ON THE BERGMAN AND STEINHAUS PROPERTIES FOR INFINITE PRODUCTS OF FINITE GROUPS

SIMON THOMAS AND JINDŘICH ZAPLETAL

Abstract. We study the relationship between the existence of nonprincipal ultrafilters over ω and the failure of the Bergman and Steinhaus properties for infinite products of finite groups.

1. Introduction

In this paper, we will investigate the Bergman and Steinhaus properties for various infinite products of finite groups, focusing especially on groups of the form $\prod SL(2, p_n)$, where $(p_n \mid n \in \omega)$ is an increasing sequence of primes. This may initially seem a strange choice, given that the results of Saxl-Shelah-Thomas [24] and Thomas [29] already imply that the Bergman and Steinhaus properties always fail for $\prod SL(2, p_n)$. However, this is not the end of the story. The arguments in both [24] and [29] make use of an ultraproduct $\prod_{\mathcal{U}} SL(2, p_n)$, where $\mathcal{U}$ is a nonprincipal ultrafilter over ω; and it is natural to ask whether the existence of such an ultrafilter is either necessary or sufficient in order to establish the failure of the Bergman and Steinhaus properties for $\prod SL(2, p_n)$.

We will begin by reminding the reader of the definitions of the Bergman and Steinhaus properties. Throughout this paper, a subset A of a group G is said to be *symmetric* if $A = A^{-1}$ is closed under taking inverses.

Definition 1.1 (Macpherson-Neumann [19], Bergman [2])**.** Suppose that G is a non-finitely generated group.

(a) G has *countable cofinality* if $G = \bigcup_{n \in \omega} G_n$ can be expressed as the union of a countable increasing chain of proper subgroups. Otherwise, G has *uncountable cofinality*.

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(b) G is *Cayley bounded* if for every symmetric generating set S, there exists an integer $n \geq 1$ such that every element $g \in G$ can be expressed as a product $g = s_1 \cdots s_n$, where each $s_i \in S \cup \{ 1 \}$.

(c) G has the *Bergman property* if G has uncountable cofinality and is Cayley bounded.

By de Cornulier [6], a group G has the Bergman property if and only if whenever G acts isometrically on a metric space, every G-orbit has a finite diameter. For this reason, groups with the Bergman property are often said to be "strongly bounded". The class of groups with the Bergman property includes the symmetric groups over infinite sets [2], automorphism groups of various infinite structures [9, 13] and oligomorphic groups with ample generics [14] The following easy observation is essentially contained in Bergman [2, Lemma 10].

Lemma 1.2. *If G is a non-finitely generated group, then the following conditions are equivalent.*

(a) *G has the Bergman property.*

(b) *If $G = \bigcup_{n \in \omega} U_n$ is the union of an increasing chain of symmetric subsets such that $U_n U_n \subseteq U_{n+1}$ for all $n \in \omega$, then there exists an $n \in \omega$ such that $U_n = G$.*

The Steinhaus property was introduced by Rosendal-Solecki [22] in the context of the automatic continuity problem for homomorphisms between topological groups. In the following definition, a subset W of a group G is said to be *countably syndetic* if there exist elements $g_n \in G$ for $n \in \omega$ such that $G = \bigcup_{n \in \omega} g_n W$.

Definition 1.3 (Rosendal-Solecki [22])**.** Let G be a topological group. Then G has the *Steinhaus property* if there exists a *fixed* integer $k \geq 1$ such that for every symmetric countably syndetic subset $W \subseteq G$, the k-fold product W^k contains an open neighborhood of the identity element 1_G.

Proposition 1.4 (Rosendal-Solecki [22])**.** *If G is a topological group with the Steinhaus property and $\varphi : G \to H$ is a homomorphism into a separable group H, then φ is necessarily continuous.*

The class of groups with the Steinhaus property includes Polish groups with ample generics [14], $\mathrm{Aut}(\mathbb{Q},<)$, $\mathrm{Homeo}(\mathbb{R})$ [22] and full groups of ergodic countable Borel equivalence relations [15]. As we mentioned earlier, in this paper, we will be studying the Bergman and Steinhaus properties for infinite products of finite groups. There are currently very few such groups which are known to have either the Bergman property or the Steinhaus property. In [6], improving an earlier result of Koppelberg-Tits [16], de Cornulier proved that if G is a product of countably many copies of a fixed finite perfect group, then G has the Bergman property; and in [23], Rosendal pointed out that the arguments in Saxl-Shelah-Thomas [24] and Thomas [29] can be modified to prove that $\prod \mathrm{Alt}(2^n)$ also has the Bergman property. No infinite product of finite groups is currently known to have the Steinhaus property. On the other hand, there are many such groups which are known to have neither the Bergman nor the Steinhaus property. In particular, the following result holds.

Theorem 1.5. *If* $(\, p_n \mid n \in \omega \,)$ *is an increasing sequence of primes, then:*

(a) $\prod SL(2,p_n)$ *has countable cofinality;*
(b) $\prod SL(2,p_n)$ *is not Cayley bounded; and*
(c) $\prod SL(2,p_n)$ *does not have the Steinhaus property.*

Theorem (1.5)(a) is essentially contained in Saxl-Shelah-Thomas [24]. However, since certain features of the argument provide the motivation for this paper, we will quickly run through the very easy proof. (The other parts of Theorem 1.5 will be proved in Section 2.) Let $\mathcal{U}$ be a nonprincipal ultrafilter over ω and let $K = \prod_{\mathcal{U}} \mathbb{F}_{p_n}$ be the corresponding ultraproduct of the fields $\mathbb{F}_{p_n}$ of order p_n. Then K is an uncountable field and

$$\prod\nolimits_{\mathcal{U}} SL(2,p_n) \cong SL(2, \prod\nolimits_{\mathcal{U}} \mathbb{F}_{p_n}) = SL(2,K).$$

It follows that $SL(2,K)$ is a homomorphic image of $\prod SL(2,p_n)$ and hence Theorem 1.5(a) is an easy consequence of the following observation.

Proposition 1.6. *If* F *is an uncountable field, then* $SL(2,F)$ *has countable cofinality.*

Proof. Let B be a transcendence basis of F over its prime subfield. Then B is uncountable and hence we can express $B = \bigcup_{n\in\omega} B_n$ as the union of a countable strictly increasing chain of proper subsets. For each $n \in \omega$, let F_n be the algebraic closure of B_n in F. Then the strictly increasing chain of proper subgroups

$$SL(2, F) = \bigcup_{n \in \omega} SL(2, F_n)$$

witnesses that $SL(2, F)$ has countable cofinality. □

The proofs of Theorems (1.5)(b) and (1.5)(c), which are given in Section 2, also involve the use of an ultraproduct $\prod_{\mathcal{U}} SL(2, p_n)$; and it is natural to ask whether the existence of a nonprincipal ultrafilter $\mathcal{U}$ over ω is either necessary or sufficient in order to prove the various parts of Theorem 1.5. Of course, when considering this kind of question, we cannot work with the usual ZFC axioms of set theory since these already imply the existence of nonprincipal ultrafilters over arbitrary infinite sets. Instead we will work with the axiom system $ZF + DC$, where DC is the following weak form of the Axiom of Choice.

Axiom of Dependent Choice (DC)**.** Suppose that X is a nonempty set and that R is a binary relation on X such that for all $x \in X$, there exists $y \in X$ with $x \, R \, y$. Then there exists a function $f : \omega \to X$ such that $f(n) \, R \, f(n+1)$ for all $n \in \omega$.

The axiom system $ZF + DC$ is sufficient to develop most of real analysis and descriptive set, but is insufficient to prove the existence of pathologies such as nonmeasurable sets. (For example, see Moschovakis [21].) In particular, since nonprincipal ultrafilters over ω are nonmeasurable subsets of $2^{\mathbb{N}}$, it follows that $ZF + DC$ does not prove the existence of such ultrafilters. The following result, which will be proved in Section 3, shows that the existence of a nonprincipal ultrafilter over ω is indeed necessary in order to prove either Theorem (1.5)(a) or Theorem (1.5)(b). It is currently not known whether Theorem (1.5)(c) implies the existence of a nonprincipal ultrafilter over ω. However, in Section 3, we will show that the failure of a weak form of the Steinhaus property does imply the existence of such ultrafilters.

Theorem 1.7 ($ZF + DC$)**.** *Let* $(\, p_n \mid n \in \omega \,)$ *be an increasing sequence of primes. If* $\prod SL(2, p_n)$ *does not have the Bergman property, then there exists a nonprincipal ultrafilter over* ω.

On the other hand, we will also show that the existence of a nonprincipal ultrafilter over ω is not sufficient to prove any of the parts of Theorem 1.5. In order to explain this result, it will be necessary in the remainder of this section to assume the existence of suitable large cardinals. We will not specify the precise large cardinal hypothesis that we need until it becomes necessary to do so in Section 6. (This paper has been written so that the first five sections can be read by mathematicians with no knowledge of advanced set theory, such as forcing, large cardinals, etc. It is only in the final section that some knowledge of advanced set theory is needed and this section can be omitted by mathematicians without the necessary background.) Following the usual convention [31], we will indicate the use of a large cardinal hypothesis by writing (LC) before the statement of the relevant theorem.

Recall that an *inner model* of ZF is a transitive class which contains all the ordinals and satisfies the ZF axioms. (For example, it is well-known that the constructible universe L is the smallest inner model of ZF.) Let $L(\mathbb{R})$ be the smallest inner model which contains all of the reals. Then $L(\mathbb{R})$ is a model of $ZF + DC$; and, assuming the existence of suitable large cardinals, $L(\mathbb{R})$ satisfies the Axiom of Determinacy, which rules out the existence of pathologies such as nonmeasurable set of reals, etc. In particular, it follows that $L(\mathbb{R})$ does not contain any nonprincipal ultrafilters over ω. Let $L(\mathbb{R})[\mathcal{U}]$ be the generic extension of $L(\mathbb{R})$ obtained by forcing with $\mathcal{P}(\omega)/\mathrm{Fin}$ to adjoin the nonprincipal ultrafilter $\mathcal{U}$ over ω. Then Di Prisco-Todorcevic [7] have shown that many of the regularity properties of $L(\mathbb{R})$ also hold in $L(\mathbb{R})[\mathcal{U}]$. For example, in $L(\mathbb{R})[\mathcal{U}]$, every uncountable set of reals has a perfect subset. Thus it seems natural to regard $L(\mathbb{R})[\mathcal{U}]$ as a canonical model of $ZF + DC$ in which a minimal number of the pathological consequences of the Axiom of Choice hold, modulo the existence of a nonprincipal ultrafilter $\mathcal{U}$ over ω. The following result, which we will prove in Section 4, provides yet more evidence for this point of view.

Theorem 1.8 (LC)**.** *If* $(\, p_n \mid n \in \omega \,)$ *is any increasing sequence of primes, then* $\prod SL(2, p_n)$ *has the Bergman property in* $L(\mathbb{R})[\mathcal{U}]$.

It is currently not known whether the analogous result holds for the Steinhaus property. However, in Section 4 we will prove the following result; and we will also prove that if $(p_n \mid n \in \omega)$ is any increasing sequence of primes, then $\prod SL(2, p_n)$ satisfies a weak form of the Steinhaus property.

Theorem 1.9 (LC)**.** *If $(p_n \mid n \in \omega)$ is a sufficiently fast growing sequence of primes, then $\prod SL(2, p_n)$ has the Steinhaus property in $L(\mathbb{R})[\mathcal{U}]$.*

Examining the above proof of Theorem (1.5)(a), we see that it relies upon the following three consequences of the Axiom of Choice:

(i) the existence of a nonprincipal ultrafilter $\mathcal{U}$ over ω;
(ii) the existence of a transcendence basis B of the field $\prod_{\mathcal{U}} \mathbb{F}_{p_n}$; and
(iii) the existence of an expression of B as the union of a countable strictly increasing chain of proper subsets.

Clearly $L(\mathbb{R})[\mathcal{U}]$ satisfies (i); and since DC implies that every infinite set has a denumerably infinite subset, it follows easily that every infinite set can be expressed as the union of a countable strictly increasing chain of proper subsets in $L(\mathbb{R})[\mathcal{U}]$. Consequently, assuming LC, if $(p_n \mid n \in \omega)$ is any increasing sequence of primes, then (ii) must fail in $L(\mathbb{R})[\mathcal{U}]$.

Corollary 1.10 (LC)**.** *If $(p_n \mid n \in \omega)$ is any increasing sequence of primes, then the field $\prod_{\mathcal{U}} \mathbb{F}_{p_n}$ does not have a transcendence basis in $L(\mathbb{R})[\mathcal{U}]$.*

This paper is organized as follows. Let $(p_n \mid n \in \omega)$ be an increasing sequence of primes. In Section 2, we will prove that $\prod SL(2, p_n)$ is not Cayley bounded and does not have the Steinhaus property. In Section 3, working with the axiom system $ZF + DC$, we will prove that if $\prod SL(2, p_n)$ does not have the Bergman property, then there exists a nonprincipal ultrafilter over ω; and we will show that the failure of a weak form of the Steinhaus property also implies the existence of such an ultrafilter. In Section 4, we will present a partition property PP for products of finite sets with measures; and we will show that $ZF + DC + PP$ implies that if $(p_n \mid n \in \omega)$ is a sufficiently fast growing sequence of primes, then $\prod SL(2, p_n)$ has both the Bergman property and the Steinhaus property. In Section 5, we will briefly discuss the questions of which infinite products of nonabelian finite simple groups have either the Bergman property or the Steinhaus property in the actual set-theoretic universe V. Finally, in Section 6, assuming the existence of suitable large cardinals, we will prove that $L(\mathbb{R})[\mathcal{U}]$ satisfies PP.

Notation 1.11. Let $(H_n \mid n \in \omega)$ be a sequence of finite groups and let $H = \prod H_n$. Suppose that $A \subseteq \omega$.

(i) $\prod_{n \in A} H_n$ denotes the subgroup of H consisting of those elements $(h_n) \in H$ such that $h_n = 1$ for all $n \in \omega \setminus A$.

(ii) If $h = (h_n) \in H$, then $h \restriction A$ denotes the element $(g_n) \in \prod_{n \in A} H_n$ such that $g_n = h_n$ for all $n \in A$.

Recall that a subgroup $G \leqslant H$ is open if and only if there exists a cofinite subset $A \subseteq \omega$ such that $\prod_{n \in A} H_n \leqslant G$.

2. On the failure of the Bergman and Steinhaus properties

In this section, we will prove Theorems (1.5)(b) and (1.5)(c). Once again, let $\mathcal{U}$ be a nonprincipal ultrafilter over ω and let $K = \prod_{\mathcal{U}} \mathbb{F}_{p_n}$ be the corresponding ultraproduct of the fields $\mathbb{F}_{p_n}$ of order p_n. The arguments in this section depend upon the existence of a suitable valuation $\upsilon : K \to \mathbb{Q} \cup \{ \infty \}$.

Definition 2.1. Let F be a field and let t be an indeterminate over F. Then $F((t))$ denotes the corresponding field of formal power series; and

$$\mathbf{P}(F) = \bigcup_{n \geq 1} F((t^{1/n}))$$

denotes the corresponding *field of Puiseux expansions*. Let $\upsilon_F : \mathbf{P}(F) \to \mathbb{Q} \cup \{ \infty \}$ be the valuation which is defined as follows. If

$$0 \neq a = \sum_{k \geq M}^{\infty} a_k t^{k/n} \in \mathbf{P}(F)$$

where $a_k \in F$, $a_M \neq 0$, k, $M \in \mathbb{Z}$ and $n \geq 1$, then $\upsilon_F(a) = M/n$. (As usual, we set $\upsilon_F(0) = \infty$.)

It is well-known that if F is an algebraically closed field of characteristic 0, then $\mathbf{P}(F)$ is algebraically closed. (For example, see Chevalley [4].) In particular, if $\overline{\mathbb{Q}}$ is the field of algebraic numbers, then $\mathbf{P}(\overline{\mathbb{Q}})$ is an algebraically closed field of cardinality 2^ω. Hence, since $K = \prod_{\mathcal{U}} \mathbb{F}_{p_n}$ is a field of characteristic 0 and $|K| = 2^\omega$, we can suppose that K is a subfield of $\mathbf{P}(\overline{\mathbb{Q}})$. Furthermore, since K is uncountable and the automorphism group of $\mathbf{P}(\overline{\mathbb{Q}})$ acts transitively on non-algebraic elements, we can suppose that $t \in K$. From now on, we let $\upsilon = \upsilon_{\overline{\mathbb{Q}}} \restriction K$

denote the corresponding valuation of K and let $R = \{ a \in K \mid \upsilon(a) \geq 0 \}$ be the corresponding valuation ring.

The fact that $\prod SL(2, p_n)$ does not have the Steinhaus Property is an easy consequence of the following result, which was proved in Thomas [29, Section 2].

Theorem 2.2. *$[\, SL(2, K) : SL(2, R)\,] = \omega$.*

Sketch proof. For each $n \in \omega$, let $K_n = K \cap \overline{\mathbb{Q}}((t^{1/n!}))$ and let $R_n = R \cap K_n$. Then $\upsilon \restriction K_n$ is a nontrivial discrete valuation on K_n with countable residue field and hence $[\, SL(2, K_n) : SL(2, R_n)\,] = \omega$. (For example, see Serre [26, Chapter II].) Since $K = \bigcup_{n \in \omega} K_n$, it follows that $[\, SL(2, K) : SL(2, R)\,] = \omega$. □

Corollary 2.3. *$\prod SL(2, p_n)$ does not have the Steinhaus Property.*

Proof. Let $\pi : \prod SL(2, p_n) \to SL(2, K)$ be the canonical surjective homomorphism and let $H = \pi^{-1}[SL(2, R)]$. Then $[\, \prod SL(2, p_n) : H\,] = \omega$ and it follows that H is a symmetric countably syndetic subset of $\prod SL(2, p_n)$. Obviously $H^k = H$ for all $k \geq 1$ and it is clear that H does not contain a neighborhood of the identity, since this would include a subgroup of finite index. □

In the remainder of this section, we will prove that $\prod SL(2, p_n)$ is not Cayley bounded. By the following easy observation, it is enough to show that $SL(2, K)$ is not Cayley bounded.

Lemma 2.4. *Suppose that G is a group and that $N \trianglelefteq G$ is a normal subgroup. If G is Cayley bounded, then $H = G/N$ is also Cayley bounded.*

Proof. Suppose that the symmetric generating set $S \subseteq H$ witnesses that H is not Cayley bounded. Let $\pi : G \to H$ be the canonical surjective homomorphism and let $T = \pi^{-1}(S)$. Then T witnesses that G is not Cayley bounded. □

Recall that after identifying K with its image under a suitable embedding into the field $\mathbf{P}(\overline{\mathbb{Q}})$ of Puiseux series in the indeterminate t, we are supposing that $t \in K = \prod_{\mathcal{U}} \mathbb{F}_{p_n}$. Also note that $\upsilon(t) = 1$ and that $\upsilon(t^{-1}) = -1$. For each $k \in K^* = K \smallsetminus \{0\}$, let

$$x(k) = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \qquad y(k) = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix} \qquad d(k) = \begin{pmatrix} k & 0 \\ 0 & k^{-1} \end{pmatrix}$$

Then it is well-known that $T = \{ x(k) \mid k \in K^* \} \cup \{ y(k) \mid k \in K^* \}$ generates $SL(2, K)$. (For example, see Lang [17, Lemma XIII.8.1].) Let

$$U = \{ d(t), d(t^{-1}) \} \cup \{ x(k) \mid 0 \leq \upsilon(k) \leq 2\} \cup \{ y(k) \mid 0 \leq \upsilon(k) \leq 2\}.$$

Since $\upsilon(-k) = \upsilon(k)$ for all $k \in K^*$, it follows that U is a symmetric subset of $SL(2, K)$. We claim that U generates $SL(2, K)$. To see this, note that

$$d(t)x(k)d(t)^{-1} = x(t^2k) \qquad d(t)^{-1}x(k)d(t) = x(t^{-2}k)$$

and that

$$\upsilon(t^2k) = \upsilon(t^2) + \upsilon(k) = \upsilon(k) + 2 \qquad \upsilon(t^{-2}k) = \upsilon(t^{-2}) + \upsilon(k) = \upsilon(k) - 2.$$

Hence if $k \in K^*$, then there exists $m \in \mathbb{Z}$ such that $d(t)^m x(k)d(t)^{-m} \in U$; and similarly, there exists $m \in \mathbb{Z}$ such that $d(t)^m y(k)d(t)^{-m} \in U$. It follows that $T \subseteq \langle U \rangle$ and hence $\langle U \rangle = SL(2, K)$. Next for each matrix

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \in SL(2, K),$$

we define

$$\tau(A) = \min\{ \upsilon(a_i) \mid 1 \leq i \leq 4, a_i \neq 0 \}.$$

Notice that since

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} = \begin{pmatrix} a_1b_1 + a_2b_3 & a_1b_2 + a_2b_4 \\ a_3b_1 + a_4b_3 & a_3b_2 + a_4b_4 \end{pmatrix}$$

and since, for example,

$$\begin{aligned} \upsilon(a_1b_1 + a_2b_3) &\geq \min\{ \upsilon(a_1b_1), \upsilon(a_2b_3) \} \\ &= \min\{ \upsilon(a_1) + \upsilon(b_1), \upsilon(a_2) + \upsilon(b_3) \}, \end{aligned}$$

it follows that $\tau(AB) \geq \tau(A) + \tau(B)$ for all A, $B \in SL(2, K)$. Finally recall that for each $m \in \mathbb{N}$, we have that $\upsilon(t^{-m}) = -m$ and so $\tau(d(t^m)) = -m$. It now follows easily that for each $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $d(t^m)$ is not a product of n elements of $U \cup \{ 1 \}$. Thus $SL(2, K)$ is not Cayley bounded and it follows that $\prod SL(2, p_n)$ is not Cayley bounded.

3. On the existence of nonprincipal ultrafilters

In this section, working with the axiom system $ZF + DC$, we will first prove that if $\prod SL(2, p_n)$ does not have the Bergman property, then there exists a nonprincipal ultrafilter over ω. It is currently not known whether the failure of the Steinhaus property also implies the existence of a nonprincipal ultrafilter over ω. However, we will show that the failure of a weak form of the Steinhaus property does imply the existence of such an ultrafilter. We will make repeated use of the following algebraic result.

Proposition 3.1 (Ellers-Gordeev-Herzog [10]). *If K is any field such that $|K| > 5$ and C is any noncentral conjugacy class of $SL(2, K)$, then $C^8 = SL(2, K)$.*

Proof of Theorem 1.7 ($ZF + DC$). Suppose that $G = \prod SL(2, p_n)$ does not have the Bergman property. Then we can express $G = \bigcup_{k\in\omega} U_k$ as the union of a strictly increasing chain of symmetric proper subsets such that $U_k U_k \subseteq U_{k+1}$ for all $k \in \omega$. Consider

$$\mathcal{I} = \{\, A \subseteq \omega \mid \prod_{n\in A} SL(2, p_n) \subseteq U_k \text{ for some } k \in \omega \,\}.$$

Then clearly $\mathcal{I}$ is an ideal which contains all the finite subsets of ω. Hence it is enough to prove that there exists a set $B \notin \mathcal{I}$ such that $\mathcal{I} \cap \mathcal{P}(B)$ is a prime ideal over B.

Suppose that no such set B exists. Then for each $A \notin \mathcal{I}$, there exists $A' \subseteq A$ such that $A' \notin \mathcal{I}$ and $A \smallsetminus A' \notin \mathcal{I}$; and hence we can inductively find pairwise disjoint subsets $\{\, A_k \mid k \in \omega \,\}$ of ω such that $A_k \notin \mathcal{I}$ and $\omega \smallsetminus \bigcup_{\ell\leq k} A_\ell \notin \mathcal{I}$ for all $k \in \omega$.

Claim 3.2. *There exists $k \in \omega$ such that for every $h \in \prod_{n\in A_k} SL(2, p_n)$, there exists $g \in U_k$ such that $g \restriction A_k = h$.*

Proof of Claim 3.2. If not, then there exists $h \in G$ such that for all $k \in \omega$ and $g \in U_k$, we have that $g \restriction A_k \neq h \restriction A_k$. But this means that $h \notin \bigcup_{k\in\omega} U_k$, which is a contradiction. □

Fix some such $k \in \omega$ and let $h = (\, h_n \,) \in \prod_{n\in A_k} SL(2, p_n)$ be such that h_n is a noncentral element of $SL(2, p_n)$ for all $n \in A_k$. Let $h \in U_\ell$ and let $m = \max\{\, k, \ell \,\}$. Then it follows that the conjugacy class C of h in $\prod_{n\in A_k} SL(2, p_n)$ is contained in U_m^3; and hence by Proposition 3.1, $\prod_{n \in A_k} SL(2, p_n)$ is contained in U_m^{24}. But this means that $\prod_{n \in A_k} SL(2, p_n) \subseteq U_{m+5}$, which contradicts the fact that $A_k \notin \mathcal{I}$. This completes the proof of Theorem 1.7. □

In the remainder of this section, we will consider the following weak form of the Steinhaus property.

Definition 3.3. The Polish group G is said to have the *weak Steinhaus property* if for every symmetric countably syndetic subset $W \subseteq G$, there exists an integer $k \geq 1$ such that W^k contains an open neighborhood of the identity element 1_G.

For example, if the Polish group G has a non-open subgroup of countable index, then clearly G does not have the weak Steinhaus property. In particular, if we work with ZFC, then Theorem 2.2 implies that $\prod SL(2, p_n)$ does not have the weak Steinhaus property. The rest of this section is devoted to the proof of the following result.

Theorem 3.4 $(ZF + DC)$**.** *Let* $(\, p_n \mid n \in \omega \,)$ *be an increasing sequence of primes. If* $\prod SL(2, p_n)$ *does not have the weak Steinhaus property, then there exists a non-principal ultrafilter over* ω.

Most of our effort will go into proving the following special case of Theorem 3.4.

Theorem 3.5 $(ZF + DC)$**.** *Let* $(\, p_n \mid n \in \omega \,)$ *be an increasing sequence of primes. If there exists a subgroup* $H \leqslant \prod SL(2, p_n)$ *such that* $[\, \prod SL(2, p_n) : H \,] = \omega$, *then there exists a nonprincipal ultrafilter over* ω.

The proof of Theorem 3.5 makes use of some of the basic properties of primitive permutation groups. Recall that if Ω is any nonempty set and $G \leqslant \mathrm{Sym}(\Omega)$, then G is said to act *primitively* on Ω if:

(i) G acts transitively on Ω; and

(ii) there does not exist a nontrivial G-invariant equivalence relation on Ω.

It is well-known that if $G \leqslant \mathrm{Sym}(\Omega)$ is a transitive subgroup, then G acts primitively on Ω if and only if the stabilizer $G_\alpha = \{\, g \in G \mid g(\alpha) = \alpha \,\}$ is a maximal subgroup of G for some (equivalently every) $\alpha \in \Omega$. Also if G acts primitively on Ω and $1 \neq N \trianglelefteq G$ is a nontrivial normal subgroup, then it follows that N must act transitively on Ω. (For example, see Cameron [3, Theorem 1.7].)

The proof of Theorem 3.5 also makes use of the following easy consequence of Proposition 3.1.

Lemma 3.6 ($ZF + DC$)**.** *If* $(p_n \mid n \in \omega)$ *is an increasing sequence of primes, then every normal subgroup* N *of countable index in* $\prod SL(2, p_n)$ *is open.*

Proof. Let $G = \prod SL(2, p_n)$ and let $\mathcal{F} = \{ g_\tau = (g_{\tau(n)}) \mid \tau \in 2^{\mathbb{N}} \} \subseteq G$ be a family such that for each $\tau \neq \sigma \in 2^{\mathbb{N}}$, there exists an integer $n_0 \geq 0$ such that

- $g_{\tau(n)} = g_{\sigma(n)}$ for all $n < n_0$; and
- $g^{-1}_{\tau(n)} g_{\sigma(n)}$ is a noncentral element of $SL(2, p_n)$ for all $n \geq n_0$.

Since $[\, G : N \,] \leq \omega$, there exist $\tau \neq \sigma \in 2^{\mathbb{N}}$ such that $g_\tau N = g_\sigma N$ and hence $g = g^{-1}_{\tau(n)} g_{\sigma(n)} \in N$. Since N is a normal subgroup, the conjugacy class $C = g^G$ is contained in N. Applying Proposition 3.1, it follows easily that N contains the open subgroup $\prod_{n \geq n_0} SL(2, p_n)$ and hence N is open. □

Proof of Theorem 3.5. Let $G = \prod SL(2, p_n)$ and let $\{ P_j \mid j \in J \}$ be the set of open subgroups of G such that $H \leqslant P_j$. Since $H \leqslant \bigcap_{j \in J} P_j$ and the intersection of infinitely many open subgroups of G has index 2^ω, it follows that J is finite. Let

$$G' = \prod_{n \geq n_0} SL(2, p_n) \leqslant \bigcap_{j \in J} P_j.$$

Then after replacing G by G' and H by its projection H' into G' if necessary, we can suppose that H is not contained in any proper open subgroups of G.

Let $G = \bigsqcup_{n \in \omega} g_n H$ be the coset decomposition of H in G. Then we can construct a strictly increasing chain H_n of proper subgroups of G as follows.

- $H_0 = H$.
- Suppose inductively that H_n has been defined and that $H \leqslant H_n < G$. If H_n is a maximal proper subgroup of G, then the construction terminates with H_n. Otherwise, let k_n be the least integer k such that $H_n < \langle\, H_n, g_k \,\rangle < G$ and let $H_{n+1} = \langle\, H_n, g_{k_n} \,\rangle$.

If the construction does not terminate after finitely many steps, then $G = \bigcup_{n \in \omega} H_n$ has countable cofinality; and hence, by Theorem 1.7, there exists a nonprincipal ultrafilter over ω. Thus we can suppose that there exists an integer n such that H_n is a maximal proper subgroup of G. We claim that $[\, G : H_n \,] = \omega$. If not, then $[\, G : H_n \,] < \omega$ and hence $N = \bigcap_{g \in G} g H_n g^{-1}$ is a normal subgroup of G such that $N \leqslant H_n$ and $[\,G : N\,] < \omega$. Applying Lemma 3.6, we see that N is an open subgroup of G and hence H_n is also an open subgroup of G. But this contradicts the fact that H is not contained in any proper open subgroups of G.

In order to simplify notation, we will suppose that H is a maximal subgroup of G. Hence, by considering the left translation action of G on the set $\{\, g_n H \mid n \in \mathbb{N}\,\}$, we obtain a homomorphism

$$\psi : G \to \mathrm{Sym}(\mathbb{N})$$

such that $\psi(G)$ acts primitively on $\mathbb{N}$. It follows that if $N \trianglelefteq G$ is any normal subgroup, then either $\psi(N) = 1$ or else $\psi(N)$ acts transitively on $\mathbb{N}$. Let

$$\mathcal{I} = \{\, A \subseteq \omega \mid \psi(\textstyle\prod_{n\in A} SL(2,p_n)) = 1\,\}.$$

Then $\mathcal{I}$ is clearly an ideal on ω. Furthermore, if $F \subseteq \omega$ is a finite subset, then $\psi(\prod_{n\in F} SL(2,p_n))$ cannot act transitively on $\mathbb{N}$ and so $F \in \mathcal{I}$. We will show that $\mathcal{I}$ is a prime ideal.

So suppose that there exists a subset $A \subseteq \omega$ such that both $A \notin \mathcal{I}$ and $\omega \smallsetminus A \notin \mathcal{I}$. Let $P = \prod_{n\in A} SL(2,p_n)$ and let $Q = \prod_{n\in\omega\smallsetminus A} SL(2,p_n)$. Then both $\psi(P)$ and $\psi(Q)$ act transitively on $\mathbb{N}$. Suppose that $g \in P$ is such that $\psi(g)$ fixes some integer $n \in \mathbb{N}$. If $k \in \mathbb{N}$ is arbitrary, then there exists $h \in Q$ such that $\psi(h)(n) = k$; and since g and h commute, it follows that

$$\psi(g)(k) = (\,\psi(g)\circ\psi(h)\,)(n) = (\,\psi(h)\circ\psi(g)\,)(n) = \psi(h)(n) = k.$$

Thus $g \in \ker\psi$. It follows that $N = \ker\psi \cap P$ is a normal subgroup of P such that $[\,P : N\,] = \omega$, which contradicts Lemma 3.6. □

Proof of Theorem 3.4. Let $G = \prod SL(2,p_n)$ and suppose that the symmetric countably syndetic subset $W \subseteq G$ witnesses the failure of the weak Steinhaus property. Let $H = \langle\, W\,\rangle$ be the subgroup generated by W. Then clearly $[\,G : H\,] \leq \omega$. If $[\,G : H\,] = \omega$, then the result follows from Theorem 3.5 and so we can suppose that $[\,G : H\,] < \omega$. Applying Lemma 3.6, it follows easily that H is an open subgroup of G. Let

$$G' = \prod_{n\geq n_0} SL(2,p_n) \leqslant H$$

and let $\pi : G \to G'$ be the canonical projection. Consider the set $W' = \pi(W)$ of generators of G'. If W' witnesses that G' is not Cayley bounded, then the result follows from Theorem 1.7. Hence we can suppose that there exists an integer $k \geq 1$ such that $(W')^k = G'$. Let $g = (g_n) \in G'$ be such that g_n is a noncentral element of $SL(2, p_n)$ for all $n \geq n_0$. Then $g \in W^\ell$ for some $\ell \geq 1$; and Proposition 3.1 implies that

$$G' \subseteq \underbrace{W^k W^\ell W^k \cdots W^k W^\ell W^k}_{8 \text{ times}} = W^{16k+8\ell}.$$

But this contradicts the assumption that W witnesses the failure of the weak Steinhaus property. □

4. The Bergman and Steinhaus Properties in $L(\mathbb{R})[\mathcal{U}]$

In this section, assuming the existence of suitable large cardinals (LC), we will prove that if $(p_n \mid n \in \omega)$ is a sufficiently fast growing sequence of primes, then $\prod SL(2, p_n)$ has both the Bergman property and the Steinhaus property in $L(\mathbb{R})[\mathcal{U}]$. More precisely, we will present a partition property PP for products of finite sets with measures; and we will show that $ZF + DC + PP$ implies that if $(p_n \mid n \in \omega)$ is a sufficiently fast growing sequence of primes, then $\prod SL(2, p_n)$ has both the Bergman property and the Steinhaus property. Then in Section 6, assuming the existence of suitable large cardinals, we will prove that PP holds in $L(\mathbb{R})[\mathcal{U}]$.

The Partition Property (PP)**.** If $(\langle a_n, \mu_n \rangle \mid n \in \omega)$ is a sufficiently fast growing sequence of finite sets a_n with measures μ_n, then for every partition

$$\prod a_n = \bigsqcup_{m \in \omega} X_m,$$

there exists an integer $m \in \omega$ such that $\prod b_n \subseteq X_m$ for some sequence of subsets $b_n \subseteq a_n$ such that $\lim_{n \to \infty} \mu_n(b_n) = \infty$.

Here the words "sufficiently fast growing" should be interpreted in the sense that there is a *fixed* function f that assigns a natural number to every finite sequence of finite sets with measures $(\langle a_m, \mu_m \rangle \mid m < n)$ and that an infinite sequence $(\langle a_n, \mu_n \rangle \mid n \in \omega)$ is sufficiently fast growing if

$$\mu_n(a_n) > f((\langle a_m, \mu_m \rangle \mid m < n))$$

for all $n \in \omega$. The exact formula for the function f is immaterial for the purposes of this paper. We will only mention that it is primitive recursive with a growth rate approximately that of a tower of exponentials of linear height.

The partition property PP fails in ZFC, since the Axiom of Choice can be used to construct highly irregular partitions. However, it does hold in ZFC if we restrict our attention to partitions into Borel sets; and it also hold for arbitrary partitions in many models of set theory in which the Axiom of Choice fails. In particular, we will prove the following result in Section 6.

Theorem 4.1 (LC)**.** *$L(\mathbb{R})[\mathcal{U}]$ satisfies PP.*

We will also make use of the following recent result of Babai-Nikolov-Pyber [1] in the newly flourishing area of "arithmetic combinatorics".

Definition 4.2. If H is a finite group, then $d(H)$ denotes the minimal dimension of a nontrivial complex representation of H; i.e. the least d such that there exists a nontrivial homomorphism $\theta : H \to GL(d, \mathbb{C})$.

Theorem 4.3 (Babai-Nikolov-Pyber [1])**.** *Let H be a nontrivial finite group and let k be an integer such that $1 \leq k^3 \leq d(H)$. If $A \subseteq H$ is a subset such that $|A| \geq |H|/k$, then $A^3 = H$.*

Proof. Let $d'(H)$ be the minimal dimension of a nontrivial real representation of H. Then Babai-Nikolov-Pyber [1, Corollary 2.6] implies that if $1 \leq k^3 \leq d'(H)$ and $A \subseteq H$ with $|A| \geq |H|/k$, then $A^3 = H$. Since $d(H) \leq d'(H)$, the result follows. □

Theorem 4.4 ($ZF + DC + PP$)**.** *If $(H_n \mid n \in \omega)$ is a sequence of finite groups such that $(d(H_n) \mid n \in \omega)$ grows sufficiently fast, then $\prod H_n$ has both the Bergman property and the Steinhaus property.*

Proof. For each $n \in \omega$, let $k_n = \lfloor d(H_n)^{1/3} \rfloor$ and let μ_n be the measure on H_n defined by $\mu_n(A) = k_n \, (|A|/|H_n|)$. To see that $G = \prod H_n$ has the Steinhaus property, suppose that $W \subseteq G$ is a symmetric countably syndetic subset and let $G = \bigcup_{m \in \omega} g_m W$. Since $\mu_n(H_n) = k_n$ grows sufficiently fast, PP implies that there exists $m \in \omega$ such that $\prod A_n \subseteq g_m W$ for some sequence of subsets $A_n \subseteq H_n$ such that $\lim_{n \to \infty} \mu_n(A_n) = \infty$; and after replacing $\prod A_n$ by $g_m^{-1} \prod A_n$, we can suppose that $\prod A_n \subseteq W$. Let $n_0 \in \omega$ be such that $\mu_n(A_n) \geq 1$ and hence $|A_n| \geq |H_n|/k_n$ for all $n \geq n_0$. Clearly we can suppose that $A_n = \{ a_n \}$ is a singleton for each $n < n_0$. Applying Theorem 4.3, it follows that $W^3 \supseteq gG'$, where

- $g = (\, a_0^3, \cdots, a_{n_0-1}^3, 1, 1, \cdots)$ and
- G' is the open subgroup $\{ \, (h_n) \in \prod H_n \mid h_n = 1 \text{ for all } n < n_0 \, \}$.

Since W is symmetric, it follows that $W^6 \supseteq (gG')^{-1} gG' = G'$. This completes the proof that $\prod H_n$ has the Steinhaus property.

To see that $G = \prod H_n$ has the Bergman property, suppose that we can express $G = \bigcup_{m \in \omega} U_m$ as the union of an increasing chain of symmetric proper subsets such that $U_m U_m \subseteq U_{m+1}$ for all $m \in \omega$. Arguing as above, it follows that there exists $m \in \omega$ such that U_m^6 contains an open subgroup G' and hence $G' \subseteq U_{m+3}$. Since $[\, G : G' \,] < \infty$, this implies that there exists $k \in \omega$ such that $G = U_k$, as required. □

It is perhaps worth pointing out that the proof of following corollary does *not* make use of the classification of the finite simple groups.

Corollary 4.5 (*LC*)**.** *If* $(\, S_n \mid n \in \omega \,)$ *is a sufficiently fast growing sequence of nonabelian finite simple groups, then* $\prod S_n$ *has both the Bergman property and the Steinhaus property in* $L(\mathbb{R})[\mathcal{U}]$.

Proof. By Jordan's Theorem, there exists a function $\varphi : \mathbb{N} \to \mathbb{N}$ such that if H is a finite subgroup of $GL(n, \mathbb{C})$, then H contains an abelian normal subgroup N with $[\, H : N \,] \leq \varphi(n)$. (For example, see Curtis-Reiner [5, Theorem 36.13].) Hence if $|S_n|$ grows sufficiently fast, then $d(S_n)$ also grows sufficiently fast. □

Corollary 4.6 (*LC*)**.** *If* $(\, p_n \mid n \in \omega \,)$ *is a sufficiently fast growing sequence of primes, then* $\prod SL(2, p_n)$ *has both the Bergman property and the Steinhaus property in* $L(\mathbb{R})[\mathcal{U}]$.

Proof. It is well-known that if p is an odd prime, then the minimal dimension of a nontrivial complex representation of $SL(2,p)$ is $(p-1)/2$. (For example, see Humphreys [12].) □

The growth condition on $(\, p_n \mid n \in \omega \,)$ in the statement of Corollary 4.6 is almost certainly not necessary.

Theorem 4.7 (*LC*)**.** *If* $(\, p_n \mid n \in \omega \,)$ *is* any *increasing sequence of primes, then* $\prod SL(2, p_n)$ *has both the Bergman property and the weak Steinhaus property in* $L(\mathbb{R})[\mathcal{U}]$.

The proof of Theorem 4.7 makes use of the following two simple observations.

Lemma 4.8. *If $p_1 < \cdots < p_t$ are odd primes and $H = SL(2, p_1) \times \cdots \times SL(2, p_t)$, then $d(H) = (p_1 - 1)/2$.*

Proof. Clearly $d(H) \leq d(\, SL(2, p_1)\,) = (p_1 - 1)/2$. So suppose that $d < (p_1 - 1)/2$ and that $\theta : H \to GL(d, \mathbb{C})$ is a homomorphism. Then $\theta \restriction SL(2, p_i)$ is the trivial homomorphism for each $1 \leq i \leq n$ and hence θ is trivial. □

Lemma 4.9. *If $(\, p_n \mid n \in \omega\,)$ is an increasing sequence of primes, then there exists an increasing sequence of integers $0 = a_0 < a_1 < \cdots < a_n < \cdots$ such that if*

$$H_n = \prod_{a_{2n} \leq i < a_{2n+1}} SL(2, p_i) \quad \text{and} \quad K_n = \prod_{a_{2n+1} \leq i < a_{2n+2}} SL(2, p_i),$$

then both $(\, d(H_n) \mid n \in \omega\,)$ and $(\, d(K_n) \mid n \in \omega\,)$ grow sufficiently fast.

Proof. First let $a_1 = 1$. Now suppose that $n \geq 1$ and that a_ℓ has been defined for all $\ell \leq n$. Suppose, for example, that $n = 2m+1$ is odd, so that the groups $H_0, \cdots, H_m$ have already been determined. Then we can choose a_{2m+2} so that $(p_{2m+2} - 1)/2$ is sufficiently large with respect to $(\, d(H_0), \cdots, d(H_m)\,)$. Applying Lemma 4.8, it follows that for any choice of a_{2m+3}, we will have that $d(H_{m+1}) = (p_{2m+2} - 1)/2$ is sufficiently large with respect to $(\, d(H_0), \cdots, d(H_m)\,)$. □

Proof of Theorem 4.7. Let $(\, p_n \mid n \in \omega\,)$ be any increasing sequence of primes. Working inside $L(\mathbb{R})[\mathcal{U}]$, first suppose that $\prod SL(2, p_n)$ does not have the Bergman property and express $G = \bigcup_{k \in \omega} U_k$ as the union of a strictly increasing chain of symmetric proper subsets such that $U_k U_k \subseteq U_{k+1}$ for all $k \in \omega$. Then

$$\mathcal{I} = \{\, A \subseteq \omega \mid \textstyle\prod_{n \in A} SL(2, p_n) \subseteq U_k \text{ for some } k \in \omega\,\}.$$

is an ideal which contains all the finite subsets of ω; and the proof of Theorem 1.7 shows that there exists a set $B \notin \mathcal{I}$ such that $\mathcal{I} \cap \mathcal{P}(B)$ is a prime ideal over B. In order to simplify notation, suppose that $B = \omega$ and let $\mathcal{D}$ be the dual nonprincipal ultrafilter over ω. Let $(\, a_n \mid n \in \omega\,)$ be the increasing sequence of natural numbers given by Lemma 4.9. Then we can suppose that

$$A = \{\, i \mid a_{2n} \leq i < a_{2n+1} \text{ for some } n \in \omega\,\} \in \mathcal{D}.$$

Since $(d(H_n) \mid n \in \omega)$ grows sufficiently fast, it follows that

$$\prod H_n = \prod_{n \in A} SL(2, p_n)$$

has the Bergman property. For each $k \in \omega$, let

$$W_k = \{ g \restriction A \mid g \in U_k \} \subseteq \prod_{n \in A} SL(2, p_n).$$

Then there exists $k \in \omega$ such that $W_k = \prod_{n \in A} SL(2, p_n)$. Arguing as in the proof of Theorem 1.7, it follows that $\prod_{n \in A} SL(2, p_n) \subseteq U_\ell$ for some $\ell \geq k$, which contradicts the fact that $A \notin \mathcal{I}$. Thus $\prod SL(2, p_n)$ has the Bergman property.

To show that $\prod SL(2, p_n)$ has the weak Steinhaus property, we will first prove that $\prod SL(2, p_n)$ has no subgroups H such that $[\prod SL(2, p_n) : H] = \omega$. So suppose that such a subgroup H exists. Then, arguing as in the proof of Theorem 3.5 and using the fact that $\prod SL(2, p_n)$ has the Bergman property, we can suppose that H is a maximal proper subgroup. Hence, by considering the left translation action of $\prod SL(2, p_n)$ on the set of cosets of H in $\prod SL(2, p_n)$, we obtain a homomorphism

$$\psi : \prod SL(2, p_n) \to \mathrm{Sym}(\mathbb{N})$$

such that $\psi(\prod SL(2, p_n))$ acts primitively on $\mathbb{N}$. In particular, it follows that if $N \trianglelefteq \prod SL(2, p_n)$ is any normal subgroup, then either $\psi(N) = 1$ or else $\psi(N)$ acts transitively on $\mathbb{N}$. Arguing as in the proof of Theorem 3.5, we see that

$$\mathcal{D} = \{ A \subseteq \omega \mid \psi(\prod_{n \in A} SL(2, p_n)) \neq 1 \}.$$

is a nonprincipal ultrafilter over ω. Arguing as in the previous paragraph, it follows that there exists a subset $A \in \mathcal{D}$ such that $\prod_{n \in A} SL(2, p_n)$ has the Steinhaus property. But since $\psi(\prod_{n \in A} SL(2, p_n))$ acts transitively on $\mathbb{N}$, there exists a subgroup K such that $[\prod_{n \in A} SL(2, p_n) : K] = \omega$, which is a contradiction.

At this point, we know that $\prod SL(2, p_n)$ has the Bergman property and that $\prod SL(2, p_n)$ has no subgroups H with $[\prod SL(2, p_n) : H] = \omega$. Arguing as in the proof of Theorem 3.4, it follows easily that $\prod SL(2, p_n)$ has the weak Steinhaus property. □

We will conclude this section with a result which shows that it is necessary to impose some condition on the growth rate of the sequence $(d(H_n) \mid n \in \omega)$ if we wish to obtain the conclusion of Theorem 4.4.

Theorem 4.10 *($ZF + DC$).* *Suppose that there exists a nonprincipal ultrafilter over ω. Then whenever $(H_n \mid n \in \omega)$ is a sequence of finite groups such that $\liminf d(H_n) < \infty$, then $\prod H_n$ does not have the Steinhaus property.*

Proof. Recall that every complex representation of a finite group is similar to a unitary representation. (For example, see Curtis-Reiner [5, Exercise 10.6].) Hence there exists an infinite subset $I \subseteq \omega$ and a fixed integer $d \geq 1$ such that for each $n \in I$, there exists a nontrivial homomorphism $\varphi_n : H_n \to \mathrm{U}(d, \mathbb{C})$, where $\mathrm{U}(d, \mathbb{C})$ denotes the compact group of $d \times d$ unitary matrices. In order to simplify notation, we will suppose that $I = \omega$.

For each $g_n \in H_n$ and $1 \leq i, j \leq d$, let $\varphi_n(g_n)_{ij}$ denote the ij entry of the matrix $\varphi_n(g_n) \in \mathrm{U}(d, \mathbb{C})$. Then if $\mathcal{U}$ is a nonprincipal ultrafilter over ω, we can define a homomorphism

$$\psi : \prod H_n \to \mathrm{U}(d, \mathbb{C})$$
$$(g_n) \mapsto (z_{ij}),$$

where $z_{ij} = \lim_{\mathcal{U}} \varphi_n(g_n)_{ij}$. By Proposition 1.4, in order to prove that $\prod H_n$ does not have the Steinhaus property, it is enough to show that ψ is not continuous. So suppose that ψ is continuous and let $W \subseteq \mathrm{U}(d, \mathbb{C})$ be an open neighborhood of the identity element which contains no nontrivial subgroups. (For the existence of such a neighborhood, see Helgason [11, II.B.5].) Then there exists an open subgroup $H \subseteq \prod H_n$ such that $\psi(H) \subseteq W$ and clearly this implies that $H \leqslant \ker \psi$. In particular, there exists a cofinite subset $K \subseteq \omega$ such that $\prod_{k \in K} H_k \leqslant \ker \psi$. For each $k \in K$, choose $g_k \in H_k$ such that $\varphi_k(g_k) \notin W$. Then, letting $g = (g_k) \in \prod_{k \in K} H_k$, we have that $\psi(g) \notin W$, which is a contradiction. □

Remark 4.11. Recall that de Cornulier [6] has shown that if G is an infinite product of countably many copies of a *fixed* finite perfect group H, then G has the Bergman property. Thus the analogue of Theorem 4.10 is false for the Bergman property.

5. The Bergman and Steinhaus Properties in V

Suppose that $(S_n \mid n \in \omega)$ is a sequence of nonabelian finite simple groups. Then, applying Theorem 4.1 and Corollary 4.5, it follows that if $(S_n \mid n \in \omega)$ is sufficiently fast growing, then $\prod S_n$ has both the Bergman property and the

Steinhaus property in $L(\mathbb{R})[\mathcal{U}]$. In this section, we will briefly discuss the question of when $\prod S_n$ has either the Bergman property or the Steinhaus property in the actual set-theoretic universe V. In particular, throughout this section, we will work with the usual ZFC axioms of set theory.

Recall that the classification of the finite simple groups says that if S is a non-abelian finite simple group, then one of the following cases must hold.

(i) S is one of the 26 sporadic finite simple groups.
(ii) S is an alternating group $\mathrm{Alt}(n)$ for some $n \geq 5$.
(iii) S is a group $L(q)$ of (possibly twisted) Lie type L over a finite field $\mathbb{F}_q$ for some prime power q.

The following condition is the key to understanding when the product $\prod S_n$ has countable cofinality.

Definition 5.1. A sequence $(S_n \mid n \in \omega)$ of nonabelian finite simple groups satisfies the *Malcev condition* if there exists an infinite subset I of ω such that the following properties hold.

(a) There exists a fixed (possibly twisted) Lie type L such that for all $n \in I$, $S_n = L(q_n)$ for some prime power q_n.
(b) If n, $m \in I$ and $n < m$, then $q_n < q_m$.

Arguing as in the proof of Theorem 1.5(a), it follows easily that if $(S_n \mid n \in \omega)$ satisfies the Malcev condition, then $\prod S_n$ has countable cofinality. Conversely, by Saxl-Shelah-Thomas [24, Theorem 1.9], if $(S_n \mid n \in \omega)$ does not satisfy the Malcev condition, then $\prod S_n$ has uncountable cofinality. Furthermore, as we mentioned earlier, Rosendal [23] has checked that the arguments in Saxl-Shelah-Thomas [24] and Thomas [29] can be modified to prove that $\prod \mathrm{Alt}(2^n)$ has the Bergman property. Thus it seems natural to make the following conjecture.

Conjecture 5.2. If $(S_n \mid n \in \omega)$ is a sequence of nonabelian finite simple groups, then the following are equivalent:

(a) $(S_n \mid n \in \omega)$ does not satisfy the Malcev condition.
(b) $\prod S_n$ has the Bergman property.

The proof of Theorem 1.5(c) can be adapted to show that if $(S_n \mid n \in \omega)$ satisfies the Malcev condition, then there exists a subgroup $H \leqslant \prod S_n$ with $[\prod S_n : H] = \omega$

and hence $\prod S_n$ does not have the Steinhaus property. (See Thomas [29].) Also, it is easily seen that if $(S_n \mid n \in \omega)$ satisfies the following condition, then $\prod S_n$ has a non-open subgroup of finite index and so once again the Steinhaus property fails. (For example, see Saxl-Wilson [25].)

Definition 5.3. A sequence $(S_n \mid n \in \omega)$ of nonabelian finite simple groups satisfies the *Saxl-Wilson condition* if there exists an infinite subset I of ω and a fixed group S such that $S_n = S$ for all $n \in I$.

In Thomas [29], it was shown that $\prod S_n$ has a non-open subgroup H such that $[\prod S_n : H] < 2^\omega$ if and only if $(S_n \mid n \in \omega)$ satisfies neither the Malcev condition nor the Saxl-Wilson condition. Consequently, it seems natural to make the following conjecture.

Conjecture 5.4. If $(S_n \mid n \in \omega)$ is a sequence of nonabelian finite simple groups, then the following are equivalent:

(a) $(S_n \mid n \in \omega)$ satisfies neither the Malcev condition nor the Saxl-Wilson condition.

(b) $\prod S_n$ has the Steinhaus property.

6. The Partition Property (PP)

Theorem 6.1. *Suppose that κ is an inaccessible cardinal and that $G \subseteq \mathrm{Coll}(\omega, <\kappa)$ is a V-generic filter. If $V(\bar{\mathbb{R}})$ is the corresponding Solovay model and the Ramsey ultrafilter $\bar{\mathcal{U}}$ is $V(\bar{\mathbb{R}})$-generic for $\mathcal{P}(\omega)/\mathrm{Fin}$, then $V(\bar{\mathbb{R}})[\bar{\mathcal{U}}]$ satisfies PP.*

Proof of Theorem 4.1 (LC). Let κ be the least inaccessible cardinal and let $V(\bar{\mathbb{R}})[\bar{\mathcal{U}}]$ be as in Theorem 6.1. Then the Ramsey ultrafilter $\bar{\mathcal{U}}$ is also $L(\bar{\mathbb{R}})$-generic for $\mathcal{P}(\omega)/\mathrm{Fin}$. Working inside $L(\bar{\mathbb{R}})[\bar{\mathcal{U}}]$, suppose that $(\langle a_n, \mu_n \rangle \mid n \in \omega)$ is a sufficiently fast growing sequence of finite sets a_n with measures μ_n and that

$$\prod a_n = \bigsqcup_{m \in \omega} X_m,$$

is any partition. Since $V(\bar{\mathbb{R}})[\bar{\mathcal{U}}]$ satisfies PP, there exists an integer $m \in \omega$ and a sequence of subsets $(b_n \subseteq a_n \mid n \in \omega) \in V(\bar{\mathbb{R}})[\bar{\mathcal{U}}]$ such that $\prod b_n \subseteq X_m$ and $\lim_{n \to \infty} \mu_n(b_n) = \infty$. Since $L(\bar{\mathbb{R}})[\bar{\mathcal{U}}]$ and $V(\bar{\mathbb{R}})[\bar{\mathcal{U}}]$ have the same reals, it follows that $(b_n \subseteq a_n \mid n \in \omega) \in L(\bar{\mathbb{R}})[\bar{\mathcal{U}}]$. Thus $L(\bar{\mathbb{R}})[\bar{\mathcal{U}}]$ satisfies PP. Finally since the theory of $L(\mathbb{R})[\mathcal{U}]$ is not altered by forcing with $\mathrm{Coll}(\omega, <\kappa)$, it follows that $L(\mathbb{R})[\mathcal{U}]$ satisfies also PP. $\square$

The remainder of this section will be devoted to the proof of Theorem 6.1. The key element of the proof is the work of Shelah-Zapletal [28] showing that for every sufficiently fast growing sequence $(\langle a_n, \mu_n\rangle \mid n \in \omega)$ of finite sets a_n with measures μ_n, there is a notion of forcing $\mathbb{P}$ such that:

1. $\mathbb{P}$ adds a new element $\dot{x} \in \prod_n a_n$;
2. $\mathbb{P}$ is proper, ${}^\omega\omega$-bounding and adds no independent reals;
3. $\mathbb{P}$ is defined in a way that quantifies over real numbers only;
4. for every transitive model M of set theory and every condition $p \in \mathbb{P}^M$ there are sets $(b_n \mid n \in \omega)$ with $b_n \subseteq a_n$ and $\mu_n(b_n) \to \infty$ such that the product $\prod_n b_n$ consists of M-generic points for the poset $\mathbb{P}^M_p = \{ q \in \mathbb{P}^M \mid q \leq p \}$.

Here an independent real is an infinite set $a \subseteq \omega$ in the generic extension such that neither a nor $\omega \setminus a$ contains a ground model subset.

Let κ is an inaccessible cardinal and let $G \subseteq \mathrm{Coll}(\omega, <\kappa)$ be a V-generic filter. Suppose that $V(\bar{\mathbb{R}})$ is the corresponding Solovay model and that the Ramsey ultrafilter $\bar{\mathcal{U}}$ is $V(\bar{\mathbb{R}})$-generic for $\mathcal{P}(\omega)/\mathrm{Fin}$. Let $(\langle a_n, \mu_n\rangle \mid n \in \omega)$ be a sufficiently fast growing sequence of finite sets a_n with measures μ_n and let $\prod_n a_n = \bigsqcup_m B_m$ be a partition of the product into countably many pieces in the model $V(\mathbb{R})[\mathcal{U}]$. By standard homogeneity arguments regarding the poset $\mathrm{Coll}(\omega, <\kappa)$, we can assume that the sequence is in the ground model V and that the partition is definable from the elements of the ground model and the ultrafilter $\mathcal{U}$.

Working inside the ground model V, consider the product of the forcing $\mathbb{P}$ with $\mathbb{Q} = \mathcal{P}(\omega)/\mathrm{Fin}$. Then the poset $\mathbb{Q}$ adds a Ramsey ultrafilter $\mathfrak{u}$ and $\mathbb{P}$ adds a point $x \in \prod_n a_n$. Since the definition of the forcing $\mathbb{P}$ only depends on the real numbers, it follows that $\mathbb{P}^V = \mathbb{P}^{V[\mathfrak{u}]}$. Hence if $\mathfrak{u}$,x are mutually generic, then x will be $\mathbb{P}^{V[\mathfrak{u}]}$-generic over the model $V[\mathfrak{u}]$.

Lemma 6.2. *In $V[\mathfrak{u}][x]$, $\mathfrak{u}$ still generates a Ramsey ultrafilter.*

Proof. By Shelah [27, VI.5.1], since $\mathbb{P}^V = \mathbb{P}^{V[\mathfrak{u}]}$ is proper and ${}^\omega\omega$-bounding in $V[\mathfrak{u}]$, it is enough to show that $\mathfrak{u}$ still generates an ultrafilter in $V[\mathfrak{u}][x]$. First note that since $\mathbb{Q}$ is σ-closed and $\mathbb{P}$ is proper, it follows that $\mathcal{P}(\omega) \cap V[\mathfrak{u}][x] = \mathcal{P}(\omega) \cap V[x]$.

(Since $\mathbb{P}$ is proper, each real $r \in V[\mathfrak{u}][x]$ is obtained from a countable collection $\mathcal{C} = \{ C_n \mid n \in \omega \} \in V[\mathfrak{u}]$ of countable subsets $C_n \subseteq \mathbb{P}$ such that each C_n is predense below some condition $p \in \mathbb{P}$; and since $\mathbb{Q}$ is σ-closed, it follows that $\mathcal{C} \in V$ and hence $r \in V[x]$.) Now suppose that $p \in \mathbb{P}, q \in \mathbb{Q}$ are conditions and that $p \Vdash \tau \subseteq \omega$. Since $\mathbb{P}$ does not add any independent reals, there exists a condition $p' \leq p$ and an infinite subset $q' \subseteq q$ such that either $p' \Vdash q' \subseteq \tau$ or $p' \Vdash \tau \cap q' = \emptyset$. Hence either $\langle q', p' \rangle \Vdash \tau \in \dot{\mathfrak{u}}$ or $\langle q', p' \rangle \Vdash \omega \smallsetminus \tau \in \dot{\mathfrak{u}}$. It follows that $\mathfrak{u}$ still generates an ultrafilter in $V[\mathfrak{u}][x]$. $\square$

Let $\mathbb{D} \in V[\mathfrak{u}]$ be the poset consisting of the conditions (s, S), where $s \in [\omega]^{<\omega}$ and $S \in \mathfrak{u}$, partially ordered by

$$(s, S) \leq (t, T) \quad \Longleftrightarrow \quad s \supseteq t \text{ and } s \smallsetminus t \subseteq T.$$

Then $\mathbb{D}$ adjoins an infinite subset $\dot{c} \subseteq \omega$ which diagonalizes the Ramsey ultrafilter $\mathfrak{u}$; i.e. a subset $\dot{c}$ such that $|\dot{c} \smallsetminus S| < \infty$ for all $S \in \mathfrak{u}$. In fact, by Mathias [20], every set diagonalizing $\mathfrak{u}$ is $V[\mathfrak{u}]$-generic for the poset $\mathbb{D}$. By Lemma 6.2, if $\bar{\mathfrak{u}}$ is the upwards closure of $\mathfrak{u}$ in the model $V[\mathfrak{u}][x]$, $\bar{\mathfrak{u}}$ is a Ramsey ultrafilter in $V[\mathfrak{u}][x]$. Hence if $\bar{\mathbb{D}} \in V[x, \mathfrak{u}]$ is the corresponding poset diagonalizing $\bar{\mathfrak{u}}$, then $\mathbb{D}$ is dense in $\bar{\mathbb{D}}$ and every set diagonalizing $\mathfrak{u}$ is $V[\mathfrak{u}][x]$-generic for both $\bar{\mathbb{D}}$ and $\mathbb{D}$.

Lemma 6.3. *In $V[\mathfrak{u}][x]$, there is a natural number $n \in \omega$ such that*

$$\mathbb{D} \Vdash \mathrm{Coll}(\omega, < \kappa) \Vdash \dot{c} \Vdash \check{x} \in \dot{B}_n.$$

Proof. Suppose for contradiction that there are distinct numbers m_0, m_1 and conditions $\langle r_0, s_0, \dot{d}_0 \rangle$, $\langle r_1, s_1, \dot{d}_1$ in the three step iteration below $\langle 1, 1, \dot{c} \rangle$ such that the first one forces $\check{x} \in \dot{B}_{m_0}$ and the other $\check{x} \in \dot{B}_{m_1}$. Choose mutually $V[x, \mathfrak{u}]$-generic filters $H_0 \subseteq \mathbb{D}$, $K_0 \subseteq \mathrm{Coll}(\omega, < \kappa)$ meeting the conditions r_0, s_0 and note that $d = \dot{d}_0 / H_0, K_0 \subseteq \dot{c}/H$ is a $V[x, \mathfrak{u}]$-generic set for the poset $\mathbb{D}$. Thus, with a finite adjustment of the set d if necessary, we can find a filter $H_1 \subseteq \mathbb{D}$ meeting the condition r_1 such that $d = \dot{c}/H_1$. Standard homogeneity facts about the Levy collapse then show that there is a $V[x, \mathfrak{u}][H_1]$-generic filter $K_1 \subseteq \mathrm{Coll}(\omega, < \kappa)$ meeting the condition s_1 such that $V[x, \mathfrak{u}][H_0, K_0] = V[x, \mathfrak{u}][H_1, K_1]$. Now in this latter model, in the forcing $\mathcal{P}(\omega)/\mathrm{Fin}$, the forcing theorem on the 0 side says that $d \Vdash \check{x} \in \dot{B}_{m_0}$, and the forcing theorem on the 1 side says that $\dot{d}_1/H_1, K_1 \Vdash \check{x} \in \dot{B}_{m_1}$. However, $\dot{d}_1/H_1, K_1 \subseteq \dot{c}/H_1 = d$, a contradiction. $\square$

In the model $V[\mathfrak{u}]$, find a condition $p \in \mathbb{P}$ that identifies the number n from the previous claim. In the model $V(\mathbb{R})$, find sets $(b_n \mid n \in \omega)$ such that $b_n \subseteq a_n$ and the numbers $\phi_n(b_n)$ tend to infinity, so that the product $\prod_n b_n$ consists solely of points $\mathbb{P}$-generic for the model $V[\mathfrak{u}]$ meeting the condition p. Let $c \subseteq \omega$ be an infinite set diagonalizing the ultrafilter $\mathfrak{u}$. By the previous claim and the forcing theorem, the condition c forces in the poset $\mathcal{P}(\omega)/\text{Fin}$ $\check{x} \in \dot{B}_n$ for every $x \in \prod_n b_n$. This completes the proof of Theorem 6.1.

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Mathematics Department, Rutgers University, 110 Frelinghuysen Road, Piscataway, New Jersey 08854-8019, USA

E-mail address: `sthomas@math.rutgers.edu`

Institute of Mathematics, Czech Academy of Sciences, Prague, Czech Republic

Department of Mathematics, University of Florida, Gainesville, Florida 32611-8105, USA

E-mail address: zapletal@math.cas.cz