions $(\boldsymbol{P})$ and whose matrix $\mathbf{B}$ of the linear terms has moreover the following properties:

(i) $\boldsymbol{B} = \boldsymbol{S}^{-1}\boldsymbol{AS}$, $\boldsymbol{B} = (b_{i,j})_{i,j=1,2,\ldots,n}$.

(ii) $b_{i,j} = 0$ if $j > i$ or $i > j + 1$, $b_{i,i} = \lambda_i$ $(i = 1, 2, \ldots, n)$.

(iii) Each $b_{j,j-1}$ $(j = 2, 3, \ldots, n)$ either equals 0 or 1, the fact whether for a certain $j = 2, 3, \ldots, n$ there holds $b_{j,j-1} = 0$ or $b_{j,j-1} = 1$ being uniquely determined by the matrix $\boldsymbol{A}$ (or more precisely by the multiplicity of the elementary divisors of the matrix $(\lambda_j \boldsymbol{I} - \boldsymbol{A})$). We shall use the notation $b_{j,j-1} = \varepsilon_j$ $(j = 2, 3, \ldots, n)$ and we shall call the $(n-1)$-vector $\boldsymbol{\varepsilon} = (\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_n)$ the first subdiagonal of the matrix $\boldsymbol{B}$. The n-vector $(b_{1,1}, b_{2,2}, \ldots, b_{n,n}) = (\lambda_1, \lambda_2, \ldots, \lambda_n) = \boldsymbol{\lambda}$ is called the diagonal of the matrix $\boldsymbol{B}$ and the matrix $\boldsymbol{B}$ is called the Jordan canonical form of the matrix $\boldsymbol{A}$.

In the sequel we shall use the following convention: The roots of the characteristic equation of the system (1,1) having been enumerated in some manner (which is not in a contradiction with the rule from sec. 2,1, of course), the diagonal of the Jordan canonical form $\boldsymbol{B}$ of the matrix $\boldsymbol{A}$ is assumed to be ordered so that $b_{j,j} = \lambda_j$ $(j = 1, 2, \ldots, n)$. Let $\boldsymbol{\varepsilon}' = (\varepsilon'_2, \varepsilon'_3, \ldots, \varepsilon'_n)$ be an arbitrary n-vector from $\boldsymbol{K}_{n-1}$ such that for any $j = 2, 3, \ldots, n$ ε'_j equals 0 if and only if ε_j equals 0. Then there exists a regular $n \times n$-matrix $\boldsymbol{R}$ of complex numbers such that for the matrix $\boldsymbol{C} = \boldsymbol{R}^{-1}\boldsymbol{AR} = (c_{i,j})_{i,j=1,2,\ldots,n}$ there holds: $c_{j,j} = \lambda_j$ $(j = 1, 2, \ldots, n)$, $c_{j,j-1} = \varepsilon'_j$ $(j = 2, 3, \ldots, n)$ and $c_{i,j} = 0$ if and only if $b_{i,j} = 0$. The matrix $\boldsymbol{C}$ is then called the Jordan canonical form of the matrix $\boldsymbol{A}$ with the first subdiagonal $\boldsymbol{\varepsilon}'$.

Let λ be an arbitrary root of the characteristic equation of the system (1,1). Let $\mu_1, \mu_2, \ldots, \mu_r$ be all mutually distinct roots of the characteristic equation of the system (1,1) with the algebraical multiplicity $n_1, n_2, \ldots, n_r$, respectively. Let $\boldsymbol{\varepsilon} = (\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_n)$ be the first subdiagonal of some arbitrary Jordan canonical form of the matrix $\boldsymbol{A}$. Then the root λ is said to be diagonable (see [2]) if, being $\lambda = \mu_j$ for some $j = 1, 2, \ldots, r$, there holds $\varepsilon_{n_1+n_2+\ldots+n_{j-1}+k} = 0$ for all $k = 1, 2, \ldots, n_j$. (It also means that the root λ of the characteristic equation of the system (1,1) is diagonable if and only if the matrix $(\lambda \boldsymbol{I} - \boldsymbol{A})$ has only simple elementary divisors.)

2,3. Basic definitions and theorems. Let $\boldsymbol{Z}(t, \boldsymbol{x})$ be an arbitrary complex-valued n-vector function of n complex variables $x_1, x_2, \ldots, x_n$ and one real variable t defined for every $t \geqq 0$ and for every $\boldsymbol{x}$ belonging to some open neighbourhood $\boldsymbol{\Omega}_1$ of the origin $\boldsymbol{O}$ in $\boldsymbol{K}_n$ and such that there holds $\boldsymbol{Z}(t, \boldsymbol{O}) = \boldsymbol{O}$ for each $t \geqq 0$. Then the stability of the trivial solution $\boldsymbol{x} \equiv \boldsymbol{O}$ of the system

$$\dot{\boldsymbol{x}} = \boldsymbol{Z}(t, \boldsymbol{x}) \tag{2,3}$$

will be defined in the usual Ljapunov way. (Clearly our system (1,1) is a special case of the system (2,3).)

Definition 2,3. *a) If for any $\varepsilon > 0$ there exists $\delta > 0$ such that every solution $\boldsymbol{x}(t)$ of the system (2,3) with $\|\boldsymbol{x}(0)\| < \delta$ is defined on $\langle 0, \infty)$ and satisfies for every $t \geqq 0$ the inequality $\|\boldsymbol{x}(t)\| < \varepsilon$, then the trivial solution of the system (2,3) is called stable.*

b) If the trivial solution of the system (2,3) is stable and if there exists $\delta_1 > 0$ such that for any solution $\mathbf{x}(t)$ of the system (2,3) with $\|\mathbf{x}(0)\| < \delta_1$ there holds $\lim_{t \to \infty} \|\mathbf{x}(t)\| = 0$, *then the trivial solution of the system (2,3) is called asymptotically stable.*

c) If the trivial solution of the system (2,3) is not stable, it is called unstable.

The following theorem is well known (see [1]).

Theorem 2,3A. *Let the right-hand side of the system (2,3) satisfy the following conditions ($\boldsymbol{P}'$):*

$\mathbf{Z}(t, \mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{X}(t, \mathbf{x})$ *where $\mathbf{A}$ is a constant $n \times n$-matrix of complex numbers and $\mathbf{X}(t, \mathbf{x})$ is a complex-valued n-vector function defined and continuous on $\langle 0, \infty) \times \times \boldsymbol{\Omega}_1$ such that*

$$\lim_{\|\mathbf{x}\| \to 0, \mathbf{x} \in \boldsymbol{\Omega}_1} \frac{\mathbf{X}(t, \mathbf{x})}{\|\mathbf{x}\|} = \mathbf{O}$$

uniformly with respect to $t \geqq 0$.

Then there holds:

a) If all roots of the characteristic equation of the system (2,3) have negative real parts, then the trivial solution of the system (2,3) is asymptotically stable.

b) If at least one root of the characteristic equation of the system (2,3) has a positive real part, then the trivial solution of the system (2,3) is unstable.

The case that the system (2,3) fulfils conditions ($\boldsymbol{P}'$) and that at least one root of the characteristic equation of the system (2,3) has a zero real part, the others having negative real parts, is called the critical case. We shall deal with some critical cases of the system (1,1) (with the properties ($\boldsymbol{P}$)). The list of theorems which are necessary for our investigation will be given below.

All these theorems are due to O. Götz. (See [2].)

Theorem 2,3B. *The trivial solution of the system (1,1) is stable if and only if there exists a domain of boundedness of the system (1,1) for $t \geqq 0$.*

Theorem 2,3C. *A necessary condition for the trivial solution of the system (1,1) to be stable is that all roots of the characteristic equation of the system (1,1) have nonpositive real parts while those having zero real parts are diagonable.*

Theorem 2,3D. *The trivial solution of the system (1,1) is asymptotically stable if and only if all roots of the characteristic equation of the system (1,1) have negative real parts.*

Theorem 2,3E. *If the characteristic equation of the system (1,1) has an n-tuple zero root, then the trivial solution of the system (1,1) is stable if and only if there* *holds*: $\mathbf{A}$ *is the zero matrix* (*i.e.* $a_{i,j} = 0$ *for every* $i, j = 1, 2, \ldots, n$) *and* $\mathbf{X}(\mathbf{x}) = \mathbf{O}$ *on* $\mathbf{\Omega}_1$.

Besides the results of O. Vejvoda, who treated some critical cases of the system (1,1) as we remarked above, some results of I. G. MALKIN (see [11]) concerning the critical cases of real systems can be also used for the investigation of some critical cases of complex systems. This concerns particularly the case of two couples: $\lambda_1, \bar{\lambda}_1$ and $\lambda_2, \bar{\lambda}_2$ of purely imaginary roots of the characteristic equation such that $\lambda_1\lambda_2^{-1}$ is an irrational number. This critical case is evidently equivalent to the critical case of the complex system (1,1) with two purely imaginary roots λ_1, λ_2 of the characteristic equation (the other roots having negative real parts), $\lambda_1\lambda_2^{-1}$ being an irrational number.

3. PSEUDONORMAL FORM OF THE SYSTEM OF DIFFERENTIAL EQUATIONS

3,1. Some remarks concerning formal power series. Expressions of the type

$$h(\mathbf{z}) = \sum_{\mathbf{p}\in\mathcal{M}} \{h\}_{\mathbf{p}}\, \mathbf{z}^{\mathbf{p}}$$

where $\{h\}_{\mathbf{p}}$ ($\mathbf{p} \in \mathcal{M}$) are some complex constants and where an arbitrary n-vector from $\boldsymbol{K}_n$ can be substituted for $\mathbf{z}$ are called formal power series (of n complex variables $z_1, z_2, \ldots, z_n$). Their convergence for some $\mathbf{z} \neq \mathbf{O}$ is not generally supposed.

Notation 3,1. Let h be an arbitrary formal power series of n complex variables $(z_1, z_2, \ldots, z_n)$. Throughout the paper we shall make use of the following notation:

a) Given $\mathbf{p} \in \mathcal{M}$, the coefficient of $\mathbf{z}^{\mathbf{p}}$ in the series h will be denoted by $\{h\}_{\mathbf{p}}$.

b) The formal power series (of the variables $z_1, z_2, \ldots, z_n$) derived from the series h by omitting the terms of the first order will be denoted by h^*.

c) The formal power series (of the variables $z_1, z_2, \ldots, z_n$) derived from the series h by replacing every coefficient by its absolute value will be denoted by $\| h \|$, i.e.

$$h(\mathbf{z}) = \sum_{\mathbf{p}\in\mathcal{M}} \{h\}_{\mathbf{p}}\, \mathbf{z}^{\mathbf{p}};\ h^*(\mathbf{z}) = \sum_{\mathbf{p}\in\mathcal{M}_2} \{h\}_{\mathbf{p}}\, \mathbf{z}^{\mathbf{p}},\ \| h \| (\mathbf{z}) = \sum_{\mathbf{p}\in\mathcal{M}} |\{h\}_{\mathbf{p}}|\, \mathbf{z}^{\mathbf{p}}.$$

d) Given a natural number k less than $n + 1$, the formal power series g (of the variables $z_1, z_2, \ldots, z_n$) with $\{g\}_{\mathbf{p}} = (p_k + 1)\{h\}_{\mathbf{p}+\mathbf{e}_k}$ for each $\mathbf{p} \in \mathcal{M}$ will be denoted by $\partial h/\partial z_k$ (and called the formal partial derivative of h with respect to z_k). It is obvious that if h is absolutely convergent on some (open) neighbourhood $\boldsymbol{U}$ of the origin $\mathbf{O}$ in $\boldsymbol{K}_n$ (i.e. h is a function holomorphic on $\boldsymbol{U}$), then for any $k = 1, 2, \ldots, n$ the series $\partial h/\partial z_k$ is absolutely convergent on $\boldsymbol{U}$ and for every $\mathbf{y} \in \boldsymbol{U}$ the value of the partial derivative of h with respect to z_k in $\mathbf{y}$ equals the sum of the series $\partial h/\partial z_k(\mathbf{y})$.

e) The formal power series $u(1 + v + v^2 + \ldots)$ of the variables u and v will be denoted by $u\,.\,(1 - v)^{-1}$.

f) Let f be a formal power series of m complex variables $y_1, y_2, \ldots, y_m$ and let $g_1, g_2, \ldots, g_m$ be formal power series of n complex variables $z_1, z_2, \ldots, z_n$. Let us denote by F the formal power series of n complex variables $z_1, z_2, \ldots, z_n$ derived from the series f by the substitution $y_j = g_j(z_1, z_2, \ldots, z_n)$ $(j = 1, 2, \ldots, m)$. Then given $\mathbf{p} \in \mathscr{M}$, the coefficient of $\mathbf{z}^{\mathbf{p}}$ in the series $F(\mathbf{z})$ will be denoted also by $\{f(g)\}_{\mathbf{p}}$.

g) If the right-hand side of the system (1,1) is a formal power series (of n complex variables $x_1, x_2, \ldots, x_n$) which is not generally absolutely convergent on any neighbourhood of the origin $\mathbf{O}$ in K_n, we shall call the system (1,1) a formal system of differential equations.

h) We shall say that the formal power series f and g (of n complex variables) are equal to each other if there holds $\{f\}_{\mathbf{p}} = \{g\}_{\mathbf{p}}$ for any $\mathbf{p} \in \mathscr{M}$. Let us define in the usual way (see [14], pp. 3–9) the addition and multiplication on the set of all formal power series. The sum and the product of formal power series f and g will be denoted $f + g, fg$, respectively.

Given a complex number α, an integer k and a formal power series f, we define in the obvious manner the α-multiple of the series f (αf) and the k^{th} power of the series f (f^k).

Definition 3,1A. *We say that the formal power series g (of n complex variables) is majorant to the formal power series f (of n complex variables) if, given an arbitrary n-tuple $\mathbf{p} \in \mathscr{M}$, there holds $|\{f\}_{\mathbf{p}}| \leqq \{g\}_{\mathbf{p}}$. (This will be denoted by $h \lll g$ or $h(\mathbf{z}) \lll g(\mathbf{z})$.)*

Let us give (without proofs) two trivial assertions justifying some operations with formal power series utilized in the proof of the main theorem (Theorem 3,4) of this chapter.

Lemma 3,1A. *Let f, g, h be formal power series of n complex variables $(z_1, z_2, \ldots \ldots, z_n)$, the series h having all nonnegative coefficients and f being majorant to g $(g \lll f)$. Then there holds:*

$$a)\quad g + h \lll f + h$$
$$b)\quad g \,.\, h \lll f \,.\, h$$

Lemma 3,1B. *a) Let g and h be formal power series of n complex variables $z_1, z_2, \ldots, z_n$ and let $f_1, f_2, \ldots, f_n$ be formal power series of k complex variables $x_1, x_2, \ldots, x_k$ with zero absolute terms and with all nonnegative coefficients. Let us denote by G, H the formal power series of the variables $x_1, x_2, \ldots, x_k$ obtained by the substitution $z_j = f_j(x_1, x_2, \ldots, x_k)$ $(j = 1, 2, \ldots, n)$ into the series $g(\mathbf{z})$ and $h(\mathbf{z})$, respectively. Let $g(\mathbf{z}) \lll h(\mathbf{z})$, then there holds also $G(\mathbf{x}) \lll H(\mathbf{x})$.*

b) Let f be a formal power series of n complex variables $z_1, z_2, \ldots, z_n$ with all nonnegative coefficients. Let g_j, h_j $(j = 1, 2, \ldots, n)$ be formal power series of k

complex variables $x_1, x_2, \ldots, x_k$ *with zero absolute terms. Let* $g_j \ll h_j$ $(j = 1, 2, \ldots$ $\ldots, n)$. *Let us denote by* F_g, F_h *the formal power series of the variables* $x_1, x_2, \ldots, x_k$ *obtained by inserting* $z_j = g_j(x_1, x_2, \ldots, x_k)$ *or* $z_j = h_j(x_1, x_2, \ldots, x_k)$ $(j = 1, 2, \ldots,$ $\ldots, n)$ *into* $f(\mathbf{z})$, *respectively. Then there holds* $F_g(\mathbf{x}) \ll F_h(\mathbf{x})$.

Remark 3,1. Let us give several consequences of Lemma 3,1B.

a) Given formal power series g_1, g_2, h_1, h_2 of n complex variables $z_1, z_2, \ldots, z_n$ not including the absolute term and such that $g_1 \ll h_1$, $g_2 \ll h_2$ hold, there holds $G(\mathbf{z}) = g_1(\mathbf{z}) \,.\, (1 - g_2(\mathbf{z}))^{-1} \ll h_1(\mathbf{z}) \,.\, (1 - h_2(\mathbf{z}))^{-1} = H(\mathbf{z})$. ($G$ and H are formal power series of the variables $z_1, z_2, \ldots, z_n$.)

b) It is evident that if the power series g of the variables $z_1, z_2, \ldots, z_n$ is absolutely convergent for $|z_j| < \varrho_j$ $(\varrho_j > 0, j = 1, 2, \ldots, n)$ and majorant to the formal power series h (of the variables $z_1, z_2, \ldots, z_n$), then also the series h is absolutely convergent for $|z_j| < \varrho_j$ $(j = 1, 2, \ldots, n)$.

c) It is well known that to any function g of n complex variables $z_1, z_2, \ldots, z_n$ holomorphic on some open neighbourhood of the origin $\mathbf{O}$ in K_n there exist such positive numbers c_1, c_2 that, considering $g(\mathbf{z})$ as a power series of the variables $z_1, z_2, \ldots, z_n$ there holds

$$g(\mathbf{z}) \ll \frac{c_1}{\left(1 - \frac{z_1}{c_2}\right)\left(1 - \frac{z_2}{c_2}\right) \ldots \left(1 - \frac{z_n}{c_2}\right)} . \qquad \text{(See [13].)}$$

Thus (by Lemma 3,1B)

$$g(\mathbf{z}) \ll \frac{c_1}{1 - \frac{z_1 + z_2 + \ldots + z_n}{c_2}},$$

the right-hand side series being absolutely convergent for $|z_1 + z_2 + \ldots + z_n| < c_2$.

d) It is obvious that for arbitrary formal power series f, g, h of n complex variables $(z_1, z_2, \ldots, z_n)$,

$$\frac{h(\mathbf{z})}{1 - g(\mathbf{z})} + f(\mathbf{z}) = \frac{h(\mathbf{z}) + f(\mathbf{z}) \,.\, (1 - g(\mathbf{z}))}{1 - g(\mathbf{z})}$$

holds.

3,2. The pseudonormal form of the system of differential equations.

Definition 3,2A. *The mapping* $\mathbf{f}$ *from* K_n *into* K_n *which fulfils the following conditions*:

(i) $\mathbf{f}$ *is a complex-valued n-vector function of n complex variables each component*

f_j $(j = 1, 2, \ldots, n)$ *of which is holomorphic on some open neighbourhood of the origin* $\mathbf{O}$ *in* K_n (*i.e.* $\mathbf{f}$ *is a holomorphic in* $\mathbf{O}$ *mapping*),

(ii) $\mathbf{f}(\mathbf{O}) = \mathbf{O}$,

(iii) *The Jacobian* $\mathscr{I}_{\mathbf{f}}$ *of the mapping* $\mathbf{f}$ *does not vanish at* $\mathbf{O}$, *will be called the regular mapping.*

Remark 3,2A. Let $\mathbf{f}$ be an arbitrary regular mapping (from K_n into K_n). It is well known that then there are such open neighbourhoods V, V' of the origin $\mathbf{O}$ in K_n that $\mathbf{f}$ is a one-to-one mapping of V onto V' and that the mapping $\mathbf{g}$ inverse to $\mathbf{f}$ is a one-to-one regular mapping of V' onto V. It is possible to write

$$(3,1) \qquad \mathbf{y} = \mathbf{f}(\mathbf{x}) = \mathbf{F}\mathbf{x} + \mathbf{f}^*(\mathbf{x}) \quad (\text{for } \mathbf{x} \in V)$$

and

$$(3,1') \qquad \mathbf{x} = \mathbf{g}(\mathbf{y}) = \mathbf{G}\mathbf{y} + \mathbf{g}^*(\mathbf{y}) \quad (\text{for } \mathbf{y} \in V')$$

where $\mathbf{F}$ and $\mathbf{G}$ are constant regular $n \times n$-matrices of complex numbers and where $\mathbf{f}^*$ and $\mathbf{g}^*$ are complex-valued n-vector functions any component of which is of the type $[\mathbf{x}]_2$, $[\mathbf{y}]_2$, respectively. (See [13].)

Given $\mathbf{x} \in V$, there holds

$$\mathbf{x} = \mathbf{g}(\mathbf{f}(\mathbf{x})) = \mathbf{G}\mathbf{F}\mathbf{x} + \mathbf{G}\,\mathbf{f}^*(\mathbf{x}) + \mathbf{g}^*(\mathbf{F}\mathbf{x} + \mathbf{f}^*(\mathbf{x})) = \mathbf{G}\mathbf{F}\mathbf{x} + \mathbf{h}^*(\mathbf{x})$$

where $\mathbf{h}^*(\mathbf{x})$ is some complex-valued n-vector function of the variables $x_1, x_2, \ldots, x_n$ any component of which is a function of the type $[\mathbf{x}]_2$. Thus clearly

$$(3,2) \qquad \mathbf{G}\mathbf{F} = \mathbf{I} \quad \text{and} \quad \mathbf{h}^*(\mathbf{x}) \equiv \mathbf{O} \quad \text{on } V.$$

Let us substitute (3,1) into (1,1). Then we obtain a new system of differential equations for $\mathbf{y}$

$$(3,3) \qquad \dot{\mathbf{y}} = \mathbf{B}\mathbf{y} + \mathbf{Y}(\mathbf{y})$$

where (according to (3,1') and (3,2)) $\mathbf{B} = \mathbf{G}^{-1}\mathbf{A}\mathbf{G} = \mathbf{F}\mathbf{A}\mathbf{F}^{-1}$ and any component $Y_j(\mathbf{y})$ $(j = 1, 2, \ldots, n)$ of $\mathbf{Y}(\mathbf{y})$ is of the type $[\mathbf{y}]_2$.

Hence the mapping $\mathbf{f}$ generates the transformation of the given system of differential equations onto some other system. The following lemma can be easily proved.

Lemma 3,2A. *The transformation of the system* (1,1) *generated* (*in the above mentioned manner*) *by a regular mapping transforms the system* (1,1) *onto a system which also fulfils conditions* ($\boldsymbol{P}$) *and whose trivial solution is stable or unstable or asymptotically stable if and only if the trivial solution of* (1,1) *is stable, unstable or asymptotically stable, respectively.* (*The transformation of the system* (1,1) *generated by a regular mapping will be called the regular transformation.*)

Thus, without any loss of generality we can investigate an arbitrary system transformable by a regular transformation onto the system (1,1) instead of (1,1). Let us try to find the regular transformation which transforms the system (1,1) onto another system as simple as possible.

First of all (see the remark f) in sec. 2,2) we may assume the system (1,1) to be of the form

$$(3,4) \qquad \dot{x}_k = \lambda_k x_k + \varepsilon_k x_{k-1} + \hat{X}_k(x_1, x_2, \dots, x_n) \quad (k = 1, 2, \dots, n)$$

where $\lambda_1, \lambda_2, \dots, \lambda_n$ are the roots of the characteristic equation of the system (1,1), $\varepsilon_1 = 0$ and $\boldsymbol{\varepsilon} = (\varepsilon_2, \varepsilon_3, \dots, \varepsilon_n)$ is the first subdiagonal of some Jordan canonical form $\hat{\boldsymbol{A}}$ of the matrix $\boldsymbol{A}$ (thus the matrix of the linear terms of the system (3,4) equals $\hat{\boldsymbol{A}}$) and, finally, $\hat{X}_k(\mathbf{x}) = [\mathbf{x}]_2$ $(k = 1, 2, \dots, n)$. Let us substitute (3,1′) (where $\boldsymbol{G} = \boldsymbol{I}$) and (3,3) (where $\boldsymbol{B} = \hat{\boldsymbol{A}}$ and Y_k $(k = 1, 2, \dots, n)$ will be determined later, of course) into (3,4). Then we get a system of partial differential equations for g_k^* and Y_k $(k = 1, 2, \dots, n)$

$$(3,5) \qquad \sum_{j=1}^{n} \frac{\partial g_k^*}{\partial y_j}(\lambda_j y_j + \varepsilon_j y_{j-1} + Y_j) + Y_k =$$
$$= \varepsilon_k g_{k-1}^* + \lambda_k g_k^* + \hat{X}_k(g_1, g_2, \dots, g_n) \quad (k = 1, 2, \dots, n)\,.$$

Definition 3,2B. a) *The formal power series Y_k, g_k^* $(k = 1, 2, \dots, n)$ will be said to solve formally the system (3,5) if after substituting these series into (3,5) there holds:*

Given $k = 1, 2, \dots, n$ and $\boldsymbol{p} \in \mathcal{M}$, the coefficient of $\mathbf{y}^{\boldsymbol{p}}$ in the right-hand side of (3,5) formal power series (of the variables $y_1, y_2, \dots, y_n$) equals to that in the left-hand side of (3,5) formal power series.

b) *Let Y_k $(k = 1, 2, \dots, n)$ be some given formal power series (of the varibles $y_1, y_2, \dots, y_n$). Then, if the formal power series Y_k, g_k^* $(k = 1, 2, \dots, n)$ solve formally the system (3,5), we shall say that the mapping (3,1′) generates the formal transformation of the system (1,1) onto the formal system (3,4).*

Let us now look for the formal solution g_k^*, Y_k $(k = 1, 2, \dots, n)$ of the system (3,5) such that the series Y_k $(k = 1, 2, \dots, n)$ are as simple as possible. The procedure of seeking such formal power series will be described only in such details as it is necessary for our purposes. (Since it does not differ from that given by A. D. Brjuno in [5].)

Let m be an arbitrary nonnegative integer less than $n + 1$. Let us suppose henceforth g_j^* $(j = m + 1, m + 2, \dots, n)$ to be arbitrary but fixed holomorphic on some open neighbourhood of the origin $\boldsymbol{O}$ in $\boldsymbol{K}_n$ functions of the variables $y_1, y_2, \dots, y_n$. From the system (3,5) the following system of algebraical equations for the coefficients $\{g_k\}_{\boldsymbol{p}}$ $(k = 1, 2, \dots, m;\ \boldsymbol{p} \in \mathcal{M}_2)$, $\{Y_k\}_{\boldsymbol{p}}$ $(k = 1, 2, \dots, n;\ \boldsymbol{p} \in \mathcal{M}_2)$ of the series g_k $(k = 1, 2, \dots, m)$ and Y_k $(k = 1, 2, \dots, n)$ can be easily derived:

$$(3,6) \qquad \{g_k\}_{\mathbf{p}}\,[(\mathbf{p},\boldsymbol{\lambda}) - \lambda_k] + \{Y_k\}_{\mathbf{p}} = \varepsilon_k\{g_{k-1}\}_{\mathbf{p}} + \{\hat{X}_k(\mathbf{g})\}_{\mathbf{p}} -$$
$$- \sum_{j=2}^{n} (p_j + 1)\,\varepsilon_j\{g_k\}_{\tilde{\mathbf{p}}(j)} - \sum_{j=1}^{n} \sum_{\substack{\boldsymbol{\omega}+\boldsymbol{\mu}=\mathbf{p}\\ \boldsymbol{\omega},\boldsymbol{\mu}\in\mathcal{M}_2}} (\omega_j + 1)\,\{g_k\}_{\hat{\boldsymbol{\omega}}(j)}\,\{Y_j\}_{\boldsymbol{\mu}} \quad (k = 1, 2, \dots, n;\ \mathbf{p} \in \mathcal{M}_2)$$

where $\tilde{\mathbf{p}}(j) = \mathbf{p} - \mathbf{e}_{j-1} + \mathbf{e}_j$ and $\hat{\omega}(j) = \boldsymbol{\omega} + \mathbf{e}_j(\{g_k\}_{\tilde{\mathbf{p}}(j)} = 0$ for $p_{j-1} = 0$, of course).

Let us order the set $\mathrm{E}[\{g_k\}_{\mathbf{p}} : k = 1, 2, \dots, n$ and $\mathbf{p} \in \mathcal{M}_2]$ by the following rule: $\{g_i\}_{\boldsymbol{\omega}} \prec \{g_j\}_{\boldsymbol{\mu}}$ if and only if the first nonzero number in the set: $\{\sum_{k=1}^{n} \mu_k - \sum_{k=1}^{n} \omega_k,$ $\mu_1 - \omega_1, \mu_2 - \omega_2, \dots, \mu_n - \omega_n, j - i\}$ is positive. Let us further order the set $\mathrm{E}[\{Y_k\}_{\mathbf{p}} : k = 1, 2, \dots, n$ and $\mathbf{p} \in \mathcal{M}_2]$ in the same way. It is easy to prove that then there holds:

Given any $k = 1, 2, \dots, n$ and any $\mathbf{p} \in \mathcal{M}_2$, the right-hand side of the corresponding equation in (3,6) includes only coefficients $\{g_j\}_{\mathbf{q}}$, $\{Y_j\}_{\mathbf{q}}$ which precede (in the ordering given above) $\{g_k\}_{\mathbf{p}}$ or $\{Y_k\}_{\mathbf{p}}$, respectively. Therefore it is possible to determine all coefficients of the series g_k $(k = 1, 2, \dots, m)$ and Y_k $(k = 1, 2, \dots, n)$ successively in the following way:

For $1 \leqq k \leqq m$ and $\mathbf{p} \notin \mathcal{N}_k(\boldsymbol{\lambda})$ (i.e. $(\mathbf{p}, \boldsymbol{\lambda}) \neq \lambda_k$) $\{g_k\}_{\mathbf{p}}$ can be computed from the corresponding equation in (3,6) (putting $\{Y_k\}_{\mathbf{p}} = 0$). For $1 \leqq k \leqq m$ and $\mathbf{p} \in \mathcal{N}_k(\boldsymbol{\lambda})$ we choose $\{g_k\}_{\mathbf{p}}$ arbitrarily and then we compute $\{Y_k\}_{\mathbf{p}}$ from the corresponding equation in (3,6). For any $k = m + 1, m + 2, \dots, n$ and $\mathbf{p} \in \mathcal{M}_2$ the value of $\{g_k\}_{\mathbf{p}}$ is supposed to be known and thus we can again compute $\{Y_k\}_{\mathbf{p}}$ from the corresponding equation in (3,6). In this manner the proof of the following assertion has been just completed.

Lemma 3,2B. *Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the roots of the characteristic equation of the system (1,1). Let m be an arbitrary nonnegative integer less than $n + 1$, let $\mathbf{B}$ be an arbitrary Jordan canonical form of the matrix $\mathbf{A}$ of the linear terms of the system (1,1) and let $\boldsymbol{\varepsilon} = (\varepsilon_2, \varepsilon_3, \dots, \varepsilon_n)$ be the first subdiagonal of the matrix $\mathbf{B}$. Then there always exists a formal transformation which transforms the system (1,1) onto generally formal system*

$$(3,7) \qquad \dot{y}_k = \lambda_k y_k + \varepsilon_k y_{k-1} + Y_k(y_1, y_2, \dots, y_n) \quad (\varepsilon_1 = 0;\ k = 1, 2, \dots, n)$$

where

$$(3,7') \qquad Y_k(\mathbf{y}) = \sum_{\mathbf{p}\in\mathcal{L}_k(\boldsymbol{\lambda})} \{Y_k\}_{\mathbf{p}}\,\mathbf{y}^{\mathbf{p}} \quad (k = 1, 2, \dots, n)$$

($\mathcal{L}_k(\boldsymbol{\lambda}) = \mathcal{N}_k(\boldsymbol{\lambda})$ for $k = 1, 2, \dots, m$ and $\mathcal{L}_k(\boldsymbol{\lambda}) = \mathcal{M}_2$ for $k = m + 1, m + 2, \dots, n$) are generally formal power series of the variables $y_1, y_2, \dots, y_n$.

Remark 3,2B. a) The coefficients $\{Y_k\}_{\mathbf{p}}$ $(k = 1, 2, \dots, n;\ \mathbf{p} \in \mathcal{L}_k(\boldsymbol{\lambda}))$ in (3,7′) are complex constants depending on the choice of the coefficients $\{g_k\}_{\mathbf{p}}$ of the series g_k^*

$(k = 1, 2, \ldots, n)$ for $\boldsymbol{p} \in \mathscr{L}_k(\boldsymbol{\lambda})$ and further on the choice of the Jordan canonical form $\boldsymbol{B}$ of the matrix $\boldsymbol{A}$ and also on the choice of the number m.

b) For $m = n$, the assertion of Lemma 3,2B is identical with that of A. D. Brjuno in [5].

Definition 3,2C. *Every system of the type* (3,7) *obtained from the given system* (1,1) *in the way described in the proof of Lemma* 3,2B *will be called a pseudonormal form of the system* (1,1) *of the type* $(m; \mathbf{B})$ (*or* $(m, \boldsymbol{\varepsilon})$).

For $m = n$ *we shall call such a system a normal form of the system* (1,1) (*of the type* $\mathbf{B}$ *or* $\boldsymbol{\varepsilon}$).

Remark 3,2C. As a result of the variety of the choice of the coefficients $\{g_k\}_{\boldsymbol{p}}$ with $\boldsymbol{p} \in \mathscr{L}_k(\boldsymbol{\lambda})$, every system of the type (1,1) has infinite number of pseudonormal forms of the type $(m; \mathbf{B})$. Those differentiate themselves in the values of the coefficients $\{Y_k\}_{\boldsymbol{p}}$ $(\boldsymbol{p} \in \mathscr{L}_k(\boldsymbol{\lambda}))$. Hence it is useful to complete Definition 3,2C:

The pseudonormal form of the system (1,1) *which arises so that* $\{g_k\}_{\boldsymbol{p}}$ *is chosen equal to* 0 *for* $\boldsymbol{p} \in \mathscr{L}_k(\boldsymbol{\lambda})$ *will be called a fundamental pseudonormal form of the system* (1,1) *of the type* $(m; \mathbf{B})$ (*or a fundamental normal form of the system* (1,1) *of the type* $\mathbf{B}$ – *if* $m = n$).

Remark 3,2D. Let us notice one interesting fact more: On the right-hand side of the pseudonormal form (3,7) of the system (1,1) of the type $(m; \boldsymbol{B})$ $(0 \leqq m \leqq n)$ let us introduce the notation $Y_k(\mathbf{y}) \equiv y_k\, \eta_k(\mathbf{y})$ $(k = 1, 2, \ldots, m)$. Thus

$$\eta_k(\mathbf{y}) = \sum_{\boldsymbol{p} \in \mathscr{P}_k(\boldsymbol{\lambda})} \{\eta_k\}_{\boldsymbol{p}}\, \mathbf{y}^{\boldsymbol{p}} \quad (k = 1, 2, \ldots, m)$$

where for an arbitrary $k = 1, 2, \ldots, m$ $\mathscr{P}_k(\boldsymbol{\lambda})$ is the set of all n-vectors $\boldsymbol{p} = (p_1, p_2, \ldots \ldots, p_n)$ of integers such that there holds: $p_k \geqq -1$, $p_j \geqq 0$ for $j \neq k$, $p_1 + p_2 + \ldots \ldots + p_n \geqq 1$ and $p_1\lambda_1 + p_2\lambda_2 + \ldots, p_n\lambda_n = 0$. Let us denote further $\tilde{\mathscr{P}}_k(\boldsymbol{\lambda}) = \mathrm{E}[\boldsymbol{p} \in \mathscr{P}_k(\boldsymbol{\lambda}) : p_k \geqq 0]$ $(k = 1, 2, \ldots, m)$ and $\tilde{\mathscr{N}}_k(\boldsymbol{\lambda}) = \mathrm{E}[\boldsymbol{p} \in \mathscr{N}_k(\boldsymbol{\lambda}) : p_k \geqq 1]$ $(k = 1, 2, \ldots, m)$. Then there holds

$$\tilde{\mathscr{P}}_1(\boldsymbol{\lambda}) = \tilde{\mathscr{P}}_2(\boldsymbol{\lambda}) = \ldots = \tilde{\mathscr{P}}_m(\boldsymbol{\lambda}) = \tilde{\mathscr{P}}(\boldsymbol{\lambda})$$

or: given an arbitrary $k = 1, 2, \ldots, m$, the n-vector $\boldsymbol{p}$ from $\mathscr{M}_2$ belongs to $\tilde{\mathscr{N}}_k(\boldsymbol{\lambda})$ if and only if for any $j = 1, 2, \ldots, m$ the n-vector $\boldsymbol{p} + \mathbf{e}_j - \mathbf{e}_k$ belongs to $\tilde{\mathscr{N}}_j(\boldsymbol{\lambda})$.

3,3. Several lemmas. The following lemmas will be useful in the proof of the main theorem (Theorem 3,4) of this chapter.

Lemma 3,3A. *Let the n-tuple* $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ *of nonzero complex numbers fulfil the following condition*:

(Q_0) *There exists such a constant $\delta > 0$ that, given an arbitrary $k = 1, 2, \ldots, n$ and an arbitrary $\boldsymbol{p} \in \mathcal{M}$ with $(\boldsymbol{p}, \boldsymbol{\lambda}) \neq \lambda_k$, there holds*

$$|(\boldsymbol{p}, \boldsymbol{\lambda}) - \lambda_k| \geqq \delta > 0 . \tag{3,8}$$

Then there exists such a constant $K_1 < \infty$ that, given an arbitrary $k = 1, 2, \ldots, n$ and an arbitrary $\boldsymbol{p} \in \mathcal{M}$ with $(\boldsymbol{p}, \boldsymbol{\lambda}) \neq \lambda_k$, there holds:

$$\left|\frac{(\boldsymbol{p}, \boldsymbol{\lambda})}{(\boldsymbol{p}, \boldsymbol{\lambda}) - \lambda_k}\right| \leqq K_1 < \infty . \tag{3,9}$$

Proof. Let k be an arbitrary natural number less than $n + 1$. Then:

a) Given $\boldsymbol{p} \in \mathcal{M}$ with $|(\boldsymbol{p}, \boldsymbol{\lambda})| \geqq 2|\lambda_k|$, there holds

$$\left|\frac{(\boldsymbol{p}, \boldsymbol{\lambda})}{(\boldsymbol{p}, \boldsymbol{\lambda}) - \lambda_k}\right| \leqq \sup_{x \geqq 2|\lambda_k|} \frac{x}{x - |\lambda_k|} = \varrho_k < \infty .$$

b) Given $\boldsymbol{p} \in \mathcal{M}$ with $(\boldsymbol{p}, \boldsymbol{\lambda}) \neq \lambda_k$ and $|(\boldsymbol{p}, \boldsymbol{\lambda})| < 2|\lambda_k|$, there holds (owing to (Q_0))

$$\left|\frac{(\boldsymbol{p}, \boldsymbol{\lambda})}{(\boldsymbol{p}, \boldsymbol{\lambda}) - \lambda_k}\right| \leqq \frac{2|\lambda_k|}{\delta} = \varrho_{k+n} < \infty .$$

Hence $K_1 = \max\limits_{k=1,2,\ldots,2n} \varrho_k < \infty$ is the required constant.

Lemma 3,3B. *Let the n-tuple $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of nonzero complex numbers fulfil the following condition:*

(Q_1) *If we regard $\lambda_1, \lambda_2, \ldots, \lambda_n$ as points of the complex plane $\boldsymbol{K}$, then there exists in the complex plane $\boldsymbol{K}$ such a straight line L passing through the origin 0 in $\boldsymbol{K}$ that all the points $\lambda_1, \lambda_2, \ldots, \lambda_n$ belong to the same closed half-plane bounded in $\boldsymbol{K}$ by L. Let l $(0 \leqq l \leqq n)$ points from the set $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ belong to the straight line L, the other belonging to the same open half-plane bounded in $\boldsymbol{K}$ by L. Let us suppose the numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ to be enumerated so that $\lambda_j \in L$ for $j = 1, 2, \ldots, l$.*

Then there exists such a constant $K_2 < \infty$ that given arbitrary $k = 1, 2, \ldots, l$ and $j = l + 1, l + 2, \ldots, n$ and an arbitrary $\boldsymbol{p} \in \mathcal{M}$ with $(\boldsymbol{p}, \boldsymbol{\lambda}) \neq \lambda_k$ there holds

$$\frac{p_j}{|(\boldsymbol{p}, \boldsymbol{\lambda}) - \lambda_k|} \leqq K_2 < \infty . \tag{3,10}$$

Proof. It can be proved easily that the condition (Q_1) is equivalent to the following condition:

(q_1) There exists such a real number ϑ that for the numbers:

$$\mu_k = e^{i\vartheta}\lambda_k \quad (k = 1, 2, \ldots, n) \tag{3,11}$$

there holds:

$$(3{,}12)\qquad \operatorname{Re}\mu_k = 0 \text{ for } k = 1, 2, \ldots, l\,; \quad \operatorname{Re}\mu_k < 0 \text{ for } k = l+1, l+2, \ldots, n\,.$$

As a simple consequence of the assumption (Q_1) (or (q_1)) we get for any $k = 1, 2, \ldots, l$; $j = l+1, l+2, \ldots, n$ and $\boldsymbol{p} \in \mathcal{M}$ with $(\boldsymbol{p}, \boldsymbol{\lambda}) \neq \lambda_k$

$$\frac{p_j}{|(\boldsymbol{p}, \boldsymbol{\lambda}) - \lambda_k|} \leqq \frac{p_j}{|(\boldsymbol{p}, \operatorname{Re}\boldsymbol{\mu}) - \operatorname{Re}\mu_k|} = \frac{p_j}{(\boldsymbol{p}, \operatorname{Re}\boldsymbol{\mu})} \leqq \frac{1}{\operatorname{Re}\mu_j} < \infty$$

(where $\operatorname{Re}\boldsymbol{\mu} = (\operatorname{Re}\mu_1, \operatorname{Re}\mu_2, \ldots, \operatorname{Re}\mu_n)$). Therefore

$$K_2 = \max_{l+1 \leqq j \leqq n} \frac{1}{\operatorname{Re}\mu_j}$$

is the required constant.

Lemma 3,3C. *Let the n-tuple* $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ *of nonzero complex numbers fulfil the condition* (Q_1) *from Lemma* 3,3B *and, moreover, the following condition:*

(Q_2) *There exists such a constant* $\delta > 0$ *that, given an arbitrary* $k = 1, 2, \ldots, l$ *and an arbitrary l-tuple* $(p_1, p_2, \ldots, p_l)$ *of nonnegative integers with* $p_1\lambda_1 + p_2\lambda_2 + \ldots + p_l\lambda_l \neq \lambda_k$, *there holds* $|p_1\lambda_1 + p_2\lambda_2 + \ldots + p_l\lambda_l - \lambda_k| \geqq \delta > 0$.

Then there exists such a constant $K_3 < \infty$ *that, given an arbitrary* $k = 1, 2, \ldots, l$ *and an arbitrary* $\boldsymbol{p} \in \mathcal{M}$ *with* $(\boldsymbol{p}, \boldsymbol{\lambda}) \neq \lambda_k$, *there holds*

$$(3{,}13)\qquad \left|\frac{p_1\lambda_1 + p_2\lambda_2 + \ldots + p_l\lambda_l}{(\boldsymbol{p}, \boldsymbol{\lambda}) - \lambda_k}\right| \leqq K_3 < \infty\,.$$

Proof. For an arbitrary $k = 1, 2, \ldots, l$ there follows from (Q_1), (Q_2), Lemma 3,3A and Lemma 3,3B:

$$\left|\frac{p_1\lambda_1 + p_2\lambda_2 + \ldots + p_l\lambda_l}{(\boldsymbol{p}, \boldsymbol{\lambda}) - \lambda_k}\right| \leqq \left|\frac{(\boldsymbol{p}, \boldsymbol{\lambda})}{|(\boldsymbol{p}, \boldsymbol{\lambda}) - \lambda_k|}\right| + \sum_{j=l+1}^{n} |\lambda_j| \frac{p_j}{|(\boldsymbol{p}, \boldsymbol{\lambda}) - \lambda_k|} \leqq K_1 + K_2\Big(\sum_{j=l+1}^{n} |\lambda_j|\Big) = K_3 < \infty\,.$$

3,4. Regular transformations of the system of differential equations to its pseudonormal form. Sufficient conditions for the regularity of a transformation of the system(1,1) to its normal form were given firstly by H. Poincaré (see [11] or [12], pp. 131–137).

Let the matrix $\boldsymbol{A}$ *of linear terms of the system* (1,1) *and the roots* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *of the characteristic equation of this system satisfy the following conditions:*

(i) *Jordan canonical form of the matrix* $\mathbf{A}$ *is a diagonal matrix.*

(ii) *Given an arbitrary* $\mathbf{q} \in \mathcal{M}_2$ *and an arbitrary* $j = 1, 2, \ldots, n$, *there always holds* $(\mathbf{q}, \boldsymbol{\lambda}) \neq \lambda_j$.

(iii) *If we regard* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *as points of the complex plane* $\boldsymbol{K}$, *then there exists in this plane such a straight line* L *passing through the origin* 0 *in* $\boldsymbol{K}$ *that all points* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *belong to the same open half-plane bounded in* $\boldsymbol{K}$ *by* L.

Then there exists a regular transformation of the system (1,1) *to its normal form* $\dot{y}_k = \lambda_k y_k$ $(k = 1, 2, \ldots, n)$.

The Poincaré conditions were generalized by H. Dulac (see [4]) who did not suppose (i) and (ii), K. L. Siegel (see [8], [9]) whose conditions are of somewhat different type (see Remark 3,4F) and above all by A. D. Brjuno (see [6], [7]) whose results include all formerly known results. He requires that the coefficients on the right-hand side of the pseudonormal form of the system (1,1) fulfil some relations. For example, in the case that for the roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the characteristic equation of the system (1,1) there holds $\operatorname{Re} \lambda_j = 0$ $(j = 1, 2, \ldots, l)$, $\operatorname{Re} \lambda_j < 0$ $(j = l + 1, l + 2, \ldots, n)$ $(2 \leqq l \leqq n)$, for at least one couple λ_j, λ_k $(1 \leqq j, k \leqq l)$ being $\operatorname{sign} \operatorname{Im} \lambda_j \neq \operatorname{sign} \operatorname{Im} \lambda_k$, some requirements on the coefficients on the right-hand sides of the equations of the pseudonormal form of the system (1,1) corresponding to λ_j with $\operatorname{Re} \lambda_j < 0$ are made, too. This is disadvantageous for the study of the stability of the trivial solution of the system (1,1). It is namely possible to expect that such coefficients will have no influence on the stability of the trivial solution of the system (1,1). Moreover, A. D. Brjuno gave his assertions without proofs. Theorem 3,4 given below is an analogy of the theorem of A. D. Brjuno from [7], but it is neither equivalent to this theorem, nor a consequence of it.

Theorem 3,4. *Let the matrix* $\mathbf{A}$ *of the linear terms of the system* (1,1) *be regular. Let the roots* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *(ordered by the rule from sec.* 2,1*) of the characteristic equation of the system* (1,1) *fulfil the conditions* $(\boldsymbol{Q}_1)$ *(see Lemma* 3,3B*) and* $(\boldsymbol{Q}_2)$ *(see Lemma* 3,3C*). Let* $\mathbf{B}$ *be an arbitrary Jordan canonical form of the matrix* $\mathbf{A}$ *and let* $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ *be the first subdiagonal of the matrix* $\mathbf{B}$. *Let*

$$(3,14) \qquad \dot{y}_k = \lambda_k y_k + \varepsilon_k y_{k-1} + Y_k(y_1, y_2, \ldots, y_n) \quad (\varepsilon_1 = 0;\ k = 1, 2, \ldots, n)$$

be the fundamental pseudonormal form of the system (1,1) *of the type* $(l; \mathbf{B})$ *(generally formal) where*

$$(3,15) \qquad Y_k(\mathbf{y}) = y_k \sum_{\mathbf{p} \in \mathscr{P}_k(\boldsymbol{\lambda})} \{\eta_k\}_{\mathbf{p}} \, \mathbf{y}^{\mathbf{p}} = y_k \, \eta_k(\mathbf{y}) \quad (k = 1, 2, \ldots, l)$$

(for the definition of $\mathscr{P}_k(\boldsymbol{\lambda})$, *see in Remark* 3,2D*)*

and

$$(3,15') \qquad Y_k(\mathbf{y}) = \sum_{\mathbf{p} \in \mathcal{M}_2} \{Y_k\}_{\mathbf{p}} \, \mathbf{y}^{\mathbf{p}} \quad (k = l + 1, l + 2, \ldots, n)\,.$$

Let the following two conditions be satisfied, moreover:

$$(Q_3) \qquad \varepsilon_2 = \varepsilon_3 = \ldots = \varepsilon_l = 0$$

and

(Q_4) *There exists such a finite real number v that, given an arbitrary $\mathbf{p} \in \mathscr{P}(\boldsymbol{\lambda}) = \bigcup_{k=1}^{l} \mathscr{P}_k(\boldsymbol{\lambda})$ with $p_1 + p_2 + \ldots + p_n \geqq v$, there exists such a complex number $\alpha_{\mathbf{p}}$ that*

$$(3{,}16) \qquad \{\eta_k\}_{\mathbf{p}} = \alpha_{\mathbf{p}} \lambda_k \quad (k = 1, 2, \ldots, l)$$

holds.

Then the transformation of the system (1,1) to this fundamental pseudonormal form of the type $(l; \boldsymbol{\varepsilon})$ is regular.

Notation 3,4. a) Given an arbitrary natural number v and an arbitrary $k = 1, 2, \ldots$ $\ldots, l$, the set of all n-tuples $\mathbf{p} \in \mathscr{P}_k(\boldsymbol{\lambda})$ with $p_1 + p_2 + \ldots + p_n \geqq v$ will be denoted by $\mathscr{Q}_{k,v}(\boldsymbol{\lambda})$. The sets $\mathscr{P}_k(\boldsymbol{\lambda}) \setminus \mathscr{Q}_{k,v}(\boldsymbol{\lambda})$ and $\mathrm{E}[\mathbf{p} \in \mathscr{Q}_{k,v}(\boldsymbol{\lambda}) : p_k \geqq 0]$ will be then denoted by $\mathscr{Q}'_{k,v}(\boldsymbol{\lambda})$ and $\mathscr{S}_{k,v}(\boldsymbol{\lambda})$, respectively. Owing to Remark 3,2D $\mathscr{S}_{1,v}(\boldsymbol{\lambda}) = \mathscr{S}_{2,v}(\boldsymbol{\lambda}) = \ldots$ $\ldots = \mathscr{S}_{l,v}(\boldsymbol{\lambda}) = \mathscr{S}_v(\boldsymbol{\lambda})$. Let us denote $\mathscr{S}'_v(\boldsymbol{\lambda}) = \tilde{\mathscr{P}}(\boldsymbol{\lambda}) \setminus \mathscr{S}_v(\boldsymbol{\lambda})$, moreover.

b) Given an arbitrary natural number v and an arbitrary $k = 1, 2, \ldots, l$, the set of all n-tuples $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ of integers such that there holds $p_j \geqq 0$ for $j \neq k$, $p_k \geqq -1$ and $1 \leqq p_1 + p_2 + \ldots + p_n \leqq v$ will be denoted by $\mathscr{M}'_{k,v}$. Let us denote the set $\bigcup_{k=1}^{l} \mathscr{M}'_{k,v}$ by $\mathscr{M}'_v$, finally.

Remark 3.4A. a) It can be easily proved that if the condition (Q_1) is satisfied, the n-tuple $\mathbf{p}$ from $\mathscr{M}$ belongs for some $k = 1, 2, \ldots, l$ to $\mathscr{P}_k(\boldsymbol{\lambda})$ if and only if there holds: $p_j \geqq 0$ for $j \neq k$, $p_j = 0$ for $j \geqq l + 1$, $p_k \geqq -1$, $p_1 + p_2 + \ldots, p_n \geqq 1$ and $(\mathbf{p}, \boldsymbol{\lambda}) = 0$. The right-hand sides of the first l equations of the system (3,15) are hence independent on $y_{l+1}, y_{l+2}, \ldots, y_n$. $(Y_k(y_1, y_2, \ldots, y_n) \equiv Y_k(y_1, y_2, \ldots, y_l) \equiv$ $\equiv y_k \eta_k(y_1, y_2, \ldots, y_l)$, $k = 1, 2, \ldots, l)$.

b) If the system (3,15) fulfils the condition (Q_4), then

$$(3{,}17) \qquad \{\eta_k\}_{\mathbf{p}} = 0 \quad \text{for} \quad \mathbf{p} \in \mathscr{Q}_{k,v}(\boldsymbol{\lambda}) \setminus \mathscr{S}_v(\boldsymbol{\lambda}) \quad (k = 1, 2, \ldots, l)$$

holds (Since $[\mathscr{Q}_{k,v}(\boldsymbol{\lambda}) \setminus \mathscr{S}_{k,v}(\boldsymbol{\lambda})] \cap \mathscr{Q}_{j,v}(\boldsymbol{\lambda}) = \emptyset$ for every $j = 1, 2, \ldots, l$ such that $j \neq k$).

Remark 3,4B. In the proof of Theorem 3,4 the rules for the operations on formal power series given in sec. 3,1 are utilized, no reference to them being made.

Proof (of Theorem 3,4). Let the assumptions of the theorem be fulfilled.

Given an arbitrary $k = 1, 2, \ldots, l$ and an arbitrary $\boldsymbol{p} \in \mathscr{Q}'_{k,\nu}(\boldsymbol{\lambda})$, let $\alpha_{\boldsymbol{p},k}$ be such a complex number that $\{\eta_k\}_{\boldsymbol{p}} = \alpha_{\boldsymbol{p},k}\lambda_k$ holds. Given an arbitrary $k = 1, 2, \ldots, l$ and an arbitrary $\boldsymbol{p} \in \mathscr{M}'_\nu \setminus \mathscr{Q}'_{k,\nu}(\boldsymbol{\lambda})$, let $\alpha_{\boldsymbol{p},k}$ be equal to 0. Let us denote then

$$C = \max_{k=1,2,\ldots,l;\, \boldsymbol{p}\in\mathscr{M}'_\nu} |\alpha_{\boldsymbol{p},k}| . \tag{3,18}$$

(Evidently $C < \infty$.)

Owing to Lemma 3,3B and Lemma 3,3C there exists such a finite positive number K that, given an arbitrary $k = 1, 2, \ldots, l$, an arbitrary $j = l + 1, l + 2, \ldots, n$ and arbitrary $\boldsymbol{p} \in \mathscr{M}$ with $(\boldsymbol{p}, \boldsymbol{\lambda}) \neq \lambda_k$, there holds:

$$\left|\frac{p_1\lambda_1 + p_2\lambda_2 + \ldots + p_l\lambda_l}{(\boldsymbol{p}, \boldsymbol{\lambda}) - \lambda_k}\right| \leqq K < \infty \tag{3,19}$$

and

$$\left|\frac{p_j}{(\boldsymbol{p}, \boldsymbol{\lambda}) - \lambda_k}\right| \leqq K < \infty . \tag{3,19'}$$

The system (1,1) can again be written in the form

$$\dot{x}_k = \lambda_k x_k + \varepsilon_k x_{k-1} + \tilde{X}_k(x_1, x_2, \ldots, x_n) \quad (\varepsilon_1 = 0;\ k = 1, 2, \ldots, n) \tag{3,4}$$

where $\varepsilon_2 = \varepsilon_3 = \ldots = \varepsilon_l = 0$ owing to the assumption $(\boldsymbol{Q}_3)$. By the substitution of (3,14) and (3,1') (where $g_j^* \equiv 0$ for $j = l + 1, l + 2, \ldots, n$, of course) into (3,4) we obtain the system (3,6) of the equations for the coefficients of the series g_k, Y_k ($k = 1, 2, \ldots, n$). (See the proof of Lemma 3,2B.) Owing to the assumptions $(\boldsymbol{Q}_3)$ and $(\boldsymbol{Q}_4)$ and owing to Remark 3,4A this system has in our case the following form:

$$\begin{aligned}
(3,20)\quad & \{g_k\}_{\boldsymbol{p}}\,[(\boldsymbol{p}, \boldsymbol{\lambda}) - \lambda_k] + \{Y_k\}_{\boldsymbol{p}} = \{\tilde{X}_k(\mathbf{g})\}_{\boldsymbol{p}} - \sum_{j=l+2}^{n} \varepsilon_j(p_j + 1)\,\{g_k\}_{\tilde{\boldsymbol{p}}(j)} - \\
& - \sum_{\substack{\boldsymbol{\omega}+\boldsymbol{\sigma}=\boldsymbol{p}\\ \boldsymbol{\omega}\in\mathscr{M}_2,\boldsymbol{\sigma}\in\mathscr{M}'_\nu}} \Big(\sum_{j=1}^{l}(\omega_j + 1)\,\lambda_j\alpha_{\boldsymbol{\sigma},j}\Big)\{g_k\}_{\boldsymbol{\omega}} - \sum_{\substack{\boldsymbol{\omega}+\boldsymbol{\sigma}=\boldsymbol{p}\\ \boldsymbol{\omega}\in\mathscr{M}_2,\boldsymbol{\sigma}\in\mathscr{S}_\nu(\boldsymbol{\lambda})}} \Big(\sum_{j=l+1}^{n}(\omega_j + 1)\,\lambda_j\Big)\alpha_{\boldsymbol{\sigma}}\{g_k\}_{\boldsymbol{\omega}} - \\
& - \sum_{\substack{\boldsymbol{\omega}+\boldsymbol{\sigma}=\boldsymbol{p}\\ \boldsymbol{\omega},\boldsymbol{\sigma}\in\mathscr{M}_2}} \Big(\sum_{j=l+1}^{n}(\omega_j + 1)\,\{g_k\}_{\hat{\boldsymbol{\omega}}(j)}\,\{Y_j\}_{\boldsymbol{\sigma}}\Big) \quad \text{for} \quad \boldsymbol{p} \in \mathscr{M}_2\,; \quad k = 1, 2, \ldots, l \\
& \{Y_k\}_{\boldsymbol{p}} = \{\tilde{X}_k(\mathbf{g})\}_{\boldsymbol{p}} \quad \text{for} \quad \boldsymbol{p} \in \mathscr{M}_2\,; \quad k = l + 1, l + 2, \ldots, n
\end{aligned}$$

where $\tilde{\boldsymbol{p}}(j) = \boldsymbol{p} - \mathbf{e}_{j-1} + \mathbf{e}_j$, $\hat{\boldsymbol{\omega}}(j) = \boldsymbol{\omega} + \mathbf{e}_j$ ($j = 1, 2, \ldots, n$); $\{Y_k\}_{\boldsymbol{p}} = 0$ for $\boldsymbol{p} \in \mathscr{M}_2 \setminus \mathscr{N}_k(\boldsymbol{\lambda})$ $\{Y_k\}_{\boldsymbol{p}} = \{\eta_k\}_{\boldsymbol{p}-\mathbf{e}_k}$ for $\boldsymbol{p} \in \mathscr{N}_k(\boldsymbol{\lambda})$ and $\{g_k\}_{\tilde{\boldsymbol{p}}(j)} = 0$ if $p_{j-1} = 0$, finally.

Given an arbitrary $k = 1, 2, \ldots, l$ and an arbitrary $\boldsymbol{p} \in \mathcal{M}_2 \setminus \mathcal{N}_k(\lambda)$, there holds (owing to (3,18), (3,19), (3,19') and (3,20))

$$(3,21) \qquad |\{g_k\}_{\boldsymbol{p}}| \leqq K'[|\{\tilde{X}_k(\mathbf{g})\}_{\boldsymbol{p}}| + \sum_{j=l+2}^{n} |\varepsilon_j| \, |\{g_k\}_{\tilde{\boldsymbol{p}}(j)}| +$$

$$+ \sum_{\substack{\boldsymbol{\omega}+\boldsymbol{\sigma}=\boldsymbol{p} \\ \boldsymbol{\omega}\in\mathcal{M}_2,\boldsymbol{\sigma}\in\mathcal{M}'_\nu}} C|\{g_k\}_{\boldsymbol{\omega}}| + \sum_{\substack{\boldsymbol{\omega}+\boldsymbol{\sigma}=\boldsymbol{p} \\ \boldsymbol{\omega}\in\mathcal{M}_2,\boldsymbol{\sigma}\in\mathcal{S}_\nu(\lambda)}} |\alpha_{\boldsymbol{\sigma}}| \, |\{g_k\}_{\boldsymbol{\omega}}| + \sum_{j=l+1}^{n} \sum_{\substack{\boldsymbol{\omega}+\boldsymbol{\sigma}=\boldsymbol{p} \\ \boldsymbol{\omega},\boldsymbol{\sigma}\in\mathcal{M}_2}} |\{Y_j\}_{\boldsymbol{\sigma}}| \, |\{g_k\}_{\hat{\boldsymbol{\omega}}(j)}|] ,$$

where K' is some positive constant.

Given an arbitrary $k = 1, 2, \ldots, l$ and an arbitrary $\boldsymbol{p} \in \mathcal{N}_k(\lambda)$, there holds (owing to (3,20))

$$(3,22) \qquad |\{Y_k\}_{\boldsymbol{p}}| \leqq |\{\tilde{X}_k(\mathbf{g})\}_{\boldsymbol{p}}| + \sum_{j=l+1}^{n} \sum_{\substack{\boldsymbol{\omega}+\boldsymbol{\sigma}=\boldsymbol{p} \\ \boldsymbol{\omega},\boldsymbol{\sigma}\in\mathcal{M}_2}} |\{Y_j\}_{\boldsymbol{\sigma}}| \, |\{g_k\}_{\hat{\boldsymbol{\omega}}(j)}| \, .$$

(Indeed, $\boldsymbol{p}$ belongs to $\mathcal{N}_k(\lambda)$ only if $p_{l+1} = p_{l+2} = \ldots = p_n = 0$ and therefore the second expression on the right-hand side of the corresponding equation of (3,20) vanishes. Moreover, if $\boldsymbol{\omega} + \boldsymbol{\sigma} = \boldsymbol{p}$, $\boldsymbol{\sigma} \in \mathcal{P}_k(\lambda)$ and $\boldsymbol{p} \in \mathcal{N}_k(\lambda)$, then $\boldsymbol{\omega} \in \mathcal{N}_k(\lambda)$ and $\{g_k\}_{\boldsymbol{\omega}} = 0$. Therefore the third and fourth expressions on the right-hand side of the corresponding equation of (3,20) vanish, too.)

There follows from (3,21) and (3,22)

$$(3,23) \qquad G + \sum_{k=1}^{l} \|Y_k\| \ll K'[\sum_{k=1}^{l} \|\tilde{X}_k(\mathbf{g})\| + \beta G + \alpha G +$$

$$+ \sum_{k=1}^{l} \sum_{j=l+1}^{n} |\varepsilon_j| \, \|\tilde{g}_{k,j}\| + \sum_{k=1}^{l} \sum_{j=l+2}^{n} \|\hat{g}_{k,j}\| \, \|\tilde{X}_j(\mathbf{g})\|]$$

where the symbols $G, \alpha, \tilde{g}_{k,j}, \hat{g}_{k,j}$ $(k = 1, 2, \ldots, l;\ j = l + 1, l + 2, \ldots, n)$ denote the formal power series of the variables $y_1, y_2, \ldots, y_n$ given by the following prescriptions:

$$(3,24) \qquad G = \sum_{j=1}^{l} \|g_j^*\| , \quad \alpha(\mathbf{y}) = \sum_{\boldsymbol{p}\in\mathcal{S}_\nu(\lambda)} |\alpha_{\boldsymbol{p}}| \mathbf{y}^{\boldsymbol{p}} ,$$

$$\tilde{g}_{k,j}(\mathbf{y}) = \sum_{\substack{\boldsymbol{p}\in\mathcal{M}_2 \\ p_{j-1}>0}} \{g_k\}_{\tilde{\boldsymbol{p}}(j)} \mathbf{y}^{\boldsymbol{p}} , \quad \hat{g}_{k,j}(\mathbf{y}) = \sum_{\boldsymbol{p}\in\mathcal{M}_1} \{g_k\}_{\hat{\boldsymbol{p}}(j)} \mathbf{y}^{\boldsymbol{p}}$$

and β is the polynomial of the variables $y_1, y_2, \ldots, y_n$

$$(3,24') \qquad \beta(\mathbf{y}) = \sum_{\boldsymbol{p}\in\mathcal{M}'_\nu} C\mathbf{y}^{\boldsymbol{p}} .$$

(Clearly $\beta(\mathbf{y}) = \sum_{j=1}^{l} y_j^{-1} \beta_j(\mathbf{y})$ where every polynomial $\beta_j(\mathbf{y})$ is of the type $[y_1, y_2, \ldots, y_l]_2$.)

Owing to (3,15), (3,16) and (3,24)

$$\sum_{k=1}^{l} \| Y_k \| = \psi + \left(\sum_{k=1}^{l} y_k|\lambda_k|\right) \alpha$$

where ψ is a (finite) polynomial of the variables $y_1, y_2, \ldots, y_l$ with all nonnegative coefficients. Then evidently

$$(3,25) \qquad \left(\sum_{k=1}^{l} y_k|\lambda_k|\right) \alpha \ll \sum_{k=1}^{l} \| Y_k \| .$$

The functions $\tilde{X}_k$ $(k = 1, 2, \ldots, n)$ are holomorphic functions of the variables $x_1, x_2, \ldots, x_n$ on some (open) neighbourhood of the origin $\mathbf{O}$ in $\boldsymbol{K}_n$ and $\tilde{X}_k(\mathbf{x}) = [\mathbf{x}]_2$ $(k = 1, 2, \ldots, n)$, moreover. Thus there exist positive constants c_1, c_2 such that

$$(3,26) \qquad \| \tilde{X}_k(\mathbf{g}) \| = \| \tilde{X}_k(g_1 g_2, \ldots, g_l, y_{l+1}, y_{l+2}, \ldots, y_n) \| \ll$$
$$\ll \frac{c_1(|g_1| + |g_2| + \ldots + |g_l| + y_{l+1} + y_{l+2} + \ldots + y_n)^2}{1 - c_2(|g_1| + |g_2| + \ldots + |g_l| + y_{l+1} + y_{l+2} + \ldots + y_n)} .$$

(See Remark 3,1.)

All coefficients in (3,23) are nonnegative. Hence if we restrict ourselves to $y_1 = y_2 = \ldots = y_n = u$, we cannot cause any loss of generality of our investigation. Then by (3,25) and (3,26) the relation (3,23) may be written as

$$(3,27) \qquad G + c_3 u\alpha \ll c_4 \left[\frac{(nu + G)^2}{1 - c_2(nu + G)} + \right.$$
$$\left. + \alpha G + \beta G + u^{-1} G \frac{(nu + G)^2}{1 - c_2(nu + G)} \right] + c_5 G .$$

where $c_3 = \sum_{k=1}^{l} |\lambda_k| > 0$, $c_4 = K'(nc_1 + 1) > 0$ and $c_5 = K' \sum_{j=l+2}^{n} |\varepsilon_j| \geqq 0$. (Since for any $k = 1, 2, \ldots, l$ and $j = 1, 2, \ldots, n$ the relations

$$\| \hat{g}_{k,j}(u) \| = \| \hat{g}_{k,j}(u, u, \ldots, u) \| \ll u^{-1} \| g_k^*(u) \| = u^{-1} \| g_k^*(u, u, \ldots, u) \|$$

and $\| \tilde{g}_{k,j}(u) \| = \| \tilde{g}_{k,j}(u, u, \ldots, u) \| \ll \| g_k^*(u) \| = \| g_k^*(u, u, \ldots, u) \|$ hold.)

Let $\varrho_1, \varrho_2 \ldots \varrho_n$ be now positive numbers such that the evidently regular linear transformation

$$(3.28) \qquad x_k = \varrho_k \xi_k \quad (k = 1, 2, \ldots, n)$$

transforms the system (3,4) to the system

$$(3,4') \qquad \dot{\xi}_k = \lambda_k \xi_k + \mu_k \xi_{k-1} + \hat{X}_k(\xi_1, \xi_2, \ldots, \xi_n) \quad (k = 1, 2, \ldots, n)$$

where $\hat{X}_k(\xi_1, \xi_2, \ldots, \xi_n) = (1/\varrho_k)\,\tilde{X}_k(\varrho_1\xi_1, \varrho_2\xi_2, \ldots, \varrho_n\xi_n)$ $(k = 1, 2, \ldots, n)$ and $\mu_1, \mu_2, \ldots, \mu_n$ are some complex numbers with the following properties:

α) $\mu_1 = 0$.

β) Given an arbitrary $j = 2, 3, \ldots, n$, $\mu_j = 0$ if and only if $\varepsilon_j = 0$. (Hence $\mu_1 = \mu_2 = \ldots = \mu_{l+1} = 0$.)

γ) $1 - K' \sum_{j=l+2}^{n} |\mu_j| > 0$.

Let

$$\dot{\eta}_k = \lambda_k\eta_k + \mu_k\eta_{k-1} + Y_k(\eta_1, \eta_2, \ldots, \eta_n) \quad (k = 1, 2, \ldots, n) \tag{3,15''}$$

be the fundamental pseudonormal form of the system (3,4') of the type $(l; \mu)$. This formal system obviously fulfils the conditions (Q_3) and (Q_4). It is easy to prove that inserting

$$\eta_k = \varrho_k^{-1} y_k \quad (k = 1, 2, \ldots, n) \tag{3,28'}$$

into (3,15'') we get just the fundamental pseudonormal form (3,15) of the system (3,4) of the type $(l; \varepsilon)$. The transformation given by (3,28') is evidently regular. Hence if we suppose

$$0 < 1 - c_5 = 1 - K'(\sum_{j=l+2}^{n} |\varepsilon_j| \tag{3,29}$$

we cannot cause any loss of generality.

According to (3,24') there exists such a positive constant c_6 that the estimate

$$\beta(u) \ll \frac{c_6 u}{1 - c_2(nu + G)} \tag{3,30}$$

holds.

By (3,28) and (3,30) the relation (3,27) may be replaced by an equivalent relation

$$G + c_7 u\alpha \ll c_8 \left[\frac{(nu + G)^2}{1 - c_2(nu + G)} + \alpha G + u^{-1}G + \frac{(nu + G)^2}{1 - c_2(nu + G)}\right]$$

where c_7 and c_8 are some positive constants.

Whence the following estimates for the formal power series $V = Gu^{-1} + c_7\alpha$ and $W = u + V + uV$ can be successively obtained:

$$V \ll c_9 \left[\frac{u(1 + V)^3}{1 - c_{10}u(1 + V)} + V^2\right]$$

and

$$W \ll c_{11} \frac{u + W^2 + W^3}{1 - c_{11}W} \tag{3,31}$$

where c_9, c_{10} and c_{11} are some positive constants.

Let us investigate the formal equation

$$P = c_{11} \frac{u + P^2 + P^3}{1 - c_{11}P} . \tag{3,32}$$

Owing to sec. 3,1 the equation (3,32) can be replaced by the equivalent equation

$$c_{11}P^3 + 2c_{11}P^2 - P + c_{11}u = 0 .$$

There is obviously just one formal power series P of the variable u which satisfies this equation (formally). By Implicit Function Theorem (see [14], p. 39) P absolutely converges on some open neighbourhood of the origin 0 in $\boldsymbol{K}$. Since obviously $g_k^*(u, u, \ldots, u) \ll G(u) \ll W(u) \ll P(u)$, all the series $g_k^*(u, u, \ldots, u)$ $(k = 1, 2, \ldots, l)$ absolutely converge on the same neighbourhood of the origin in $\boldsymbol{K}$, too. Whence the absolute convergence of the series $g_k^*(y_1, y_2, \ldots, y_n)$ $(k = 1, 2, \ldots, n)$ on some open neighbourhood of the origin $\mathbf{O}$ in $\boldsymbol{K}_n$ follows and this completes the proof of Theorem 3,4.

Corollary. *Let the matrix $\mathbf{A}$ of the linear terms of the system (1,1) be regular and let the roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the characteristic equation of the system (1,1) satisfy the following condition.*

(Q_1') There exists in the complex plane $\boldsymbol{K}$ a straight line L passing through the origin 0 in $\boldsymbol{K}$ such that all the points $\lambda_1, \lambda_2, \ldots, \lambda_n$ belong to the same open half-plane bounded in $\boldsymbol{K}$ by L. Let the enumeration of these points be performed such that for any $j = 1, 2, \ldots, n - 1$ the distance of the point λ_{j+1} from the straight line L is not less than the distance of the point λ_j from the straight line L.

Let m be an arbitrary nonnegative integer less than $n + 1$ and let $\mathbf{B}$ be an arbitrary Jordan canonical form of the matrix $\mathbf{A}$ with the first subdiagonal $\boldsymbol{\varepsilon}$.

Then the transformation of the system (1,1) to its fundamental pseudonormal form of the type $(m; \mathbf{B})$

$$\dot{y}_k = \lambda_k y_k + \varepsilon_k y_{k-1} + Y_k(y_1, y_2, \ldots, y_n) \quad (\varepsilon_1 = 0;\ k = 1, 2, \ldots, n) \tag{3,34}$$

is regular. Moreover there holds in (3,34):

$$\begin{aligned} Y_1 \equiv 0\,, \quad Y_k(y) &= \sum_{\mathbf{p} \in \mathcal{N}_k(\boldsymbol{\lambda})} \{Y_k\}_{\mathbf{p}}\, \mathbf{y}^{\mathbf{p}} && (k = 2, 3, \ldots, m) \\ Y_k(\mathbf{y}) &= [\mathbf{y}]_2 && (k = m + 1, m + 2, \ldots, n) \end{aligned} \tag{3,34'}$$

where $\mathcal{N}_k(\boldsymbol{\lambda})$ $(k = 2, 3, \ldots, m)$ are altogether finite sets.

Proof. Under given assumptions the numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ surely satisfy the conditions $(\boldsymbol{Q}_1)$ and $(\boldsymbol{Q}_2)$. Further it is clear that for any $k = 1, 2, \ldots, m$ the set $\mathcal{N}_k(\boldsymbol{\lambda})$ is finite and hence the assumption $(\boldsymbol{Q}_4)$ is also fulfilled. Finally, under given assumptions the assertion of Lemma 3,3C can be easily extended to $j = 1, 2, \ldots, l$ as well. In accordance to the proof of Theorem 3,4 we can hence leave out the assumption $(\boldsymbol{Q}_3)$. Whence our assertion immediately follows.

Remark 3,4C. a) Theorem 3,4 has been proved in the way which was used e.g. by K. L. Siegel in the book [10]. (See pp. 89–92.)

b) The assertion of Corollary is for $m = n$ identical with that of the well known theorem, given by H. Dulac in [4].

Remark 3,4D. Let us give an example of the system (1,1) which satisfies the conditions $(\boldsymbol{Q}_1)$, $(\boldsymbol{Q}_2)$ and $(\boldsymbol{Q}_3)$ and does not satisfy the condition $(\boldsymbol{Q}'_1)$:

Let the matrix $\boldsymbol{A}$ of the linear terms of the system (1,1) be again regular and let the roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the characteristic equation of the system (1,1) be such that $\operatorname{Re} \lambda_1 = \operatorname{Re} \lambda_2 = 0$, $\operatorname{Re} \lambda_j < 0$ $(j = 3, 4, \ldots, n)$ holds, $\lambda_2 \lambda_1^{-1}$ being a negative rational number. Let us suppose $\lambda_2 \lambda_1^{-1} = rs^{-1}$ where r, s are such natural numbers that the fraction rs^{-1} has its basic form. Let $\boldsymbol{B}$ be an arbitrary Jordan canonical form of the matrix $\boldsymbol{A}$ and $\boldsymbol{\varepsilon}$ the first subdiagonal of $\boldsymbol{B}$. Then the functions Y_k $(k = 1, 2, \ldots, n)$ standing on the right-hand side in the fundamental pseudonormal form of the system (1,1) of the type $(2; \boldsymbol{\varepsilon})$ are given by the following expressions:

$$Y_k(\boldsymbol{y}) = y_k \sum_{j=1}^{\infty} \{\eta_k\}_{rj,sj}\, y_1^{rj} y_1^{sj} \quad (k = 1, 2)$$

and

$$Y_k(\boldsymbol{y}) = \sum_{\boldsymbol{p} \in \mathcal{M}_2} \{Y_k\}_{\boldsymbol{p}}\, \boldsymbol{y}^{\boldsymbol{p}} \qquad (k = 3, 4, \ldots, n).$$

Then the sufficient condition for the transformation of such a system to its fundamental pseudonormal form of the type $(2, \boldsymbol{\varepsilon})$ to be regular is that the coefficients $\{\eta_k\}_{rj,sj}$ $(k = 1, 2; j = 1, 2, \ldots)$ fulfil the following relations:

$$r\{\eta_1\}_{rj,sj} + s\{\eta_2\}_{rj,sj} = 0 \quad (j = v, v + 1, \ldots)$$

where v is some natural number. (See Theorem 3,4.)

Remark 3,4E. It is obvious that if the transformation $x_k = y_k + g_k^*(y_1, y_2, \ldots, y_n)$ $(k = 1, 2, \ldots, n)$ with $g_k^* \equiv 0$ for $k = m + 1, m + 2, \ldots, n$ is a regular transformation of the system (1,1) to its fundamental pseudonormal form of the type $(m; \boldsymbol{\varepsilon})$, then every transformation $x_k = y_k + h_k(y_1, y_2, \ldots, y_n)$ $(k = 1, 2, \ldots, n)$ with $\{h_k\}_{\boldsymbol{p}} = \{g_k\}_{\boldsymbol{p}}$ for $k = 1, 2, \ldots, m$ and $\boldsymbol{p} \in \mathcal{M}_2$ and arbitrary holomorphic on some open neighbourhood of the origin $\mathbf{O}$ in K_n functions on the place of h_k $(k = m + 1, m + 2, \ldots, n)$ is regular, too.

Remark 3,4F. Sufficient conditions for the regularity of the transformation of the system (1,1) to its normal form were also given by K. L. Siegel in [8]. Let us remember them:

Let the matrix $\mathbf{A}$ *of the linear terms of the system* (1,1) *be regular and let the following conditions be satisfied*:

α) *Jordan canonical form of the matrix* $\mathbf{A}$ *is a diagonal matrix.*

β) *There is such a positive constant* δ *that for any* $\mathbf{p} \in \mathcal{M}$ *there holds*

$$|(\mathbf{p}, \boldsymbol{\lambda}) - \lambda_k| > 2n\Big(\sum_{j=1}^{n} p_j\Big)^{-\delta} \quad (k = 1, 2, \ldots, n)$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ *are the roots of the characteristic equation of the given system* (1,1).

Then the normal form of the given system (1,1) *is a linear system*

$$\dot{y}_k = \lambda_k y_k \quad (k = 1, 2, \ldots, n)$$

and the transformation of the system (1,1) *to its normal form is regular.*

In another paper of K. L. Siegel ([9]) the following important assertion has been proved:

Let $\hat{\mathcal{M}}$ *be the set of all n-tuples* $\mathbf{p}$ *of such integers* $p_1, p_2, \ldots, p_n$ *that* $|p_1| + |p_2| + \ldots + |p_n| > 0$ *holds. Then, if for the roots* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *of the characteristic equation of the system* (1,1) *the expression*

$$D_{\boldsymbol{\lambda}}(\mathbf{p}) = \frac{lg(|(\mathbf{p}, \boldsymbol{\lambda})|^{-1})}{lg(|p_1| + |p_2| + \ldots + |p_n|)}$$

is unbounded on the set $\hat{\mathcal{M}}$, *the system* (1,1) *does not satisfy the condition* β).

Thus every system of the type (1,1) with the roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the characteristic equation such that for some $\mathbf{p} \in \hat{\mathcal{M}}$ the relation $(\mathbf{p}, \boldsymbol{\lambda}) = 0$ holds does not satisfy the conditions of K. L. Siegel. In particular – systems which satisfy the assumptions of Theorem 3,4 do not satisfy generally these conditions. (In the case $n = l = 2$ these conditions are fulfilled only if the number $\lambda_1 \lambda_2^{-1}$ is irrational.)

The assertion of K. L. Siegel has been somewhat generalized by A. D. Brjuno (in [6]). A. D. Brjuno has not given the proof of his theorem, of course.

4. THE STABILITY OF THE TRIVIAL SOLUTION OF THE SYSTEM (1,1) IN SOME CRITICAL CASES

4,1. Several lemmas.

Remark 4,1. Let us remember some further properties of the system (1,1):

Given a domain $\boldsymbol{G}$ such that $\mathbf{O} \in \boldsymbol{G}$ and $\overline{\boldsymbol{G}} \subset \boldsymbol{\Omega}$ ($\boldsymbol{\Omega}$ is the domain on which the system (1,1) is defined, see sec. 2,1), there is $\tau > 0$ with the following property ($\boldsymbol{p}$) : : $\mathbf{x}(t, \xi)$ being the solution of the system (1,1) with $\mathbf{x}(0, \xi) = \xi$, $\mathbf{x}(t, \xi)$ is defined for each $\xi \in \boldsymbol{G}$ on the interval $\langle 0, \tau)$ and remains in $\boldsymbol{\Omega}$ for all $(t, \xi) \in \langle 0, \tau) \times \boldsymbol{G}$. (See [2].) Let us denote by $\mathscr{T}(\boldsymbol{G})$ the set of all $\tau \in (0, \infty)$ with the property ($\boldsymbol{p}$) and by $\tau(\boldsymbol{G})$ the least upper bound of $\mathscr{T}(\boldsymbol{G})$. (Clearly $0 < \tau(\boldsymbol{G}) \leqq \infty$, $\tau(\boldsymbol{G}) \in \mathscr{T}(\boldsymbol{G})$.) Then for each $j = 1, 2, \ldots, n$ and for each $t \in \langle 0, \tau(\boldsymbol{G}))$ the j^{th} component $x_j(t, \xi)$ of $\mathbf{x}(t, \xi)$ is a holomorphic function of the variables $\xi_1, \xi_2, \ldots, \xi_n$ on $\boldsymbol{G}$, i.e.

$$(4,1) \qquad x_j(t, \xi) = \sum_{\boldsymbol{p} \in \mathscr{M}_1} {}^{(j)}\varphi_{\boldsymbol{p}}(t)\, \xi^{\boldsymbol{p}} \quad (j = 1, 2, \ldots, n;\ \xi \in \boldsymbol{G},\ t \in \langle 0, \tau(\boldsymbol{G}))\,,$$

the right-hand side series being absolutely convergent for any $t \in \langle 0, \tau(\boldsymbol{G}))$ and $\xi \in \boldsymbol{G}$.

Lemma 4,1A. *Let the trivial solution of the system (1,1) be stable. Then there exists such a domain $\boldsymbol{B}$ that there holds $\mathbf{O} \in \boldsymbol{B}$, $\overline{\boldsymbol{B}} \subset \boldsymbol{\Omega}$ and $\tau(\boldsymbol{B}) = \infty$, every ${}^{(k)}\varphi_{\boldsymbol{p}}(t)$ ($k = 1, 2, \ldots, n$; $\boldsymbol{p} \in \mathscr{M}_1$) being a bounded function on the interval $\langle 0, \infty)$.*

Proof. The trivial solution of (1,1) being stable, there exists by Theorem 2,3B a domain $\boldsymbol{B}$ of boundedness of the system (1,1). (Hence $\mathbf{O} \in \boldsymbol{B}$, $\overline{\boldsymbol{B}} \subset \boldsymbol{\Omega}$.) Clearly $\tau(\boldsymbol{B}) = \infty$. Let ϱ be now such a positive constant that $\mathrm{E}[\mathbf{x} \in \boldsymbol{K}_n : \|\mathbf{x}\| \leqq \varrho] \subset \boldsymbol{B}$ holds. Then the Cauchy integral formula (see e.g. [13]) implies that for each $\boldsymbol{p} \in \mathscr{M}_1$, $t \geqq 0$ and $k = 1, 2, \ldots, n$ there holds

$$ {}^{(k)}\varphi_{\boldsymbol{p}}(t) = \frac{\partial^{p_1 + p_2 + \ldots + p_n} x_k(t, \mathbf{0})}{\partial \xi_1^{p_1}\, \partial \xi_2^{p_2} \ldots \partial \xi_n^{p_n}} =$$

$$= \frac{1}{(2\pi i)^n} \int_{|z_1| = \varrho} \left(\int_{|z_2| = \varrho} \cdots \left(\int_{|z_n| = \varrho} \frac{x_k(t, \mathbf{z})}{z_1^{p_1 + 1} z_2^{p_2 + 1} \ldots z_n^{p_n + 1}}\, \mathrm{d}z_n \right) \ldots \mathrm{d}z_2 \right) \mathrm{d}z_1\,.$$

Whence in virtue of the boundedness of $\mathbf{x}(t, \xi)$ on $\langle 0, \infty) \times \boldsymbol{B}$ our assertion follows.

Lemma 4,1B. *Let the system of differential equations*

$$(4,2) \qquad \begin{aligned} \dot{x}_k &= X_k(t, x_1, x_2, \ldots, x_n) \quad (k = 1, 2, \ldots, l;\ 1 \leqq l \leqq n) \\ \dot{x}_j &= a_{j,l+1} x_{l+1} + a_{j,l+2} x_{l+2} + \ldots + a_{j,n} x_n + X_j(t, x_1, x_2, \ldots, x_n) \\ &\qquad (j = l + 1, l + 2, \ldots, n) \end{aligned}$$

has the following properties:

(i) $a_{i,j}$ $(i, j = l + 1, l + 2, \ldots, n)$ *are such complex numbers that the roots of the characteristic equation of the matrix* $\hat{\mathbf{A}} = (a_{i,j})_{i,j=l+1,l+2,\ldots,n}$ *have altogether negative real parts.*

(ii) $X_j(t, \mathbf{x})$ $(j = 1, 2, \ldots, n)$ *are on* $\mathfrak{G} = \mathrm{E}[(t, \mathbf{x}) : t \geqq 0, \ \mathbf{x} \in \boldsymbol{K}_n, \ \|\mathbf{x}\| < h]$ $(h > 0)$ *holomorphic functions of the complex variables* $x_1, x_2, \ldots, x_n$ *of the type* $[\mathbf{x}]_2$ *and bounded continuous functions of the real variable t such that* $X_j(t, \mathbf{O}) \equiv 0$ *on* $\langle 0, \infty)$ $(j = 1, 2, \ldots, n)$.

Then, if there holds $X_j(t, x_1, x_2, \ldots, x_l, 0, 0, \ldots, 0) = 0$ *for all* $t \geqq 0$, $\mathbf{x} \in \mathrm{E}[\mathbf{x} \in \boldsymbol{K}_n, \|\mathbf{x}\| < h]$ *and* $j = 1, 2, \ldots, n$, *the trivial solution of the system* (4,2) *is stable.*

The proof of this assertion for real systems is given in [11] (p. 113). In [1] it is remarked that the proof for complex systems would be similar to that in [11]. Hence the proof of Lemma 4,1B is not given in this paper. This concerns Lemma 4,1C, too.

Definition 4,1. *Let the system of differential equations*

$$\dot{z}_k = Z_k(t, z_1, z_2, \ldots, z_n) \quad (k = 1, 2, \ldots, n) \tag{4,3}$$

has the following properties:

$Z_k(t, \mathbf{z})$ $(k = 1, 2, \ldots, n)$ *are on* $\mathfrak{G} = \mathrm{E}[(t, \mathbf{z}) : t \geqq 0, \ \mathbf{z} \in \boldsymbol{K}_n, \ \|\mathbf{z}\| < h]$ $(h > 0)$ *holomorphic functions of the complex variables* $z_1, z_2, \ldots, z_n$ *of the type* $[\mathbf{z}]_2$, *the coefficients* ${}^{(k)}\zeta_{\mathbf{p}}(t)$ *of their developments into power series of the variables* $z_1, z_2, \ldots$ $\ldots, z_n$

$$Z_k(t, \mathbf{z}) = \sum_{\mathbf{p} \in \mathcal{M}_2} {}^{(k)}\zeta_{\mathbf{p}}(t)\, \mathbf{z}^{\mathbf{p}} \quad (k = 1, 2, \ldots, n; \ (t, \mathbf{z}) \in \mathfrak{G})$$

being bounded and continuous functions of the real variable t on the interval $\langle 0, \infty)$.

Given an arbitrary $j = 2, 3, \ldots$, *let us denote*

$$Z_k^{(j)}(t, \mathbf{z}) = \sum_{\substack{\mathbf{p} \in \mathcal{M}_2 \\ p_1 + p_2 + \ldots + p_n = j}} {}^{(k)}\zeta_{\mathbf{p}}(t)\, \mathbf{z}^{\mathbf{p}} \quad (k = 1, 2, \ldots, n; \ (t, \mathbf{z}) \in \mathfrak{G}) \, .$$

(*Thus* $Z_k(t, \mathbf{z}) = \sum_{j=2}^{\infty} Z_k^{(j)}(t, \mathbf{z})$ $(k = 1, 2, \ldots, n; \ (t, \mathbf{z}) \in \mathfrak{G})$.)

Then given a natural number $N > 1$, *the trivial solution of the system* (4,3) *is said to be stable independently of the terms of the degree higher than N if, given an arbitrary* $\varepsilon > 0$ *and an arbitrary* $K > 0$, *there is such* $\delta > 0$ $(\delta = \delta(\varepsilon, K))$ *that for each solution* $\mathbf{z}(t)$ *of the system*

$$\dot{z}_k = \sum_{j=2}^{N} Z_k^{(j)}(t, \mathbf{z}) + \psi_k(t, \mathbf{z}) \quad (k = 1, 2, \ldots, n) \tag{4,3'}$$

with $\|\mathbf{z}(0)\| < \delta$ *there holds* $\|\mathbf{z}(t)\| < \varepsilon$ *for every* $t \geqq 0$, $\psi_k(t, \mathbf{z})$ $(k = 1, 2, \ldots, n)$ *being arbitrary functions defined on* $\mathfrak{G}$ *such that there holds*

$$\|\psi_k(t, \mathbf{z})\| \leqq K\|\mathbf{z}\|^{N+1} \quad (k = 1, 2, \ldots, n; (t, \mathbf{z}) \in \mathfrak{G}) .$$

Given a natural number $N > 1$, *the trivial solution of the system* (4,3) *is said to be unstable independently of the terms of the degree higher than* N, *if under the same assumptions concerning the functions* ψ_k *there is such* $\varepsilon > 0$ $(\varepsilon = \varepsilon(K))$ *that, given* $\delta > 0$, *there exists* $\xi \in \boldsymbol{K}_n$ *such that* $\|\xi\| < \delta$ *and,* $\mathbf{z}(t, \xi)$ *being the solution of* (4,3′) *with* $\mathbf{z}(0, \xi) = \xi$, *there holds* $\|\mathbf{z}(t_1, \xi)\| = \varepsilon$ *for at least one* $t_1 \geqq 0$.

Lemma 4,2C. *Let the system of differential equations*

$$(4,2) \qquad \dot{x}_k = X_k(t, x_1, x_2, \ldots, x_n) \quad (k = 1, 2, \ldots, l;\ 1 \leqq l \leqq n)$$
$$\dot{x}_j = a_{j,l+1}x_{l+1} + a_{j,l+2}x_{l+2} + \ldots + a_{j,n}x_n + X_j(t, x_1, x_2, \ldots, x_n)$$
$$(j = l + 1, l + 2, \ldots, n)$$

has the following properties:

(i) $a_{i,j}$ $(i, j = l + 1, l + 2, \ldots, n)$ *are complex constants such that all the roots of the characteristic equation of the matrix* $\hat{\boldsymbol{A}} = (a_{i,j})_{i,j=l+1,l+2,\ldots,n}$ *have negative real parts.*

(ii) $X_k(t, \mathbf{x})$ $(k = 1, 2, \ldots, n)$ *are on* $\mathfrak{G} = \mathrm{E}[(t, \mathbf{x}) : t \geqq 0,\ \mathbf{x} \in \boldsymbol{K}_n,\ \|\mathbf{x}\| < h]$ $(h > 0)$ *holomorphic functions of the complex variables* $x_1, x_2, \ldots, x_n$ *of the type* $[\mathbf{x}]_2$, *the coefficients of their developments into power series of the variables* $x_1, x_2, \ldots, x_n$ *being bounded continuous functions of the real variable* t *on the interval* $\langle 0, \infty)$.

Let us add to the system (4,2) *the system*

$$(4,2') \qquad \dot{x}_k = X_k(t, x_1, x_2, \ldots, x_l, 0, 0, \ldots, 0) \quad (k = 1, 2, \ldots, l)$$

which will be called the shortened system corresponding to the system (4,2).

Then, if the trivial solution of the system (4,2′) *is stable or unstable independently of the terms of the degree higher than N and the functions* $X_j(t, x_1, x_2, \ldots, x_l, 0, 0, \ldots$ $\ldots, 0)$ $(j = l + 1, l + 2, \ldots, n)$ *are altogether of the type* $[\mathbf{x}]_{N+1}$, *the trivial solution of the system* (4,2) *is stable or unstable, respectively.*

For the proof of the more general assertion for real systems, see in [11] (p. 382).

4,2. The critical case of several purely imaginary roots of the characteristic equation of the system (1,1).

Throughout the section 4,2 we shall suppose that the matrix $\boldsymbol{A}$ of the linear terms of the system (1,1) is regular and that the characteristic equation of the system

(1,1) has l (l is some integer from the interval $\langle 1, n \rangle$) purely imaginary roots, the others having negative real parts. Hence we suppose that the following condition (Q_1^*) is fulfilled.

(Q_1^*) The roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the characteristic equation of the system (1,1) being appropriately enumerated in accordance with the rule from sec. 2,1, it holds: $\operatorname{Re} \lambda_j = 0$ $(j = 1, 2, \ldots, l)$, $\operatorname{Re} \lambda_j < 0$ $(j = l + 1, l + 2, \ldots, n)$.

Notation 4,2. Let m be an arbitrary natural number less than $n + 1$. Given a nonnegative integer ν, the set of all m-tuples $(p_1, p_2, \ldots, p_m)$ of such nonnegative integers that $p_1 + p_2 + \ldots + p_m \geqq \nu$ will be denoted by $\mathscr{M}_\nu^{(m)}$ and the set of all m-tuples $(p_1, p_2, \ldots, p_m)$ of such nonnegative integers that $1 \leqq p_1 + p_2 + \ldots + p_m \leqq \leqq \nu$ by $\mathscr{R}_\nu^{(m)}$ $(\mathscr{M}_\nu^{(n)} = \mathscr{M}_\nu, \mathscr{R}_\nu^{(n)} = \mathscr{R}_\nu)$. Given a natural number k less than $m + 1$ and an m-tuple $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_m)$ of complex numbers, the set of all m-tuples $(p_1, p_2, \ldots, p_m) \in \mathscr{M}_2^{(m)}$ such that $p_1\mu_1 + p_2\mu_2 + \ldots + p_m\mu_m - \mu_k = 0$ will be denoted by $\mathscr{N}_k^{(m)}(\boldsymbol{\mu})$ $(\mathscr{N}_k^{(n)}(\boldsymbol{\mu}) = \mathscr{N}_k(\boldsymbol{\mu}))$. Given a natural number k less than $m + 1$, an arbitrary natural number ν and an m-tuple $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_m)$ of complex numbers, the set $\mathscr{N}_k^{(m)}(\boldsymbol{\mu}) \cap \mathscr{R}_\nu^{(m)}$ will be denoted by $\mathscr{N}_{k,\nu}^{(m)}(\boldsymbol{\mu})$ $(\mathscr{N}_{k,\nu}^{(n)}(\boldsymbol{\mu}) = \mathscr{N}_{k,\nu}(\boldsymbol{\mu}))$.

Given a nonnegative integer ν, the m-tuple $(p_1, p_2, \ldots, p_m)$ from $\mathscr{M}_\nu^{(m)}$ will be henceforth regarded as identical with the n-tuple $\mathbf{q}$ from $\mathscr{M}_\nu$ such that $q_j = p_j$ for $j = 1, 2, \ldots, m$ and $q_j = 0$ for $j = m + 1, m + 2, \ldots, n$.

Remark 4,2A. Let m be an arbitrary integer from the interval $\langle l, n \rangle$. Then by Lemma 3,2B the system (1,1) can be always transformed by the formal transformation (3,1′) to the formal system of the form

$$\dot{\mathbf{y}} = \mathbf{B}\mathbf{y} + \mathbf{Y}(\mathbf{y}) \tag{4,4}$$

where $Y_k(\mathbf{y})$ $(k = 1, 2, \ldots, n)$ are formal power series of the variables $y_1, y_2, \ldots, y_n$ such that for any $k = 1, 2, \ldots, m$ and for any $\mathbf{p} \in \mathscr{M}_2 \setminus \mathscr{N}_k(\boldsymbol{\lambda})$ the coefficient of $\mathbf{y}^\mathbf{p}$ in $Y_k(\mathbf{y})$ equals 0 and where $\mathbf{B}$ is some Jordan canonical form of the matrix $\mathbf{A}$. Therefore if $\boldsymbol{\varepsilon} = (\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_n)$ is the first subdiagonal of the matrix $\mathbf{B}$, the formal system (4,4) can be written in the following form

$$\dot{y}_k = \lambda_k y_k + \varepsilon_k y_{k-1} + \sum_{\mathbf{p} \in \mathscr{L}_k(\boldsymbol{\lambda})} \{Y_k\}_\mathbf{p} \mathbf{y}^\mathbf{p} \quad (k = 1, 2, \ldots, n) \tag{4,4′}$$

where $\varepsilon_1 = 0$ and $\mathscr{L}_k(\boldsymbol{\lambda}) = \mathscr{N}_k(\boldsymbol{\lambda})$ for $k = 1, 2, \ldots, m$ and $\mathscr{L}_k(\boldsymbol{\lambda}) = \mathscr{M}_2$ for $k = = m + 1, m + 2, \ldots, n$. Owing to (Q_1^*) the n-tuple $\mathbf{p} \in \mathscr{M}_2$ belongs for some $k = = 1, 2, \ldots, l$ to the set $\mathscr{N}_k(\boldsymbol{\lambda})$ if and only if it is $p_{l+1} = p_{l+2} = \ldots = p_n = 0$ and $p_1\lambda_1 + p_2\lambda_2 + \ldots + p_l\lambda_l = \lambda_k$. Hence the right-hand sides of the first l equations in (4,4) do not depend on $y_{l+1}, y_{l+2}, \ldots, y_n$.

The convergence of the series (3,1′) has been so far proved only under certain assumptions (see Chapter 3). However, there follows from the proof of Lemma

3,2 B that, given a natural number ν, there exists a transformation of the system (1,1) generated by the polynomials

$$x_k = \sum_{\boldsymbol{p}\in\mathscr{R}_{\nu_k}} \{g_k\}_{\boldsymbol{p}}\, y^{\boldsymbol{p}} \quad (k = 1, 2, \ldots, n) \tag{4,5}$$

(where ν_k $(k = 1, 2, \ldots, n)$ are some suitable natural numbers) to the system

$$\dot{\mathbf{y}} = \mathbf{B}\mathbf{y} + \mathbf{Z}(\mathbf{y}) \tag{4,6}$$

where $\mathbf{B}$ is Jordan canonical form of the matrix $\mathbf{A}$ (with the first subdiagonal $\boldsymbol{\varepsilon} = (\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_n)$), and where

$$\begin{aligned} Z_k(\mathbf{y}) &= \sum_{\boldsymbol{p}\in\mathscr{N}_{K,\nu}(\boldsymbol{\lambda})} \{Z_k\}_{\boldsymbol{p}}\, \mathbf{y}^{\boldsymbol{p}} + \sum_{\boldsymbol{p}\in\mathscr{M}_2-\mathscr{R}_\nu} \{Z_k\}_{\boldsymbol{p}}\, \mathbf{y}^{\boldsymbol{p}} \quad (k = 1, 2, \ldots, m)\,, \\ Z_k(\mathbf{y}) &= (\mathbf{y}) = [\mathbf{y}]_2 \qquad\qquad\qquad\qquad (k = m + 1, m + 2, \ldots, n)\,. \end{aligned}$$

Such a transformation is surely regular. Making use of Lemma 4,1 A, we can prove the following important assertion

Theorem 4,2A. *Let the matrix $\mathbf{A}$ of the linear terms of the system (1,1) be regular. Let the roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the characteristic equation of the system (1,1) fulfil (Q_1^*). Let for some natural number m from the interval $\langle l, n\rangle$ and for some Jordan canonical form $\mathbf{B}$ of the matrix $\mathbf{A}$ such a pseudonormal form (4,4′) of the system (1,1) of the type $(m; \mathbf{B})$ exists that, $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ being the first subdiagonal of the matrix $\mathbf{B}$, there does not hold:*

$$\varepsilon_k = 0 \quad (k = 1, 2, \ldots, l)\,, \quad \{Y_k\}_{\boldsymbol{p}} = 0 \quad (k = 1, 2, \ldots, l;\ \boldsymbol{p} \in \mathscr{N}_k(\boldsymbol{\lambda}))\,. \tag{4,7}$$

Then the trivial solution of the system (1,1) is unstable.

Proof. Let m, $\mathbf{B}$ be such a natural number from the interval $\langle l, n\rangle$ and a Jordan canonical form of the matrix $\mathbf{A}$, respectively, that there is a pseudonormal form (4,4′) of the system (1,1) of the type $(m, \mathbf{B})$ which does not fulfil (4,7).

If $\varepsilon_j \neq 0$ for some $j = 2, 3, \ldots, l$, then the trivial solution of the system (1,1) is unstable owing to Theorem 2,3C. Hence we can assume $\varepsilon_2 = \varepsilon_3 = \ldots = \varepsilon_{l+1} = 0$ henceforth.

Let us order now the set $\mathfrak{S}(\boldsymbol{\lambda}) = \mathrm{E}[\{Y_k\}_{\boldsymbol{p}} : k = 1, 2, \ldots, l;\ \boldsymbol{p} \in \mathscr{N}_k(\boldsymbol{\lambda})]$ by the prescription: $\{Y_j\}_{\boldsymbol{q}} \prec \{Y_k\}_{\boldsymbol{p}}$ if and only if the first nonzero number in the set $\{(p_1 + p_2 + \ldots + p_n) - (q_1 + q_2 + \ldots + q_n), p_1 - q_1, p_2 - q_2, \ldots, p_n - q_n, k - j\}$ is positive. Let $\{Y_j\}_{\boldsymbol{q}} = b$ be the first nonzero element of the set $\mathfrak{S}(\boldsymbol{\lambda})$. Let us modify then the system (1,1) by a regular transformation to the system

$$\begin{aligned} \dot{y}_j &= \lambda_j y_j + b y_1^{q_1} y_2^{q_2} \ldots y_l^{q_l} + Z_j(y_1, y_2, \ldots, y_n) \\ \dot{y}_k &= \lambda_k y_k + \varepsilon_k y_{k-1} + Z_k(y_1, y_2, \ldots, y_n) \quad (k = 1, 2, \ldots, n;\ k \neq j) \end{aligned} \tag{4,8}$$

where $Z_k(\mathbf{y}) = [\mathbf{y}]_{\nu+1}$ $(k = 1, 2, \ldots, j-1)$, $Z_k(\mathbf{y}) = [\mathbf{y}]_\nu$ $(k = j, j+1, \ldots, m)$ and $Z_k(\mathbf{y}) = [\mathbf{y}]_2$ $(k = m+1, m+2, \ldots, n)$ and $q_1 + q_2 + \ldots + q_l = \nu$ (see Remark 4,2 A). In order that the series (4,1) be solution of (4,8) the coefficients ${}^{(k)}\varphi_{\mathbf{p}}(t)$ $(k = 1, 2, \ldots, l;\ \mathbf{p} \in \mathscr{M}_1)$ have to fulfil some linear differential equations. In the first place those of them for which $k = 1, 2, \ldots, l$; $\mathbf{p} \in \mathscr{M}_1$ and $p_1 + p_2 + \ldots + p_n = 1$ has to fulfil the equation ${}^{(k)}\dot{\varphi}_{\mathbf{p}} = \lambda_k \cdot {}^{(k)}\varphi_{\mathbf{p}}$.

Thus ${}^{(k)}\varphi_{\mathbf{p}}(t) = {}^{(k)}\varphi_{\mathbf{p}}(0)\, e^{\lambda_k t}$ for $t \geqq 0$; $k = 1, 2, \ldots, l$; $\mathbf{p} \in \mathscr{M}_1$ and $p_1 + p_2 + \ldots + p_n = 1$.

The functions ${}^{(k)}\varphi_{\mathbf{p}}(t)$ must fulfil the following initial conditions, obviously:

$$(4,9)\quad \begin{array}{lll} {}^{(k)}\varphi_{\mathbf{p}}(0) = 0 & \text{for } \ \mathbf{p} \in \mathscr{M}_1 \ \text{ and } \ \mathbf{p} \neq \mathbf{e}_k & \\ {}^{(k)}\varphi_{\mathbf{p}}(0) = 1 & \text{for } \ \mathbf{p} = \mathbf{e}_k & (k = 1, 2, \ldots, l)\,. \end{array}$$

Thus, it is:

$$(4,10)\quad \begin{array}{ll} {}^{(k)}\varphi_{\mathbf{p}}(t) = e^{\lambda_k t} & \text{for } \ t \geqq 0 \ \text{ and } \ \mathbf{p} = \mathbf{e}_k \\ {}^{(k)}\varphi_{\mathbf{p}}(t) = 0 & \text{for } \ t \geqq 0 \ \text{ and } \ \mathbf{p} \in \mathscr{M}_1 \ \text{ with } \ p_1 + p_2 + \ldots + p_n = 1 \\ & \text{and } \ \mathbf{p} \neq \mathbf{e}_k \ \ (k = 1, 2, \ldots, l)\,. \end{array}$$

Let us order the set of all functions ${}^{(k)}\varphi_{\mathbf{p}}(t)$ $(k = 1, 2, \ldots, l;\ \mathbf{p} \in \mathscr{M}_1)$ from (4,1) in the same way as the set $\mathfrak{S}(\boldsymbol{\lambda})$ has been ordered. For the functions ${}^{(k)}\varphi_{\mathbf{p}}(t)$ such that ${}^{(k)}\varphi_{\mathbf{p}}(t) \prec {}^{(j)}\varphi_{\mathbf{q}}(t)$, (4,8) implies the following equation:

$$(4,11)\qquad {}^{(k)}\dot{\varphi}_{\mathbf{p}} = \lambda_k \cdot {}^{(k)}\varphi_{\mathbf{p}}\,.$$

The only solutions of (4,11) which fulfil also (4,9) are clearly the trivial solutions ${}^{(k)}\varphi_{\mathbf{p}}(t) \equiv 0$ on $\langle 0, \infty)$.

Finally, for ${}^{(j)}\varphi_{\mathbf{q}}(t)$ we get from (4,8)

$${}^{(j)}\dot{\varphi}_{\mathbf{q}} = \lambda_j \cdot {}^{(j)}\varphi_{\mathbf{q}} + b \prod_{i=1}^{l} \left({}^{(i)}\varphi_{\mathbf{e}_i}\right)^{q_i}.$$

Whence in virtue of $\mathbf{q} \in \mathscr{N}_j^{(l)}$, (4,9) and (4,10) we have

$${}^{(j)}\varphi_{\mathbf{q}}(t) = bte^{\lambda_j t} \quad (t \geqq 0)$$

which is not a bounded function of t on $\langle 0, \infty)$, of course. This and Lemma 4,1 A complete the proof.

Theorem 4,2B. *Let the matrix $\mathbf{A}$ of the linear terms of the system (1,1) be regular. Let the roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the characteristic equation of the system (1,1) fulfil the condition (Q_1^*) (i.e. there holds* $\operatorname{Re} \lambda_j = 0$ $(j = 1, 2, \ldots, l)$, $\operatorname{Re} \lambda_j < 0$ $(j = l+1, l+2, \ldots, n)$ *where l is some natural number less than $n + 1$) and the condition*

(Q_2) *There exists* $\delta > 0$ *such that, given* $k = 1, 2, \ldots, l$ *and an l-tuple* $(p_1, p_2, \ldots, p_l)$ *of nonnegative integers with* $p_1\lambda_1 + p_2\lambda_2 + \ldots + p_l\lambda_l \neq \lambda_k$, *there holds* $|p_1\lambda_1 + p_2\lambda_2 + \ldots + p_l\lambda_l - \lambda_k| \geqq \delta > 0$.

Then the trivial solution of the system (1,1) *is stable if and only if for some Jordan canonical form* **B** *of the matrix* **A** *the fundamental pseudonormal form of the system* (1,1) *of the type* (m; **B**) *fulfils the condition* (4,7) *from Theorem* 4,2A. *Moreover, the trivial solution of the system* (1,1) *is then never asymptotically stable.*

Proof. The last assertion follows immediately from Theorem 2,3D. The necessity of the condition (4,7) for the stability of the trivial solution of the system (1,1) follows immediately from Theorem 4,2A.

Let **B** be now the Jordan canonical form of the matrix **A** (with the first subdiagonal $\varepsilon = (\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_n)$) such that the fundamental pseudonormal form of the system (1,1) of the type (m; **B**) fulfils (4,7). By Theorem 3,4 there is a regular transformation of the system (1,1) to the system

$$(4,12)\quad \dot{y}_k = \lambda_k y_k \quad (k = 1, 2, \ldots, l)$$
$$\dot{y}_j = \lambda_j y_j + \varepsilon_j y_{j-1} + Y_j(y_1, y_2, \ldots, y_n) \quad (\varepsilon_{l+1} = 0;\ j = l + 1, l + 2, \ldots, n)$$

where $Y_j(\mathbf{y})$ $(j = l + 1, l + 2, \ldots, n)$ are holomorphic on some open neighbourhood $\boldsymbol{\Omega}^*$ of the origin **O** in $\boldsymbol{K}_n$.

Making use of the method from the proof of Theorem 3,4, it is easy to prove that the system of partial differential equations

$$\sum_{k=1}^{l} \frac{\partial v_j}{\partial y_k}(\lambda_k y_k) = \lambda_j v_j + \varepsilon_j v_{j-1} + Y_j(y_1, y_2, \ldots, y_l, v_{l+1}, v_{l+2}, \ldots, v_n)$$
$$(j = l + 1, l + 2, \ldots, n)$$

has a solution $\mathbf{v}(y_1, y_2, \ldots, y_l) = (v_{l+1}(y_1, y_2, \ldots, y_l), v_{l+2}(y_1, y_2, \ldots, y_l), \ldots \ldots, v_n(y_1, y_2, \ldots, y_l))$ whose components are holomorphic on some open neighbourhood $\boldsymbol{U}^*$ of the origin in $\boldsymbol{K}_l$ functions of the variables $y_1, y_2, \ldots, y_l$.

Inserting

$$y_j = \eta_j + v_j(y_1, y_2, \ldots, y_l) \quad (j = l + 1, l + 2, \ldots, n)$$
$$y_k = \eta_k e^{\lambda_k t} \quad (k = 1, 2, \ldots, l)$$

in the system (4,12) we get the system for η_k $(k = 1, 2, \ldots, n)$

$$(4,12')\quad \dot{\eta}_k = 0 \quad (k = 1, 2, \ldots, l)$$
$$\dot{\eta}_j = \lambda_j \eta_j + \varepsilon_j \eta_{j-1} + Z_j(t, \eta_1, \eta_2, \ldots, \eta_n) \quad (j = l + 1, l + 2, \ldots, n)$$

where

$$Z_j(t, \eta) = Y_j(\eta_1 e^{\lambda_1 t}, \eta_2 e^{\lambda_2 t}, \ldots, \eta_l e^{\lambda_l t}, \eta_{l+1} + v_{l+1}, \eta_{l+2} + v_{l+2}, \ldots, \eta_n + v_n) -$$
$$- Y_j(\eta_1 e^{\lambda_1 t}, \eta_2 e^{\lambda_2 t}, \ldots, \eta_l e^{\lambda_l t}, v_{l+1}, v_{l+2}, \ldots, v_n) \quad (j = l + 1, l + 2, \ldots, n)$$

are surely for some $h > 0$ on the set $\mathrm{E}[(t, \boldsymbol{\eta}) : t \geqq 0, \boldsymbol{\eta} \in \boldsymbol{K}_n, \|\boldsymbol{\eta}\| < h]$ holomorphic functions of the complex variables $\eta_1, \eta_2, \ldots, \eta_n$ of the type $[\boldsymbol{\eta}]_2$ and continuous and bounded functions of the real variable t.

Clearly for every $j = l + 1, l + 2, \ldots, n$ there holds

$$Z_j(t, \eta_1, \eta_2, \ldots, \eta_l, 0, 0, \ldots, 0) = 0 \quad \text{for} \quad t > 0 \quad \text{and} \quad |\eta_k| < h \quad (k = 1, 2, \ldots, l).$$

Whence by Lemma 4,1B the stability of the trivial solution of the system (1,1) readily follows and this completes the proof of Theorem 4,2B.

Corollary. *Let the matrix* $\mathbf{A}$ *of the linear terms of the system* (1,1) *fulfil the assumptions of Theorem* 4,2B. *Let* $\mathbf{B}$ *be an arbitrary Jordan canonical form of the matrix* $\mathbf{A}$ *such that the fundamental pseudonormal form of the system* (1,1) *of the type* (l; $\mathbf{B}$) *fulfils the condition* (4,7) *from Theorem* 4,2A.

Let $\mathbf{B}'$ *be an arbitrary Jordan canonical form of the matrix* $\mathbf{A}$ *and let* m *be an arbitrary natural number from the interval* $\langle l, n \rangle$. *Then any pseudonormal form of the system* (1,1) *of the type* (m; $\mathbf{B}'$) *fulfils these conditions, too.*

Remark 4,2B. Theorem 4,2B makes it possible to solve completely the problem of the stability of the trivial solution of such system (1,1) that its characteristic equation has two purely imaginary roots λ_1, λ_2 with rational quotient $\lambda_1 \lambda_2^{-1}$, the other roots of this equation having negative real parts (see Remark 3,4D). If $\lambda_1 \lambda_2^{-1}$ is an irrational number it is possible to use two ways for solving the problem in question:

α) If the conditions of K. L. Siegel (see Remark 3,4F) are fulfilled, the trivial solution of the system (1,1) is surely stable.

β) We can divide the equations (1,1) into their real and imaginary parts. Then we obtain a real system of $2n$ differential equations whose characteristic equation has two pairs of purely imaginary roots, the other roots of this equation having negative real parts. For solving problem of the stability of the trivial solution of this system we can then utilize the method given by I. G. Malkin in [11] (p. 422).

4,3. The critical case of several zero roots of the characteristic equation of the system (1,1).

Let us deal with the critical case that the characteristic equation of the system (1,1) has several (l; $1 \leqq l \leqq n$) zero roots, the other roots having negative real parts. So, we shall suppose in this section that the roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the characteristic equation of the system (1,1) being suitably enumerated in accordance with the rule given in sec. 2,1, there holds

$$(Q_1^{**}) \qquad \lambda_j = 0 \ (j = 1, 2, \dots, l), \quad \operatorname{Re} \lambda_j < 0 \ (j = l + 1, l + 2, \dots, n).$$

The given system (1,1) can be then modified by a suitable regular linear transformation to the following form:

$$(4,13) \quad \begin{aligned} \dot{y}_k &= \varepsilon_k y_{k-1} + Y_k(y_1, y_2, \dots, y_n) && (k = 1, 2, \dots, l;\ \varepsilon_1 = 0) \\ \dot{y}_j &= \lambda_j y_j + \varepsilon_j y_{j-1} + Y_j(y_1, y_2, \dots, y_n) && (j = l + 1, l + 2, \dots, n;\ \varepsilon_{l+1} = 0) \end{aligned}$$

where $\varepsilon = (\varepsilon_2, \varepsilon_3, \dots, \varepsilon_n)$ is the first subdiagonal of some Jordan canonical form $\mathbf{B}$ of the matrix $\mathbf{A}$ of the linear terms of the system (1,1) and where $Y_k(\mathbf{y}) = [\mathbf{y}]_2$ $(k = 1, 2, \dots, n)$.

If it does not hold that $\varepsilon_2 = \varepsilon_3 = \dots = \varepsilon_l = 0$, then the trivial solution of (1,1) is unstable by Theorem 2,3C. Thus we shall suppose henceforth $\varepsilon_j = 0$ $(j = 1, 2, \dots, l + 1)$ in (4,13).

Let the $(n - l)$-vector complex-valued function $\boldsymbol{u}$ of l complex variables $y_1, y_2, \dots, y_l$ be the solution of the system

$$(4,14) \quad \lambda_{j+l} u_j + \varepsilon_{j+l} u_{j-1} + Y_{j+l}(y_1, y_2, \dots, y_l, u_1, u_2, \dots, u_{n-l}) = 0 \quad (j = 1, 2, \dots, n - l),$$

the components u_j $(j = 1, 2, \dots, n - l)$ of which are holomorphic on some open neighbourhood of the origin $\mathbf{O}$ in K_l functions of the variables $y_1, y_2, \dots, y_l$. (See Implicit Function Theorem in [14], p. 39.)

Let us substitute $y_j = z_j + u_{j-l}(y_1, y_2, \dots, y_l)$ $(j = l + 1, l + 2, \dots, n)$ into (4,13). Then we obtain the system

$$(4,15) \quad \begin{aligned} \dot{y}_k &= \tilde{Y}_k(y_1, y_2, \dots, y_l, z_{l+1}, z_{l+2}, \dots, z_n) \quad (k = 1, 2, \dots, l) \\ \dot{z}_j &= \lambda_j z_j + \varepsilon_j z_{j-1} + \tilde{Y}_j(y_1, y_2, \dots, y_l, z_{l+1}, z_{l+2}, \dots, z_n) \\ &(j = l + 1, l + 2, \dots, n) \end{aligned}$$

where

$$\tilde{Y}_k(y_1, y_2, \dots, y_l, z_{l+1}, z_{l+2}, \dots, z_n) = Y_k(y_1, y_2, \dots, y_l, z_{l+1} + u_1, z_{l+2} + u_2, \dots, z_n + u_{n-l}) \quad (k = 1, 2, \dots, l)$$

and

$$\begin{aligned} &\tilde{Y}_j(y_1, y_2, \dots, y_l, z_{l+1}, z_{l+2}, \dots, z_n) = \\ &= Y_j(y_1, y_2, \dots, y_l, z_{l+1} + u_1, z_{l+2} + u_2, \dots, z_n + u_{n-l}) - \\ &\quad - Y_j(y_1, y_2, \dots, y_l, u_1, u_2, \dots, u_{n-l}) - \\ &\quad - \sum_{i=1}^{l} \frac{\partial u_j}{\partial y_j} Y_i(y_1, y_2, \dots, y_l, z_{l+1} + u_1, z_{l+2} + u_2, \dots, z_n + u_{n-l}) \\ &\qquad (j = l + 1, l + 2, \dots, n). \end{aligned}$$

The right-hand sides of (4,15) have evidently the following properties:

α) The functions $\tilde{Y}_j$ $(j = 1, 2, \ldots, n)$ are holomorphic on some open neighbourhood $\boldsymbol{\Omega}_1$ of the origin $\mathbf{O}$ in $\boldsymbol{K}_n$ functions of their variables $y_1, y_2, \ldots, y_l, z_{l+1}, z_{l+2}, \ldots \ldots, z_n$ which altogether belong to the class $[y_1, y_2, \ldots, y_l, z_{l+1}, z_{l+2}, \ldots, z_n]_2$.

β) If there holds

$$\tilde{Y}_k(y_1, y_2, \ldots, y_l, 0, 0, \ldots, 0) = [y_1, y_2, \ldots, y_l]_{\nu_k} \quad (k = 1, 2, \ldots, l)$$

where ν_k $(k = 1, 2, \ldots, l)$ are some natural numbers greater than 1, then there holds:

$$\tilde{Y}_j(y_1, y_2, \ldots, y_l, 0, 0, \ldots, 0) = [y_1, y_2, \ldots, y_l]_\nu \quad (j = l + 1, l + 2, \ldots, n)$$

where $\nu \geqq \min\limits_{1 \leqq k \leqq n} \nu_k$.

γ) The trivial solution of the shortened system

$$\dot{y}_k = \tilde{Y}_k(y_1, y_2, \ldots, y_l, 0, 0, \ldots, 0) \quad (k = 1, 2, \ldots, l)$$

is stable if and only if there holds:

$$\hat{Y}_k(y_1, y_2, \ldots, y_l) \equiv \tilde{Y}_k(y_1, y_2, \ldots, y_l, 0, 0, \ldots, 0) \equiv 0 \quad \text{on} \quad \hat{\boldsymbol{\Omega}}_1 \quad (k = 1, 2, \ldots, l)$$

where $\hat{\boldsymbol{\Omega}}_1 = \mathrm{E}[(y_1, y_2, \ldots, y_l) \in \boldsymbol{K}_l : (y_1, y_2, \ldots, y_l, 0, 0, \ldots, 0) \in \boldsymbol{\Omega}_1]$ (see Theorem 2,3E).

Now we are able to prove the following

Theorem 4,3. *Let the roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the characteristic equation of the system (1,1) be such that*

$$(Q_1^{**}) \quad \lambda_j = 0 \; (j = 1, 2, \ldots, l), \quad \mathrm{Re}\, \lambda_j < 0 \; (j = l + 1, l + 2, \ldots, n) \quad (1 \leqq l \leqq n).$$

Then the trivial solution of the system (1,1) is stable if and only if both the following conditions are satisfied:

(i) *All the zero roots λ_j $(j = 1, 2, \ldots, l)$ of the characteristic equation of the system (1,1) are diagonable (i.e. if the matrix $\mathbf{B}$ is an arbitrary Jordan canonical form of the matrix $\mathbf{A}$ of the linear terms of the system (1,1) and if $\boldsymbol{\varepsilon} = (\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_n)$ is the first subdiagonal of the matrix $\mathbf{B}$, then $\varepsilon_j = 0$ for any $j = 2, 3, \ldots, l + 1$).*

(ii) *For the functions $\hat{Y}_k$ arisen from the functions X_k in the above given way there holds:*

$$\hat{Y}_k(y_1, y_2, \ldots, y_l) \left(\equiv \tilde{Y}_k(y_1, y_2, \ldots, y_l, 0, 0, \ldots, 0)\right) \equiv 0 \quad \text{on} \quad \hat{\boldsymbol{\Omega}}_1 \quad (k = 1, 2, \ldots, l)$$

where $\hat{\boldsymbol{\Omega}}_1$ is some open neighbourhood of the origin $\mathbf{O}$ in $\boldsymbol{K}_l$.

Moreover, the trivial solution of the system (1,1) *is then never asymptotically stable.*

Proof. The last assertion of Theorem 4,3 follows immediately from Theorem 2,3D.

Making use of the foregoing, we can complete the proof of Theorem 4,3 in the same way as O. Vejvoda used in the proof of his assertion concerning the critical case of one zero root of the characteristic equation of the system (1,1) (see [1]): Instead of the system (1,1) we can investigate the system (4,15) (see above) which, as we remarked above, fulfils all assumptions of Lemma 4,1B and Lemma 4,1C. If the conditions (i) and (ii) are satisfied, then it follows immediately from Lemma 4,1B that the trivial solution of the given system is stable. The necessity of the conditions (i) and (ii) for the stability of the trivial solution of the system (1,1) follows easily from Lemma 4,1C and Theorem 2,3C.

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Author's address: Praha 1, Žitná 25, ČSSR (Matematický ústav ČSAV).