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FREDHOLM-STIELTJES INTEGRAL EQUATIONS WITH LINEAR CONSTRAINTS: DUALITY THEORY AND GREEN'S FUNCTION

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This note is devoted to the duality theory for the system of equations

$$\text{(I)} \qquad x(t) - x(0) - \int_0^1 \mathrm{d}_s[P(t,s) - P(0,s)]\, x(s) = f(t) - f(0) \quad \text{on} \quad [0,1],$$

$$\text{(II)} \qquad \int_0^1 \mathrm{d}[K(s)]\, x(s) = r,$$

where an n-vector valued function x of bounded variation on $[0, 1]$ is sought. Results analogous to those of [4], [10], [11] and [14] are obtained under less restrictive hypotheses and in a considerably simpler way. Boundary value problems for Fredholm-Stieltjes integro-differential equations which are special cases of (I) have been treated recently in [5], [12] and [13].

1. PRELIMINARIES

Given a real $p \times q$-matrix $A = (a_{i,j})_{\substack{i=1,\dots,p\\ j=1,\dots,q}}$, A^* denotes its transpose and

$$|A| = \max_{i=1,\dots,p} \sum_{j=1}^{q} |a_{i,j}|.$$

R_n denotes the space of real column n-vectors ($n \times 1$-matrices), R_n^* is the space of real row n-vectors ($1 \times n$-matrices), $R_1 = R_1^* = R$. The space of real $p \times q$-matrices is denoted by $L(R_q, R_p)$, $L(R_n, R_n) = L(R_n)$. Generally, vectors are assumed to be column. Row vectors are written as transposes of column vectors. If $a, b \in R$, $a < b$, then $[a, b]$ denotes the closed interval $a \leqq t \leqq b$, (a, b) is its interior $a < t < b$ and $[a, b)$, $(a, b]$ are the corresponding half-open intervals.

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BV_n stands for the Banach space of functions $\mathbf{x} : [0, 1] \to R_n$ of bounded variation on $[0, 1]$ equipped with the norm $\|\mathbf{x}\|_{BV} = |\mathbf{x}(0)| + \operatorname{var}_0^1 \mathbf{x}$. For $\mathbf{P} : [0, 1] \times \times [0, 1] \to L(R_n)$, $v(\mathbf{P})$ denotes its Vitali two-dimensional variation on $[0, 1] \times \times [0, 1]$. Recall that

$$v(\mathbf{P}) = \sup \sum_{i=1}^{p} \sum_{j=1}^{q} |\mathbf{P}(t_i, s_j) - \mathbf{P}(t_{i-1}, s_j) - \mathbf{P}(t_i, s_{j-1}) + \mathbf{P}(t_{i-1}, s_{j-1})|$$

where the supremum is taken over all net-type subdivisions $\sigma = \{0 = t_0 < t_1 < \ldots \ldots < t_p = 1, 0 = s_0 < s_1 < \ldots < s_q = 1\}$ of the interval $[0, 1] \times [0, 1]$. If $v(\mathbf{P}) + \operatorname{var}_0^1 \mathbf{P}(\cdot, a) + \operatorname{var}_0^1 \mathbf{P}(b, \cdot) < \infty$ for some fixed $a, b \in [0, 1]$, then there is $M < \infty$ such that $v(\mathbf{P}) + \operatorname{var}_0^1 \mathbf{P}(t, \cdot) + \operatorname{var}_0^1 \mathbf{P}(\cdot, s) + |\mathbf{P}(t, s)| \leqq M < \infty$ for all $t, s \in [0, 1]$ and $\mathbf{P}$ is called an SBV-kernel.

All integrals used are the Perron-Stieltjes integrals or equivalently (since all the functions occurring are of bounded variation) the σ-Young-Stieltjes integrals (cf. [3] and [7]). The list of properties of the Perron-Stieltjes integral may be found e.g. in [10] or [14]. Let us recall here only (for the proof see [8]) that if $\mathbf{P}$ is an SBV-kernel on $[0, 1] \times [0, 1]$, then for every $\mathbf{x}, \mathbf{y} \in BV_n$ the functions

$$\mathbf{u}(t) = \int_0^1 \mathrm{d}_\tau[\mathbf{P}(t, \tau)]\, \mathbf{x}(\tau) \quad \text{and} \quad \mathbf{v}^*(s) = \int_0^1 \mathrm{d}[\mathbf{y}^*(\tau)]\, \mathbf{P}(\tau, s)$$

are of bounded variation on $[0, 1]$. Moreover,

$$\mathbf{u}(t+) = \int_0^1 \mathrm{d}_\tau[\mathbf{P}(t+, \tau)]\, \mathbf{x}(\tau) \quad \text{and} \quad \mathbf{v}^*(s+) = \int_0^1 \mathrm{d}[\mathbf{y}^*(\tau)]\, \mathbf{P}(\tau, s+) \quad \text{for} \quad t, s \in [0, 1)$$

and

$$\mathbf{u}(t-) = \int_0^1 \mathrm{d}_\tau[\mathbf{P}(t-, \tau)]\, \mathbf{x}(\tau) \quad \text{and} \quad \mathbf{v}^*(s-) = \int_0^1 \mathrm{d}[\mathbf{y}^*(\tau)]\, \mathbf{P}(\tau, s-) \quad \text{for} \quad t, s \in (0, 1].$$

Let X and Y be linear spaces over R. The set of all linear operators $\mathscr{A}$ with values in Y and defined for all $\mathbf{x} \in X$ ($\mathscr{A} : X \to Y$) is denoted by $L(X, Y)$, $L(X, X) = L(X)$. The identity operator $\mathbf{x} \in X \to \mathbf{x} \in X$ is denoted by $\mathscr{I}$. For a linear operator $\mathscr{A} \in \in L(X, Y)$, $R(\mathscr{A})$ denotes the range of $\mathscr{A}$ and $N(\mathscr{A})$ is the null space of $\mathscr{A}$. $R(\mathscr{A})$ and $N(\mathscr{A})$ are linear subspaces of Y and X, respectively. Let us denote $\alpha(\mathscr{A}) = \dim N(\mathscr{A})$ and $\beta(\mathscr{A}) = \dim {}^{Y}|_{R(\mathscr{A})}$, where ${}^{Y}|_{R(\mathscr{A})}$ is the corresponding quotient space. It is known that if Y is a direct sum of $R(\mathscr{A})$ and $Z \subset Y$, then there is a one-to-one correspondence between ${}^{Y}|_{R(\mathscr{A})}$ and Z (cf. [2] III.20) and, in particular, $\beta(\mathscr{A}) = \dim Z$. If $\alpha(\mathscr{A})$, $\beta(\mathscr{A})$ are not both infinite, then we define the index $\operatorname{ind} \mathscr{A}$ of $\mathscr{A} \in L(X, Y)$ by the relation $\operatorname{ind} \mathscr{A} = \beta(\mathscr{A}) - \alpha(\mathscr{A})$.

Let X and X^+ be linear spaces over R and let

$$\mathbf{x} \in X\,, \quad \mathbf{x}^+ \in X^+ \to \langle \mathbf{x}, \mathbf{x}^+ \rangle \in R$$

be a bilinear form on $X \times X^+$. We say that X, X^+ form a dual pair (with respect to the bilinear form $\langle \cdot, \cdot \rangle$ if

$$\langle \mathbf{x}, \mathbf{x}^+ \rangle = 0 \quad \text{for every} \quad \mathbf{x} \in X \quad \text{implies} \quad \mathbf{x}^+ = 0 \in X^+$$

and

$$\langle \mathbf{x}, \mathbf{x}^+ \rangle = 0 \quad \text{for every} \quad \mathbf{x}^+ \in X^+ \quad \text{implies} \quad \mathbf{x} = \mathbf{0} \in X\,.$$

In [2] VI.40 the following important statement is proved:

Theorem (Heuser). *Let X, X^+ be a dual pair of linear spaces with respect to the bilinear form $\langle \cdot, \cdot \rangle$ defined on $X \times X^+$ and let the operators $\mathscr{A} \in L(X)$ and $\mathscr{A}^+ \in L(X^+)$ be such that*

$$(1,1) \qquad \langle \mathscr{A}\mathbf{x}, \mathbf{x}^+ \rangle = \langle \mathbf{x}, \mathscr{A}^+\mathbf{x}^+ \rangle \quad \textit{for every} \quad \mathbf{x} \in X \quad \textit{and} \quad \mathbf{x}^+ \in X^+$$

and

$$(1,2) \qquad \operatorname{ind} \mathscr{A} = \operatorname{ind} \mathscr{A}^+ = 0\,.$$

Then

$$(1,3) \qquad \alpha(\mathscr{A}) = \alpha(\mathscr{A}^+) = \beta(\mathscr{A}) = \beta(\mathscr{A}^+) < \infty$$

and, moreover, for given $\mathbf{y} \in X$ and $\mathbf{y}^+ \in X^+$

$$\mathscr{A}\mathbf{x} = \mathbf{y} \quad \textit{has a solution iff} \quad \langle \mathbf{y}, \mathbf{x}^+ \rangle = 0 \quad \textit{for any} \quad \mathbf{x}^+ \in N(\mathscr{A}^+)$$

and

$$\mathscr{A}^+\mathbf{x}^+ = \mathbf{y}^+ \quad \textit{has a solution iff} \quad \langle \mathbf{x}, \mathbf{y}^+ \rangle = 0 \quad \textit{for any} \quad \mathbf{x} \in N(\mathscr{A})\,.$$

Let us notice that if $\mathscr{A} \in L(X)$ is compact, then $R(\mathscr{I} - \mathscr{A})$ is closed in X, $\alpha(\mathscr{I} - \mathscr{A}) = \beta(\mathscr{I} - \mathscr{A}) < \infty$, i.e. $\operatorname{ind}(\mathscr{I} - \mathscr{A}) = 0$ (cf. [6] IV.3).

If X is a Banach space and X^* its dual space, then obviously X, X^* form a dual pair. Let us give another example of a dual pair which is of importance for our purposes.

In the following BV_n^+ denotes the set of all functions $\mathbf{z}^* : [0, 1] \to R_n$ of bounded variation on $[0, 1]$, right-continuous on $(0, 1)$ and such that $\mathbf{z}^*(1) = \mathbf{0}$. For $\mathbf{x} \in BV_n$ and $\mathbf{z}^* \in BV_n^+$ let us put

$$(1,4) \qquad \langle \mathbf{x}, \mathbf{z}^* \rangle_{BV} = \int_0^1 \mathrm{d}[\mathbf{z}^*(t)]\, \mathbf{x}(t)\,.$$

Then BV_n, BV_n^+ is a dual pair with respect to $\langle \cdot, \cdot \rangle_{BV}$. (For the proof of an analogous assertion see [8] Lemma 5.1.) When endowed with the norm $\mathbf{z}^* \in BV_n^+ \to \|\mathbf{z}^*\|_{BV^+} = |\mathbf{z}^*(0)| + \operatorname{var}_0^1 \mathbf{z}^*$, the space BV_n^+ becomes a Banach space and (1,4) defines a continuous bilinear form on $BV_n \times BV_n^+$.

2. GENERALIZED FREDHOLM-STIELTJES INTEGRO-DIFFERENTIAL OPERATOR

Let $\mathbf{P} : [0, 1] \times [0, 1] \to L(R_n)$ be an SBV-kernel. Then

$$\mathscr{P} : \mathbf{x} \in BV_n \to \int_0^1 \mathrm{d}_s[\mathbf{P}(t, s)]\, \mathbf{x}(s) \tag{2,1}$$

defines a linear compact (or completely continuous) operator on BV_n (cf. [8] Theorem 3.1).

2.1. Remark. Let $\mathbf{R} : [0, 1] \times [0, 1] \to L(R_n)$ be such that $\mathbf{R}(\cdot, s)$ is measurable on $[0, 1]$ for any $s \in [0, 1]$, $\mathrm{var}_0^1\, \mathbf{R}(t, \cdot) < \infty$ for a.e. $t \in [0, 1]$ and

$$\varrho : t \in [0, 1] \to |\mathbf{R}(t, 0)| + \mathrm{var}_0^1\, \mathbf{R}(t, \cdot)$$

is Lebesgue integrable on $[0, 1]$. Let $\mathbf{g} : [0, 1] \to R_n$ be Lebesgue integrable on $[0, 1]$. Then integrating and making use of the Cameron-Martin formula for the change of the integration order in Stieltjes integrals ([1]) we transfer the Fredholm-Stieltjes integro-differential equation for an absolutely continuous function $\mathbf{x} : [0, 1] \to R_n$

$$\dot{\mathbf{x}}(t) - \int_0^1 \mathrm{d}_s[\mathbf{R}(t, s)]\, \mathbf{x}(s) = \mathbf{g}(t) \quad \text{a.e. on} \quad [0, 1] \tag{2,2}$$

to the form (I), where

$$\mathbf{P}(t, s) = \int_0^t \mathbf{R}(\tau, s)\, \mathrm{d}\tau \quad \text{and} \quad \mathbf{f}(t) = \int_0^t \mathbf{g}(\tau)\, \mathrm{d}\tau \quad \text{for} \quad t, s \in [0, 1]\,. \tag{2,3}$$

It is easy to check that $\mathbf{P}(t, s)$ given by (2,3) is an SBV-kernel (cf. [10]). Obviously $\mathbf{f} \in BV_n$. Thus the equation (2,2) may always be rewritten as an equation of the form (I) with an SBV-kernel $\mathbf{P}(t, s)$ and $\mathbf{f} \in BV_n$. Hence the operator $\mathscr{I} - \mathscr{Q} \in L(BV_n)$, where

$$\mathscr{Q} : \mathbf{x} \in BV_n \to \mathbf{x}(0) + \int_0^1 \mathrm{d}_s[\mathbf{P}(t, s) - \mathbf{P}(0, s)]\, \mathbf{x}(s) \in BV_n \tag{2,4}$$

may be called a *generalized Fredholm-Stieltjes integro-differential operator.*

2.2. Remark. Evidently, the operator $\mathscr{Q} \in L(BV_n)$ given by (2,4) is compact ($\mathscr{Q}\mathbf{x} = \mathbf{u} + \mathscr{P}\mathbf{x}$, where

$$\mathbf{u} = \mathbf{x}(0) - \int_0^1 \mathrm{d}_s[\mathbf{P}(0, s)]\, \mathbf{x}(s) \in R_n)\,.$$

Moreover, if we put

$$\mathbf{Q}(t, s) = \begin{cases} \mathbf{P}(t, s) - \mathbf{P}(0, s) & \text{for} \quad t, s \in [0, 1]\,, \quad s > 0\,, \\ \mathbf{P}(t, 0) - \mathbf{P}(0, 0) - \mathbf{I} & \text{for} \quad t, s \in [0, 1]\,, \quad s = 0\,, \end{cases} \tag{2,5}$$

then $\mathbf{Q}(t, s)$ is also an SBV-kernel and

$$\mathscr{Q} : \mathbf{x} \in BV_n \rightarrow \int_0^1 \mathrm{d}_s[\mathbf{Q}(t, s)]\, \mathbf{x}(s) \in BV_n \,. \tag{2,6}$$

2.3. Theorem. *If $\mathbf{P} : [0, 1] \times [0, 1] \rightarrow L(R_n)$ is an SBV-kernel and the operator $\mathscr{Q} \in L(BV_n)$ is given by (2,4), then*

$$n \leqq \dim N(\mathscr{I} - \mathscr{Q}) < \infty \,, \tag{2,7}$$

while $\dim N(\mathscr{I} - \mathscr{Q}) = n$ *iff the equation* (I) *has a solution $\mathbf{x} \in BV_n$ for any $\mathbf{f} \in BV_n$.*

Proof. Since $\mathscr{Q}$ is compact, $\alpha(\mathscr{I} - \mathscr{Q}) = \beta(\mathscr{I} - \mathscr{Q}) < \infty$. Obviously

$$R(\mathscr{I} - \mathscr{Q}) \subset Z = \{\mathbf{f} \in BV_n : \mathbf{f}(0) = \mathbf{0}\} \,.$$

Hence

$$\alpha(\mathscr{I} - \mathscr{Q}) = \beta(\mathscr{I} - \mathscr{Q}) \geqq \dim {}^{BV_n}|_Z = n \,.$$

Furthermore, $\alpha(\mathscr{I} - \mathscr{Q}) = n$ iff

$$\beta(\mathscr{I} - \mathscr{Q}) = \dim {}^{BV_n}|_{R(\mathscr{I} - \mathscr{Q})} = \dim {}^{BV_n}|_Z$$

and the proof follows by means of the following lemma.

2.4. Lemma. *Given a linear space X and its linear subspaces M, N such that $M \subset N$, then* $\dim {}^X|_M = \dim {}^X|_N < \infty$ *holds iff $M = N$.*

Proof. Let $\dim {}^X|_M = \dim {}^X|_N = k < \infty$ and let $\mathbf{x} \in N \setminus M$. Let $\Xi_j = \xi^{(j)} + N$ $(j = 1, 2, \ldots, k)$ be a basis in ${}^X|_N$ and let

$$\alpha \mathbf{x} + \sum_{j=1}^{k} \lambda_j \xi^{(j)} \in M$$

for some real numbers α, λ_j $(j = 1, 2, \ldots, k)$. Since $\alpha \mathbf{x} \in N$, this may happen only if $\lambda_1 \xi^{(1)} + \lambda_2 \xi^{(2)} + \ldots + \lambda_k \xi^{(k)} \in N$, i.e. $\lambda_1 = \lambda_2 = \ldots = \lambda_k = 0$. Thus $\alpha \mathbf{x} \in M$ and $\alpha = 0$ since $\mathbf{x} \notin M$. This means that the classes $\mathbf{x} + M$, $\xi^{(j)} + M$ $(j = 1, 2, \ldots, k)$ are linearly independent in ${}^X|_M$ and $\dim {}^X|_M = k + 1 > \dim {}^X|_N$. This being contradictory to the assumption proves that $M = N$.

2.5. Remark. By 2.3 there exists an $n \times k$-matrix valued function $\mathbf{X}$ $(k = \dim N(\mathscr{I} - \mathscr{Q}))$ of bounded variation on $[0, 1]$ such that $\mathbf{x} \in BV_n$ satisfies $\mathbf{x} - \mathscr{Q}\mathbf{x} = \mathbf{0}$ iff $\mathbf{x}(t) = \mathbf{X}(t)\,\mathbf{c}$ on $[0, 1]$ for some $\mathbf{c} \in R_k$. Unfortunately, even if $k = n$, it need not be $\det \mathbf{X}(t) \neq 0$ on $[0, 1]$ as shown by the integro-differential equation

$$\dot{\mathbf{x}}(t) - 4 \int_0^1 \mathbf{x}(s)\, \mathrm{d}s = \mathbf{0} \,,$$

for which $\mathbf{X}(t) = \mathbf{I}(1 - 4t)$.

Throughout the section the following assumptions are kept:

3.1. Assumptions. (i) $\mathbf{P} : [0, 1] \times [0, 1] \to L(R_n)$ *is an SBV-kernel*, $\mathbf{f} \in BV_n$, $\mathbf{K} : [0, 1] \to L(R_n, R_m)$ *is of bounded variation on* $[0, 1]$ *and* $\mathbf{r} \in R_m$.

(ii) $\mathbf{P}(t, \cdot)$ *is right-continuous on* $(0, 1)$ *and* $\mathbf{P}(t, 1) = \mathbf{0}$ *for any* $t \in [0, 1]$, $\mathbf{P}(0, s) = \mathbf{0}$ *for any* $s \in [0, 1]$, $\mathbf{K}$ *is right-continuous on* $(0, 1)$ *and* $\mathbf{K}(1) = \mathbf{0}$.

3.2. Remark. For the investigation of the system (I), (II) the assumptions 3.1 (ii) do not cause any loss of generality. Any system (I), (II) fulfilling 3.1 (i) is equivalent to a system fulfilling also 3.1 (ii).

Let $\mathbf{Q} : [0, 1] \times [0, 1] \to L(R_n)$ and $\mathcal{Q} \in L(BV_n)$ be defined by (2,5) and (2,4), respectively. Furthermore, let us denote

(3,1)
$$\mathcal{K} : \mathbf{x} \in BV_n \to \int_0^1 \mathrm{d}[\mathbf{K}(s)]\, \mathbf{x}(s) \in R_m\,,$$

$$\mathcal{S} : \mathbf{f} \in BV_n \to \mathbf{f}(t) - \mathbf{f}(0) \in BV_n$$

and

(3,2)
$$\mathcal{T} : \begin{pmatrix} \mathbf{x} \\ \mathbf{d} \end{pmatrix} \in BV_n \times R_m \to \begin{pmatrix} \mathcal{Q}\mathbf{x} \\ \mathbf{d} - \mathcal{K}\mathbf{x} \end{pmatrix} \in BV_n \times R_m\,.$$

3.3. Proposition. *If* $\mathbf{x} \in BV_n$ *is a solution to* (I), (II), *then* $\xi = \begin{pmatrix} \mathbf{x} \\ \mathbf{d} \end{pmatrix}$ *is a solution to*

(3,3)
$$(\mathcal{I} - \mathcal{T})\, \xi = \begin{pmatrix} \mathcal{S}\mathbf{f} \\ \mathbf{r} \end{pmatrix}$$

for any $\mathbf{d} \in R_m$. *If* $\mathbf{x} \in BV_n$ *and there exists* $\mathbf{d} \in R_m$ *such that* $\xi = \begin{pmatrix} \mathbf{x} \\ \mathbf{d} \end{pmatrix}$ *verifies* (3,3) *then* $\mathbf{x}$ *is a solution to* (I), (II).

3.4. Proposition. *Under the assumptions* 3.1(i) *the operator* $\mathcal{T} \in L(BV_n \times R_m)$ *defined by* (2,4), (3,1) *and* (3,2) *is compact.*

Proof. Obviously, $\mathcal{K} \in L(BV_n, R_m)$ is bounded. As $\dim R(\mathcal{K}) \leqq m < \infty$, this implies that $\mathcal{K}$ is even compact. Since $\mathcal{Q} \in L(BV_n)$ is also compact (cf. 2.2), it is easy to see that $\mathcal{T}$ is compact.

Our wish is to establish the duality theory for the system (I), (II). Since BV_n and BV_n^+ form a dual pair with respect to the bilinear form (1,4), $BV_n \times R_m$ and $BV_n^+ \times R_m^*$ form a dual pair with respect to the bilinear form

(3,4)
$$\begin{pmatrix} \mathbf{x} \\ \mathbf{d} \end{pmatrix} \in BV_n \times R_m\,, \quad (\mathbf{z}^*, \boldsymbol{\lambda}^*) \in BV_n^+ \times R_m^* \to$$

$$\to \left\langle \begin{pmatrix} \mathbf{x} \\ \mathbf{d} \end{pmatrix}, (\mathbf{z}^*, \boldsymbol{\lambda}^*) \right\rangle = \int_0^1 \mathrm{d}[\mathbf{z}^*(t)]\, \mathbf{x}(t) + \boldsymbol{\lambda}^* \mathbf{d} \in R\,.$$

(Let us recall that the elements of BV_n^+ are row n-vector valued functions of bounded variation on $[0, 1]$, right-continuous on $(0. 1)$ and vanishing at 1.)

Let us put

$$(\mathcal{Q}'\mathbf{z})(s) = \int_0^1 \mathrm{d}[\mathbf{z}^*(t)]\,\mathbf{Q}(t, s) \quad \text{for} \quad \mathbf{z} \in BV_n \quad \text{and} \quad s \in [0, 1]\,.$$

By virtue of the assumptions 3.1 on $\mathbf{P}(t, s)$ and the definition (2,5) of $\mathbf{Q}(t, s)$ we can easily verify that $\mathbf{Q}$ is an SBV-kernel, $\mathbf{Q}(t, \cdot)$ is right-continuous on $(0, 1)$ and $\mathbf{Q}(t, 1) = \mathbf{0}$ for every $t \in [0, 1]$. Consequently (cf. [8] Lemma 3.1)

$$\mathcal{Q}'\mathbf{z} \in BV_n^+ \quad \text{for any} \quad \mathbf{z} \in BV_n\,. \tag{3,5}$$

Moreover, by Lemma 2.2 of [8] we have

$$\int_0^1 \mathrm{d}[\mathbf{z}^*(t)] \int_0^1 \mathrm{d}_s[\mathbf{Q}(t, s)]\,\mathbf{x}(s) = \int_0^1 \mathrm{d}_s\left[\int_0^1 \mathrm{d}[\mathbf{z}^*(t)]\,\mathbf{Q}(t, s)\right]\mathbf{x}(s) \tag{3,6}$$

for every $\mathbf{z} \in BV_n^+$ and $\mathbf{x} \in BV_n$.

Let us put

$$\mathcal{T}^+ : (\mathbf{z}^*, \boldsymbol{\lambda}^*) \in BV_n^+ \times R_m^* \to \tag{3,7}$$
$$\to \left(\int_0^1 \mathrm{d}[\mathbf{z}^*(t)]\,\mathbf{Q}(t, s) - \boldsymbol{\lambda}^*\,\mathbf{K}(s), \boldsymbol{\lambda}^*\right).$$

Since

$$\mathcal{T}^+(\mathbf{z}^*, \boldsymbol{\lambda}^*)(s) = ((\mathcal{Q}'\mathbf{z})(s) - \boldsymbol{\lambda}^*\,\mathbf{K}(s), \boldsymbol{\lambda}^*) \quad \text{on} \quad [0, 1]\,,$$

for each $\mathbf{z} \in BV_n$ and $\boldsymbol{\lambda} \in R_m$, it follows from 3.1 and (3,5) that

$$\mathcal{T}^+(\mathbf{z}^*, \boldsymbol{\lambda}^*) \in BV_n^+ \quad \text{for all} \quad (\mathbf{z}^*, \boldsymbol{\lambda}^*) \in BV_n^+ \times R_m^*\,. \tag{3,8}$$

Besides, we have by (3,2), (3,4) and (3,6)

$$\left\langle (\mathcal{I} - \mathcal{T})\begin{pmatrix}\mathbf{x}\\ \mathbf{d}\end{pmatrix}, (\mathbf{z}^*, \boldsymbol{\lambda}^*)\right\rangle = \tag{3,9}$$

$$= \int_0^1 \mathrm{d}_s\left[\mathbf{z}^*(s) - \int_0^1 \mathrm{d}[\mathbf{z}^*(t)]\,\mathbf{Q}(t, s) + \boldsymbol{\lambda}^*\,\mathbf{K}(s)\right]\mathbf{x}(s) + (\boldsymbol{\lambda}^* - \boldsymbol{\lambda}^*)\,\mathbf{d} =$$

$$= \left\langle \begin{pmatrix}\mathbf{x}\\ \mathbf{d}\end{pmatrix}, (\mathcal{I} - \mathcal{T}^+)(\mathbf{z}^*, \boldsymbol{\lambda}^*)\right\rangle \quad \text{for all} \quad \mathbf{x} \in BV_n\,, \quad \mathbf{d} \in R_m\,, \quad \mathbf{z} \in BV_n \quad \text{and} \quad \boldsymbol{\lambda} \in R_m$$

As mentioned above, $\mathbf{Q}(t, s)$ is an SBV-kernel and hence using Theorem 3.2 of [8] it is easy to show that $\mathcal{T}^+ \in L(BV_n^+ \times R_m^*)$ is compact. The operator $\mathcal{T}$ being compact by 3.4, we have

$$\operatorname{ind}(\mathcal{I} - \mathcal{T}) = \operatorname{ind}(\mathcal{I} - \mathcal{T}^+) = 0 \tag{3,10}$$

and we may apply Heuser's Theorem.

3.5. Theorem. *If the assumptions* 3.1 *are satisfied, then the system* (I), (II) *has a solution* $\mathbf{x} \in BV_n$ *iff*

$$\int_0^1 \mathrm{d}[\mathbf{z}^*(s)]\, \mathbf{f}(s) + \boldsymbol{\lambda}^* \mathbf{r} = 0 \tag{3,11}$$

for any $\mathbf{z}^* \in BV_n^+$ *and* $\boldsymbol{\lambda}^* \in R_m^*$ *fulfilling*

$$\mathbf{z}^*(s) - \int_0^1 \mathrm{d}[\mathbf{z}^*(t)]\, \mathbf{P}(t, s) + \boldsymbol{\lambda}^*\, \mathbf{K}(s) = \mathbf{0} \quad on \quad [0, 1]\,, \quad \mathbf{z}^*(0) = \mathbf{0}\,. \tag{3,12}$$

Proof. By (3,8)–(3,10) the operators $\mathscr{I} - \mathscr{T}$ and $\mathscr{I} - \mathscr{T}^+$ fulfil the assumptions of Heuser's Theorem. Consequently the system (I), (II) has a solution in BV_n iff

$$\int_0^1 \mathrm{d}[\mathbf{z}^*(s)]\, (\mathbf{f}(s) - \mathbf{f}(0)) + \boldsymbol{\lambda}^* \mathbf{r} = 0$$

holds for any $\mathbf{z}^* \in BV_n^+$ and $\boldsymbol{\lambda}^* \in R_m^*$ fulfilling the equation

$$\mathbf{z}^*(s) - \int_0^1 \mathrm{d}[\mathbf{z}^*(t)]\, \mathbf{Q}(t, s) + \boldsymbol{\lambda}^*\, \mathbf{K}(s) = \mathbf{0} \quad \text{on} \quad [0, 1]\,, \tag{3,13}$$

i.e. $(\mathscr{I} - \mathscr{T}^+)(\mathbf{z}^*, \boldsymbol{\lambda}^*) = \mathbf{0}$. Given $\mathbf{z}^* \in BV_n^+$, it is

$$\int_0^1 \mathrm{d}[\mathbf{z}^*(t)]\, \mathbf{Q}(t, s) = \int_0^1 \mathrm{d}[\mathbf{z}^*(t)]\, \mathbf{P}(t, s) - \begin{Bmatrix} \mathbf{z}^*(1) - \mathbf{z}^*(0) & \text{if} \quad s = 0 \\ \mathbf{0} & \text{if} \quad s > 0 \end{Bmatrix}. \tag{3,14}$$

Setting (3,14) into (3,13) we obtain

$$\begin{gathered} \mathbf{z}^*(s) - \int_0^1 \mathrm{d}[\mathbf{z}^*(t)]\, \mathbf{P}(t, s) + \boldsymbol{\lambda}^*\, \mathbf{K}(s) = \mathbf{0} \quad \text{on} \quad (0, 1]\,, \\ - \int_0^1 \mathrm{d}[\mathbf{z}^*(t)]\, \mathbf{P}(t, 0) + \boldsymbol{\lambda}^*\, \mathbf{K}(0) = \mathbf{0}\,. \end{gathered} \tag{3,15}$$

Since we assume $\mathbf{P}(0, s) = \mathbf{0}$ on $[0, 1]$, the value of each of the integrals

$$\int_0^1 \mathrm{d}[\mathbf{z}^*(t)]\, \mathbf{P}(t, s)\,, \quad s \in [0, 1]\,,$$

does not depend on the value $\mathbf{z}^*(0)$. The same holds obviously also for the integral

$$\int_0^1 \mathrm{d}[\mathbf{z}^*(t)]\, (\mathbf{f}(t) - \mathbf{f}(0))\,.$$

Consequently $(\mathbf{z}^*, \boldsymbol{\lambda}^*) \in BV_n^+ \times R_m^*$ is a solution to (3,12) iff $(\mathbf{z}_0^*, \boldsymbol{\lambda}^*)$ with $\mathbf{z}_0^*(s) = \mathbf{z}^*(s)$ on $(0, 1]$ and $\mathbf{z}_0^*(0) = \mathbf{0}$ is also its solution. Since

$$\int_0^1 \mathrm{d}[\mathbf{z}_0^*(t)]\, \mathbf{f}(0) = 0$$

for all such $\mathbf{z}_0^*$ and all $\mathbf{f} \in BV_n$, the proof is complete.

3.6. Remark. $\mathbf{P}(t, \cdot)$ and $\mathbf{K}$ being by 3.1 (ii) right-continuous on $(0, 1)$ for any $t \in [0, 1]$, $\mathbf{z}$ is also right-continuous on $(0, 1)$ for any couple $(\mathbf{z}, \boldsymbol{\lambda}) \in BV_n \times R_m$ fulfilling (3,12).

The following assertion is also a consequence of (3,8)–(3,10) and of Heuser's Theorem.

3.7. Proposition. *Let* 3.1 *hold and let* $\mathbf{h}^* \in BV_n^+$. *Then there exist* $\mathbf{z}^* \in BV_n^+$ *and* $\boldsymbol{\lambda}^* \in R_m^*$ *such that*

$$(3,16) \qquad \mathbf{z}^*(s) - \int_0^1 \mathrm{d}[\mathbf{z}^*(t)]\, \mathbf{Q}(t, s) + \boldsymbol{\lambda}^*\, \mathbf{K}(s) = \mathbf{h}^*(s) \quad \text{on} \quad [0, 1]$$

$((\mathscr{I} - \mathscr{T}^+)(\mathbf{z}^*, \boldsymbol{\lambda}^*) = (\mathbf{h}^*, \mathbf{0}))$ iff

$$\int_0^1 \mathrm{d}[\mathbf{h}^*(t)]\, \mathbf{x}(t) = 0$$

for every $\mathbf{x} \in N(\mathscr{L})$ where

$$(3,17) \qquad \mathscr{L} : \mathbf{x} \in BV_n \to \begin{pmatrix} \mathbf{x} - \mathscr{Q}\mathbf{x} \\ \mathscr{K}\mathbf{x} \end{pmatrix} \in BV_n \times R_m$$

(cf. (2,4) and (3,1)).

3.8. Theorem. *Assume* 3.1. *Then* $k = \dim N(\mathscr{L}) < \infty$ *for the operator* $\mathscr{L} \in L(BV_n, BV_n \times R_m)$ *given by* (3,17). *Furthermore, the system* (3,12) *has exactly* $k^+ = k + m - n$ *linearly independent solutions in* $BV_n^+ \times R_m$.

Proof. By 2.3 we have $k = \dim N(\mathscr{L}) < \infty$. Obviously $\dim N(\mathscr{I} - \mathscr{T}) = k + m$. Since (3,10), it is by Heuser's Theorem $\dim N(\mathscr{I} - \mathscr{T}^+) = \dim N(\mathscr{I} - \mathscr{T}) = k + m$. The set N^+ of all solutions to (3,12) consists of all $(\mathbf{z}^*, \boldsymbol{\lambda}^*) \in N(\mathscr{I} - \mathscr{T}^+)$ for which $\mathbf{z}^*(0) = \mathbf{0}$. Hence $k^+ = \dim N^+ = \dim N(\mathscr{I} - \mathscr{T}^+) - n = k + m - n$. Recall that for $(\mathbf{z}^*, \boldsymbol{\lambda}^*) \in N(\mathscr{I} - \mathscr{T}^+)$ the value $\mathbf{z}^*(0)$ may be arbitrary.

4. GREEN'S FUNCTION

We continue the investigation of the system (I), (II). In addition to 3.1 we shall suppose that it possesses a unique solution in BV_n for every $\mathbf{f} \in BV_n$ and $\mathbf{r} \in R_m$. Obviously, this is possible only if the corresponding homogeneous equation

$$\mathscr{L}\mathbf{x} = \mathbf{0} \in BV_n \times R_m$$

(cf. (3,17)) possesses only the trivial solution, i.e. $\dim N(\mathscr{L}) = 0$. On the other hand, by 3.5 the system (I), (II) has a solution in BV_n for any couple $\begin{pmatrix} \mathbf{f} \\ \mathbf{r} \end{pmatrix} \in BV_n \times R_m$ iff the system (3,12) possesses in $BV_n^+ \times R_m$ only the trivial solution. The following assertion is now a direct consequence of 3.8.

4.1. Proposition. *Provided* 3.1 *holds, the system* (I), (II) *has a unique solution in* BV_n *for every* $\mathbf{f} \in BV_n$ *and* $\mathbf{r} \in R_m$ *iff*

$$m = n \quad and \quad \dim N(\mathscr{L}) = 0 . \tag{4,1}$$

Let us suppose that (4,1) holds and let us try to express the solutions $\mathbf{x} \in BV_n$ of the systems (I), (II) in the form

$$\mathbf{x}(t) = \mathbf{H}(t)\,\mathbf{r} + \int_0^1 \mathrm{d}_s[\mathbf{G}(t, s)]\,(\mathbf{f}(s) - \mathbf{f}(0)) , \quad t \in [0, 1] , \tag{4,2}$$

where $\mathbf{H} : [0, 1] \to L(R_n)$ and $\mathbf{G} : [0, 1] \times [0, 1] \to L(R_n)$ are such that

(4,3) for every $t \in [0, 1]$, the functions $\mathbf{G}(t, \cdot)$ and $\mathbf{H}$ are of bounded variation on $[0, 1]$, right-continuous on $(0, 1)$ and $\mathbf{G}(t, 1) = \mathbf{0}$, $\mathbf{H}(1) = \mathbf{0}$.

Clearly, the function (4,2) is for any $\mathbf{f} \in BV_n$ and $\mathbf{r} \in R_m$ a solution of (I), (II) iff

$$\varphi(t) = \mathbf{H}(t) \int_0^1 \mathrm{d}[\mathbf{K}(s)]\,\varphi(s) + \int_0^1 \mathrm{d}_s[\mathbf{G}(t, s)] \left(\varphi(s) - \int_0^1 \mathrm{d}_\sigma[\mathbf{Q}(s, \sigma)]\,\varphi(\sigma) \right)$$

for every $\varphi \in BV_n$ and $t \in [0, 1]$, i.e. iff

$$\int_0^1 \mathrm{d}_s \left[\mathbf{G}(t, s) - \int_0^1 \mathrm{d}_\sigma[\mathbf{G}(t, \sigma)]\,\mathbf{Q}(\sigma, s) + \mathbf{H}(t)\,\mathbf{K}(s) - \Delta(t, s) \right] \varphi(s) = \mathbf{0} \tag{4,4}$$

$$\text{for each} \quad \varphi \in BV_n \quad \text{and} \quad t \in [0, 1] ,$$

where

$$\Delta(t, s) = \begin{cases} -\mathbf{I} & \text{for} \quad 0 < t < 1 \quad \text{and} \quad 0 < s < t , \\ \mathbf{0} & \text{for} \quad 0 < t < 1 \quad \text{and} \quad t \leqq s \leqq 1 , \\ -\mathbf{I} & \text{for} \quad 0 = t \quad \text{and} \quad 0 = s \\ \mathbf{0} & \text{for} \quad 0 = t \quad \text{and} \quad 0 < s \leqq 1 , \\ -\mathbf{I} & \text{for} \quad t = 1 \quad \text{and} \quad 0 \leqq s < 1 , \\ \mathbf{0} & \text{for} \quad t = 1 \quad \text{and} \quad s = 1 . \end{cases} \tag{4,5}$$

Provided (4,3) holds, (4,4) holds iff

$$\mathbf{G}(t, s) - \int_0^1 \mathrm{d}_\sigma[\mathbf{G}(t, \sigma)]\,\mathbf{Q}(\sigma, s) + \mathbf{H}(t)\,\mathbf{K}(s) = \Delta(t ,s) \quad \text{on} \quad [0, 1] \times [0, 1] . \tag{4,6}$$

Our wish now is to find functions $\mathbf{G}(t, s)$ and $\mathbf{H}(t)$ fulfilling (4,3) and (4,6).

Let us put

$$\mathscr{L}^+ : (\mathbf{z}^*, \lambda^*) \in BV_n^+ \times R_n^* \to \left(\mathbf{z}^*(s) - \int_0^1 \mathrm{d}[\mathbf{z}^*(t)]\,\mathbf{Q}(t, s) + \right. \tag{4,7}$$

$$\left. + \lambda^*\,\mathbf{K}(s), \mathbf{z}^*(0)\right) \in BV_n^+ \times R_n^* .$$

Since $\dim N(\mathscr{L}) = 0$, 3.7 implies that the equation (3.16) has a solution in $BV_n^+ \times R_n^*$ for any $\mathbf{h}^* \in BV_n^+$. Under our assumptions the expressions

$$\mathbf{z}^*(s) - \int_0^1 \mathrm{d}[\mathbf{z}^*(t)]\, \mathbf{Q}(t, s) + \boldsymbol{\lambda}^*\, \mathbf{K}(s)\; (\mathbf{z}^* \in BV_n^+,\ \boldsymbol{\lambda}^* \in R_n^*)$$

do not depend on the values $\mathbf{z}^*(0)$ (cf. the proof of 3.5). This means that the system

$$\mathbf{z}^*(s) - \int_0^1 \mathrm{d}[\mathbf{z}^*(t)]\, \mathbf{Q}(t, s) + \boldsymbol{\lambda}^*\, \mathbf{K}(s) = \mathbf{h}^*(s)\,, \quad s \in [0, 1]\,,$$

$$\mathbf{z}^*(0) = \boldsymbol{\delta}^*$$

has a solution $(\mathbf{z}^*, \boldsymbol{\lambda}^*) \in BV_n^+ \times R_n^*$ for every $(\mathbf{h}^*, \boldsymbol{\delta}^*) \in BV_n^+ \times R_n^*$, i.e.

$$R(\mathscr{L}^+) = BV_n^+ \times R_n^*\,. \tag{4,8}$$

Moreover, since the equations $\mathscr{L}^+(\mathbf{z}^*, \boldsymbol{\lambda}^*) = \mathbf{0} \in BV_n^+ \times R_n^*$ and (3,12) coincide, we have

$$\dim N(\mathscr{L}^+) = 0\,. \tag{4,9}$$

Taking into account (4,8) and (4,9) we conclude from the Bounded Inverse Theorem ([6] III.4.1) that the operator $\mathscr{L}^+$ possesses a bounded inverse $(\mathscr{L}^+)^{-1} \in L(BV_n^+ \times R_n^*)$. In particular, given a column $\Delta_i^*(t, \cdot) \in BV_n^+$ $(i = 1, 2, \ldots, n)$ of $\Delta(t, \cdot)$, there exists a unique couple $(\mathbf{g}_i^*(t, \cdot), \mathbf{h}_i^*(t)) \in BV_n^+ \times R_n^*$ such that

$$\mathbf{g}_i^*(t, s) - \int_0^1 \mathrm{d}_\sigma[\mathbf{g}_i^*(t, \sigma)]\, \mathbf{Q}(\sigma, s) + \mathbf{h}_i^*(t)\, \mathbf{K}(s) = \Delta_i^*(t, s) \quad \text{on} \quad [0, 1] \times [0, 1]\,,$$

$$\mathbf{g}_i^*(t, 0) = \mathbf{0} \quad \text{on} \quad [0, 1]\,, \qquad i = 1, 2, \ldots, n\,.$$

Moreover, there is $M < \infty$ such that

$$\|\mathbf{g}_i^*(t, \cdot)\|_{BV} + |\mathbf{h}_i^*(t)| \leqq M \|\Delta_i^*(t, \cdot)\|_{BV} \leqq M \quad \text{for any} \quad t \in [0, 1] \quad \text{and} \quad i = 1, 2, \ldots, n\,. \tag{4,10}$$

This completes the proof of the following

4.2. Theorem. *If 3.1 and (4,1) hold, then there exist functions*

$$\mathbf{G} : [0, 1] \times [0, 1] \to L(R_n)\,, \quad \mathbf{H} : [0, 1] \to L(R_n)$$

and a constant $\varkappa < \infty$ such that

$$\|\mathbf{G}(t, \cdot)\|_{BV} + |\mathbf{H}(t)| \leqq \varkappa \quad \text{on} \quad [0, 1]\,,$$

$\mathbf{G}(t, \cdot)$ is for any $t \in [0, 1]$ right-continuous on $(0, 1)$, $\mathbf{G}(t, 0) = \mathbf{G}(t, 1) = \mathbf{0}$ on $[0, 1]$ and (4,6) is satisfied.

4.3. Theorem. *Assume* 3.1 *and* (4,1). *Given* $\mathbf{f} \in BV_n$ *and* $\mathbf{r} \in R_n$, *the unique solution* $\mathbf{x}$ *of the system* (I), (II) *in* BV_n *is given by* (4,2) *where* $\mathbf{G}(t, s)$ *and* $\mathbf{H}(t)$ *are defined by* 4.2.

5. GENERALIZED LINEAR DIFFERENTIAL EQUATIONS

Of a special interest is the case

$$(5,1) \qquad \mathbf{P}(t, s) = \begin{cases} \mathbf{A}(0) \quad - \mathbf{A}(t) & \text{for} \quad 0 = s < t \leqq 1\,, \\ \mathbf{A}(s+) - \mathbf{A}(t) & \text{for} \quad 0 < s < t \leqq 1\,, \\ \mathbf{0} & \text{for} \quad 0 \leqq t \leqq s \leqq 1\,, \end{cases}$$

where $\mathbf{A} : [0, 1] \to L(R_n)$ is of bounded variation on $[0, 1]$. The integral equation (I) then reduces to the generalized linear differential equation

$$(5,2) \qquad \mathbf{x}(t) - \mathbf{x}(0) - \int_0^t \mathrm{d}[\mathbf{A}(s)]\, \mathbf{x}(s) = \mathbf{f}(t) - \mathbf{f}(0)\,, \quad t \in [0, 1]\,.$$

It is easy to verify that for any $\mathbf{A} : [0, 1] \to L(R_n)$ of bounded variation on $[0, 1]$, the function $\mathbf{P} : [0, 1] \times [0, 1] \to L(R_n)$ defined by (5,1) fulfils all the corresponding assumptions from 3.1. Moreover, for any $\mathbf{z}^* \in BV_n^+$ with $\mathbf{z}^*(0) = \mathbf{z}^*(1) = \mathbf{0}$ we have

$$\int_0^1 \mathrm{d}[\mathbf{z}^*(t)]\, \mathbf{P}(t, s) = \begin{cases} - \int_0^1 \mathrm{d}[\mathbf{z}^*(t)]\, \mathbf{A}(t) & , \quad \text{if} \quad s = 0\,, \\ - \int_s^1 \mathrm{d}[\mathbf{z}^*(t)]\, \mathbf{A}(t) - \mathbf{z}^*(s)\, \mathbf{A}(s+)\,, & \text{if} \quad 0 < s < 1\,, \\ \mathbf{0} & , \quad \text{if} \quad s = 1\,. \end{cases}$$

Thus the adjoint system (3.12) to (I), (II) reduces in the case (5,1) to the system for $(\mathbf{z}^*, \boldsymbol{\lambda}^*) \in BV_n^+ \times R_m^*$

$$(5,3) \qquad \mathbf{z}^*(s) + \int_s^1 \mathrm{d}[\mathbf{z}^*(t)]\, \mathbf{A}(t) + \mathbf{z}^*(s)\, \mathbf{A}(s+) + \boldsymbol{\lambda}^*\, \mathbf{K}(s) = \mathbf{0} \quad \text{on} \quad [0, 1]\,,$$
$$\mathbf{z}^*(0) = \mathbf{z}^*(1) = \mathbf{0}\,.$$

Furthermore, in the previous section we have proved the existence of Green's function for the boundary value problem (5,2), (II) if $m = n$ and $\dim N(\mathscr{L}) = 0$ for the corresponding operator $\mathscr{L} : BV_n \to BV_n \times R_n$.

The equation (5,3) resembles the generalized linear differential equation (5,2). However, in general its basic theory is not available directly from the basic theory of equations of the form (5,2). In [9] the problems (5,2), (II) are dealt with in detail. As a proper adjoint the system of equations for $(\mathbf{z}^*, \boldsymbol{\lambda}^*) \in BV_n^+ \times R_m^*$

$$(5,4) \qquad \mathbf{z}^*(s) + \int_s^1 \mathrm{d}[\mathbf{z}^*(t)]\, \mathbf{A}(t) + \mathbf{z}^*(s)\, \mathbf{A}(s) + \boldsymbol{\lambda}^*\, \mathbf{K}(s) = \mathbf{0} \quad \text{on} \quad [0, 1]\,,$$
$$\mathbf{z}^*(0) = \mathbf{z}^*(1) = \mathbf{0}$$

is derived provided $\det(I - \Delta^{-}A(t)) \neq 0$ for $t \in (0, 1]$ and $\det(I + \Delta^{+}A(t)) \neq 0$ for $t \in [0, 1)$. Under these assumptions also usual basic results for the equation (5,4) (as the existence and uniqueness of a solution, fundamental matrix, variation-of-constants formula) have been derived. In the same paper it was shown that the systems (5,3) and (5,4) are equivalent in a certain sense.

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