Ramsey theorems for product of finite sets with submeasures*

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Abstract

We prove parametrized partition theorem on products of finite sets equipped with submeasures, improving the results of DiPrisco, Llopis, and Todorcevic.

1 Introduction

The polarized partition theorems have a long history. The behavior of finite products of finite sets is governed by the positive answer to the Zarankiewicz problem:

Fact 1.1. *[6, Theorem 5, Section 5.1] For every number $k \in \omega$, every $m \in \omega$ and every sequence $r_n : n \in m$ of natural numbers there is a sequence of finite sets $a_n : n \in m$ such that for every partition of the product $\Pi_n a_n$ into k many pieces, one of the pieces contains a product $\Pi_n b_n$, where $b_n \subset a_n$ are sets of respective cardinality at least r_n.*

It is not difficult to provide a precise formula for the necessary size of the sets a_n. The infinite version of this theorem holds as well.

Fact 1.2. *[1] For every number $k \in \omega$ and every sequence $r_n : n \in \omega$ of natural numbers there is a sequence $a_n : n \in \omega$ of finite sets such that for every partition of the product $\Pi_n a_n$ into k many Borel pieces, one of the pieces contains a product of the form $\Pi_n b_n$ where $b_n \subset a_n$ are sets of respective size at least r_n.*

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Here, the space $\Pi_n a_n$ is equipped with the product topology of the discrete topologies on the finite sets a_n. The computation of the sequence of needed sizes of the finite sets $a_n : n \in \omega$ turned out to be more complicated, and the first non-primitive-recursive estimate appeared in [2]. One can parametrize this theorem with one more infinite dimension:

Fact 1.3. *[3] Suppose k is a number and $r_n : n \in \omega$ are natural numbers. Then there is a sequence $a_n : n \in \omega$ of finite sets such that for every partition of the product $\Pi_n a_n \times \omega$ into k many Borel pieces, one of the pieces contains a subset of the form $\Pi_n b_n \times c$ where $b_n \subset a_n$ are sets of size at least r_n, and $c \subset \omega$ is an infinite set.*

The difficult proof contains a reference to the Galvin-Prikry partition theorem [5], and it provides no estimate for the growth of the sequence necessary for the partition property to hold. We will greatly improve on these efforts. Our theorems are more general, they offer many more applications, and the argument yields direct primitive recursive computations. The arguments differ greatly from those of [3]; they employ the powerful method of *creature forcing* of [12] which promises many more applications to Ramsey theory in the future.

One can attempt to measure the size of homogeneous products not in terms of the cardinality of the finite sets in the product, but in terms of a different measure or submeasure. The arguments of [3] do not work in such a case. However, we can provide a nearly complete information.

Theorem 1.4. *Suppose $k \in \omega$ is a number and $r_n : n \in \omega$ is a sequence of real numbers. Then for every sequence of submeasures $\phi_n : n \in \omega$ on finite sets, increasing fast enough, and for every partition $B_i : i \in k$ of the product $\Pi_n \mathrm{dom}(\phi_n) \times \omega$ into Borel pieces, one of the pieces contains a product of the form $\Pi_n b_n \times c$ where $c \subset \omega$ is an infinite set, and $b_n \subset \mathrm{dom}(\phi_n)$ and $\phi_n(b_n) > r_n$ for every number $n \in \omega$.*

Here, the phrase "for every fast enough increasing sequence of submeasures" means that Player I has a winning strategy in the infinite game in which he indicates real numbers $s_n : n \in \omega$, to which Player II responds with submeasures ϕ_n on finite sets such that $\phi_n(\mathrm{dom}(\phi_n)) \geq s_n$. Player I wins if for the resulting sequence of submeasures, the partition property holds. It will be clear from the proof that a rate of growth corresponding to a stack of exponentials of linear height is sufficient for the partition property to hold. The proof also shows that a number of other effects can be achieved. For example, if $f : \Pi_n \mathrm{dom}(\phi_n) \to 2^\omega$ is a Borel function, then the sets $b_n : n \in \omega$ can be found such that $g \restriction \Pi_n b_n$ is continuous.

This fairly general theorem allows for several variations. One of them deals with the size of the homogeneous set in the infinite coordinate. An abstract argument based on Theorem 1.4 will show

Theorem 1.5. *Suppose $k \in \omega$ is a number, K a F_σ-ideal on ω, and $r_n : n \in \omega$ is a sequence of real numbers. Then for every sequence of submeasures $\phi_n : n \in \omega$ on finite sets, increasing fast enough, and for every partition $B_i : i \in k$ of the* *product* $\Pi_n \mathrm{dom}(\phi_n) \times \omega$ *into Borel pieces, one of the pieces contains a product of the form* $\Pi_n b_n \times c$ *where* $c \subset \omega$ *is a* K*-positive set, and* $b_n \subset \mathrm{dom}(\phi_n)$ *and* $\phi_n(b_n) > r_n$ *for every number* $n \in \omega$*.*

Another possible variation arises from adding another axis to the partitions. We will state and prove a measure parametrized version:

Theorem 1.6. *Suppose that* $\varepsilon > 0$ *is a real number and* $r_n : n \in \omega$ *is a sequence of real numbers. Then for every sequence of measures* $\phi_n : n \in \omega$ *on finite sets increasing fast enough, and every Borel set* $B \subset \Pi_n \mathrm{dom}(\phi_n) \times \omega \times [0, 1]$ *with vertical sections of Lebesgue mass at least* ε*, there are sets* $b_n \subset \mathrm{dom}(\phi_n)$ *with* $\phi_n(b_n) > r_n$*, an infinite set* $c \subset \omega$ *and a point* $z \in [0, 1]$ *such that* $\Pi_n b_n \times c \times \{z\} \subset B$*.*

For the second author, the stated theorems are really results about forcing, and their main applications also lie in the realm of forcing theory. They seem to be the strongest tool known to date for proving that various bounding forcings do not add independent reals. Here, a set $a \subset \omega$ in a generic extension is independent if neither it nor its complement contain a ground model infinite subset. We get

Corollary 1.7. *Suppose that* I_n *is a* σ*-ideal on a Polish space* X_n *generated by a compact family of compact sets, this for every number* $n \in \omega$*. The countable support product of posets* $P_{I_n} : n \in \omega$ *does not add an independent real.*

Here, the symbol P_I stands for the poset of I-positive Borel sets ordered by inclusion. The partial orders of the form described in the corollary have been studied in [13, Theorem 4.1.8]; they include for example the Sacks forcing, or all the tree limsup infinity forcings of [12]. Thus the corollary can be understood as a far-reaching generalization of Laver's theorem on independent reals and product of Sacks reals [9].

Corollary 1.8. *The Halpern-Läuchli forcing, the* E_0 *and* E_2 *forcings do not add independent reals.*

The notation in this paper follows the set theoretic standard of [7]. An *atom* of a partial order is an element with no elements below it. An independent real over a transitive model of set theory is a set $a \subset \omega$ such that neither it nor its complement contain an infinite subset from the model. All logarithms in this paper are evaluated with base 2. Theorems 1.3 and 1.4 can be stated in a stronger form: with an axis $[\omega]^{\aleph_0}$ and homogeneous combinatorial cubes $[c]^{\aleph_0}$ instead of the infinite axis and a homogeneous infinite set c. However, no such reasonable stronger form exists for Theorems 1.5 and 1.6.

2 The creature forcing

In order to prove theorems from the introduction, we need to consider a forcing from the family of creature forcings introduced in [12]. The general approach of that book may seem daunting to many readers; our special case is fairly simple and still quite useful.

Definition 2.1. Let a be a nonempty finite set. A *setup* on a is an atomic partially ordered set C, with a =the set of atoms of C, and an order-preserving function $\mathtt{nor} : C \to \mathbb{R}$ which is constantly zero on the set a.

In the nomenclature of Rosłanowski-Shelah, the nonatomic elements of a setup are called *creatures*. The set of atoms below a given creature $c \in C$ is a set of its *possibilities*, $\mathtt{pos}(c)$.

Definition 2.2. Let a_n be pairwise disjoint finite sets, and $C_n, \mathtt{nor}_n$ a setup on each. The forcing P consists of all functions p with domain ω such that $\forall n\ p(n) \in C_n$ and the numbers $\mathtt{nor}_n(p(n))$ tend to infinity. The ordering is that of coordinatewise strengthening.

The partial order P will add a function $\dot{x}_{gen} \in \Pi_n a_n$ defined as the unique function in the product which is coordinatewise below every condition in the generic filter. In the specific cases discussed in this paper, the whole generic filter can be reconstructed from this function. Note that partitioning ω into finitely many disjoint infinite sets one can present P as a product of finitely many similar forcings; this feature makes P a natural tool for the investigation of product forcing. The forcing P is not separative. If $p, q \in P$ are two conditions such that for every $n \in \omega$, $\mathtt{pos}(q(n)) \subset \mathtt{pos}(p(n))$, and for all but finitely many $n \in \omega$ $q(n) \leq p(n)$, then there is no strengthening of q incompatible with p, even though $q \leq p$ may fail. This feature appears to be essential, and it will be exploited in several places.

The forcing properties of P depend on subtle combinatorial properties of the setups. We will need the following notions.

Definition 2.3. Let $\varepsilon > 0$ be a real number. The setup C has ε-*bigness* if for every $c \in C$ and every partition of the set a into two parts, there is $d \leq c$ with $\mathtt{nor}(d) > \mathtt{nor}(c) - \varepsilon$ such that all atoms below d fall into the same piece of the partition.

The simplest example of a setup with ε-bigness arises from a submeasure ϕ on the set a. Define $C = \mathcal{P}(a)$, $\mathtt{nor}(b) = \varepsilon \log(1 + \phi(b))$ if $b \subset a$ is not a singleton, and $\mathtt{nor}$ of the singletons equal to zero. Another example starts with an arbitrary partially ordered set C with a finite set a of atoms such that every nonatomic element has at least two atoms below it. For every nonatomic $c \in C$ and $n \in \omega$ consider the game of length n in which Player I plays partitions of the set a into two parts and player II plays a descending chain of nonatomic creatures below c such that the atoms below i-th condition are all contained in the same set of i-th partition. Player II wins if he survives all rounds. Now let $\mathtt{nor}(c) = \varepsilon\cdot$the largest number n such that player II has a winning strategy in the game of length n below c, and norm of the atoms will be again zero.

The setups we will use will have to be a little more complicated, since they have to satisfy the following subtle condition.

Definition 2.4. Let $\varepsilon > 0$ be a real number. The setup C has ε-*halving* if for every $c \in C$ there is $d \leq c$ (so called *half of* c) such that $\mathtt{nor}(d) > \mathtt{nor}(c) - \varepsilon$ and for every nonatomic $d' \leq d$ there is $c' \leq c$ such that $\mathtt{nor}(c') > \mathtt{nor}(c) - \varepsilon$ and every atom below c' is also below d'.

This may sound mysterious, but in fact there is a mechanical procedure to adjust any setup to a setup with halving. Suppose C is a setup with a norm $\mathtt{nor}_C$. Let D be the partial order whose nonatomic elements are of the form $\langle c, r\rangle$, where $c \in C$ is not an atom and $\mathtt{nor}(c) \geq r$. The ordering is defined by $\langle d, s\rangle \leq_D \langle c, r\rangle$ if $d \leq_C c$ and $r \leq s$. The atoms of D are exactly the atoms of C, and if i is such an atom then $i \leq_D \langle c, r\rangle$ if and only if $i \leq_C c$. The norm on D is defined by $\mathtt{nor}_D(c, r) = \varepsilon \log(\mathtt{nor}_C(c) - r + 1)$, where ε is a real number; the norm of atoms is again zero. The adjusted setup D has ε-halving: the half of the creature $\langle c, r\rangle$ is the creature $\langle c, r + \frac{\mathtt{nor}_C(c)-r}{2}\rangle$. It is not difficult to see that if $\langle d, s\rangle$ is a creature below the half, the creature $\langle d, r\rangle$ has norm ε-close to $\langle c, r\rangle$ and the same set of possibilities as $\langle d, s\rangle$.

Another approach for building a norm function with ε-halving on a given partial order C uses a two player game of length n. In i-th round Player I produces nonatomic creatures $c_i \geq d_i$ and Player II responds with a nonatomic creature $e_i \leq d_i$. If Player II chooses $e_i = d_i$ then Player I must choose c_{i+1} smaller than d_i, and if $e_i < d_i$ then $\mathtt{pos}(c_{i+1})$ must be a subset of $\mathtt{pos}(e_i)$. Player I wins if he survives all rounds. One can then define $\mathtt{nor}(c) = \varepsilon\cdot$ the largest number n for which Player I has a winning strategy in the game of length n with the first move equal to c.

In spite of the grammar used in this paper, the half of a creature is not necessarily unique.

Definition 2.5. Let $\varepsilon > 0$ be a real number. The setup C has ε-*Fubini property* if for every creature $c \in C$ with $\mathtt{nor}(c) > 2$ and every Borel set $B \subset a \times [0, 1]$ with vertical sections of Lebesgue mass at least ε there is a creature $d \leq c$ such that $\mathtt{nor}(d) > \mathtt{nor}(c) - 1$ and a point $z \in [0, 1]$ such that $\mathtt{pos}(d) \times \{z\} \subset B$.

This is a property used for preservation of outer Lebesgue measure. One possible way to obtain a setup with the ε-Fubini property for a given real number $0 < \varepsilon < 1$ starts with a measure ϕ on a finite set a and defines $C = \mathcal{P}(a)$, with $\mathtt{nor}(b) = \frac{\log(\phi(a)+1)}{-\log\varepsilon}$ for a non-singleton set $b \subset a$.

The following proposition is the heart of this paper.

Proposition 2.6. *Let $a_n : n \in \omega$ be a collection of pairwise disjoint finite sets, with a setup $C_n, \mathtt{nor}_n$ on each, and let P be the resulting partial order.*

1. *Let $\varepsilon_n = 1/\Pi_{m\in n}|a_m|$. If every setup C_n has ε_n-halving and bigness, the forcing P is proper and bounding.*

2. *Let $\varepsilon_n = 1/\Pi_{m\in n}2^{|a_m|}$. If every setup C_n has ε_n-halving and bigness, the forcing P adds no independent reals.*

3. *Let* $\varepsilon_n = 1/\Pi_{m\in n} 2^{2^n |a_m|}$. *If every setup* C_n *has* ε_n*-halving, bigness and Fubini property, then the forcing* P *adds no* V*-independent sequence of sets of positive mass.*

The first item is just a rehash of [12]. The third item introduces a new forcing preservation property.

Definition 2.7. A *V-independent sequence of sets of positive mass* in the generic extension is a collection $D_i : i \in \omega$ of closed subsets of some Borel probability space with masses bounded away from zero, such that for no grown model infinite set $c \subset \omega$ and no ground model element z of the probability space it is the case that $z \in \bigcap_{i\in c} D_i$.

This is a property that implies adding no independent reals and preservation of outer Lebesgue measure. The estimates for $\varepsilon_n : n \in \omega$ in this proposition as well as other assumptions are almost certainly not the best possible.

The proofs are essentially just careful fusion arguments. We will need several pieces of notation and terminology. For a condition $p \in P$ let $[p] = \Pi_n \mathtt{pos}(p(n))$. If moreover $a \subset \omega$ is a finite set, then $[p] \restriction a = \Pi_{n\in a} \mathtt{pos}(p(n))$. For every sequence $t \in [p] \restriction a$, $p \restriction\restriction t$ is the condition $q \leq p$ defined by $q \restriction a = t$ and $\forall n \in \omega \setminus a\ q(n) = p(n)$.

We will proceed with a sequence of simple claims.

Claim 2.8. *(the halving trick) Suppose that* $D_i : i \in k$ *are open subsets of* P *invariant under the inseparability equivalence. Suppose that all setups have* $1/k$*-halving, and suppose that* $p \in P$ *is a condition on which all the norms are equal to at least* $r > 3$*. Then there is* $q \leq p$ *on which all the norms are at least* $r - 1$*, and for every* $i \in k$ *either* $q \in D_i$ *or there is no* $q' \leq q$ *with all norms nonzero and* $q' \leq q$*.*

Here, an open set is invariant under inseparability if, whenever p, q are conditions such that q has no extension incompatible with p, and $p \in D$, then $q \in D$. Note that if π is the natural map of P into the separative quotient of P, and D' is an open subset of the separative quotient, then $\pi^{-1}D$ is invariant under inseparability. Thus, in forcing we only need to care about the open sets that are invariant under inseparability.

Proof. By induction on $i \in k$ construct a sequence of conditions $p_i : i \leq k$ starting with $p_0 = p$ using the following rules.

- if there is a condition $q \leq p_i$ in D_i whose norms are at least $r - (i+1)/k$, then let p_{i+1} be such a condition;
- otherwise let p_{i+1} be the half of p_i; that is, for every $n \in \omega$ $p_{i+1}(n)$ is the half of $p_i(n)$.

In the end, the condition $q = p_k$ will satisfy the conclusion of the claim. To see that, pick $i \in k$. If the first case occurred at i, then $q \in D_i$ and we are done.

If the second case occurred, there is no $q' \leq q$ with all norms nonzero in the set D_i. Since if such a condition q' existed, we could find $m \in \omega$ such that $\forall n \geq m\ \mathtt{nor}_n(q'(n)) \geq r - i/k$, and use the properties of halving to find a condition $q'' \leq p_i$ such that $\forall n < m\ \mathtt{pos}(q''(n)) \subset \mathtt{pos}(q'(n))$, $\forall n < m\ \mathtt{nor}(q''(n) \geq r - i/k$, and $\forall n \geq m\ q''(n) = q'(n)$. Such a condition is inseparable from q', it therefore must be in D_i, and it contradicts the assumption that the first case failed at i. □

Claim 2.9. *(the bigness trick) Suppose that $O_i : i \in k$ are clopen sets covering the space $\Pi_n a_n$. Suppose that all setups have $1/k$ bigness, and suppose that $p \in P$ is a condition with all norms greater than 3. Then there is a condition $q \leq p$ in which all the norms decreased by at most one, and such that the set $[q]$ belongs to at most one piece of the partition.*

Proof. Let $m \in \omega$ be a number such that the membership of any point $x \in [p]$ in the given clopen sets depends only on $x \restriction m$. By downward induction on $i \in m$ construct a decreasing chain $p_i : i \leq m$ of conditions such that $p = p_m$

- $p_i = p_{i+1}$ at all entries except i and there the norm is decreased by at most one;
- the membership of $x \in [p_i]$ in the clopen sets depends only on $x \restriction i$.

This is easily done using the bigness property. In the end, $q = p_0$ is the requested condition. □

Now we need to introduce standard fusion terminology. Suppose that $p, q \in P$ and $r \in \mathbb{R}$. Say that $q \leq^r p$ if $q \leq p$ and for every $n \in \omega$ such that $\mathtt{nor}_n(p(n)) \leq r$ it is the case that $p(n) = q(n)$, and for all other $n \in \omega$ it is the case that $\mathtt{nor}_n(q(n) \geq r$. A fusion sequence is a sequence $p_i : i \in \omega$ such that for some numbers $r_i \in \mathbb{R}$ tending to infinity, $p_{i+1} \leq^{r_i} p_i$. It is immediate to verify that a fusion sequence in the poset P has a lower bound. Finally, a condition $p \in P$ is *almost contained* in a set D if there is a number $m \in \omega$ such that for every $t \in [p] \restriction m$, $p \restriction t \in D$.

Claim 2.10. *Suppose that $D \subset P$ is an open dense subset invariant under inseparability, $p \in P$, and $r \in \mathbb{R}$. Then there is $q \leq^r p$ such that q is almost contained in D.*

Proof. Fix D, p, and r and suppose that the claim fails. By induction on $i \in \omega$ construct conditions p_i and numbers m_i so that $p_0 = p$ and m_0 is such that $\forall n \geq m_0\ \mathtt{nor}_n(p(n)) \geq r + 1$ and for all $i \in \omega$,

- $p_{i+1} \restriction m_i = p_i \restriction m_i$;
- for all $m_i \leq n < m_{i+1}$, $\mathtt{nor}_n(p_i(n)) \geq r + i$ and for all $n \geq m_{i+1}$, $\mathtt{nor}_n(p_i(n)) > r + i + 2$;
- for all $t \in [p_{i+1}] \restriction [m_0, m_{i+1})$, no condition $q' \leq p_{i+1} \restriction t$ with $\forall n \geq m_{i+1}\ \mathtt{nor}_n(q'(n)) > 0$ is almost contained in D.

If this has been done, consider the condition q which is the natural limit of the sequence $p_i : i \in \omega$. The first and second item imply that indeed, q exists as an element of the forcing P. Find a condition $q' \leq q$ and a number $i \in \omega$ such that $\forall n \geq m_{i+1}\ \mathtt{nor}_n(q'(n)) > 0$ and $\forall t \in [p] \restriction m_0\ q' \restriction t \in D$. Then certainly the condition q' is almost contained in the set D and therefore contradicts the third item above.

In order to perform the induction, suppose that p_i, m_i have been defined. Find $m_{i+1} \in \omega$ such that $\forall n \geq m_{i+1}\ \mathtt{nor}_n(p_i(n)) \geq r + i + 1$. Use Claim 2.8 and halving to find a condition $p'_i \leq p_i$ so that $\forall n \leq m_i\ p'_i(n) = p_i(n)$ and $\forall n \geq m_{i+1}\ \mathtt{nor}_n(p'_i(n)) > r+i$ such that for every sequence $t \in [p'_n] \restriction [m_0, m_{i+1})$, either (1) $p'_i \restriction t$ is almost contained in D or else (2) there is no $q' \leq p'_i \restriction t$ such that $\forall n < m_0\ q'(n) = p'(n)$, $\forall n \geq m_{i+1}\ \mathtt{nor}_n(q'(n)) > 0$, and q' is almost contained in D. Use the bigness and Claim 2.9 to thin out the condition p'_i in the interval $[m_i, m_{i+1})$ to find a condition $p_{i+1} \leq p'_i$ such that $\forall n < m_i\ p_{i+1}(n) = p_i(n)$, $\forall m_i \leq n < n_{i+1}\ \mathtt{nor}_n(p_{i+1}(n) > r + i$, $\forall n \geq m_{i+1}\ p_{i+1}(n) = p'_i(n)$, and for every sequence $t \in p_{i+1} \restriction [m_0, m_{i+1})$, whether case (1) or (2) above takes place depends only on $t \restriction [m_0, m_i)$. Now, the induction hypothesis implies that for no such t case (1) can hold: the condition $p_{i+1} \restriction\restriction (t \restriction [m_0, m_i))$ would then violate the third item of the induction hypothesis at i. Reviewing the resulting situation, we see that the condition p_{i+1} and the number m_{i+1} have successfully been chosen in a way that makes the induction hypothesis hold at $i + 1$.

$\square$

The properness of the forcing P now immediately follows. Suppose that $p \in P$ is a condition and M is a countable elementary submodel. Let $D_i : i \in \omega$ be a list of all open dense subsets of the poset P in the model M. Construct a fusion sequence $p_i : i \in \omega$ of conditions in the model M such that p_i is almost contained in the set D_i. The fusion q will be a master condition for the model M stronger than P. Note that every element $x \in [q]$ defines an M-generic filter; namely, the filter of those conditions $p \in M$ such that there exists $n \in \omega$ such that the condition q with the first n coordinates replaced with the first n coordinates of x is below p.

The bounding property of the forcing is proved in exactly the same way. Note that if a condition almost belongs to an open dense set, then there is a finite subset of the dense set which is predense below the condition. Not adding splitting reals is more sophisticated. Also note the stronger requirement on the growth of the numbers $1/\varepsilon_n : n \in \omega$. Suppose that $p \in P$ is a condition and $\dot{x}$ a name for an infinite binary sequence. We need to find a condition $q \leq p$ deciding infinitely many values of the name $\dot{x}$.

Strengthening the condition p as in the previous paragraphs we may assume that for every number $i \in \omega$ the condition p is almost contained in the set of conditions deciding the value $\dot{x}(i)$. Now use Claim 2.8 repeatedly to build a fusion sequence $p_i : i \in \omega$ and numbers $m_i \in \omega$ in such a way that for every $i \in \omega$ and every sequence of sets $c_m : n_p \leq m < m_i$ if there is a condition $q \leq p_i$ with $\forall n < m_i\ \mathtt{pos}(q(n)) = c_m$, $\forall n \geq m_i\ \mathtt{nor}(q(n)) > 0$ and and q decides a value of $\dot{x}(j)$ for some $j > i$, then there is such a condition q with $\forall n > m_i\ q(n) = p_i(n)$.

Let $q \leq p$ be the fusion of this sequence. For every number $j \in \omega$, use Claim 2.9 to find a condition $q_j \leq q$ such that q_j decides the value of the bit $\dot{x}_j$, and $q_j(n) = q(n)$ for all but finitely many n, and $\mathtt{nor}(q_j(n)) \geq \mathtt{nor}(q(n))$ for all $n \in \omega$. Use a compactness argument to find a condition $r \leq q$ and an infinite set $a \subset \omega$ such that the sequences $\langle \mathtt{pos}(q_j(n)) : n \in \omega \rangle : j \in a$ converge in the natural topology to the sequence $\langle \mathtt{pos}(r(n)) : n \in \omega \rangle$. We claim that the condition r decides infinitely many values of the name $\dot{x}$.

To see this, let $i \in \omega$ be a number. Let $j \in a$ be a number such that $\mathtt{pos}(q_j(n)) = \mathtt{pos}(r(n))$ for all $n < m_i$. Consider the condition q_j. It witnesses that there is a condition $s \leq q$ deciding a value of the name $\dot{x}(j)$ such that $\forall n < m_i$ $\mathtt{pos}(r(n)) = \mathtt{pos}(s(n))$ and $\forall n$ $\mathtt{nor}(s(n)) > 0$. By the fusion construction, it must be the case that already q is such a condition, and therefore $r \leq q$ is such a condition!

The second item of the theorem can be improved to the following.

Claim 2.11. *Suppose that $p \in P$, $r \in \mathbb{R}$, $u \subset \omega$ is infinite, and $p \Vdash \dot{A} \subset \mathcal{P}(\omega)/\mathrm{Fin}$ is open dense. Then there is $q \leq^r p$ and an infinite set $v \subset u$ such that $q \Vdash \check{v} \in \dot{A}$.*

Proof. We will provide an abstract argument in the spirit of [13] which can be applied in many similar situations. There is also an alternative argument which proceeds through tightening the fusion process above.

Let Q be the quotient partial order of $\mathcal{P}(\omega)/\mathrm{Fin}$ below u. Consider the partial order $P \times Q$, with respective generic filters $G \subset P$, $H \subset Q$. The following is easy to check.

- In $V[H]$, H is a Ramsey ultrafilter on ω containing u;
- In $V[H]$, P is a proper bounding forcing adding no independent reals;
- $\mathbb{R} \cap V[G][H] = \mathbb{R} \cap V[G]$;
- H still generates a Ramsey ultrafilter in $V[G][H]$.

The first item is entirely standard. For the second item, repeat the proof of (1) and (2) of the theorem in the model $V[H]$. For the third item, note that by a properness argument, every real in $V[H][G]$ is obtained from a countable collection of countable sets predense below some condition in P which exists in the model $V[H]$. But countable subsets of P are the same in V as in $V[H]$, and therefore the real belongs to $V[G]$. For the last item, use mutual genericity and the no independent real property to show that H indeed generates an ultrafilter in $V[G][H]$. To check that this ultrafilter is selective, use the bounding property of the poset P to find, for every partition π of ω into finite sets in the model $V[G][H]$, a partition π' of ω into finite sets in $V[H]$ such that every set in π is contained in the union of two successive pieces of π'. use the selectivity of H in the model $V[H]$ to find a set $u \in H$ that meets every set in π' in at most one point. Either the set of even indexed numbers in u or the set of odd indexed numbers in u belongs to H, and it meets every set in π in at most one point.

Now note that every Ramsey ultrafilter meets every analytic open dense subset of $\mathcal{P}(\omega)/\mathrm{Fin}$ [13, Claim 4.3.4]. Working in $V[H]$, $p \Vdash$there is an element $v \in H \cap \dot{A}$. The proof of the bounding property shows that there is $q \leq^r p$ and a finite set $h \subset H$ such that $q \Vdash h \cap \dot{A} \neq 0$. Clearly, $q \Vdash \check{u} \cap \bigcap \check{h} \in \dot{A}$ as required! $\square$

Towards the proof of Proposition 2.6(3), suppose $p \in P$, $\varepsilon > 0$, and $p \Vdash \dot{B}_n : n \in \omega$ is a sequence of Borel subsets of 2^ω of Lebesgue mass greater than ε. Passing to subsets, we may assume that all the sets on the sequence are forced to be closed. We may also assume that there is a continuous function $f : [p] \to K(2^\omega)^\omega$ such that $p \Vdash$the sequence is recovered as the functional value at the generic point $\dot{x}_{gen}$. Find a number m_0 such that $\varepsilon_{m_0} < \varepsilon$ and $\forall n \geq m_0\ \mathtt{nor}(p(n)) > 3$. Thinning out the condition p if necessary, assume that $p(n) \in a_n$ for all $n \in m_0$.

By induction on $i \in \omega$ build conditions $p_i \in P$, infinite sets $u_i \subset \omega$, finite sets $v_i \subset \omega$, and binary sequences $s_i : i \in \omega$ so that

- p_i form a fusion sequence: $p_{i+1} \leq^i p_i$. The limit of the sequence will be a condition q;
- v_i strictly increase, u_i strictly decrease, and $v_i \subset u_i$. Thus $u = \bigcap_i u_i \subset \omega$ will be an infinite set;
- the sequences s_i are linearly ordered by the initial segment relation. The union $y = \bigcup_i s_i$ will be a point in 2^ω.

We want to achieve $q \Vdash \check{y} \in \bigcap_{n \in u_i} \dot{B}_n$. For that, another induction assumption is necessary. A piece of notation: whenever a P-generic filter is overwritten on a finite set of coordinates with a sequence t of atoms, the result is again a P-generic filter. Whenever τ is a P-name, then τ/t is the name for the evaluation of τ according to the overwritten generic filter. Here is the last item of the induction hypothesis.

- for every number $k \in u_i$, the condition p_i forces the closed set $\dot{B}^i_k = O_{t_i} \cap \bigcap \{\dot{B}_n/t : n \in v_i \cup \{k\}, t \in [p_i] \restriction [m_0, m_0 + i)\}$ to have Lebesgue measure larger than ε_{m_0+i}.

Suppose that p_i, u_i, v_i have been constructed. Fix $k \in u_i$. For every element $e \in [p_i] \restriction \{m_0 + i\}$ and every $k \in u_i$, the set $\dot{B}^i_k/e$ is forced by p_i to have mass at least ε_{m_0+i}. Therefore, by the Fubini property of the setup C_{m_0+i}, the condition p_i forces that there is a creature $c \leq p_i(m_0+i)$ with a large norm such that the set $\bigcap_{e \in \mathtt{pos}(c)} B^i_k/e$ has mass at least $\varepsilon_{m_0+1}/2^{|a_n|}$. Now we will apply the previous Claim 2.11 successively three times. First, there is a condition $p'_i \leq p$ and an infinite set $u'_i \subset u$ such that there is a creature $c \leq p_i(m_0 + 1)$ which is forced to work for all $k \in u'$ simultaneously. Second, there is an infinite set $u''_i \subset u'_i$ and a one-step extension t_{i+1} of of the binary sequence t_i such that it is forced that the sets $O_{t_{i+1}} \cap \bigcap_{e \in \mathtt{pos}(c)} B^i_k/e$ have mass at least

$\varepsilon_{m_0+1}/2^{|a_n|+1}$ for all $k \in u_i''$. And finally, and most importantly, by a theorem of [4] applied in the gemeric extension, these infinitely many sets are going to have an infinite subcollection with pairwise intersections of mass bounded away from zero: there is a condition $p_i''' \leq p_i''$ and an infinite set $u_i''' \subset u_i''$ such that the sets $O_{t_{i+1}} \cap \bigcap_{e \in \mathtt{pos}(c)} B^i_k : k \in u_i'''$ are forced to have pairwise intersections of mass at least $\frac{1}{2}(\frac{\varepsilon_{m_0+1}}{2^{|a_n|+1}})^2 > \varepsilon_{m_0+i+1}$. Claim 2.11 shows that the conditions p_i', p_i'', and p_i''' can be chosen $\leq^i p_i$. Now let p_{i+1} be the condition p_i''' with its m_0+i-th coordinate replaced by c, $u_{i+1} = u_i''' \cup v_i$, and $v_{i+1} = v_i \cup \min(u_{i+1} \setminus v_i)$. It is not difficult to see that the induction hypothesis is satisfied.

In the end, let $u = \bigcap_i u_i$ and let $q \leq p$ be the lower bound of the conditions p_i. Let $y = \bigcup_i t_i$. The last item of the induction hypothesis shows that indeed, $\forall x \in [q] \forall n \in u\ y \in f(x)(n)$, and therefore $q \Vdash \check{y} \in \bigcap_{n \in u} \dot{B}_n$, as desired.

3 The proofs of the parametrized theorems

With the key properties of the creature forcing at hand, the parametrized theorems follow fairly easily. Suppose that $k \in \omega$ is a natural number and $r_n : n \in \omega$ is a sequence of real numbers. Suppose that $\phi_n : n \in \omega$ is a sequence of submeasures on finite sets $a_n : n \in \omega$ such that, writing $\varepsilon_n = 1/\Pi_{m \in n} 2^{|a_m|}$, the numbers $\varepsilon_n \log(\log(1 + s\phi_n(a_n)) - \log(1 + r_n) + 1)$ are all defined, larger than k, and tend to infinity. We will prove that every partition of the Polish space $\Pi_n a_n \times \omega$ into k many Borel pieces $D_i : i \in k$, one of the pieces contains a product of the form $\Pi_n b_n \times c$, where $b_n \subset a_n$ are sets of respective ϕ_n-mass at least r_n and $c \subset \omega$ is an infinite set. This will prove theorem 1.4.

For every number $n \in \omega$, define a setup C_n on the set a_n with a norm $\mathtt{nor}_n$. Nonatomic elements of C_n are pairs $\langle b, r \rangle$ where $b \subset a_n$, $r \in \mathbb{R}^+$ and $\log(1 + \phi_n(b)) \geq r$; the norm is defined by $\mathtt{nor}_n(b, r) = \varepsilon_n \log(\log(1 + \phi_n(b) - r + 1)$. The ordering is defined by $\langle c, s \rangle \leq \langle b, r \rangle$ if $c \subset b$ and $s \geq r$. Define the creature forcing P derived from the setups C_n on the sets a_n and consider the condition $p \in P$ such that $p(n) = \langle a_n, \log(1 + r_n) \rangle$. Consider the P-name for a partition of ω into k pieces $(D_i)_{\dot{x}_{gen}} : i \in k$ obtained as a vertical section of the Borel partition of $\Pi_n a_n \times \omega$ above the generic sequence $\dot{x}_{gen}$. The forcing P does not add independent reals, and therefore there is a condition $q \leq p$ and an infinite set $c \subset \omega$ and an index $i \in k$ such that $q \Vdash \check{c}$ is a subset of i-th piece of this partition. Reviewing the proof of Proposition 2.6 (2), or using Claim 2.11, it becomes clear that the condition q can be found in such a way that $\forall n\ \mathtt{nor}_n(q(n)) > 0$. Now let M be a countable elementary submodel of a large enough structure containing the condition q, and find an M-master condition $q' \leq q$. The proof of Proposition 2.6 (1) in fact shows that the master condition q' can be chosen so that $\forall n \in \omega\ \mathtt{nor}_n(q'(n)) > 0$ and all points in $[q']$ are M-generic in the sense that for every $x \in [q']$ the filter $g_x = \{r \in M \cap P : \exists n\ q' \upharpoonright\upharpoonright (x \upharpoonright n) \leq r\} \subset P$ is M-generic. An absoluteness argument between $M[g_x]$ and V will show that $\langle x, n \rangle \in D_i$ and so $[q'] \times c \subset D_i$. Theorem 1.4 follows.

Theorem 1.5 can now be derived abstractly. Suppose that K is an F_σ-ideal and $r_n : n \in \omega$ are real numbers. Use a theorem of Mazur [10] to find a lower semicontinuous submeasure μ on ω such that $K = \{a \subset \omega : \mu(a) < \infty\}$. Suppose that $\phi_n : n \in \omega$ is a fast increasing sequence of submeasures on finite sets a_n, and $\Pi_n a_n \times \omega = \bigcup_{i\in k} B_i$ is a partition of the product into finitely many Borel sets. There will be pairwise disjoint finite subsets $b_n : n \in \omega$ of ω such that the sequence $\phi_n, \mu \restriction b_n : n \in \omega$ of submeasures still increases fast enough to apply Theorem 1.4. Let $\Pi_n a_n \times \Pi_n b_n \times \omega = \bigcup_{i\in k} C_i$ be the partition defined by $\langle x, y, n\rangle \in C_i \leftrightarrow \langle x, y(n)\rangle \in B_i$. Use Theorem 1.4 to find sets $a'_n \subset a_n$, $b'_n \subset b_n$ and $c \subset \omega$ such that $\phi_n(a'_n) \geq r_n$, $\mu(b'_n) \geq n$, and c is infinite, and the product $\Pi_n a'_n \times \Pi_n b'_n \times c$ is wholly contained in one of the pieces of the partition, say C_i. The review of the definitions reveals that $c' = \bigcup_{n\in c} b'_n$ is a K-positive set, and $\Pi_n b'_n \times c' \subset B_i$. This proves Theorem 1.5.

To derive Theorem 1.6, suppose that $\varepsilon > 0$ is a real number, $\phi_n : n \in \omega$ is a fast increasing sequence of measures on finite sets, and $B \subset \Pi_n \mathrm{dom}(\phi_n) \times \omega \times 2^\omega$ is a Borel set with vertical sections of mass at least ε. Choose the setups as in the proof of Theorem 1.4 and observe that they do have the Fubini property. It follows from Proposition 2.6(3) that the derived forcing does not add a V-independent sequence of sets of mass $> \varepsilon$. In fact, the proof shows that there is a condition $p \in P$, a point $z \in 2^\omega$, and an infinite set $u \subset \omega$ such that $p \Vdash \check{z}$ belongs to the vertical section of the set B corresponding to $\dot{x}_{gen}$ and any $n \in u$. Moreover, the condition p can be chosen with all norms nonzero. Let M be a countable elementary submodel of a large structure and find a condition $q \leq p$ with all norms nonzero such that the set $[q]$ consists of M-generic points only. Then $[q] \times u \times \{z\} \subset B$, and Theorem 1.6 follows.

4 Applications

Theorem 1.4 is one of the strongest tools available to prove that certain bounding forcings do not add independent reals.

The first application concerns the independent reals in countable support products. Suppose that for every number $n \in \omega$, I_n is a σ-ideal on a compact space X_n generated by a σ-compact collection of compact sets in the hyperspace $K(X_n)$. The quotient forcings P_{I_n} of Borel I_n-positive sets ordered by inclusion have been studied in [13, Theorem 4.1.8]. They include such posets as Sacks forcing, $c_{\min}$-forcing, the limsup ∞ tree forcings of [12], as well as some more mysterious entities such as the quotient forcing of Borel non-σ-finite packing mass sets ordered by inclusion. They are proper, bounding, and do not add independent reals. The proof of [13, Theorem 4.1.8] easily generalizes to show that even their finite products share these properties. The infinite product is proper, bounding, and preserves category [13, Theorem 5.2.6]. The question of independent reals in the infinite product is more subtle:

Proposition 4.1. *Countable product of quotient forcings $P_{I_n} : n \in \omega$, where each I_n is σ-generated by a σ-compact collection of compact sets, does not add independent reals.*

In fact, it is not difficult to argue that the product has the weak Laver property, which in conjuction with this proposition and [14] shows that the product preserves P-points.

Proof. For the simplicity of notation assume that the underlying space X_n is always equal to the Cantor space 2^ω. For every number $n \in \omega$, fix compact sets $K_{n,i} \subset K(2^\omega) : i \in \omega$ whose union σ-generates the ideal I_n; assume that these sets are closed under taking compact subsets. Let $O_{n,i} = \{a \subset 2^{<\omega}$ finite: there is a set $b \in K_{n,i}$ such that $\forall t \in a \exists x \in b\ t \subset x\}$. In order to be able to apply Theorem 1.4 in this context, we must identify the submeasures on finite sets. Suppose $T \subset 2^{<\omega}$ is a tree, $n, i \in \omega$, and $m_0 < m_1$ are natural numbers. We will define a submeasure $\phi = \phi(T, n, i, m_0, m_1)$ on the finite set $a(T, m_0, m_1) = \Pi_{s \in 2^{m_0} \cap T}\{t \in 2^{m_1} \cap T : s \subset t\}$. Denote the elements of the set $a(T, m_0, m_1)$ as functions with the domain $2^{m_0} \cap T$. Define $W = \{b \subset a : \exists s \in 2^{m_0} \cap T\ \{w(s) : w \in b\} \in O_{n,i}\}$, and then define $\phi(b) =$the number of sets in W necessary for covering the set b. The main claim:

Claim 4.2. *For every $n, i \in \omega$ every I_n-positive tree T, every number m_0 and every number r there is m_1 such that the submeasure $\phi(T, n, i, m_0, m_1)$ assigns a value larger than r to its domain.*

Proof. Suppose this fails for T, n, i, m_0 and r. For every number $m > m_0$ find a partition of the set $a(T, m_0, m)$ into sets b^m_j in the set $W^m = \{b \subset a(T, m_0, m) : \exists s \in 2^{m_0} \cap T\ \{w(s) : w \in b \in O_{n,i}\}$. Consider the product forcing $\Pi_{s \in 2^{m_0} \cap T} P_{I_n} \restriction T \restriction s$. Consider the name for the function $\dot f : \omega \to r$ defined by $\dot f(m) =$ that value of $j \in r$ for which $\vec x_{gen} \restriction m \in b^m_j$, where $\vec x_{gen}$ is the product name for the finite sequence of generic points. Since the finite product does not add independent reals, there must be an infinite set $c \subset \omega$, a number $j \in r$, and a condition $\langle S_s : s \in 2^{m_0} \cap T\rangle$ which forces $\dot f \restriction \check c$ to return the constant value j. Find a number $m \in C$ such that the sets $S_s \restriction m : s \in 2^{m_0} \cap T$ all fall out of $O_{n,i}$. It must be the case then that $\Pi_{s \in 2^{m_0} \cap T} S_s \restriction m \subset b^m_j$, which contradicts the assumption that $b^m_j \in W^m$. $\square$

Now suppose that $T_n : n \in \omega$ is a condition in the product of the quotient posets, forcing that $\dot y \in 2^\omega$ is a point. We must find a stronger condition deciding infinitely many values of the point $\dot y$. Strengthening the condition if necessary, we may assume that there is a continuous function $f : \Pi_n [T_n] \to 2^\omega$ such that the condition forces the point $\dot y$ to be the functional value of f at the generic point. Using the claim, it is not difficult to find natural numbers $m_{n,i} : n, i \in \omega$ so that

- $m_{n,0} = 0 < m_{n,1} < m_{n,2} < \dots$;
- the submeasures $\phi_{n,i} = \phi(T_n, n, i, m_i, m_{i+1})$ on the sets $a_{n,i} = a(T_n, m_i, m_{i+1})$ form a sufficiently fast increasing sequence of submeasures under a suitable enumeration of all pairs $(n, i) \in \omega \times \omega$ in ordertype ω.

Note that every sequence $x \in \Pi_{n,i} a_{n,i}$ defines a point $z(x) \in \Pi_n [T_n]$ as the unique point z such that $\forall n, i\ z(n) \restriction m_{n,i+1} = (x(n))(z(n) \restriction m_{n,i})$. Moreover, if $b_{n,i} \subset a_{n,i}$ are sets with ϕ_{n_i}-mass at least 2 for every pair $(n,i) \in \omega \times \omega$, then $\{z(x) : x \in \Pi_{n,i} b_{n,i}\} = \Pi_n [S_n]$ for some I_n-positive trees $S_n \subset T_n : n \in \omega$. Consider the partition $\Pi_{n,i} a_{n,i} \times \omega = C_0 \cup C_1$ defined by $\langle x, n\rangle \in C_0 \leftrightarrow f(z(x))(n) = 0$. Theorem 1.4 yields sets $b_{n,i} \subset a_{n,i}$ of respective $\phi_{n,i}$ mass greater than 2, an infinite set $c \subset \omega$ and a bit $b \in 2$ such that $\forall x \in \Pi_{n,i} b_{n_i} \forall n \in c\ f(z(x))(n) = b$. For every $n \in \omega$ let $S_n \subset T_n$ be the I_n-positive trees such that $\{z(x) : x \in \Pi_{n,i} b_{n,i} = \Pi_n S_n\}$. Then the condition $\langle S_n : n \in \omega\rangle$ decides the values of the point $\dot{y}$ to be equal to b at all numbers $n \in c$. This completes the proof. □

It is possible to use Theorem 1.4 to show that some other, otherwise intractable, forcings do not add independent reals or preserve P-points. All the arguments also show that the countable support products of arbitrary combinations of these forcings add no independent reals. It is in general not true that not adding an independent real is preserved in product, as an example in [14] shows.

Rosłanowski [11] and others considered the *Halpern-Läuchli forcing.* A tree $T \subset 2^{<\omega}$ is said to be strongly embedded if there is an infinite set $c \subset \omega$ such that a node of T is a splitnode iff its length belongs to the set c. The Halpern-Lauchli forcing is just the poset of strongly embedded trees with inclusion. This is a proper and bounding forcing with the Sacks property, as simple fusion arguments immediately show.

Proposition 4.3. *The Halpern-Läuchli forcing does not add independent reals.*

Proof. Suppose $T \Vdash \dot{y} \in 2^\omega$. We must produce a stronger condition S deciding infinitely many values of $\dot{y}$. Strengthening the condition T if necessary, we can find a continuous function $f : [T] \to 2^\omega$ such that T forces $\dot{y}$ to be the functional value of f at the canonical generic point. By a simple homogeneity argument we may assume that in fact $T = 2^{<\omega}$. Choose a fast increasing sequence $m_n : n \in \omega$ of natural numbers, and consider the sets $a_n = 2^{m_{n+1} \setminus m_n}$ with the counting measure on each. For every point $x \in \Pi_n a_n$ let $z(x) = \bigcup_n x(n)$ and note that whenever $b_n : n \in \omega$ are subsets of the respective sets a_n of cardinality at least 2, there is a strongly embedded tree $S \subset 2^{<\omega}$ such that $[S] = \{z(x) : x \in \Pi_n b_n$. Consider the partition $\Pi_n a_n \times \omega = C_0 \cup C_1$ into two Borel parts defined by $\langle x, n\rangle \in C_0 \leftrightarrow f(z(x))(n) = 0$. Theorem 1.4 provides sets $b_n \subset a_n : n \in \omega$ of size at least 2 and an infinite set $c \subset \omega$ such that the product $\Pi_n b_n \times c$ is wholly contained in one piece of the partition. Obviously, if $S \subset 2^{<\omega}$ is the strongly embedded tree such that $[S] = \{z(x) : x \in \Pi_n b_n\}$, then S decides all values of the sequence $\dot{y}$ on the infinite set c. □

[13] and [8] considered the E_2 *forcing.* E_2 is the Borel equivalence relation on 2^ω defined by xE_2y iff $x_n = y_n$ for all numbers $n \in \omega$ except for a summable set of exceptions; that is, $\Sigma\{1/n + 1 : x(n) \neq y(n)\} < \infty$. This is a standard example of a Borel equivalence relation not Borel reducible to an orbit equivalence relation of an action of the infinite symmetric group. Consider the collection I of all Borel subsets $B \subset 2^\omega$ such that $E \restriction B$ is reducible to such an orbit

equivalence relation. This turns out to be a σ-ideal, and a Borel set $B \subset 2^\omega$ is in I if and only if $E \restriction B$ is Borel reducible to a Borel equivalence relation with countably many classes, if and only if E_2 is reducible to $E_2 \restriction B$. The quotient partial order P_I has been investigated in [13]. It is proper and has the Sacks property. Its combinatorial presentation is fairly complicated.

Proposition 4.4. *The E_2 forcing does not add independent reals.*

Proof. The method of proof depends on a simple fact. Let E_2' be the equivalence relation on ω^ω of equality modulo the summable ideal. Then the equivalence relations E_2 and E_2' are Borel bireducible to each other.

Suppose $B \in P_I$ is a condition and $\dot{y}$ is an name for an infinite binary sequence. We must find a stronger condition that decides inifnitely many values of the sequence $\dot{y}$. Strengthening the condition B if necessary, we may assume that there is a Borel function $f : B \to 2^\omega$ such that B forces $\dot{y}$ to be the f-image of the canonical generic point. Since E_2 is reducible to $E_2 \restriction B$, so is E_2', via a Borel reduction $g : \omega^\omega \to B$. Choose a fast increasing sequence $k_n : n \in \omega$ of natural numbers, each equipped with a counting measure. Note that whenever $b_n \subset k_n : n \in \omega$ are sets of size at least 2, then E_2 reduces to $E_2' \restriction \Pi_n b_n$, and therefore to the set $g''\Pi_n b_n \subset B$, and this set will be I-positive. Now consider the partition of the product $\Pi_n k_n \times \omega = C_0 \cup C_1$ into two Borel parts defined by $\langle x, i\rangle \in C_0 \leftrightarrow f(g(x))(n) = 0$. Theorem 1.4 yields sets $b_n \subset k_n : n \in \omega$, each containing at least two elements, and an infinite set $c \subset \omega$ such that $g \restriction \Pi_n b_n$ is continuous and the set $\Pi_n b_n \times c$ is wholly contained in one of the pieces of the partition. Since E_2 clearly Borel reduces to $E_2' \restriction \Pi_n b_n$, it also reduces to $g''\Pi_n b_n$. A review of the definitions shows that the set $g''\Pi_n b_n \subset B$ is a compact I-positive set that decides all values $\dot{y}(n) : n \in c$ as desired. $\square$

The E_0 forcing studied in [13] offers a similar story. E_0 is the equivalence relation on 2^ω defined by xE_0y if the two binary sequences x, y agree on all but finitely many entries. This is a canonical example of a Borel equivalence relation that is not Borel reducible to the identity. Let I be the collection of Borel sets such that $E_0 \restriction B$ is reducible to the identity. It turns out that I is a σ ideal, and the quotient forcing is proper and has the Sacks property.

Proposition 4.5. *The E_0 forcing does not add independent reals.*

Proof. Observe that the equivalence relation E_0' on ω^ω, defined by $xE_0'y$ if the two sequences differ only at finitely many entries, is bireducible with E_0. The remainder of the proof is exactly the same as before. $\square$

The measure parametrized theorem can be used to prove the preservation of outer Lebesgue measure in various products.

Proposition 4.6. *The product of Miller forcing with countably many copies of Sacks forcing preserves outer Lebesgue measure.*

Proof. The key observation for this proposition is the fact that the product of countably many copies of the Sacks forcing does not add a V-independent sequence of sets of positive measure. To see this, suppose that p is a condition in the product, forcing $\dot{B}_i : i \in \omega$ to be a sequence of closed sets of mass at least $\varepsilon > 0$. We must find an infinite set $a \subset \omega$ and a point $z \in 2^\omega$ as well as a condition $q \leq p$ forcing $\check{z} \in \bigcap_{i \in a} \dot{B}_i$. By a usual proper forcing argument, strengthening the condition p if necessary we can find a continuous function $f : (2^\omega)^\omega \to K(2^\omega)^\omega$ such that p forces $\dot{B}_i = \dot{f}(\vec{x}_{gen})(i)$. By a homogeneity argument, we may assume that p is in fact the largest condition in the product. Now, it is not difficult to find numbers $m^l_k : k \in \omega, l \in \omega$ so that $m^l_k : l \in \omega$ form an increasing sequence starting with zero for every $k \in \omega$, and moreover the sets $2^{m^l_k \setminus m^{l-1}_k} : k, l \in \omega$ equipped with counting measures form a sequence increasing fast enough so that Theorem 1.6 can be applied. There will be sets b^l_k of size at least 2 each for $k, l \in \omega$, a point $z \in 2^\omega$, and an infinite set $a \subset \omega$ such that for every $i \in a$ and every sequence $\langle x_k : k \in \omega \rangle \in (2^\omega)^\omega$ with $\forall k, l\ x_k \restriction [m^l_k, m^{l+1}_k) \in b^l_k$ it is the case that $z \in f(x_k : k \in \omega)(i)$. Consider the condition $q \leq p$ in the product in which the k-th tree $q(k) \subset 2^{<\omega}$ consists of those finite binary sequences s such that for every $l \in \omega$, $s \restriction [m^l_k, m^{l+1}_k) \in b^l_k$. Clearly, $q \Vdash \check{z} \in \bigcap_{i \in a} \dot{B}_i$ as desired.

Suppose that p is a condition in the product forcing $\dot{O} \subset 2^\omega$ to be an open set of mass $\varepsilon < 1$. We must produce a condition $q \leq p$ and a point $z \in 2^\omega$ such that $q \Vdash \check{z} \notin \dot{O}$. The proposition then follows from [13, Proposition 3.2.11].

For a Miller tree $T \subset \omega^{<\omega}$ let $\pi_T : \omega^{<\omega} \to T$ be the natural order-preserving map with the range consisting of all the splitnodes of T. Let $\varepsilon_t : t \in \omega^{<\omega}$ be positive real numbers whose sum is less than $\frac{1-\varepsilon}{2}$. A standard fusion argument will yield a condition $p' = \langle T, S_n : n \in \omega \rangle \leq p$ such that for every sequence $t \in \omega^{<\omega}$ there is a Sacks product name $\dot{O}_t$ for a clopen set such that $\langle T \restriction \pi_T(t), S_n : n \in \omega \rangle$ forces $\dot{O}_t \subset \dot{O}$ and $\lambda(\dot{O} \setminus \dot{O}_t) < \varepsilon_t$; moreover, $\dot{O}_t$ is forced to contain all basic clopen sets of radius $< 2^{-|t|}$ that are a subset of $\dot{O}$. For the simplicity of notation, we will assume that $T = \omega^{<\omega}$ and $S_n = 2^{<\omega}$ for every number $n \in \omega$.

For every number $i \in \omega$, consider the product Sacks name $\dot{P}_i$ for the set $\bigcup_{t \in \omega^{<\omega}} (\dot{O}_{t^\frown i}) \setminus \dot{O}_t) \cup \dot{O}_0$. The Lebesgue measure of the set $\dot{P}_i$ is forced to be smaller than $\varepsilon + \Sigma_t \varepsilon_t < \frac{1+\varepsilon}{2}$. Since the product Sacks forcing does not add a V-independent sequence of sets of positive measure, there must be a condition $\langle S'_n : n \in \omega \rangle$ in the Sacks product, a point $z \in 2^\omega$, and an infinite set $a \subset \omega$ such that the condition forces $\check{y} \notin \bigcup_{i \in a} \dot{P}_i$. Consider the condition $q = \langle T', S'_n : n \in \omega \rangle$ where T' is the tree of all sequences whose entries come from the infinite set a. A review of the definitions shows that indeed, $q \Vdash \check{y} \notin \dot{O}$ as required.

□

The previous arguments can be repeated with a large class of other forcing notions. Borderline unclear cases include the following:

Question 4.7. Does the product of Laver and Sacks forcing preserve the outer Lebesgue measure?

Question 4.8. Let I be the σ-ideal generated by finite Hausdorff 1/2-dimensional mass sets on 2^ω with the minimum difference metric. Does the product $P_I \times P_I$ add an independent real?

Question 4.9. Is not adding a V-independent sequence of sets of positive mass equivalent to the conjunction of not adding an independent real and preserving outer Lebesgue measure for some large class of proper forcing notions?

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