Canonical Ramsey Theory on Polish Spaces

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Chapter 1

Introduction

1.1 Motivation

The Ramsey theory starts with a classical result:

Fact 1.1.1. *For every partition of pairs of natural numbers into two classes there is a* homogeneous *infinite set: a set $a \subset \omega$ such that all pairs of natural numbers from a belong to the same class.*

It is not difficult to generalize this result for partitions into any finite number of classes. An attempt to generalize further, for partitions into infinitely many classes, hits an obvious snag: every pair of natural numbers could fall into its own class, and then certainly no infinite homogeneous set can exist for such a partition. Still, there seems to be a certain measure of regularity in partitions of pairs even into infinitely many classes. This is the beginning of canonical Ramsey theory.

Fact 1.1.2. *For every equivalence relation E on pairs of natural numbers there is an infinite* homogeneous *set: a set $a \subset \omega$ on which one of the following happens:*

1. *$pEq \leftrightarrow p = q$ for all pairs $p, q \in [a]^2$;*
2. *$pEq \leftrightarrow \min(p) = \min(q)$ for all pairs $p, q \in [a]^2$;*
3. *$pEq \leftrightarrow \max(p) = \max(q)$ for all pairs $p, q \in [a]^2$;*
4. *pEq for all pairs $p, q \in [a]^2$.*

In other words, there are four equivalence relations on pairs of natural numbers such that any other equivalence can be *canonized*: made equal to one of the four equivalences on the set $[a]^2$, where $a \subset \omega$ is judiciously chosen infinite set. It is not difficult to see that the list of the four primal equivalence relations is irredundant: it cannot be shortened for the purposes of this theorem. It is also not difficult to see that the usual Ramsey theorems follow from the canonical version.

Further generalizations of these results can be sought in several directions. An exceptionally fruitful direction considers partitions and equivalences of substructures of a given finite or countable structure, such as in [32]. Another direction seeks to find homogeneous sets of larger cardinalities. In set theory with the axiom of choice, the search for uncountable homogeneous sets of arbitrary partitions leads to large cardinal axioms [20], and this is one of the central concerns of modern set theory. A different approach will seek homogeneous sets for partitions that have a certain measure of regularity, typically expressed in terms of their descriptive set theoretic complexity in the context of Polish spaces. This is the path this book takes. Consider the following classical result:

Fact 1.1.3 ([39]). *For every partition $[\omega]^{\aleph_0} = B_0 \cup B_1$ into two Borel pieces, one of the pieces contains a set of the form $[a]^{\aleph_0}$, where $a \subset \omega$ is some infinite set.*

Here, the space $[\omega]^{\aleph_0}$ of all infinite subsets of natural numbers is considered with the usual Polish topology which makes it homeomorphic to the space of irrational numbers. This is the most influential example of a Ramsey theorem on a Polish space. It deals with Borel partitions only as the Axiom of Choice can be easily used to construct a partition with no homogeneous set of the requested kind.

Are there any canonical Ramsey theorems on Polish spaces concerning sets on which Borel equivalence relations can be canonized? A classical example of such a theorem starts with an identification of Borel equivalence relations E_γ on the space $[\omega]^{\aleph_0}$ for every function $\gamma : [\omega]^{<\aleph_0} \to 2$ (the exact statement and definitions are stated in Section 3.11) and then proves

Fact 1.1.4 ([34, 30]). *If $f : [\omega]^{\aleph_0} \to 2^\omega$ is a Borel function then there is γ and an infinite set $a \subset \omega$ such that for all infinite sets $b, c \subset a$, $f(b) = f(c) \leftrightarrow b E_\gamma c$.*

Thus, this theorem deals with *smooth* equivalence relations on the space $[\omega]^{\aleph_0}$, i.e. those equivalences E for which there is a Borel function $f : [\omega]^{\aleph_0} \to 2^\omega$ such that $b \; E \; c \leftrightarrow f(b) = f(c)$, and shows that such equivalence relations can be canonized to a prescribed form on a Ramsey cube. Other similar results can be found in the work of Otmar Spinas [45, 44, 28].

The starting point of this book consists of three simple sociological observations:

1. Among the known canonization theorems on Polish spaces, most deal with smooth equivalence relations. However, there are many Borel equivalence relations that are not smooth, and there is also the fast-growing area of descriptive set theory ordering the Borel equivalence relations according to their complexity in the sense of Borel reducibility, in which the smooth ones serve as the simplest case only. Perhaps it is possible to connect the canonical Ramsey theory with the reducibility complexity of the equivalence relations in question?

2. The known canonization theorems all seek a homogeneous set which in retrospect is a Borel set positive with respect to a suitable σ-ideal on the underlying Polish space. Perhaps there is something to be gained by looking at many different σ-ideals, attempting to prove a suitable canonical Ramsey theorem for each.

3. For every σ-ideal I on a Polish space X there is the quotient algebra P_I of Borel subsets of X modulo the ideal I, and it can be considered as a notion of forcing. A quick look shows that every smooth equivalence relation corresponds to an intermediate forcing extension of the P_I-generic extension. Perhaps it is possible to connect canonization properties of Borel equivalences with forcing properties of P_I?

The conjunction of these three points opens a whole fascinating new landscape, to which this book can only be a short introduction. It turns out that there is a whole array of canonization results depending on the Borel reducibility properties of E and forcing properties of the quotient poset P_I, for a Borel equivalence E and a σ-ideal I on a Polish space X. The techniques range from the Borel reducibility theory [8, 22], Shelah's theory of proper forcing [37, 2] and the theory of definable forcing [49] to such concepts as concentration of measure [33].

The basic setup for a problem addressed by this book can then be described as follows. Let I be a σ-ideal on a Polish space X and E a Borel equivalence relation on X. Is there a Borel I-positive set $B \subset X$ on which the equivalence relation E is significantly simpler than on the whole space? In a fairly small but significant number of cases, we show that great simplification is possible–we find a finite or countable collection of Borel equivalences such that the Borel set B can be found so that $E \restriction B$ is equal to one of the equivalences on this short list. In particularly advantageous circumstances, we even prove a strong Silver type dichotomy: either the whole space breaks into a countable collection of equivalence classes and an I-small set, or there is a Borel I-positive set consisting of pairwise inequivalent elements. In a typical case though, one cannot hope to prove anything so informative, so we will at least attempt to find a Borel I-positive set $B \subset X$ such that the restricted equivalence relation $E \restriction B$ is in the Borel reducibility sense strictly less complex than E itself. A negative result will say that an equivalence relation F is in the *spectrum* of the ideal I: there is an equivalence relation E on a Borel I-positive subset $B \subset X$ bireducible with F such that for every smaller Borel I-positive set $C \subset B$, $E \restriction C$ is still bireducible with F. Both positive and negative canonization results have their worth and their applications. Several sample theorems are in order:

Theorem 1.1.5. *For every equivalence relation E classifiable by countable structures on the Hilbert cube, there are perfect sets $\{P_n : n \in \omega\}$ of reals such that $E \restriction \Pi_n P_n$ is smooth.*

This follows from the analysis of the spectrum of the countable support of Sacks forcing in Theorem 3.8.7.

Theorem 1.1.6. *Let E be a Borel equivalence relation on the unit circle X. Either X decomposes into countably many equivalence classes and a set of multiplicity, or there is a compact set of uniqueness consisting of pairwise E-inequivalent points.*

This is a consequence of the general Theorem 3.2.3. The preexisting knowledge on the σ-ideal of sets of multiplicity shows that this theorem is indeed applicable to that ideal.

Theorem 1.1.7. *Let E be a Borel equivalence relation on 2^ω. There is a compact set $C \subset 2^\omega$ such that $E_0 \restriction C$ is not smooth, and $E \restriction C$ is equal either to the identity, or to E_0, or to $[C]^2$.*

This striking canonization feature of the equivalence E_0 is proved in Theorem 3.4.9.

1.2 Navigation

Chapter 2 introduces basic techniques and concepts useful for the canonization results in Polish spaces in general. It starts with the Trichotomy Theorem 2.1.3, showing that every Borel equivalence relation E on a Polish space X defines in a rather mysterious way an intermediate forcing extension of the P_I-extension, where I is a σ-ideal on the space X. In all cases that we have been able to compute explicitly, this extension comes from the σ-algebra P_I^E of Borel E-invariant sets modulo the ideal I or its close relatives; in all these cases P_I^E happens to be a regular subalgebra of P_I. The most pressing open issue in this book is to find natural examples in which this regularity fails badly.

Chapter 2 then introduces an array of possible weakenings and strengthenings of canonization on Polish spaces that can serve to attack the canonization problems more efficiently and to formulate the strongest possible results. The most curious of them are the generalizations of the classical Silver dichotomy [40]. It turns out that in many cases, a canonization result for a Borel equivalence relation E and a σ-ideal I on a Polish space X can be abstractly converted into the strongest possible form: the space X breaks down into countably many pieces, on each of which the equivalence relation E is very simple, and a remainder which is small in the sense of the ideal I.

The whole landscape is indexed by two variables, E and I, and in order to exhibit its main features efficiently, one needs to fix one of these variables and look at the resulting cross-section. The various sections of Chapter 3 fix a σ-ideal I–typically, P_I is equivalent to a classical notion of forcing such as Sacks or Laver forcing–and attempts to prove canonization theorems for various classes of equivalence relations and I. In the case that such canonization results are not forthcoming, we attempt to evaluate the *spectrum* of the σ-ideal–the set of those Borel equivalence relations that cannot be simplified in a Borel reducibility sense by passing to a Borel I-positive set. Surprisingly, even the most thoroughly exploited notions of forcing all of a sudden spring back to life with new unexpected features and open problems.

Chapter 4 proceeds in the perpendicular direction. In its various sections, we fix a Borel equivalence relation and study the consequences of possible canonization features connected to it, mainly for the forcing properties of the quotient posets. Thus smooth equivalence relations correspond to intermediate forcing extensions given by a single real, Borel equivalence relations with countable classes correspond to specific σ-closed intermediate extensions, and equivalence relations classifiable by countable structures correspond to choiceless intermediate models of a certain kind. We do stay fairly low in the Borel reducibility hierarchy though, and do not have much to say about complicated equivalences such as isometry of Polish spaces or isomorphism of Banach spaces.

The last chapter deals with the study of equivalence cardinals in choice-free context. Whenever X is a Polish space and E is an equivalence relation on it, then one can consider the set X/E of all E-equivalence classes. With the Axiom of Choice, these sets typically all have the same cardinality, and therefore from the cardinality point of view they are uninteresting. The situation changes profoundly when the Axiom of Choice is dropped and replaced with principles such as the Axiom of Determinacy. Then the comparison of cardinalities of these sets for various equivalence relations E, F is very similar to the comparison of E, F modulo Borel reducibility. Canonization and anti-canonization results offer a very good tool for the study of these cardinalities. We will show that a number of these cardinals are measurable and their respective measures have great degree of completeness, that the comparison does not change much even with a help of an ultrafilter etc.

1.3 Background facts

1.3a Descriptive set theory

Familiarity with basic concepts of descriptive set theory is assumed throughout. [27] serves as a standard reference.

Definition 1.3.1. A collection I of subsets of a Polish space X is $\mathbf{\Pi}^1_1$ *on* $\mathbf{\Sigma}^1_1$ if for every analytic set $A \subset 2^\omega \times X$ the set $\{y \in 2^\omega : \{x \in X : \langle y, x\rangle \in A\} \in I\}$ is coanalytic.

Fact 1.3.2. *(The first reflection theorem, [27, Theorem 35.10]) If I is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ then every analytic set in I has a Borel superset in I.*

Fact 1.3.3. *(Luzin-Novikov) If $A \subset X \times Y$ is a Borel set with countable vertical sections, then A is the countable union of graphs of Borel functions from X to Y.*

Fact 1.3.4. *(Luzin-Suslin) A Borel one-to-one image of a Borel set is Borel.*

In several places, we will need the following folklore uniformization theorem. It does not seem to have been published in print and so we include the proof.

Theorem 1.3.5. *Let Y be a Polish space and let $A \subset 2^\omega \times Y$ be a Π^1_1 set with a ZFC-provably Π^1_2 projection. For every perfect set $B \subset p[A]$ there is a perfect set $C \subset B$ and a continuous function $f : C \to Y$ such that $f \subset A$.*

The odd assumption on the projection of A is necessary for the theorem to work in ZFC; it really turns the theorem into a theorem scheme, one theorem for every ZFC proof. If there is a Woodin cardinal, the theorem works without this assumption. The proof uses the parlance introduced in the Definable Forcing subsection below.

Proof. Note that $B \subset p[A]$ is a $\mathbf{\Pi^1_2}$ statement, and so for every forcing P and every P-name $\dot{x}$ for an element of B, $P \Vdash \dot{x} \in p[A]$ by Shoenfield absoluteness. Consider B as a condition in Sacks forcing with its name $\dot{x}$ for the Sacks real, and choose a name $\dot{y}$ such that $B \Vdash \langle \dot{x}, \dot{y} \rangle \in \dot{A}$. Let M be a countable elementary submodel of a large structure containing B and $\dot{y}$. Use a standard fusion argument to find a perfect set $C \subset B$ such that for every dense open subset $D \in M$ of the Sacks forcing there are finitely many clopen sets $\{O_i : i \in n\}$ such that $C \subset \bigcup_{i \in n} O_i$ and for every $i \in n$, $C \cap O_i$ is a subset of some set in $D \cap M$. It follows that C consists only of points which are M-generic for the Sacks forcing, and the function f assigning any point $x \in C$ the evaluation of the name $\dot{y}$ according to the Sacks generic point x, is continuous. By the forcing theorem, $M[x] \models \langle x, f(x) \rangle \in A$, and by analytic absoluteness between transitive models of set theory $\langle x, f(x) \rangle \in A$. Thus C, f are as desired. $\square$

1.3b Invariant descriptive set theory

If E, F are Borel or analytic equivalence relations on Polish spaces X, Y then $E \leq_B F$ (E is *Borel reducible* to F) denotes the fact that there is a Borel function $f : X \to Y$ such that $x_0 \; E \; x_1 \leftrightarrow f(x_0) \; F \; f(x_1)$. Bireducibility is an equivalence relation on the class of all Borel equivalence relations. $\leq_B$ turns into a complicated ordering on the bireducibility classes. There are several equivalences occupying an important position in this ordering:

Definition 1.3.6. E_0 is the equivalence on 2^ω defined by $x \; E_0 \; y \leftrightarrow x \Delta y$ is finite. E_1 is the equivalence relation on $(2^\omega)^\omega$ defined by $\vec{x} \; E_1 \; \vec{y} \leftrightarrow \{n \in \omega : \vec{x}(n) \neq \vec{y}(n)\}$ is finite. E_2 is the equivalence relation on 2^ω defined by $x \; E_2 \; y \leftrightarrow \Sigma\{1/(n+1) : x(n) \neq y(n)\} < \infty$. F_2 is the equivalence relation on $(2^\omega)^\omega$ defined by $\vec{x} \; F_2 \; \vec{y} \leftrightarrow \mathrm{rng}(\vec{x}) = \mathrm{rng}(\vec{y})$. E_{K_σ} is the equivalence on $\Pi_n(n+1)$ defined by $x \; E_{K_\sigma} \; y \leftrightarrow \exists m \forall n \; |x(n) - y(n)| < m$. E_{c_0} is the equivalence on $[0,1]^\omega$ defined by $x \; E_{c_0} \; y \leftrightarrow \lim(x(n) - y(n)) = 0$.

Definition 1.3.7. An equivalence E on a Polish space X is *classifiable by countable structures* if there is a countable relational language L such that E is Borel reducible to the equivalence relation of isomorphism of models of L with universe ω.

Definition 1.3.8. An equivalence E on a Polish space X is *essentially countable* if it is Borel reducible to a *countable* Borel equivalence relation: one all of whose classes are countable.

The countable Borel equivalences are governed by the Feldman–Moore theorem:

Fact 1.3.9. *For every countable Borel equivalence relation E on a Polish space X there is a countable group G and a Borel action of G on X whose orbit equivalence relation is equal to E.*

Fact 1.3.10 (Silver dichotomy, [40], [22, Section 10.1])**.** *For every coanalytic equivalence relation E on a Polish space, either E is covered by countably many equivalence classes or there is a perfect set of mutually inequivalent points.*

One consequence often used in this book: if E is a coanalytic equivalence relation on X and $A \subset X$ is an E-invariant analytic set then either A can be covered by countably many classes or contains a perfect set of pairwise incompatible elements. To see this, apply the Silver dichotomy to the equivalence relation $\bar{E}$ defined by $x \mathrel{\bar{E}} y \leftrightarrow (x \notin A \wedge y \notin I) \vee x \mathrel{E} y$.

Fact 1.3.11 (Glimm-Effros dichotomy, [11], [22, Section 10.4])**.** *For every Borel equivalence relation E on a Polish space, either $E \leq_B \mathrm{ID}$ or $E_0 \leq_B E$.*

Fact 1.3.12. *[14] If $E \leq E_2$ is a Borel equivalence relation then either E is essentially countable, or $E_2 \leq_B E$.*

Fact 1.3.13. *[24], [22, Section 11.3] For every Borel equivalence relation $E \leq E_1$ on a Polish space, either $E \leq_B E_0$ or $E_1 \leq_B E$.*

Fact 1.3.14. *[35], [22, Section 6.6] Every K_σ equivalence relation on a Polish space is Borel reducible to E_{K_σ}.*

Fact 1.3.15. *[35], [22, Chapter 18] For every Borel equivalence relation E there is a Borel ideal I on ω such that $E \leq_B =_I$.*

Here, $=_I$ is the equivalence on 2^ω defined by $x =_I y \leftrightarrow \{n \in \omega : x(n) \neq y(n)\} \in I$. It is not difficult to construct an F_σ-ideal I on a countable set such that $=_I$ is bireducible with E_{K_σ}, and such an ideal will be useful in several arguments in the book. Just let $\mathrm{dom}(I)$ be the countable set of all pairs $\langle n, m\rangle$ such that $n \in \omega$ and $m \leq n$ and let $a \in I$ if there is a number $k \in \omega$ such that for every number $n \in \omega$, there are at most k many numbers m with $\langle n, m\rangle \in a$. This ideal is clearly F_σ and so $=_I \leq_B E_{K_\sigma}$ by the above fact. On the other hand, $E_{K_\sigma} \leq =_I$, as the reduction $f : \Pi_n(n+1) \to \mathrm{dom}(I)$ defined by $f(x) = \{\langle n, m\rangle : m \leq x(n)\}$ shows.

1.3c Forcing

Familiarity with basic forcing concepts is assumed throughout the book. If $\langle P, \leq\rangle$ is a partial ordering and $Q \subset P$ then Q is said to be *regular* in P if every maximal antichain of Q is also a maximal antichain in P; restated, for every element $p \in P$ there is a *pseudoprojection* of p into Q, a condition $q \in Q$ such that every strengthening of q in Q is still compatible with p.

Whenever κ is a cardinal, $\mathrm{Coll}(\omega, \kappa)$ is the poset of finite functions from ω to κ ordered by reverse extensions. If κ is an inaccessible cardinal, then $\mathrm{Coll}(\omega, <\kappa)$ is the finite support product of the posets $\mathrm{Coll}(\omega, \lambda)$ for all cardinals $\lambda \in \kappa$. The following fact sums up the homogeneity properties of these partial orders.

Fact 1.3.16. *Let κ be a cardinal, P a poset of size $< \kappa$, and $G \subset \mathrm{Coll}(\omega, \kappa)$ be a generic filter. In $V[G]$, for every V-generic filter $H \subset P$ there is a $V[H]$-generic filter $K \subset \mathrm{Coll}(\omega, \kappa)$ such that $V[G] = V[H][K]$. Identical statement holds true if κ is an inaccessible cardinal and $\mathrm{Coll}(\omega, \kappa)$ is replaced by the poset $\mathrm{Coll}(\omega, <\kappa)$.*

The following definition sums up the common forcing properties used in this book:

Definition 1.3.17. A forcing P is *proper* if for every condition $p \in P$ and every countable elementary submodel M of large enough structure containing P, p there is a *master* condition $q \leq p$ which forces the generic filter to meet all dense subsets in M in conditions in M. The forcing is *bounding* if every function in ω^ω in its extension there is a ground model function with larger value on every entry. The forcing *preserves outer measure* if every ground model set of reals has the same outer measure whether evaluated in the ground model or in the extension. The forcing *preserves Baire category* if every ground model set of reals is meager in the ground model if and only if it is meager in the extension. The forcing *does not add independent reals* if every sequence in 2^ω in the extension has an infinite ground model subsequence.

Fact 1.3.18. *(Shoenfield) Let $x \in X$ be an element of a Polish space, and ϕ a $\mathbf{\Pi^1_2}$ formula with one variable. Then the truth value of $\phi(x)$ is the same in all forcing extensions.*

As one application of Shoenfield's absoluteness, observe that if I is a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal on some Polish space X, then its definition yields a σ-ideal in any forcing extension. All $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ classes of sets are closed under analytic subsets by ???, so the only problem is that the definition of I applied in the extension may yield a class of analytic sets no longer closed under countable unions. This is not the case though: let $A \subset 2^\omega \times (\omega \times X)$ be a universal analytic set for subsets of $\omega \times X$. The set $B \subset 2^\omega \times X$ be the analytic set given by $\langle y, x\rangle \in B \leftrightarrow \exists n\ \langle y, n, x\rangle \in A$. The analytic sets in I form a σ-ideal if and only if for every point $y \in 2^\omega$ such that $B_y \notin I$ there is $n \in \omega$ such that $A_{\langle y,n\rangle} \notin I$. This is a $\mathbf{\Pi^1_2}$ statement and therefore absolute throughout all forcing extensions.

1.3d Idealized forcing

The techniques of the book depend heavily on the theory of idealized forcing as developed in [49]. Let X be a Polish space and I be a σ-ideal on it. The symbol P_I stands for the partial ordering of all Borel I-positive subsets of X ordered by inclusion; we will alternately refer to it as the *quotient forcing* of the σ-ideal I.i Its separative quotient is the σ-complete Boolean algebra of Borel sets modulo I. The forcing extension is given by a unique point $x_{\mathrm{gen}} \in X$ such that the generic filter consists of exactly those Borel sets in the ground model containing x_{gen} as an element. One can also reverse this operation and for every forcing P adding a generic point $\dot{x} \in X$, one may form the σ-ideal J associated with the forcing P, generated by those Borel sets $B \subset X$ such that $P \Vdash \dot{x} \notin \dot{B}$.

If I is a σ-ideal on a Polish space X, $x \in X$ is a point and M is a wellfounded model of a large part of ZFC, we will say that the point x is *M-generic* for P_I if for every $D \in M$ for which $M \models D$ is an open dense subset of P_I, there is a $B \in M$ such that $B \in D$ and $x \in B$. (Here, Borel sets are identified with their Borel codes.) In such a case, the collection $g \subset P_I^M$ consisting of those $B \in P_I^M$ for which $x \in B$ is an M-generic filter on P_I^M. We then consider $M[x]$ as the transitive model obtained from the transitive isomorph of M by adjoining the filter g. If $\tau \in M$ is a P_I-name then τ/x denotes the evaluation of the transitive collapse image of τ according to the filter g.

Fact 1.3.19. *Let I be a σ-ideal on a Polish space X. The following are equivalent:*

1. *P_I is proper;*
2. *for every Borel I-positive set $B \subset X$ and every countable elementary submodel of a large structure, the set $C \subset B$ of all M-generic points in B for the poset P_I is I-positive.*

Fact 1.3.20. *Suppose that P_I is proper and x_{gen} is a generic real. Whenever $y \in V[x_{\mathrm{gen}}]$ is a real then $V[y] \cap 2^\omega = \{f(y) : f$ is a ground model Borel function with $y \in \mathrm{dom}(f)$ and $\mathrm{rng}(f) \subset 2^\omega\}$.*

The following general result will be convenient in several places in this book. The proof is nontrivial and unfortunately does not appear in [49]. It is not in any way related to equivalence relations and so it is included in this section.

Theorem 1.3.21. *Let I be a suitably definable σ-ideal such that the forcing P_I is proper; let B be an I-positive Borel set. Let $V[G]$ be some forcing extension in which $(2^{\mathfrak{c}})$ of V is countable. In $V[G]$, there is an I-positive Borel set $C \subset B$ consisting only of V-generic points for the poset P_I.*

We will prove this in ZFC with the assumption that the ideal I is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$. The argument for more complicated ideals uses the determinacy of the associated properness game, similar to [49, Proposition 3.10.5], and uses large cardinal assumptions. One can obtain ZFC proofs for other classes of ideals. The reader is advised to use the standard fusion arguments to prove the theorem for such forcings as Sacks or Laver forcing.

Proof. Assume that the ideal I is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$, fix a partition $\omega = \bigcup_k a_k$ of ω into infinite sets, fix a universal analytic set $A \subset 2^\omega \times X$, and fix a closed set

$Z \subset (2^\omega)^\omega \times \omega^\omega$ projecting into the analytic set $\{\vec{y} : B \cap \bigcap_n \bigcup_{m \in a_n} A_{\vec{y}(n)} \notin I\} \subset (2^\omega)^\omega$. Note that the latter set is analytic by the definability assumptions on the ideal I.

Consider the game (an unraveled version of the properness game) between Player I and II in which at round $n \in \omega$ Player I chooses an open dense subset D_n of the poset P_I, and Player II responds with a point $y(n) \in 2^\omega$ and numbers $k_n \geq n, i_n \in \omega$. Player II has to satisfy the following extra requirements at his n-th move: $A_{\vec{y}(n)}$ is an I-positive Borel set, and if $n \in a_{k_m}$ for some $m \in n$ then $A_{\vec{y}(n)} \in D_m$; moreover, there still is a point $\langle \vec{y}, z \rangle \in Z$ such that $\forall m \leq n\ \vec{y}(m) = y(m)$ and $y(m) = i_m$. Player II wins if he can pass all rounds. Note that if Player II wins then the *result* of the game, the set $B \cap \bigcap_k \bigcup_{n \in a_k} A_{\vec{y}(n)}$, is a Borel I-positive subset of the set B, since the sequence $\langle y(n) : n \in \omega, i_n : n \in \omega \rangle$ must be in the closed set Z. The game is closed for Player II and therefore determined.

Player I cannot have a winning strategy in this game. If σ was such a strategy, find a countable elementary submodel M of a large enough structure containing the strategy, enumerate the open dense subsets of P_I in the model M by $E_k : k \in \omega$ with infinite repetitions, and let Player II play against the strategy σ in such a way that the moves $\vec{y}(n) : n \in a_k$ enumerate the set $E_k \cap M$. Since the poset P_I is proper, it must be the case that $B \cap \bigcap_k \bigcup_{n \in a_k} A_{y(n)} \notin I$, and therefore there must be a point $z \in \omega^\omega$ such that $\langle y(n) : n \in \omega, z \rangle \in Z$. Let Player II choose $i_n = z(n)$. Observe that if Player II challenges the strategy σ in this way, all of the moves will be in the model M no matter what the k numbers are, and therefore at each stage n there must be a number $k_n \geq n$ such that $D_n = E_{k_n}$. Let Player II play these numbers at each round. It is clear that such a counterplay is going to defeat the strategy σ.

Thus, Player II must have a winning strategy σ. The fact that σ is winning is a wellfoundedness statement, and therefore σ will be a winning strategy in every transitive model containing it for the game with the same set of allowed moves and the same closed winning condition for Player II. In particular, this strategy is still a winning strategy in the generic extension $V[G]$. In this model, the cardinal $2^\mathfrak{c}$ of V is countable, therefore there are only countably many open dense subsets of P_I in V. Consider the counterplay against the strategy σ in which Player I enumerates these sets. All moves of such a play will belong to the ground model, the strategy σ wins, and the result of the play is the required set $C \subset B$. $\square$

1.3e Ramsey theory on Polish spaces

Let $(\omega)^\omega$ be the space of all infinite sequences a of finite sets of natural numbers with the property that $\min(a_{n+1}) > \max(a_n)$ for every number $n \in \omega$. Define a partial ordering $\leq$ on this space by $b \leq a$ if every set on b is a union of several sets on a.

Fact 1.3.22. *[46, Corollary 4.49] (Bergelson-Blass-Hindman) For every partition $(\omega)^\omega = B_0 \cup B_1$ into Borel sets there is $a \in (\omega)^\omega$ such that the set $\{b \in (\omega)^\omega : b \leq a\}$ is wholly included in one of the pieces.*

Suppose that $\langle a_n, \phi_n : n \in \omega$ is a sequence of finite sets and submeasures on them such that the numbers $\phi_n(a_n)$ increase to infinity very fast. The necessary rate of increase is irrelevant for the applications in this book; suffice it to say that it is primitive recursive in n and the sizes of the sets $\{a_m : m \in n\}$.

Fact 1.3.23. *[38] For every partition $\Pi_n a_n \times \omega = B_0 \cup B_1$ into two Borel pieces, one of the pieces contains a product $\Pi_n b_n \times c$ where each $b_n \subset a_n$ is a set of ϕ_n mass at least 1 and $c \subset \omega$ is infinite.*

There is an similar partition theorem parametrized by measure. For the given $\varepsilon > 0$ in the following theorem, the necessary rate of growth of the numbers $\phi_n(a_n)$ must be adjusted.

Fact 1.3.24. *[38] Suppose in addition that the submeasures ϕ_n are measures. For every Borel set $D \subset \Pi_n a_n \times \omega \times [0, 1]$ such that the sections of the set D in the last coordinate have Lebesgue mass at least ε, there are sets $\{b_n : n \in \omega\}$, $c \subset \omega$ and a point $z \in [0, 1]$ such that $b_n \subset a_n$, $\phi_n(b_n) \geq 1$, $c \subset \omega$ is inifnite, and $\Pi_n b_n \times c \times \{z\} \subset D$.*

1.4 Notation

We employ the set theoretic standard notation as used in [18]. If E is an equivalence relation on a set X and $x \in X$ then $[x]_E$ denotes the equivalence class of x. If $B \subset X$ is a set then $[B]_E$ denotes the *saturation* of the set B, the set $\{x \in X : \exists y \in B\ xEy\}$. If E, F are equivalence relations on respective Polish spaces X, Y, we write $E \leq_B F$ if E is *Borel reducible* to F, in other words if there is a Borel function $f : X \to Y$ such that $x_0 F x_1 \leftrightarrow f(x_0) E f(x_1)$. If $f : X \to Y$ is a Borel function between two Polish spaces and F is an equivalence relation on the space Y, then $f^{-1}F$ is the *pullback* of F, the equivalence relation E on the space X defined by $x_0 E x_1 \leftrightarrow f(x_0) F f(x_1)$; so $E \leq F$. If T is a tree then $[T]$ denotes the set of all of its infinite branches. ID is the identity equivalence relation on any underlying set, EE is the equivalence relation making every two points of the underlying set equivalent. If $x \in 2^\omega$ and $m \in \omega$ then $x \odot m$ is the element of 2^ω that differs from x exactly and only at its m-th entry. If $h \in 2^{<\omega}$ is a finite binary sequence then $x \oslash h$ is the sequence in 2^ω obtained from x by rewriting its initial segment of corresponding length with h.

Theorem is a self-standing statement ready for applications outside of this book. Claim is an intermediate result within a proof of a theorem. Fact is a result that has been obtained elsewhere and is not going to be proved in this book; this is not in any way to intimate that it is an unimportant or easy or peripheral result.

Chapter 2

Basic tools

2.1 The trichotomy theorem

Let I be a σ-ideal on a Polish space X. The key idea underlying much of the current of thought in this book is the correspondence between Borel equivalence relations on X and intermediate forcing extensions of the generic extension given by the quotient poset P_I of Borel I-positive sets ordered by inclusion. The correspondence is easiest to illustrate on smooth equivalence relations. If E is a smooth equivalence on X and $f : X \to 2^\omega$ is a Borel function reducing it to the identity, then the model $V[f(\dot{x}_{gen})]$ depends only on E and not on the choice of the reduction. However, we need to find a sufficiently general definition that covers the nonsmooth case, where most of interest and difficulty lies. If $x \in X$ is a point generic over the ground model V, we consider a ZFC model $V[x]_E$ that depends on the E-equivalence class of x, not on x itself. The correct formalization:

Definition 2.1.1. Let E be a Borel equivalence relation on a Polish space X. Let $x \in X$ be a point set-generic over the ground model V. Let κ be a cardinal greater than $(2^{|P|})^V$ where $P \in V$ is a poset from which the generic extension is derived, and let $H \subset \mathrm{Coll}(\omega, \kappa)$ be a $V[x]$-generic filter. The model $V[x]_E$ is the collection of all sets hereditarily definable from parameters in V and from the equivalence class $[x]_E$ in the model $V[x][H]$.

It is clear that $V[x]_E$ is a model of ZFC by basic facts about HOD type models. The basic observation:

Proposition 2.1.2. *Let X be a Polish space, E an analytic equivalence relation on it, and let $x \in X$ be a set generic point over the ground model.*

1. *$V[x]_E \subset V[x]$;*
2. *the definition of $V[x]_E$ does not depend on the choice of the cardinal κ;*
3. *if $y \in X$ is a point such that $x \mathrel{E} y$ and the pair $\langle x, y\rangle$ is set generic over the ground model, then $V[x]_E = V[y]_E$.*

Proof. Fix P, a P-name $\tau \in V$ for the real x and a cardinal $\kappa > 2^{|P|}$. Let $\phi(\vec{v}, [x]_E, \alpha\})$ be a formula with parameters $\vec{v} \in V$ defining a set of ordinals in the $\mathrm{Coll}(\omega, \kappa)$ extension: $a = \{\alpha \in \beta : \phi(\vec{v}, [x]_E, \alpha)\}$. Homogeneity arguments such as Fact 1.3.16 show that $a = \{\alpha \in \beta : V[x] \models \mathrm{Coll}(\omega, \kappa) \Vdash \phi(\vec{v}, [x]_E, \check{\alpha})\} \in V[x]$. As ϕ was arbitrary, the first item follows.

For the second item, we will find a different formula $\theta(\vec{D}, \kappa, [x]_E, \alpha)$ with parameters in the ground model and another parameter $[x]_E$, which defines a in any other transitive model extending V containing x. This will show that the $\mathrm{Coll}(\omega, \kappa)$ extension of $V[x]$ has the smallest possible HOD among all transitive models extending V containing x. As $\kappa > 2^{|P|}$ was arbitrary, this will prove the second item.

Let $\vec{D} = \langle D_\alpha : \alpha \in \beta\rangle$ be the sequence of subsets of P in V defined by $p \in D_\alpha$ if $V \models p \Vdash \mathrm{Coll}(\omega, \kappa) \Vdash \phi(\vec{v}, [\tau]_E, \alpha)$. Consider the formula $\psi(\vec{D}, [x]_E, \alpha)$ =there exists a V-generic filter $g \subset P$ such that $\tau/g \in [x]_E$ and $g \cap D_\alpha \neq 0$. The formula $\theta(\vec{D}, \kappa, [x]_E, \alpha)$ says $\mathrm{Coll}(\omega, \kappa) \Vdash \psi(\vec{D}, [x]_E, \alpha)$; we claim that θ works as desired.

First, observe that in any transitive model extending V, containing x, and containing an enumeration of $\mathcal{P}(P) \cap V$ in ordertype ω, the formula $\psi(\vec{D}, [x]_E, \alpha)$ is a statement about illfoundedness of a certain tree for any fixed ordinal α. Namely, let $C_n : n \in \omega$ enumerate all open dense subsets of P in V, let $A \subset X \times X \times \omega^\omega$ be a closed set coded in V that projects to the equivalence E, and let T be the tree of all finite sequences $\langle p_i, O_i : i \in j\rangle$ such that the conditions $p_i \in P$ form a decreasing sequence such that $p_0 \in D_\alpha$, $p_i \in C_i$, and $O_i \subset X \times X \times \omega^\omega$ form a decreasing sequence of basic open neighborhoods coded in V of radius $< 2^{-i}$ with nonempty intersection with A, such that x belongs to the projection of O_i to the first coordinate, and $p_i \Vdash \tau$ belongs to the projection of O_i in the second coordinate. It is immediate that $\psi(\vec{D}, [x]_E, \alpha)$ is equivalent to the statement that T contains an infinite branch.

Second, observe that the validity of the illfoundedness statement does not depend on the particular enumeration of open dense subsets of P chosen–if one gets an infinite branch with one enumeration, then a subsequence of the conditions in P used in this branch will yield a branch for a different enumeration. Therefore, the formula $\psi(\vec{D}, [x]_E, \alpha)$ is absolute between all transitive models of set theory containing V, x satisfying "$\mathcal{P}(P) \cap V$ is countable". Ergo, the formula $\theta(\vec{D}, [x]_E, \alpha)$ is absolute between all transitive models containing V and x.

Lastly, the formula θ does define the set a in all such models. It is enough to check that this is so in $V[x]$. If $\alpha \in a$ then $\theta(\vec{D}, [x]_E, \alpha)$ is satisfied: if $H \subset \mathrm{Coll}(\omega, \kappa)$ is a $V[x]$-generic filter, then ψ will be witnessed by the V-generic filter on the poset P obtained from the point x. On the other hand, if $\theta(\vec{D}, [x]_E, \alpha)$ holds, then for every $V[x]$-generic filter $H \subset \mathrm{Coll}(\omega, \kappa)$ there is a V-generic filter $g \subset P$ in $V[x][H]$ such that $\tau/g\ E\ x$ and $g \cap D_\alpha \neq 0$, by Fact 1.3.16 there is a $V[g]$-generic filter $K \subset \mathrm{Coll}(\omega, \kappa)$ such that $V[g][K] = V[x][H]$, and by the forcing theorem applied in $V[g]$, the model $V[g][K] = V[x][H]$ satisfies $\phi(\vec{v}, [\tau/g]_E, \alpha)$, which is the same as $\phi(\vec{v}, [x]_E, \alpha)$. Thus $\alpha \in a$ as required!

For the third item, let κ be a cardinal larger than the density of the poset in the ground model that yields the pair $\langle x, y\rangle$, let $H \subset \mathrm{Coll}(\omega, \kappa)$ be a $V[x, y]$-

generic filter, and let W be the class consisting of all sets hereditarily definable from parameters in V and the additional parameter $[x]_E = [y]_E$ in the model $V[x, y][H]$. Standard homogeneity arguments using Fact 1.3.16 show that the model $V[x, y][H]$ is a $\mathrm{Coll}(\omega, \kappa)$-extension of both $V[x]$ and $V[y]$. By the second item above, $W = V[x]_E = V[y]_E$ and the third item follows! $\square$

The definition and properties of the model $V[x]_E$ may be somewhat mysterious. Several more or less trivial examples will serve as a good illustration of what can be expected. Let $E = E_0$ and consider the cases of the Sacks real, the Cohen real, and the Silver real. In the case of Sacks real $x \in 2^\omega$, a simple density argument will show that there will be a perfect tree T in the Sacks generic filter consisting of E_0-inequivalent points. Then, x can be defined from $[x]_{E_0}$ as the only point of $[T] \cap [x]_{E_0}$ and therefore $V[x] = V[x]_{E_0}$. In the case of Cohen generic real $x \in 2^\omega$, the rational translations of 2^ω that generate E_0 can transport any Cohen condition to any other one, and the equivalence class $[x]_{E_0}$ is invariant under all of them, leading to the conclusion that $V[x]_E = V$. In the case of Silver generic real x, the model $V[x]_{E_0}$ is a nontrivial σ-closed generic extension of V, roughly speaking generated by the quotient of Silver forcing in which E_0-equivalent conditions are identified. On general grounds, $V[x]_E$ is a generic extension of V, but it is not so easy to compute the forcing that induces it. In a typical case, it is the poset of Borel E-invariant sets positive with respect to some σ-ideal as in Theorem 2.2.7.

The importance of the model $V[x]_E$ is clarified in the main result of this section, a trichotomy theorem:

Theorem 2.1.3. *Suppose that X is a Polish space, I is a suitably definable σ-ideal on it such that the quotient forcing P_I is proper, and E is a Borel equivalence relation on X. Let $V[G]$ be a P_I-generic extension of V and $x_{\mathrm{gen}} \in X$ its associated generic point. One of the following holds:*

1. *(ergodicity) if $V = V[x_{\mathrm{gen}}]_E$ then there is a Borel I-positive set $B \subset X$ such that any two Borel I-positive sets $C, D \subset B$ contain E-equivalent points $x \in C, y \in D$;*

2. *(intermediate extension) the model $V[x_{\mathrm{gen}}]_E$ is strictly between V and $V[G]$;*

3. *(canonization) if $V[x_{\mathrm{gen}}]_E = V[G]$ then there is a Borel I-positive set $B \subset X$ such that $E \restriction B = \mathrm{ID}$.*

The wording of the theorem must be discussed more closely. The argument presented works in ZFC for ideals I which are $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$. Small changes in the proof of Theorem 1.3.21 will produce a ZFC argument for other ideals as well. If one desires to use ideals with much more complicated projective definitions, suitable large cardinal assumptions will push the proof of that theorem through as explained after its statement.

The first item is fairly close to saying that the E-saturation of every Borel set either has an I-small intersection with B, or contains all of B up to an I-small set. This is indeed equivalent to (1) if every coanalytic set is either in I or contains a Borel I-positive set, or if every positive Borel set has a positive Borel subset whose E-saturation is Borel. Both of these assumptions frequently hold: the former perhaps on the basis of the first dichotomy of the ideal I [49] or a redefinition of the ideal I, and the latter perhaps on the basis of the quotient P_I being bounding and the equivalence E being K_σ.

A trivial way to satisfy the first item is to find an I-positive equivalence class of E. A typical special case in which the ergodicity case holds nontrivially is that of Laver forcing and the E_{K_σ} equivalence relation, see Section 3.10. The Laver forcing generates a minimal forcing extension, prohibiting the second item from ever occurring, but still there is an equivalence relation that cannot be simplified to ID or EE on any Borel I-positive set. Another interesting special case is the Cohen forcing and the F_2 equivalence relation, see Theorem 3.2.22. There, the ergodicity is not at all surprising, but we get more information looking at *choiceless* intermediate models of ZF as in Section 4.3.

In the second case we get a somewhat mysterious intermediate forcing extension strictly between V and the P_I-extension $V[G]$. For many σ-ideals, the existence of such an intermediate model is excluded purely on forcing grounds, such as the ideal of countable sets or σ-compact subsets of ω^ω, or the more general Theorem 2.4.5. In such cases, we get a neat dichotomy. It is not true that every intermediate extension is necessarily obtained from a Borel equivalence relation in this way, as Section 3.3 shows. In special cases though (such as the countable length $\alpha \in \omega_1$ iteration of Sacks forcing), the structure of intermediate models is well-known (each of them is equal to $V[x_\gamma : \gamma \in \beta]$ for some ordinal $\beta \leq \alpha$) and they exactly correspond to certain critical equivalence relations (such as the relations E_β on $(2^\omega)^\alpha$ connecting two sequences if they are equal on their first β entries). We cannot compute the forcing responsible for the extension $V[x_{\mathrm{gen}}]_E$ directly, but in all specific cases we *can* compute, it is the poset of E-invariant I-positive Borel sets ordered by inclusion, as described in Theorem 2.2.7 and its section. In the common situation that the model $V[x_{\mathrm{gen}}]_E$ does not contain any new reals, we can compute the forcing leading from $V[x_{\mathrm{gen}}]_E$ to $V[G]$–it is a quotient poset of the form P_{I^*} for a suitable σ-ideal $I^* \supset I$ which has the ergodicity property of the first item of the theorem. As Asger Tornquist remarked, in this context the theorem can be viewed as a general counterpart to Farrell-Varadarajan ergodic decomposition theorem [25, Theorem 3.3] for actions of countable groups by measure preserving Borel automorphisms.

Proof. To prove the theorem, choose a cardinal $\kappa > 2^{\mathfrak{c}}$ and choose a $V[G]$-generic filter $H \subset \mathrm{Coll}(\omega, \kappa)$. There are the three cases.

Either, $V[x_{\mathrm{gen}}]_E = V$. In this case, in the model $V[G][H]$ consider the collection $K = \{C \in P_I^V : C \cap [x_{\mathrm{gen}}]_E \neq 0\}$. This collection is certainly in the model $V[x_{\mathrm{gen}}]_E$, and therefore it is in V. There must be conditions $B \in P_I$ and $p \in \mathrm{Coll}(\omega, \kappa)$ and an element $L \in V$ such that $\langle B, p\rangle \Vdash \dot{K} = \check{L}$. If there were two Borel I-positive sets $C_0, C_1 \subset B$ with $[C_0]_E \cap [C_1]_E = 0$, then only one, say C_0 of these sets can belong to the set L. But then $\langle C_1, p\rangle \Vdash C_1 \in \dot{K} \setminus \check{L}$! This contradiction shows that we are in the first case of the theorem.

In fact, $V[x_{\text{gen}}]_E = V$ is equivalent to the statement that the filter G contains an ergodic condition. If $V[x_{\text{gen}}]_E = V$ then an ergodic condition can be found in the generic filter by the argument in the previous paragraph, and a genericity argument. On the other hand, suppose that $B \in P_I$ is an ergodic condition and $B \Vdash \dot{a}$ is a set of ordinals in $V[x_{\text{gen}}]_E$; we must prove that $B \Vdash a \in V$. Thinnig out the condition B if necessary find a formula ϕ with parameters in the ground model or with the parameter $[\dot{x}_{gen}]_E$ defining the set a and show that B decides the statement $\beta \in \dot{a}$ for every ordinal β. And indeed, if there were Borel sets $C_0, C_1 \in P_I$ below the set B, one forcing $\beta \in \dot{a}$ and the other forcing the opposite, use Claim 1.3.21 below to find conditions $\bar{C}_0 \subset C_0, \bar{C}_1 \subset C_1$ in the forcing P_I in the model $V[G, H]$ consisting purely of V-generic points. Note that the ergodicity of the set B is a $\mathbf{\Pi^1_2}$ statement and use Shoenfield's absoluteness 1.3.18 to transport the ergodicity of the condition B from the ground model to the model $V[G, H]$ and find points $x_0 \in \bar{C}_0$ and $x_1 \in \bar{C}_1$ which are E-equivalent. Use a standard homogeneity argument, Fact 1.3.16, to find generic filters H_0, H_1 on it so that $V[x_0, H_0] = V[x_1, H_1] = V[G, H]$. Note that $V[x_0, H_0] \models \phi(\beta, [x_0]_E)$ if and only if $V[x_1, H_1] \models \phi(\beta, [x_1]_E)$ simply because the two models are the same and the two E-equivalence classes are the same. This contradicts the forcing theorem and the assumption on the conditions C_0, C_1.

The second possibility is that $V \subset V[x_{\text{gen}}]_E \subset V[G]$ and these inclusions are proper. In this case, we are content to fall into the second item of the theorem.

Lastly, assume that $V[x_{\text{gen}}]_E = V[G]$; in particular, $x_{\text{gen}} \in V[x_{\text{gen}}]_E$. Then, there must be a condition $B \in P_I$ forcing $\dot{x}_{gen} \in \dot{V}[x_{\text{gen}}]_E$, say $\langle B, 1\rangle \Vdash \dot{x}_{gen}$ is the only element $x \in X$ satisfying $\phi(x, [\dot{x}_{gen}]_E, \vec{v})$ for some sequence of parameters $\vec{v} \in V$ and a formula ϕ. Let $C \subset B$ be a Borel I-positive set of V-generic points as guaranteed by Theorem 1.3.21.

Claim 2.1.4. $E \restriction C = \text{ID}$.

Proof. Work in the model $V[G, H]$. Suppose for contradiction that $x, y \in V[G, H]$ are two distinct E-equivalent points in the set C. The usual homogeneity properties of $\text{Coll}(\omega, \kappa)$ as in Fact 1.3.16 imply that there are filters $H_x, H_y \subset \text{Coll}(\omega, \kappa)$ such that $G_x \times H_x, G_y \times H_y \subset P_I \times \text{Coll}(\omega, \kappa)$ are V-generic filters, where $G_x = \{D \in P_I^V : x \in D\}$ and $G_y = \{D \in P_I^V : y \in D\}$, and moreover $V[G, H] = V[G_x, H_x] = V[G_y, H_y]$. Since both filters G_x, G_y meet the condition $B \in P_I$, the forcing theorem implies that x is the only point in X satisfying $\phi(x, [x]_E, \vec{v})$, and similarly y is the only point satisfying $\phi(y, [y]_E, \vec{v})$. However, $[x]_E = [y]_E$, reaching a contradiction! □

Thus, we see that in $V[G, H]$ there is an I-positive Borel set $C \subset B$ such that $E \restriction C = \text{ID}$. If the ideal I is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$, this is a $\mathbf{\Sigma^1_2}$ statement, and therefore pulls back into the ground model by Shoenfield's absoluteness 1.3.18. If the σ-ideal I is more complicated, we can use a large cardinal assumption and a corresponding absoluteness argument to find such a set C in the ground model. We are safely in the third case of the theorem, and the proof is complete!

Again, it is not difficult to see that the statement $V[x_{\text{gen}}]_E = V[G]$ is equivalent with the existence in the generic filter G of a set B such that $E \restriction B = \text{ID}$. $\square$

As with most models of set theory, the user wants to know which reals belong to the model $V[x]_E$. In the most interesting cases, $2^\omega \cap V = 2^\omega \cap V[x]_E$. Then, the remainder forcing leading from $V[x]_E$ to $V[x]$ can be computed via idealized forcing.

Definition 2.1.5. Let I be a σ-ideal on a Polish space X, let E be a Borel equivalence relation, and let $x \in X$ be a P_I-generic point. In $V[x]$, let I^* be the collection of ground model coded Borel sets that are forced by a large collapse to contain no P_I-generic reals E-equivalent to x.

It should be clear that I^* is a σ-ideal of Borel sets extending I, and $I^* \in V[x]_E$. Moreover, if $2^\omega \cap V[x]_E = 2^\omega \cap V$, the ideal I^* has the ergodicity property in $V[x]_E$: any pair of I^*-positive Borel sets contains a pair of E-connected points. This follows from the fact that in some large collapse, each of the sets in the pair contains some points equivalent to x; in particular, such points are equivalent to each other, and analytic absoluteness transports this feature back to $V[x]_E$.

Theorem 2.1.6. *Suppose that X is a Polish space, I a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal on X such that the quotient forcing P_I is proper. Let x be a P_I-generic point. Then, $2^\omega \cap V[x_{\text{gen}}]_E = \{f(x_{\text{gen}}) : f : X \to 2^\omega$ is a Borel function coded in V which is E-invariant on its domain$\}$. Moreover, if $2^\omega \cap V = 2^\omega \cap V[x]_E$, then $V[x]_E = V[I^*]$ and $V[x]$ is a P_{I^*}-extension of $V[x]_E$.*

Again, the theorem holds in ZFC for many ideals with a definition more complicated than $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$.

Proof. For the first part, suppose that $\langle B, p\rangle \in P_I \times \text{Coll}(\omega, \kappa)$ is a condition that forces $\dot{y} \in V[\dot{x}_{gen}]_E \cap 2^\omega$. Strengthening the condition we may assume that there is a formula ϕ and a sequence $\vec{v} \in V$ of parameters such that it is forced that $\dot{y}$ is the unique element of 2^ω satisfying $\phi(\dot{y}, [\dot{x}_{gen}]_E, \vec{v})$. Since $V[x_{\text{gen}}]_E \subset V[x_{\text{gen}}]$, we may also strengthen the condition if necessary to find a Borel function $g : X \to 2^\omega$ such that it is forced that $\dot{y} = \dot{g}(\dot{x}_{gen})$.

We will now find a Borel I-positive set $C \subset B$ on which the function g is E-invariant. This will certainly complete the proof of the proposition, since then $\langle C, p\rangle$ forces that $\dot{y}$ is the image of the generic real by an E-invariant function. We will in fact find such a set in a model $V[H]$ where $H \subset \text{Coll}(\omega, \kappa)$ is generic, and then pull it back to V using Shoenfield absoluteness 1.3.18.

Indeed, in $V[H]$ consider the set $C \subset B$ of all V-generic points for the poset P_I in the set B. As in Claim 1.3.21, the set C is Borel and I-positive. Suppose that $x_0, x_1 \in C$ are two E-equivalent points. By standard homogeneity arguments, find mutually V-generic filters $G_0 \subset P_I, H_0 \subset \text{Coll}(\omega, \kappa)$ and $G_1 \subset$

$P_I, H_1 \subset \mathrm{Coll}(\omega, \kappa)$ such that x_0 is the point associated with G_0, x_1 is the filter associated with G_1, $p \in H_0, H_1$, and $V[H] = V[G_0, H_0] = V[G_1, H_1]$. It is clear that the formula ϕ must define the same point in $V[H]$ whether x_0 or x_1 are plugged into the generic real, since these two points share the same equivalence class. By the forcing theorem, it must be the case that $f(x_0) = f(x_1)$, and we are done!

Now assume that $V[x]_E$ contains the same reals as V; we will first show that x is a P_{I^*} generic point over $V[x]_E$. Move back to the ground model, and assume for contradiction that a condition $\langle B, p\rangle \in P_I \times \mathrm{Coll}(\omega, \kappa)$ forces that $\dot{D} \subset P_{I^*}$ is an open dense set in the model $V[x_{\mathrm{gen}}]_E$ such that the point $\dot{x}_{gen}$ does not belong to any of its elements. Strengthening the condition if necessary, we may find a formula $\phi(u, [\dot{x}_{gen}]_E, \vec{v})$ with parameters in $V \cup \{[\dot{x}_{gen}]_E\}$ defining the set $\dot{D}$: $\dot{D} = \{C \in P_{I^*} : \phi(C, [\dot{x}_{gen}]_E, \vec{v})\}$. Of course, the condition $\langle B, p\rangle$ forces $B \in P_{I^*}$, so there must be a strengthening $B' \subset B, p' \leq p$ and a Borel set $C \subset B$ such that $\langle B', p'\rangle \Vdash C \in \dot{D}$. Note that $C \in P_I$ must be an I-positive Borel set in the ground model.

Find mutually generic filters $G \subset P_I, H \subset \mathrm{Coll}(\omega, \kappa)$ with $B' \in G$, $p' \in H$, let $x_{\mathrm{gen}} \in X$ be the point associated with the filter G, use the definition of the ideal I^* to find a point $y \in C \cap V[G, H]$ which is V-generic for the poset P_I and equivalent to x_{gen}, and use the homogeneity features of the poset $\mathrm{Coll}(\omega, \kappa)$ as ijn Fact 1.3.16 to find mutually generic filters $\bar{G} \subset P_I, \bar{H} \subset \mathrm{Coll}(\omega, \kappa)$ such that y is the generic point associated with $\bar{G}$ (so $C \in \bar{G}$), $p \in \bar{H}$, and $V[G, H] = V[\bar{G}, \bar{H}]$.

Since the points x_{gen}, y are E-equivalent, the definition of the set $\dot{D}$ is evaluated in the same way with either x_{gen} or y plugged into the definition. Applying the forcing theorem to the filters $\bar{G}, \bar{H}$, there must be a condition in that filter stronger than $\langle B, p\rangle$ which forces $C \in \dot{D}$ and $\dot{x}_{gen} \in C$. However, this directly contradicts the statement forced by the weaker condition $\langle B, p\rangle$!

To show that $V[x]_E = V[I^*]$, note that $I^* \in V[x_{\mathrm{gen}}]_E$, so the right to left inclusion is immediate. For the opposite inclusion, assume that $a \in V[x]_E$ is a set of ordinals, perhaps defined as $a = \{\alpha : \phi(\alpha, \vec{v}, [x]_E)\}$ in the model $V[x, H]$, where $\vec{v}$ are parameters in the ground model and $H \subset \mathrm{Coll}(\omega, \kappa)$. Let $b = \{\alpha : \exists B \in P_I^V, B \notin I^* \ B \Vdash \mathrm{Coll}(\omega, \kappa) \Vdash \phi(\check{\alpha}, \vec{v}, [x_{\mathrm{gen}}]_E)\} \in V[I^*]$ and show that $a = b$. It is immediate that $a \subset b$ since the filter defined by the P_I-generic point $x \in X$ has empty intersection with I^*. On the other hand, if $\alpha \in b$ as witnessed by a set B then there is a V-generic point $y \in B \cap [x]_E$, by the usual homogeneity argument there is a $V[y]$-generic filter $H_y \subset \mathrm{Coll}(\omega, \kappa)$ such that $V[y][H_y] = V[x][H]$, by the forcing theorem $V[y][H_y] \models \phi(\alpha, \vec{v}, [x]_E)$, and therefore $\alpha \in a$. The theorem follows.

□

The model $V[x]_E$ has a larger, less well understood, possibly choiceless companion $V[[x]]_E$. In spirit, this larger model is the intersection of all models containing V and a point E-equivalent to x. In formal language,

Definition 2.1.7. Let E be a Borel equivalence relation on a Polish space X. Let $x \in X$ be a point set-generic over the ground model V. The model $V[[x]]_E$

consists of all sets $a \in V[x]$ such that for every ordinal κ, $\mathrm{Coll}(\omega, \kappa)$ forces that for every $y \in X$ with $x\ E\ y$, $a \in V[y]$.

Proposition 2.1.8. *$V[[x]]_E$ is a model of ZF with $V[x]_E \subseteq V[[x]]_E \subseteq V[x]$.*

Proof. Let $\phi(x)$ be the definition of the model $V[[x]]_E$ as described above. The first observation is that for every ordinal κ, $\mathrm{Coll}(\omega, \kappa)$ forces that for every point $y \in X$ E-equivalent to x, the formula $\phi(y)$ defines the same class in $V[y]$ as $\phi(x)$ defines in $V[x]$. We will show that $\{a \in V[x] : V[x] \models \phi(a, x)\} \subset \{a \in V[y] : V[y] \models \phi(a, y)\}$; the other inclusion has the same proof. Suppose that $a \in V[y]$ and $V[y] \models \phi(a, y)$. Then $a \in V[x]$, since otherwise $V[x]$ forms a counterexample to $V[y] \models \phi(a, y)$ inside some large common generic extension of both models $V[y]$ and $V[x]$. Also, $V[x] \models \phi(a, x)$, because otherwise in some common forcing extension of $V[x]$ and $V[y]$, there is a counterexample to $\phi(a, x)$ which is also a counterexample to $\phi(a, y)$.

To show that $V[[x]]_E$ is a model of ZF, argue that $L(V[[x]]_E) = V[[x]]_E$. Indeed, for every ordinal κ it is the case that $\mathrm{Coll}(\omega, \kappa)$ forces that $L(V[[x]]_E)$ is contained in every model $V[y]$ where $y\ E\ x$, since as argued in the previous paragraph, the models $V[y]$ all contain $V[[x]]_E$ as a class. But then, every element of $L(V[[x]]_E)$ belongs to $V[[x]]_E$ by the definition of $V[[x]]_E$.

To show that $V[x]_E$ is a subset of $V[[x]]_E$, let $a \in V[x]_E$ be a set of ordinals and let $V[g]$ be a set generic extension of V inside a set generic extension of $V[x]$ containing a point $y \in X$ equivalent to x; we must argue that $a \in V[g]$. There is a common $\mathrm{Coll}(\omega, \kappa)$ extension $V[h]$ of both $V[g]$ and $V[x]$ for a sufficiently large cardinal κ. Then a is definable in $V[H]$ from parameters in V and perhaps the additional parameter $[x]_E = [y]_E$. Since $V[h]$ is a homogeneous extension of the model $V[g]$ and all parameters of the definition are themselves definable from parameters in $V[g]$, it must be the case that $a \in V[g]$ as desired, and the proof is complete. $\square$

We do not know much more about the comparison of $V[x]_E$ and $V[[x]]_E$ at this point. If the equivalence E is smooth and reduced to the identity by a Borel function $f : X \to 2^\omega$, then both of the models are equal to $V[f(x)]$. If the equivalence E has countable classes, then every generic extension containing a representative of an equivalence class contains the whole class by an absoluteness argument, and therefore $V[[x]]_E = V[x]$ in such a case, while the model $V[x]_E$ does not contain x unless x belongs to a ground model coded Borel set on which the equivalence is smooth by Theorem realstheorem. If the Borel equivalence E is classifiable by countable structures then $V[[x]]_E = V(a)$ for a hereditarily countable set a, and the model fails the axiom of choice unless x belongs to a ground model coded Borel set on which the equivalence is essentially countable. Other cases are much less clear.

Theorem 2.1.9. *Let I be a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal on a Polish space X such that the quotient forcing P is proper. Let E be an analytic equivalence relation on X, and let $x \in X$ be a P_I-generic point. Then $2^\omega \cap V[[x]]_E = \{f(x) : f$ is a Borel function coded in V which on its domain attains only countably many values on each equivalence class$\}$.*

An important observation complementary to the wording of the theorem: the statement that a given Borel function f attains only countably many values on each E-equivalence class is Π^1_1 in the code for the function and the equivalence relation, and therefore it is invariant under forcing by Shoenfield's absoluteness. To see this, note that every E-equivalence class is analytic, its image under f is analytic, and so if countable it contains only reals hyperarithmetic in any parameter that can define it as an analytic set. Thus, f attains only countably many values on each E-equivalence class if and only if for every pair of points $x, y \in \mathrm{dom}(f)$, either x, y are not equivalent or $f(x)$ is hyperarithmetic in y, the code for f and the code for E. This is a coanalytic formula by [22, Theorem 2.8.6]

Proof. Suppose that $B \in P_I$ is a Borel I-positive set and τ a name for an element for the set $2^\omega \cap V[[x]]_E$; by thinning out the condition B if necessary we may assume that τ is represented by a Borel function $f : B \to 2^\omega$. We will find a condition $C \subset B$ such that $f \restriction C$ attains only countably many values on each equivalence class; this will complete the proof.

Let $H \subset \mathrm{Coll}(\omega, \kappa)$ be a generic filter for a sufficiently large cardinal κ. It will be enough to find the set C in the model $V[H]$ and then pull it back to the ground model using Shoenfield absoluteness. Let $C = \{x \in B : x \text{ is } V\text{-generic}\}$. By Theorem 1.3.21 this is a Borel I-positive set. If there is an E-equivalence class in C on which f takes uncountably many values, then choose a point $x \in C$ in such an equivalence class, note that the model $V[x]$ contains only countably many reals as viewed from the model $V[H]$, and find a point y in the same equivalence class such that $f(y) \notin V[x]$. Now, the forcing theorem implies that $V[y] \models f(y) \in V[[x]]_E$; however, in the forcing extension $V[H]$ of $V[y]$ there is a forcing extension $V[x]$ of V which contains an equivalent of y but does not contain $f(y)$. A contradiction.

□

2.2 Associated forcings

In all special cases investigated in this book, the intermediate extension $V[x_{\mathrm{gen}}]_E$ of Definition 2.1.1 is generated by a rather natural regular subordering of P_I:

Definition 2.2.1. (A somewhat incorrect attempt) Let X be a Polish space and I a σ-ideal on it. P^E_I is the partial ordering of I-positive E-saturated Borel sets ordered by inclusion.

The problem here is that E-saturations of Borel sets are in general analytic, and in principle there could be very few Borel E-saturated sets. There are two possibilities for fixing this. The preferred one is to work only with σ-ideals I satisfying the third dichotomy [49, Section 3.9.3]: every analytic set is either in the ideal I or it contains an I-positive Borel subset. Then the quotient P_I is dense in the partial order of analytic I-positive sets, and we can define P^E_I as the ordering of analytic E-saturated I-positive sets.

Another possibility is to prove that in a particular context, E-saturations of many Borel sets are Borel. This is for example true if E is a countable Borel equivalence relation. It is also true if the poset P_I is bounding and the equivalence E is reducible to E_{K_σ}. There, if $B \subset X$ is a Borel I-positive set and $f : B \to \Pi_n(n+1)$ is the Borel function reducing E to E_{K_σ}, the bounding property of P_I can be used to produce a Borel I-positive compact set $C \subset B$ on which the function f is continuous, and then $[C]_E$ is Borel. However, in general the question whether every analytic I-positive E-saturated set has a Borel I-positive E-saturated subset seems to be quite difficult even if one assumes that every I-positive analytic set contains a Borel I-positive subset.

Definition 2.2.2. (The correct version) Let X be a Polish space and I a σ-ideal on it such that every analytic I-positive set contains a Borel I-positive subset. P_I^E is the poset of I-positive E-saturated analytic sets ordered by inclusion. If $C \subset X$ is a Borel I-positive set then $P_I^{E,C}$ is the poset of relatively E-saturated analytic subsets of C ordered by inclusion.

By a slight abuse of notation, if the ideal I has the third dichotomy, we will write P_I for the poset of all analytic I-positive sets ordered by inclusion, so that $P_I^E \subset P_I$. The previous discussion seems to have little to offer for the solution of the main problem associated with the topic of this book:

Question 2.2.3. Is $P_I^{E,C}$ a regular subposet of P_I for some Borel I-positive set $C \subset X$?

Even in quite natural cases, it may be necessary to pass to an I-positive subset even in fairly simple cases to find the requested regularity:

Example 2.2.4. Consider $X = \omega^\omega \times 2$ and the σ-ideal I generated by sets $B \subset X$ such that $\{y \in \omega^\omega : \langle y, 0\rangle \in B\}$ is compact and $\{y \in \omega^\omega : \langle y, 1\rangle \in B\}$ is countable. The quotient forcing is clearly just a disjoint union of Sacks and Miller forcing. Consider the equivalence relation E on X defined as the equality on the first coordinate. Then E has countable classes, indeed each of its classes has size 2, and P_I^E is not a regular subposet of P_I. To see that, consider a maximal antichain $A \subset \omega^\omega$ consisting of uncountable compact sets, and let $\bar{A} = \{C \times 2 : C \in A\}$. It is not difficult to see that $\bar{A}$ is a maximal antichain in P_I^E, but the set $\omega^\omega \times 1 \subset X$ is Borel, I-positive, and has an I-small intersection with every set in $\bar{A}$.

The answer to the previous question turns out to be positive in all specific cases investigated here, and the set C is invariably equal to the whole space. Even more, it always so happens that the model $V[x_{\mathrm{gen}}]_E$ introduced in Theorem 2.1.3 is exactly P_I^E extension of the ground model. We will use only one way to prove that the forcing P_I^E is a regular subforcing of P_I. Namely, we will show that there is a dense set of conditions $B \in P_I$ such that the set $[B]_E$ is a pseudoprojection of B into P_I^E. This is to say that for every E-saturated set $C \subset [B]_E$ in P_I the intersection $C \cap B$ is I-positive; restated, for every set $C \subset [B]_E$ in P_I the set $B \cap [C]_E$ is I-positive.

It will be often the case that the forcing P_I^E is $\aleph_0$-distributive. We will always prove this through the following property of the equivalence:

Definition 2.2.5. The equivalence E on a Polish space X is *I-dense* if for every I-positive Borel set $B \subset X$ there is a point $x \in B$ such that $[x]_E \cap B$ is dense in B.

Proposition 2.2.6. *If the poset P_I has the continuous reading of names, P_I^E is a regular subposet of P_I, and E is I-dense, then P_I^E is $\aleph_0$-distributive.*

Proof. Since P_I is proper, so is P_I^E, and it is enough to show that P_I^E does not add reals. Suppose that $B \in P_I$ forces that τ is a name for a real in the P_I^E-extension. Let M be a countable elementary submodel of a large enough structure, and let $C \subset B$ be the I-positive set of all M-generic points for P_I. If $x \in C$ is a point, write $H_x \subset P_I^E$ for the M-generic filter of all sets $D \in P_I^E$ such that $[x]_E \subset D$. It is clear that the map $x \mapsto H_x$ is constant on equivalence classes, and so is the Borel map $x \mapsto \tau/H_x$. Thin out the set C if necessary to make sure that the map $x \mapsto \tau/H_x$ is continuous on C. By the density property, there is a point $x \in C$ such that $[x]_E \cap C$ is dense in C. Thus, the map $x \mapsto \tau/H_x$ is continuous on C and constant on a dense subset of C, therefore constant on C. The condition C clearly forces τ to be equal to the single point in the range of this function. $\square$

Note that typically the poset P_I^E is fairly simply definable. If suitable large cardinals exist, then the descending chain game on the poset P_I^E is determined. Thus, when the poset is $\aleph_0$-distributive, Player I has no winning strategy in this game, so it must be Player II who has a winning strategy and the poset is strategically σ-closed. It is not entirely clear whether there may be cases in which P_I^E is $\aleph_0$-distributive but fails to have a σ-closed dense subset. In many situations, one can find forcing extensions in which P_I^E is even $\aleph_1$-distributive or more.

If the poset P_I^E is regular in P_I, we will want to show that it generates the model $V[x_{\mathrm{gen}}]_E$ introduced in Definition 2.1.1. This will always be done through the following theorem:

Theorem 2.2.7. *Suppose that I is a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal on a Polish space X such that the quotient forcing P_I is proper and every analytic I-positive set contains a Borel I-positive subset. Suppose that E is a Borel equivalence relation such that P_I^E is a regular poset of P_I. Suppose moreover that for all I-positive analytic sets $B, C \subset X$,*

- *either there is a Borel I-positive subset $B' \subset B$ such that $[B']_E \cap C \in I$;*
- *or there is a Borel I-positive subset $B' \subset B$ and an injective Borel map $f : B' \to C$ such that $f \subset E$ and f preserves the ideal I: images of I-small sets are I-small, and images of I-positive sets are I-positive.*

Then the model $V[\dot{x}_{gen}]_E$ is forced to be equal to the P_I^E-extension.

Proof. Let $G \subset P_I$ be a generic filter and $H = G \cap P_I^E$. It is clear that $V[H] \subset V[\dot{x}_{gen}]_E$, since H is definable from $[\dot{x}_{gen}]_E$ as the collection of those Borel I-positive E-saturated sets coded in the ground model containing the calss $[\dot{x}_{gen}]_E$ as a subset. The other inclusion is harder.

Suppose that $\phi(\alpha, \vec{v}, [\dot{x}_{gen}]_E)$ is a formula with an ordinal parameter, other parameters in V, and another parameter $[\dot{x}_{gen}]_E$, and let κ be a cardinal. A density argument shows that there must be a condition $B \in P_I$ such that the condition $\langle P, 1\rangle \in P_I \times \mathrm{Coll}(\omega, \kappa)$ decides the truth value of ϕ, and H contains a condition below the pseudoprojection of B to P_I^E. We will show that for every two such conditions, the decision of the truth value must be the same. Thus, the set of ordinals defined by ϕ in the $P_I \times \mathrm{Coll}(\omega, \kappa)$ must be already in the model $V[H]$, defined as the set of those ordinals α such that for some condition $B \in P_I$ whose pseudoprojection to P_I^E belongs to the filter H, $\langle B, 1\rangle \Vdash \phi(\alpha, \vec{v}, [\dot{x}_{gen}]_E)$.

So suppose for contradiction that there are conditions $C_0, C_1 \in P_I$ and $B \in P_I^E$ such that B is below the pseudoprojection of both C_0, C_1 into P_I^E, and $\langle C_0, 1\rangle \Vdash \phi(\alpha, \vec{v}, [\dot{x}_{gen}]_E)$ and $\langle C_1, 1\rangle \Vdash \neg\phi$. Since B is below the pseudoprojections, the first item of the assumptions is excluded for both $C = C_0$ and $C = C_1$–the set $[B']_E$ would be an extension of B in P_I^E incompatible with the condition C_0 or C_1. So the second case must happen, and strengthening twice, we get a Borel I-positive set $B' \in P_I$ below B and Borel injections $f_0 : B' \to C_0$ and $f_1 : B' \to C_1$ as postulated in the assumptions. Since these functions preserve the ideal I and are subsets of the equivalence E, B' forces in the forcing P_I that the points $\dot{x}_{gen} \in \dot{B}'$, $\dot{f}_0(\dot{x}_{gen}) \in \dot{C}_0$ and $\dot{f}_1(\dot{x}_{gen}) \in \dot{C}_1$ are E-equivalent V-generic points for the poset P_I, giving the same generic extension. Thus, if $G \subset P_I$ is a V-generic filter containing the condition B' and $K \subset \mathrm{Coll}(\omega, \kappa)$ is a $V[G]$-generic filter, the forcing theorem applied to $f_0(x_{\mathrm{gen}})$ and $f_1(x_{\mathrm{gen}})$ implies that $V[G, K] \models \phi(\alpha, \vec{v}, [f_0(x_{\mathrm{gen}})]_E) \wedge \neg\phi(\alpha, \vec{v}, [f_1(x_{\mathrm{gen}})]_E)$, which is impossible in view of the fact that $f_0(x_{\mathrm{gen}})\ E\ f_1(x_{\mathrm{gen}})$. □

Another forcing feature of the arguments in this book is the use of a *reduced product* of quotient posets. There is a general definition of such products:

Definition 2.2.8. Let I be a σ-ideal on a Polish space X, and let E be a Borel equivalence relation on X. The reduced product $P_I \times_E P_I$ is defined as the poset of those pairs $\langle B, C\rangle$ of Borel I-positive sets such that for every sufficiently large cardinal κ, in the $\mathrm{Coll}(\omega, \kappa)$ extension there are E-equivalent points $x \in B$ and $y \in C$, each of them V-generic for the poset P_I. The ordering is that of coordinatewise inclusion.

It is not difficult to see that the left coordinates of conditions in a reduced product generic filter form a P_I-generic filter, and similarly for the right coordinates. To see this, assume that $\langle B, C\rangle$ is a condition in $P_I \times_E P_I$ and $D \subset P_I$ is a dense set. The poset $\mathrm{Coll}(\omega, \kappa)$ forces that there exists a pair of V-generic E-equivalent points $\dot{x} \in \dot{B}, \dot{y} \in \dot{C}$. Find a condition $p \in \mathrm{Coll}(\omega, \kappa)$ and conditions $B' \subset B$ and $C' \subset C$ in the dense set D such that $p \Vdash \dot{x} \in B', \dot{y} \in C'$ and observe that $\langle B', C'\rangle$ is still a condition in the reduced product $P_I \times_E P_I$ and it clearly forces that both filters on the left hand side and the right hand side meet the dense set D. Thus, the reduced produc generic filter is given by two points $\dot{x}_{\mathrm{lgen}}, \dot{x}_{\mathrm{rgen}} \in X$, each of which is V-generic for the poset P_I. These two points are typically not E-related. The general definition of the reduced product is certainly puzzling, and in all special cases, a good deal of effort is devoted to the elimination of the $\mathrm{Coll}(\omega, \kappa)$ forcing relation from it. The reduced products are always used in the intermediate case of the trichotomy 2.1.3:

Proposition 2.2.9. *If I is a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal on a Polish space X such that the quotient forcing P_I is proper, and E is a Borel equivalence relation on X such that the ergodic case of Theorem 2.1.3 happens, then the E-reduced product of P_I with itself is equal to product.*

Proof. Let $B \subset X$ be a Borel I-positive subset such that every two Borel I-positive subsets $C_0, C_1 \subset B$ contain a pair of equivalent points. We will show that the reduced product below the condition $\langle B, B\rangle$ consists of all pairs of Borel I-positive subsets of B. And indeed, if C_0, C_1 are two such sets, then pass to a $\mathrm{Coll}(\omega, \kappa)$ extension, apply in it Theorem 1.3.21 to find Borel I-positive sets $D_0 \subset C_0, D_1 \subset C_1$ consisting only of P_I-generic points, use Shoenfield absoluteness to transfer ergodicity from the ground model to the extension and find E-equivalent points, one in D_0, the other in D_1. These points show that the pair $\langle C_0, C_1\rangle$ is a condition in the reduced product as desired. □

Proposition 2.2.10. *Let I be a σ-ideal on a Polish space X such that the quotient P_I is proper. Suppose that E is a Borel equivalence relation on X such that the model $V[\dot{x}_{gen}]_E$ is forced to contain no new reals. Then the reduced product forces that $V[\dot{x}_{gen}]_E$ is equal to the intersection of $V[\dot{x}_{\mathrm{lgen}}]$ and $V[\dot{x}_{\mathrm{rgen}}]$.*

2.3 The spectrum of an ideal

To the untrained eye, the most typical situation regarding a Borel equivalence relation and a σ-ideal may appear to be that the equivalence relation cannot be significantly simplified by passing to a Borel I-positive set. This is the contents of the central definition in this book:

Definition 2.3.1. Let X be a Polish space, I a σ-ideal on X, and F an equivalence relation on another Polish space. We say that F is in the *spectrum* of I if there is a Borel I-positive set $B \subset X$ and an equivalence relation E on B bireducible with F such that for every Borel I-positive set $C \subset B$, the equivalence $E \restriction C$ is still bireducible with F.

If the quotient forcing P_I is more commonly known than the σ-ideal itself, as is the case with the Laver forcing for example, we will refer to the spectrum of the quotient forcing.

The notion of a spectrum is natural from several points of view. The study of the interplay of the quotient forcing and the equivalence relation will gravitate towards the cases that cannot be simplified by passing to a stronger condition.

The study of structures that a given σ-ideal imparts on the quotient space of E-equivalence classes will tend towards structures that disappear when restricted to a simpler quotient space. And finally, identifying an equivalence relation that is in a spectrum of a given σ-ideal is the strongest kind of negative canonization results possible within the calculus developed in this book. Thus, how do we investigate the notion of a spectrum?

One possible line of research is the attempt to evaluate the spectrum for a given σ-ideal. Some ideals have trivial spectrum consisting of just the identity and the equivalence relation with a single class. For others, the spectrum is nontrivial, but still fairly easy to identify. Still some others may have extremely complicated spectrum, rich in features or quite chaotic. In another direction, one may fix an equivalence relation and ask what it means for a σ-ideal to have it in the spectrum, especially in terms of forcing properties of the quotient poset P_I. These are fairly new and fine forcing properties mostly concerning the nature of intermediate forcing extensions as the Trichotomy Theorem would lead one to believe. There do not seem to be any obvious general forcing operation preservation theorems dealing with the properties obtained in this way.

The notion of spectrum is informative, but not without its faults. In particular, there is no guarantee that there will be a Borel I-positive set $B \subset X$ such that the equivalence E will be as simple as possible in the sense of the reducibility ordering on the set B. There just may be a decreasing chain of sets $B_0 \subset B_1 \subset \dots$ such that the equivalence relations $E \restriction B_0, E \restriction B_1 \dots$ strictly decrease in complexity, without a possibility of reaching a stable point. While such a situation may appear to be somewhat exotic, it does occur in the important case of the Lebesgue null ideal and countable Borel equivalence relations by a result of Hjorth [16, Lemma 3.12]. The notion of spectrum will not detect this type of behavior.

The spectrum does not appear to have any general monotonicity or closure properties beyond the following simple observation.

Proposition 2.3.2. *Suppose that I is a σ-ideal on a Polish space X and E is in the spectrum of I as witnessed by some equivalence F on X bireducible with E. If $J \supset I$ is a larger σ-ideal on X then E is again in the spectrum of J as witnessed by J.*

2.4 Canonization of equivalence relations

The simplest situation that one may hope to encounter is that when a given equivalence relation simplifies either to identity on a Borel I-positive set, or makes all elements of an I-positive Borel set equivalent.

Definition 2.4.1. A σ-ideal I on a Polish space X has *total canonization* for a certain class of equivalence relations if for every I-positive Borel set $B \subset X$ and every equivalence E on B in the given class there is a Borel I-positive set $C \subset B$ such that either $E \restriction C = \mathrm{ID}$ or $E \restriction C = \mathrm{EE}$.

As an example, the σ-ideal on ω^ω generated by compact sets has total canonization for all Borel equivalence relations, and the Laver ideal has total canonization for all equivalence relations classifiable by countable structures. In most cases though, the total canonization is way too much to hope for, and one has to settle for lesser, but still quite informative conclusion:

Definition 2.4.2. Let $\mathbf{E}$, $\mathbf{F}$ be two classes of equivalence relations, and let I be a σ-ideal on a Polish space X. $\mathbf{E} \to_I \mathbf{F}$ denotes the statement that for every I-positive Borel set $B \subset X$ and an equivalence $E \in \mathbf{E}$ on B there is a Borel I-positive set $C \subset B$ such that $E \restriction C \in \mathbf{F}$.

For example, the ideal I associated with Silver forcing satisfies classifiable by countable structures$\to_I \{\subset E_0, \mathrm{EE}\}$, meaning that every equivalence relation on a Borel I-positive set classifiable by countable structures either has an I-positive equivalence class or else can be simplified to an equivalence relation which is a subset of E_0 on an I-positive Borel subset–Theorem 3.6.6.

The total canonization is often proved using the free set property, see below. The trichotomy theorem 2.1.3 allows us to argue for total canonization from fairly common abstract properties of the ideal I. In order to state a comprehensible theorem, we must introduce two properties of σ-ideals.

Definition 2.4.3. A σ-ideal I on a Polish space X has the *rectangular property* if for Borel I-positive sets $B, C \subset X$ and a Borel partition $B \times C = \bigcup_n D_n$ of their product into countably many pieces one of the pieces contains a product $B' \times C'$ of Borel I-positive sets.

This property is useful for the study of the product forcing $P_I \times P_I$. It has been verified for all definable σ-ideals such that the quotient P_I is proper, bounding and Baire category preserving [49, Theorem 5.2.6], for the σ-ideal generated by compact subsets of ω^ω [45], as well as for some other cases.

Definition 2.4.4. A σ-ideal I on a Polish space X has the *transversal property* if for every Borel set $D \subset 2^\omega \times X$ such that the vertical sections are pairwise disjoint I-positive sets there is an I-positive Borel set $B \subset X$ which is covered by the vertical sections of D and visits each of the vertical sections in at most one point.

This property is a ZFC version of the determinacy dichotomies such as the first dichotomy of [49, Section 3.9.1]. It holds true of all definable σ-ideals generated by closed sets and many others, as in Proposition 3.2.1; its proof may use an infinite integer game associated with the σ-ideal as in Theorem 3.10.1(3). It fails for the likes of the E_0 ideal and the ideal of subsets of $\mathcal{P}(\omega)$ nowhere dense in the algebra $\mathcal{P}(\omega)$ modulo finite. Consider the set $D \subset 2^\omega \times \mathcal{P}(2^{<\omega})$ defined by $\langle y, x\rangle \in D$ if and only if x is a set of initial segments of y. The vertical sections of this set are somewhere dense in the algebra $\mathcal{P}(2^{<\omega})$ modulo finite; on the other hand, sets in distinct vertical sections have finite intersections, and so every transversal must be nowhere dense. We do not have an example of a σ-ideal with basis consisting of Borel sets of bounded complexity which does not have the transversal property.

The following theorem is not explicitly referred to in any further proofs. However, it offers an initial guidance as to which canonization results are possible and isolates the connection between canonization and the rectangular and transversal properties. Its proof also highlights the role of the model $V[x]_E$ and the trichotomy theorem 2.1.3.

Theorem 2.4.5. *(CH) Suppose that I is a suitably definable σ-ideal such that the quotient forcing P_I is $< \omega_1$-proper, has the rectangular and transversal properties. Then I has total canonization.*

The wording of the theorem must be scrutinized more closely. The continuum hypothesis assumption can be dropped if the quotient forcing is $< \omega_1$-proper in all forcing extensions, which is the case for all σ-ideals investigated in this book. A forcing P is $< \omega_1$-*proper* if for every countable ordinal α and every ε-tower $\langle M_\beta : \beta \in \alpha\rangle$ of countable elementary submodels of a large structure such as $\langle V_\kappa, \varepsilon\rangle$ for a sufficiently large ordinal κ, with $P \in M_0$, for every condition $p \in P \cap M_0$ there is a strengthening $q \leq p$ which is simultaneously M_β-master for each ordinal $\beta \in \alpha$. This is a technical strengthening of properness due to Shelah; it is satisfied for all σ-ideals in this book as proved by a standard transfinite induction argument in each case. Ishiu [17] proved that $< \omega_1$-properness of a forcing is equivalent to the Axiom A property of the complete Boolean algebra of regular open subsets of P. For the purposes of the theorem, this property does not seem to be replaceable by anything descriptive set theoretic in nature.

We will prove the theorem for $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideals, but small adjustments will result in a proof for other specific ideals in ZFC and for larger definability classes under large cardinal assumptions. The rectangular property is used to rule out the nontrivial ergodicity clause of the trichotomy theorem, and the example of Laver ideal and the E_{K_σ} equivalence–see Section 3.10–shows that some such assumption is again necessary.

Proof. The main point of the proof is the analysis of the possible intermediate extensions of the P_I extension. We will show that every intermediate extension must be c.c.c.–this uses the $< \omega_1$-properness and the transversal property. Then, the possibilities for c.c.c. intermediate extensions will be discussed, revealing that no such extension can exist either–this uses the rectangular property and the definability. Looking back at the Trichotomy theorem 2.1.3, the second option is ruled out, the nontrivial ergodicity is ruled out as well by the rectangular property applied to the given equivalence relation E, and the total canonization is what is left from our options.

We start with showing that all intermediate extensions must be c.c.c. Suppose that $\dot{f}$ is a name for a function from ω_1 to ω_1 which is not bounded by any ground model function; we will show that there is a condition $B \in P_I$, an ordinal $\alpha \in \omega_1$ and an injective Borel function $h : B \to \alpha^\alpha$ such that $B \Vdash h(\dot{x}_{gen}) = \dot{f} \restriction \alpha$. The injectivity of the function h is a coanalytic statement, therefore absolute to the P_I extension and so B forces that $\dot{x}_{gen}$ is recovered in $V[\dot{f}]$ as the unique element of the set B which is sent to $\dot{f} \restriction \alpha$ by h. Since the forcing P_I has density $\aleph_1$, every nowhere c.c.c. intermediate extension of

P_I must contain such an unbounded function $\dot{f}$, and therefore must be equal to the whole P_I-extension.

To obtain the condition B, let θ be a large regular cardinal, $\langle M_\beta : \beta \in \omega_1\rangle$ a continuous ε-tower of countable elementary submodels of H_θ containing $X, I, \dot{f}$, and let N be a countable elementary submodel containing this tower. Note that no point $x \in X$ can be simultaneously N-generic and M_β-generic for all $\beta \in N$: if it is N-generic then $\dot{f}/x$ must not be bounded by the function $\beta \mapsto \omega_1 \cap M_{\beta+1}$ which belongs to N, but this means that x cannot be $M_{\beta+1}$-generic for that $\beta \in N$ for which $\dot{f}(\beta)/x > \omega_1 \cap M_{\beta+1}$. Now build ε-towers $\vec{M}_n : n \in \omega$ of elementary submodels and submodels $N_n : n \in \omega$ so that $\vec{M}_n \in N_n \in \vec{M}_{n+1}(0)$. For every binary sequence $y \in 2^\omega$ let T_y be the unique continuous ε-tower of models containing the model N_n if $y(n) = 0$ and the models $\{\vec{M}_n(\beta) : \beta \in N_n \cap \omega_1 + 1\}$ if $y(n) = 1$ (and no other models). Let $B_y = \{x \in X : x$ is generic for all models on the tower $T_y\}$. This is a Borel set by its definition, and it is I-positive by $< \omega_1$-properness: it is the set of generic points for a certain tower of models of countable length. By the previous observation, the set $D \subset 2^\omega \times X$ defined by $\langle y, x\rangle \in D$ if $x \in B_y$ is Borel, and has pairwise disjoint I-positive sections by $< \omega_1$-properness. Use the transversal property of the ideal I to find a Borel I-positive set B which visits each section in at most one point, and is covered by the vertical sections of the set D.

Let $\alpha = \bigcup_n N_n \cap \omega_1$; this is the common value of $\bigcup T_y \cap \omega_1$ for every point $y \in 2^\omega$. Whenever $y \in 2^\omega$ and $x \in B$ are points such that $x \in B_y$, then the point x is generic for all models on the tower T_y and therefore we may define $h(x) = \dot{f}/x$, or in other words $h(x)$ is the function from $\alpha \to \alpha$ defined by $h(x)(\beta) = \gamma$ iff x belongs to some set in $P_I \cap \bigcup T_y$ which forces $\dot{f}(\check{\beta}) = \check{\gamma}$. The function $h : B \to \alpha^\alpha$ is certainly Borel. We will show that $B \Vdash \dot{f} \restriction \alpha = \dot{h}(\dot{x}_{gen})$ and that h is an injection. This will complete the first part of the proof.

For the injectivity, if $x_0 \neq x_1 \in B$ are distinct points, they must come from distinct vertical sections of the set D indexed by some distinct $y_0 \neq y_1 \in 2^\omega$. Let $n \in \omega$ be such that $y_0(n) = 0$ and $y_1(n) = 1$, let $\gamma = N_n \cap \omega_1$ and let $k \in \gamma \to \gamma$ be the function defined by $k(\beta) = \vec{M}_n(\beta + 1) \cap \omega_1$; then $h(x_1) \restriction \gamma$ is bounded by the function k since x_1 is generic for the models $\vec{M}_n(\beta) : \beta \in \gamma$, and $h(x_0) \restriction \gamma$ is not bounded by this function since x_0 is generic for the model M_n and $\dot{f}$ was forced to be unbounded. Thus $h(x_0) \neq h(x_1)$.

To show $B \Vdash \dot{f} \restriction \alpha = \dot{h}(\dot{x}_{gen})$, let $C \subset B$ be a Borel I-positive set and $\beta \in \alpha$ be an ordinal. We must find an ordinal γ and a stronger condition $C' \subset C$ forcing $\dot{f}(\beta) = \gamma$ and such that $\forall x \in C'\ h(x)(\beta) = \gamma$. Note that every point $x \in C$ is generic for all models on a tower T_y for some point $y \in 2^\omega$. The countable set $\bigcup T_y$ is the same for all these towers. Thus, the set C is covered by the countably many sets in $\bigcup T_y \cap P_I$ which force a specific value to $\dot{f}(\beta)$, and one of these countably many sets will have positive intersection C' with C. It is not difficult to see that the set $C' \subset C$ works.

The second part of the proof discusses the possibility of intermediate c.c.c. extension, and the aim is to show that there cannot be one. First argue that it must add a real. Suppose for contradiction that it is effected by a c.c.c. forcing Q which does not add a real. Let M be a countable elementary submodel of a large enough structure containing Q. Since no real is added by Q, Q forces that the intersection of its generic filter with M belongs to the ground model. By c.c.c. of Q, there are only countably many possibilities $\{g_n : n \in \omega\}$ for this intersection, and each of them is in fact M-generic. Let $h \subset \mathrm{Coll}(\omega, \kappa) \cap M$ be an M-generic filter which is mutually generic with all the filters $g_n : n \in \omega$, and work in the model $M[h]$. By Claim 1.3.21, $M[h]$ contains (a code for) a Borel I-positive set $B \subset X$ consisting only of M-generic reals. Let $f : B \to \mathcal{P}(Q) \cap M$ be the function defined by $f(x) = \{q \in Q \cap M : \exists C \in P_I \cap M\ x \in C \wedge C$ forces q to belong to the Q-generic filter$\}$. This is a Borel function with code in the model $M[h]$. In that model, there are obviously two cases: either the f-preimage of every subset of $Q \cap M$ is in the ideal I, or there is a subset of $Q \cap M$ with Borel I-positive preimage. In the first case, the property of B and f is a coanalytic statement since the ideal I is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$, therefore transfers to the model V, and there B forces that the intersection of the Q-generic filter with M is not in the ground model, contradicting the assumptions. In the latter case, $B \Vdash$ the intersection of the Q-generic filter with M is in the model $M[h]$ and therefore not on the list $\{g_n : n \in \omega\}$, reaching a contradiction again.

Thus, suppose that P_I adds a nontrivial c.c.c. real; let $B \in P_I$ be a condition forcing that $f(\dot{x}_{gen})$ is such a real for some fixed Borel function $f : X \to Y$. Let J be a σ-ideal on the Polish space Y defined by $A \in J \leftrightarrow f^{-1}A \in I$. Since $f(\dot{x}_{gen})$ is forced to be a c.c.c. real, falling out of all J-small sets, it must be the case that it is P_J generic. If the ideal I is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$, then so is J. There are two cases.

If P_J adds an unbounded real, then it in fact adds a Cohen real by a result of Shelah [2, Theorem 3.6.47]. One has to adjust the final considerations of that proof very slightly to conclude that it works for all $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ c.c.c. σ-ideals. Let $g : Y \to 2^\omega$ be a Borel function representing the Cohen real. Now, the meager ideal does not have the rectangular property, and it is easy to transport this feature to the ideal I. Just let $D \subset B \times B$ be the set of all points $\langle x_0, x_1 \rangle$ such that $g(f(x_0))\ E_0\ g(f(x_1))$ and observe that neither D nor its complement can contain a rectangle with Borel I-positive sides.

If P_J is bounding, then Player II has a winning strategy in the bounding game [49, Theorem 3.10.7]. By a result of Fremlin [19, Theorem 7.5], there is a continuous submeasure ϕ on the space Y such that $J = \{A \subset Y : \phi(A) = 0\}$. The relevant properties of such ideals were investigated in [7] and it turns out that they cannot have rectangular property. They in fact fail to have even the Fubini property with each other unless both of the submeasures in the product share their null ideal with a σ-additive Borel probability measure. In this last case, the rectangular property fails again since every Borel probability measure is ergodic with respect to some hyperfinite Borel equivalence relation.

The conclusion is that there are no intermediate forcing extensions strictly between the ground model and the P_I extension, c.c.c. or otherwise. The first paragraph of this proof then completes the argument for the theorem. $\square$

The total canonization of Borel equivalence relations is not the strongest possible statement one can obtain. A significant strengthening is the Silver type dichotomy introduced in the next section. A strengthening in a different direction concerns arbitrary graphs:

Definition 2.4.6. A σ-ideal I on a Polish space X has *total canonization of Borel graphs* if for every Borel I-positive set $B \subset X$ and every Borel graph $G \subset [B]^2$ there is a Borel I-positive set $C \subset B$ such that $[C]^2 \subset G$ or $[C]^2 \cap G = 0$.

This can be restated in a somewhat different language: the collection of those Borel sets $G \subset [X]^2$ for which there is no Borel I-positive $C \subset X$ with $[C]^2 \subset G$, is an ideal. Where the canonization of Borel equivalence relations is often attained simply by an analysis of the forcing properties of the poset P_I, the canonization of Borel graphs invariably needs strong partition theorems for Polish spaces such as the Milliken theorem.

2.5 A Silver-type dichotomy for a σ-ideal

In fairly common circumstances, the total canonization takes up an even stronger form:

Definition 2.5.1. A σ-ideal I on a Polish space X has the *Silver property* for a class $\mathbf{E}$ of equivalence relations if for every I-positive Borel set $B \subset X$ and every equivalence relation $E \in \mathbf{E}$ on the set B, either B can be decomposed into countably many equivalence classes and an I-small set or there is a Borel I-positive set $C \subset B$ such that $E \restriction C = \mathrm{ID}$. If $\mathbf{E}$ is equal to the class of all Borel equivalence relations, it is dropped from the terminology and we speak of the Silver property of I instead.

This should be compared with the classical Silver dichotomy [40], which establishes the Silver property for the ideal of countable sets. Observe that unlike total canonization, the Silver property introduces a true dichotomy: the two options cannot coexist. It also has consequences for undefinable sets: if an equivalence relation $E \in \mathbf{E}$ has an I-positive set $A \subset X$ consisting of pairwise E-inequivalent points, then it has an I-positive *Borel* set $B \subset X$ consisting of pairwise E-inequivalent points, simply because the first clause of the dichotomy cannot hold in such circumstances.

Let us offer a perhaps artificial reading of the dichotomy which nevertheless fits well with the techniques developed in this book or [49]. Given a σ-ideal I on a Polish space X and a Borel equivalence E on X, consider the ideal $I^* \supset I$ σ-generated by E-equivalence classes and sets in I. Then either I^* is trivial, containing the whole space, or else the quotient forcing P_{I^*} is equal to P_I below some condition.

Proposition 2.5.2. *Suppose that total canonization holds for I and the ideal I is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ and has the transversal property. Then the ideal I has the Silver property.*

Proof. Let E be a Borel equivalence relation, and consider the set $C = \{x \in X : [x]_E \notin I\}$. This is an analytic set. Either there are only countably many equivalence classes in the set C. In this case, the set C is even Borel. If the complement of C is in the ideal I then we are in the first clause of the Silver property, and if the complement of C is I-positive then we can apply total canonization to it: the equivalence classes of E below the complement of C are in the ideal I and so the application of total canonization must yield a Borel I-positive set consisting of E-inequivalent points.

Or, there are uncountably many equivalence classes in the set C. The classical Silver dichotomy [40] then provides a perfect set $P \subset C$ of pairwise inequivalent points. Let $D \subset P \times X$ be the set defined by $\langle x, y\rangle \in D \leftrightarrow xEy$. The transversal theorem yields an I-positive Borel set C covered by the vertical sections of D, visiting each of its sections in at most one point. We are in the second clause of the Silver property! $\square$

A quick perusal of the argument shows that the proposition in fact holds even if restricted to any class of Borel equivalence relations closed under reduction. Thus for example the ideal $I(Fin \times Fin)$ introduced in Section 3.2b is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$, has the transversal property, and has canonization for equivalences below E_{K_σ}, even though it perhaps does not have total canonization. The proof shows that in fact $I(Fin \times Fin)$ has the Silver property for the class of equivalences reducible to E_{K_σ}.

Note that the failure of the transversal property as witnessed by some set $D \subset 2^\omega \times X$ implies the failure of the Silver property in the common case that the projection of D to the X axis is a Borel set. Just consider the equivalence relation E on X connecting two points if they are from the same section of the set D or if they both fail to be in any section of the set D. This is Borel, since $x \; E \; y$ if and only if x, y are not in the projection of D or else $\exists z \; \langle z, x\rangle, \langle z, y\rangle \in D$, which is equivalent to $\forall z_0, z_1 \; \langle z_0, x\rangle, \langle z_1, y\rangle \in D \rightarrow z_0 = z_1$, therefore analytic coanalytic. The equivalence relation E witnesses the failure of the Silver property as the I-positive transversal for D does not exist; note that in fact E is smooth.

2.6 Free set property

Definition 2.6.1. An ideal I on a Polish space X has the *free set property* if for every I-positive Borel set $B \subset X$ and every Borel set $D \subset B \times B$ with all vertical sections in the ideal I there is a Borel I-positive set $C \subset B$ such that $(C \times C) \cap D \subset \mathrm{ID}$.

This should be viewed as a generalization of the various free set properties in combinatorics to the Borel context: if one assigns a small set to every point, then there will be a large set which is free for this assignment. The search for free sets of various sizes on uncountable cardinals is always present in the work of Péter Komjáth, for example [29]. One nontrivial instance of the free set property in the Borel context was obtained by Solecki and Spinas [43]. The free set property implies total canonization of all Borel equivalences. Namely, if E is a Borel equivalence on a Borel I-positive set $B \subset X$, then either one of its equivalence classes is I-positive (in which case this class is an I-positive set on which $E = \mathrm{EE}$), or one can use the free set property with $D = E$ to find a positive Borel set on which $E = \mathrm{ID}$.

The free set property does not seem to be equivalent to the total canonization of Borel equivalences, one such candidate is discussed in Section 3.2b. The most important way to argue for the free set property is via the mutual generics property.

2.7 Mutual genericity

Definition 2.7.1. The ideal I has the *mutual generics property* if for every countable elementary submodel M of a large structure and every Borel I-positive set $B \in M$ there is a Borel I-positive set $C \subset B$ such that every pair of distinct points in the set C is generic for the product of the forcings P_I.

The mutual generics property holds for a variety of ideals. One important class of ideals with this property is the class of all suitably definable ideals generated by closed sets for which the quotient poset is bounding–Theorem 3.2.3.

Mutual generics property may fail for a variety of reasons. It may fail even though the free set property holds–Section 3.3. However, the typical reason for its failure is that there is a Borel equivalence relation E on the space X which cannot be completely canonized. In such a case, one can consider variations of the mutual generics property asserting that there is a Borel I-positive set such that every pair of nonequivalent points is suitably generic etc. It is here where a variety of reduced products of the poset P_I may enter the scene. In the simplest case, a reduced product $P_I \times_E P_I$ is the set of all pairs $\langle B_0, B_1\rangle \in P_I \times P_I$ such that $[B_0]_E = [B_1]_E$, ordered by coordinatewise inclusion–Definition 2.2.8.

Mutual genericity is not equivalent to the free set property, as the example of symmetric Sacks forcing in Section 3.3 shows. Another potential example is the Miller forcing, where Solecki and Spinas [43] proved the free set property, while the status of mutual genericity remains unknown.

One way to disable most versions of the mutual generics property is to find a Borel function $f : X^2 \to 2^\omega$ such that for every Borel I-positive set $B \subset X$, the image $f''B^2$ contains a nonempty open set. This means that a pair of points in any given Borel I-positive set can code anything, in particular it can code a wellordering of length greater than that of ordinals of any give countable transitive model, and so it cannot be suitably generic.

2.8 Functions on squares

Thus, instead of equivalences on Polish spaces we may want to consider general Borel functions on squares of Polish spaces and investigate their behavior on squares of positive Borel sets. This turns out to have a close relationship with the behavior of Borel equivalence relations:

Proposition 2.8.1. *If E_0 is in the spectrum of I then there is a Borel I-positive set $B \subset X$ and a Borel function $f : B^2 \to \omega^\omega$ such that $f''C^2 \subset \omega^\omega$ is unbounded for every Borel I-positive set $C \subset B$.*

Proof. Suppose that E_0 is in the spectrum of the ideal I, as witnessed by a Borel function $g : B \to 2^\omega$ on a Borel I-positive set $B \subset X$. Define $f : B^2 \to \omega^\omega$ by $g(x, y)(n)$ =the least $m > n$ such that $g(x)(m) \neq g(y)(m)$ if $\neg g(x)\ E_0\ g(y)$, and $f(x, y)$ =trash otherwise. We claim that this function works.

Suppose for contradiction that the conclusion fails for some Borel I-positive set $C \subset B$; the set $g''C \subset \omega^\omega$ would be modulo finite bounded by some function $h \in \omega^\omega$. In such a case, for any points $x, y \in C$ it would be the case that $f(x)\ E_0\ f(y)$ if and only if $\forall^\infty n\ f(x) \restriction [n, h(n)) = f(y) \restriction [n, h(n))$ if and only if $\exists^\infty n\ f(x) \restriction [n, h(n)) = f(y) \restriction [n, h(n))$. This means that the equivalence E_0 on the analytic set $f''C$ is relatively G_δ, and by the argument of [8, Theorem 6.4.4] it is in fact smooth on this set. This contradicts the assumption that E_0 is in the spectrum of the ideal I.

□

The conclusion cannot be strengthened to obtain dominating images, as a basic example shows. Consider the σ-ideal I associated with the E_0-forcing as in Section 3.4; thus E_0 is in the spectrum of I. Let $B \subset 2^\omega$ be an I-positive Borel set, and $f : [B]^2 \to \omega^\omega$ be a function. Let M be a countable elementary submodel of a large structure containing all this information. The proof of Theorem 3.4.5 yields a Borel I-positive subset $C \subset B$ such that every pair of non-E_0-equivalent points of C is reduced product generic for the model M, and every single point in C is P_I-generic. Let $g \in \omega^\omega$ be a function dominating all the functions in M. The reduced product does not add a dominating real by Theorem 3.4.7, and so if $x, y \in C$ are not E_0 related then $f(x, y)$ cannot modulo finite dominate g. And, if the points $x, y \in C$ are E_0-related, then the functional value $f(x, y)$ belongs to $M[x]$, and as the P_I forcing is bounding, $f(x, y)$ cannot modulo finite dominate g in this case either.

Proposition 2.8.2. *If E_2 is in the spectrum of I then there is a Borel I-positive set $B \subset X$ and a Borel function $f : B^2 \to [0, 1]$ such that $f''C^2 \subset [0, 1]$ is somewhere dense for every Borel I-positive set $C \subset B$.*

Proof. Recall the summable metric d on 2^ω, $d(x, y) = \Sigma\{1/n+1 : x(n) \neq y(n)\}$; the distance of two points may be infinite. E_2 is the equivalence relation on 2^ω connecting points of finite distance. Now suppose that E_2 is in the spectrum of I as witnessed by a Borel function $g : B^2 \to 2^\omega$. Define a Borel function $f : B^2 \to [0, 1]$ by setting $f(x, y) = d(g(x), g(y))$ if this number is < 1, and $f(x, y)$ =trash otherwise. We claim that this function works. Consider an arbitrary Borel I-positive set $C \subset B$.

Indeed, since E_2 is in the spectrum of the ideal I, it must be that $E_2 \restriction f''C$ is not essentially countable, and therefore the analytic set $f''C$ cannot be grainy as described in Section 3.5. Thus, for every real $\varepsilon > 0$ there is a finite walk through the set C such that its steps are shorter than ε and the endpoints have

d-distance at least 1. If the set $f''C \subset [0,1]$ was not dense, perhaps avoiding a basic open neighborhood of diameter ε, no such walk could exist for that ε! $\square$

Proposition 2.8.3. *Suppose that there is a Borel function $f : X^2 \to 2^\omega$ such that $f''C^2 \subset 2^\omega$ contains a nonempty open set for every Borel I-positive set $C \subset B$. Then I does not have the mutual generics property.*

Proof. The mutual generics property most certainly fails since for every countable elementary submodel M of a large structure and a Borel I-positive set $C \subset B$ there are two points $x, y \in C$ such that $f(x, y)$ codes the transitive collapse of the model M, and therefore this pair cannot be (mutually or otherwise) generic over the model M. $\square$

Example 2.8.4. Consider the Laver forcing with the associated σ-ideal I on ω^ω and the function $f : (\omega^\omega)^2 \to 2^\omega$ defined by $f(x,y)(n) = 1 \leftrightarrow x(n) > y(n)$. Every I-positive Borel set $B \subset \omega^\omega$ contains all branches of some Laver tree T with trunk of length n, and then it is easy to find, for every binary sequence $z \in \omega^\omega$, two branches $x, y \in [T]$ so that $f(x, y) = z$ on all entries past n. So $f''B^2$ contains a nonempty open set.

Example 2.8.5. Consider the Silver forcing with the associated ideal I and the function $f : (2^\omega)^2 \to 2^\omega$ defined by $f(x, y) = x \circ \pi^{-1}$ where π is the increasing enumeration of the set $\{n : x(n) \neq y(n)\}$ if this set is infinite, and $f(x, y)$ =trash if this set is finite. Every I-positive Borel set B contains all total extensions of some fixed partial function $g : \omega \to 2$ with co-infinite domain. Whenever $z \in 2^\omega$ is a binary sequence, let $x, y \in 2^\omega$ be the unique points such that $g \subset x, y$ and $z = x \circ \pi^{-1}$ and $1 - z = y \circ \pi^{-1}$ where π is the increasing enumeration of the complement of the domain of g. Clearly then, $f(x, y) = z$ and $f''B^2 = 2^\omega$.

Both Silver and Laver forcing contain E_{K_σ} in their spectrum, and also other approaches in this book that insert E_{K_σ} into the spectrum of a σ-ideal immediately lead to a construction of a Borel function $f : X \times X$ such that the image of any Borel square with an I-positive side contains an open set. This brings up the obvious question.

Question 2.8.6. Suppose that E_{K_σ} is in the spectrum of a σ-ideal I on a Polish space X. Does there have to exist a Borel I-positive set $B \subset X$ and a Borel function $f : B^2 \to 2^\omega$ such that for every Borel I-positive subset $C \subset B$, the image $f''C^2$ contains a nonempty open set?

Chapter 3

Particular forcings

3.1 Sacks forcing and variations

The exposition of canonization properties of various σ-ideals is best started with the simplest and most instructive example, that of Sacks forcing, its finite products and countable iterations. Sacks forcing is the poset of all perfect binary trees ordered by inclusion. The associated σ-ideal is the ideal of countable subsets of 2^ω, as the perfect set theorem shows. Every uncountable Borel set has a perfect subset, and therefore the map $T \mapsto [T]$ is a dense embedding of the Sacks forcing into P_I. Restating the classical Silver dichotomy 1.3.10,

Fact 3.1.1. *The ideal of countable sets has the Silver property.*

Let us now move to a finite product of Sacks forcing of dimension $n \in \omega$. The associated ideal I_n on the space $(2^\omega)^n$ is generated by those Borel sets which do not contain a product $\Pi_{i \in n} C_i$ of nonempty perfect sets, as the rectangular property of the ideal of countable sets shows [49, 5.2.6]. The map $\pi_n : \langle C_i : i \in n\rangle \mapsto \Pi_{i \in n} C_i$ then constitutes a dense embedding of the n-fold product of the Sacks forcing into the poset P_{I_n}. The ideal I_n is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ and every positive analytic set contains a positive Borel subset. These fairly well-known facts and more follow from [49, Section 5.2.1].

There are some new equivalence relations on the product as opposed to the single Sacks forcing. Let $a \subset n$ be a set, and define the equivalence relation E_a on $(2^\omega)^n$ by setting $\vec{x} E_a \vec{y}$ if and only if $\vec{x} \restriction a = \vec{y} \restriction a$. This is obviously a smooth equivalence relation which is not equal to identity or everything on any Borel I_n-positive set unless $a = 0$ or $a = n$. However, these equivalence relations are the only obstacles to total canonization:

Theorem 3.1.2. *Let $n \in \omega$ be a natural number. Then Borel$\rightarrow_{I_n}$ $\{E_a : a \subset n\}$.*

The much more complicated case of the infinite product of Sacks forcing will be treated in its own section.

Proof. Let $a \subset n$ be a set. Define the *reduced product* P_a as the product of $n \setminus a$ many copies of Sacks forcing (indexed with elements of the set $n \setminus a$) with a many copies of Sacks forcing (indexed with $(i, 0) : i \in a$) and further with a many more copies of Sacks forcing (indexed with $(i, 1) : i \in a$). Clearly, the reduced product is isomorphic to the product of $n + |a|$ many copies of Sacks forcing, and it adds $n + |a|$ many mutually generic reals, indexed by the set $(n \setminus a) \cup \{(i, j) : i \in a, j \in 2\}$. We will organize these reals into sequences $\vec{x}_{\mathrm{lgen}} \in (2^\omega)^n$ and $\vec{x}_{\mathrm{rgen}} \in (2^\omega)^n$ with $\vec{x}_{\mathrm{lgen}} \restriction (n \setminus a) = \vec{x}_{\mathrm{rgen}} \restriction (n \setminus a)$.

If M is a countable elementary submodel of a large structure and $\vec{y}, \vec{x} \in (2^\omega)^n$ are two sequences, we will say that they are reduced product generic for M, if writing $a = \{i \in n : \vec{x}(i) = \vec{y}(i)\}$, the sequence $\vec{z}$ defined by $\vec{z}(i) = \vec{x}(i)$ if $i \in a$, $\vec{z}(i, 0) = \vec{x}(i)$ and $\vec{z}(i, 1) = \vec{y}(i)$ if $i \in n \setminus a$, is P_a generic for the model M. The following is the key claim:

Claim 3.1.3. *Let M be a countable elementary submodel of a large structure, and let $B_i : i \in n$ be perfect subsets of 2^ω in the model M. There are perfect subsets $C_i \subset B_i : i \in n$ of the Cantor space such that the product $\Pi_{i \in n} C_i$ consists of pairwise reduced product generic sequences for the model M.*

This is to say that any two sequences $\vec{x}, \vec{y} \in \Pi_{i \in n} C_i$ satisfying $x(i) = y(i) \leftrightarrow i \notin a$, are reduced product generic for the model M.

The theorem immediately follows from the claim. Let E be a Borel equivalence relation on the space $(2^\omega)^n$, and find an inclusion minimal set $a \subset n$ such that some condition in P_a forces $\vec{x}_{\mathrm{lgen}}\ E\ \vec{x}_{\mathrm{rgen}}$. If a condition $p = \langle B_i : i \in (n \setminus a), B_i : i \in a, C_i : i \in a\rangle$ in P_a forces this, then so does $q = \langle B_i : i \in (n \setminus a), B_i : i \in a, B_i : i \in a\rangle$: whenever $\vec{x}_{\mathrm{lgen}}, \vec{x}_{\mathrm{rgen}}$ are generic sequences meeting the condition q, in a further generic extension one can find a condition $\vec{y}_{gen}$ such that the pairs $\vec{x}_{\mathrm{lgen}}, \vec{y}_{gen}$ and $\vec{x}_{\mathrm{rgen}}, \vec{y}_{gen}$ are generic sequences meeting the condition q, so these pairs consist of E-equivalent sequences by the forcing theorem, and consequently $\vec{x}_{\mathrm{lgen}}\ E\ \vec{x}_{\mathrm{rgen}}$ by the transitivity of the relation E. Now let M be a countable elementary submodel of a large structure and use the claim to find sets $C_i \subset B_i : i \in n$ whose product consists of pairwise reduced product generic sequences for the model M. We claim that $E \restriction \Pi_{i \in n} C_i = E_a$.

To show this, suppose that $\vec{x}, \vec{y} \in \Pi_{i \in n} C_i$ are sequences and let $b = \{i \in n : \vec{x}(i) = \vec{y}(i)\}$. Find sequences $\vec{x}'$, $\vec{y}'$ in the product such that $a = \{i \in n : \vec{x}(i) = \vec{x}'(i)\} = \{i \in n : \vec{y}(i) = \vec{y}'(i)\}$ and $a \cap b = \{i \in n : \vec{x}'(i) = \vec{y}'(i)\}$. The forcing theorem applied to the reduced product P_a together with Borel absoluteness implies that $\vec{x}\ E\ \vec{x}'$ and $\vec{y}\ E\ \vec{y}'$. Now if $a \cap b = a$ then the forcing theorem also implies that $\vec{x}'\ E\ \vec{y}'$ and so $\vec{x} E \vec{y}$. on the other hand, if $a \cap b \neq a$ then the minimal choice of the set a together with the forcing theorem applied to $P_{a \cap b}$ give $\neg \vec{x}'\ E\ \vec{y}'$ and $\neg \vec{x}\ E\ \vec{y}$!

The claim is proved by a standard fusion argument.

□

The countable support iteration of Sacks forcing of countable length is another natural subject of study. Let $\alpha \in \omega_1$ be a countable ordinal, and consider the Polish space $X = (2^\omega)^\alpha$ with the product topology. The ideal I_α associated with the countable support iteration of the Sacks forcing of length α is the transfinite Fubini power of the countable ideal, as described in [49].

There are obvious obstacles to total canonization in this case. Let $\beta \leq \alpha$ be an ordinal and consider the equivalence relation E_β on $(2^\omega)^\alpha$ given by $\vec{x}\ E_\beta\ \vec{y}$ if and only if $\vec{x} \restriction \beta = \vec{y} \restriction \beta$. Again, it turns out that these are exactly the optimal irreducible list of obstacles:

Theorem 3.1.4. *Let $\alpha \in \omega_1$ be a countable ordinal. Then Borel$\rightarrow_{I_\alpha} \{E_\beta : \beta \leq \alpha\}$.*

Proof. Given ordinals $\beta \leq \alpha \in \omega_1$, consider the reduced product P^α_β consisting of pairs $\langle p, q\rangle \in P^\alpha \times P_\alpha$ for which $p \restriction \beta = q \restriction \beta$. The reduced product adds sequences $\vec{x}_{\mathrm{lgen}}, \vec{x}_{\mathrm{rgen}} \in (2^\omega)^\alpha$ which coincide on their first β many coordinates. If M is a countable elementary submodel of a large structure, we call sequences $\vec{x}, \vec{y} \in (2^\omega)^\alpha$ reduced product generic if the set $\beta = \{\gamma \in \alpha : \vec{x}(\gamma) = \vec{y}(\gamma)\}$ is an ordinal and the sequences are P^α_β-generic for the model M. As in the product case, there is a key claim:

Claim 3.1.5. *Let M be a countable elementary submodel of a large enough structure, and $B \in P_{I_\alpha} \cap M$ is a condition. There is a Borel I_α-positive set $C \subset B$ consisting of pairwise reduced product generic sequences.*

The theorem follows from the claim exactly as in the previous argument. The claim itself is proved by a standard fusion process.

□

3.2 σ-ideals σ-generated by closed sets

Following the exposition of [49], the σ-ideals that should be easiest to deal with are those σ-generated by closed sets. Indeed, there is a wealth of relevant information available:

Proposition 3.2.1. *Suppose that I is a σ-ideal on a Polish space σ-generated by closed sets. Then*

1. *the poset P_I is $< \omega_1$-proper and preserves Baire category;*
2. *if I is a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ then I has the transversal property;*
3. *moreover, if I is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ then every intermediate extension of the P_I extension is generated by a single Cohen real.*

Proof. (1) is contained in [49, Theorem 4.1.2] and (3) is in [49, Theorem 4.1.7]. That leaves (2) to be proved.

Let $D \subset 2^\omega \times X$ be a Borel set with pairwise disjoint I-positive vertical sections. Solecki's theorem [41] shows that every vertical section contains a G_δ I-positive subset. Further manipulation even yields a homeomorphic copy of ω^ω

such that every nonempty relatively open set is I-positive. Note that for every such a set the sets in the ideal I must be relatively meager.

Use the Sacks uniformization to find a perfect set $C \subset 2^\omega$ and a continuous injective map $\pi : C \times \omega^\omega \to X$ fixing the first coordinate such that for every $y \in C$, $\pi_y : \omega^\omega \to X$ is a homeomorphic embedding with range included in D_y, such that relatively open subsets of the range are I-positive. Find an F_σ set $F \subset P \times X$ such that its vertical sections range over all F_σ-subsets of X. The set $G \subset C \times \omega^\omega$, $G = \{\langle y, z\rangle : \langle y, \pi(y, z)\rangle \in F\}$ is Borel.

The set $C' = \{y \in C : G_y \text{ is meager in } \omega^\omega\}$ is Borel by [27, Theorem 16.1]. Use uniformization [27, Theorem 18.6] to find a Borel map $f : C' \to \omega^\omega$ such that for every point $y \in C'$, $f(y) \notin G_y$. Consider the set $B = \{\pi(y, f(y)) : y \in C'\} \subset X$. As a Borel one-to-one image of a Borel set it is Borel. It intersects only the vertical sections $D_y : y \in C'$, and each of them in exactly one point, namely $\pi(y, f(y))$. Finally, the set B is I-positive: every I-small set is covered by an F_σ set in the ideal I, indexed as F_y for some point $y \in C$. The set F_y must be relatively meager in $\mathrm{rng}(\pi_y)$, so $y \in C'$ and $\pi(y, f(y)) \in B \setminus F_y$ and $B \notin I$ as desired! $\square$

Now, the spectrum of the Cohen forcing (associated with the meager ideal, which certainly is σ-generated by closed sets) is extremely complicated, it is treated as a separate case. We first concentrate on the treatment of those σ-ideals whose quotient does not add a Cohen real. For these ideals, the intermediate generic extension case of Theorem 2.1.3 is ruled out by Proposition 3.2.1, and a natural conjecture appears:

Conjecture 3.2.2. If I is a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal on a Polish space X σ-generated by closed sets, then either the quotient forcing adds a Cohen real, or the ideal I has the Silver property.

This section should be understood as a work towards the decision of this conjecture by consideration of a number of special cases. Note that if the quotient forcing adds no Cohen reals, then we have a minimal forcing extension, and total canonization of equivalences classifiable by countable structures follows by Corollary 4.3.8. Proposition 2.5.2 then yields the Silver property of I for Borel equivalences classifiable by countable structures. Thus, the difficulty lies on the other side of the Borel equivalence relation map.

3.2a The bounding case

The initial suspicions voiced in Conjecture 3.2.2 are fully confirmed in the case of bounding quotient forcing. Recall that a forcing is *bounding* if every function in ω^ω in its extension is forced to be pointwise dominated by a ground model function.

Theorem 3.2.3. *Suppose that I is a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal σ-generated by closed sets such that the quotient forcing P_I is bounding. Then I has the mutual generics property.*

Corollary 3.2.4. *In this case, the ideal I has the Silver property.*

To prove the corollary, look at the trichotomy theorem 2.1.3. For every given Borel equivalence relation on a Borel I-positive set $B \subset X$, the intermediate model case is excluded by the bounding property and Proposition 3.2.1(3). Nontrivial ergodicity is excluded as well by the free set property, which is implied by the mutual generics property. Thus, we have total canonization. The transversal property 3.2.1(2) and proposition 2.5.2 then close the deal.

The assumptions do not imply total canonization for Borel graphs, as the basic example of the c_{min}-graph and the associated ideal shows [49, Section 4.1.5].

Proof. The argument is a sort of fusion of fusions, and it requires some preliminary considerations and notation. [49, Section 5.2.1] shows that Player I has a winning strategy in a certain game. The game between Players I and II starts with Player II indicating an initial condition $B_{ini} \in P_I$. After that, there are infinitely many mega-rounds, the n-th ending with a set $B_n \in P_I$. The n-th megaround proceeds in the following way. Player I indicates one by one conditions $C_i \in P_I$, and Player II responds with subsets $D_i \subset C_i$, $D_i \in P_I$. After some finite number of rounds, Player I decides to end the megaround n, and the set B_n equals to $\bigcup_i D_i$. Player I wins if $B_{ini} \cap \bigcap_n B_n \notin I$.

We will need the following notation. If τ is a finite play of the game ending after Player I finished the n-th megaround, we will write $\tau(\text{ini})$ for the initial condition indicated, and $\tau(\text{end}) = \tau(\text{ini}) \cap \bigcap_{m \in n} B_m$. If τ is an infinite play then $\tau(\text{end}) = \tau(\text{ini}) \cap \bigcap_n B_n$. The following is the essence of the proofs in [49, Claim 5.2.7]:

Fact 3.2.5. *If σ is a strategy for Player I and τ_0, τ_1 are finite play respecting the strategy σ, and $D \subset P_I \times P_I$ is an open dense set, then there are finite extensions τ_0', τ_1' of the plays, still respecting the strategy σ, such that $\tau_0'(\text{end}) \times \tau_1'(\text{end})$ is covered with finitely many elements of D.*

Now we are ready to start the argument for the theorem. Let $B \in P_I$ be a set, let σ be a winning strategy for Player I in the game, let M be a countable elementary submodel of a large enough structure. We must produce an I-positive Borel subset of B consisting of points mutually generic for the model M. Let $t_n : n \in \omega$ enumerate $\omega^{<\omega}$ with infinite repetitions, let $O_i : i \in \omega$ enumerate basic open sets of X, and let $D_n : n \in \omega$ enumerate dense open subsets of $P_I \times P_I$ in the model M. By induction on $n \in \omega$ build finite trees $T_n \subset \omega^{<\omega}$ and maps f_n with $\text{dom}(f_n) = T_n$ such that

- $T_0 \subset T_1 \subset T_2 \dots$
- for every node $t \in T_n$, $f_n(t)$ is a finite play of the game G in the model M respecting the strategy σ with $f_n(t)(\text{ini}) \subset B$, and if $m > n$ then $f_n(t) \subset f_m(t)$;
- whenever $t \neq s \in T_n$ are nodes then $f_n(t)(\text{end}) \times f_n(s)(\text{end})$ is covered by finitely many sets in the set D_n;

- if $t_n \in T_n$ then $T_{n+1} = T_n \cup t_n^\frown i$ from the smallest possible $i \in \omega$ such that $t_n^\frown i \notin T_n$, $f_{n+1}(t_n^\frown i)(\text{ini}) \subset f_n(t_n)(\text{end})$ and if $f_n(t_n)\text{end})(\text{ini}) \cap O_i \notin I$ then even $f_n(t_n^\frown i) \subset f_n(t_n)(\text{end}) \cap O_i$;
- if $t_n \notin T_n$ then $T_{n+1} = T_n$.

The induction is not difficult to perform using the previous Fact. Let $x \in \omega^\omega$ be a point. The third item implies that the sets $f_n(t)(\text{ini}) : t \subset x, t \in T_n$ form a system of compact sets linearly ordered by inclusion, with a single point $g(x)$ in the intersection. The point $g(x)$ is M-generic. The map $g : \omega^\omega \to X$ is a Borel injection, and its range must be Borel. The following two claims show that $\text{rng}(g) \subset B$ is the required set.

Claim 3.2.6. $\text{rng}(g) \notin I$.

Proof. Suppose that $C_j : j \in \omega$ are closed sets in the ideal I. By induction on $j \in \omega$ build nodes $s_j \in \omega^{<\omega}$ as follows: note that the set $\bigcup_n f_n(s_j)(\text{end})$ is I-positive, and let i be a number such that $C_j \cap O_i = 0$ while $\bigcup_n f_n(s_j)(\text{end}) \cap O_i \notin I$. In the end, let $x = \bigcup_j s_j$ and observe that $g(x) \notin \bigcup_j C_j$. □

Claim 3.2.7. *If $x \neq y$ then $g(x)$, $g(y)$ are mutually P_I-generic points for the model M.*

Proof. Let $D = D_n \in M$ be an open dense subset of $P_I \times P_I$ in the model M. Find a large enough number $m > n$ such that the longest initial segments $t \subset x, u \subset y$ still in the tree T_m are already distinct. The third item of the induction construction shows that $\langle g(x), g(y)\rangle \in f_m(t)(\text{end}) \times f_m(u)(\text{end})$. At the same time, $f_m(t)(\text{end}) \times f_m(u)(\text{end}) \in M$ is a set covered by finitely many elements of D; these elements can be found in the model M as well. Thus, the pair $\langle g(x), g(y)\rangle$ belongs to some element of $D \cap M$ as required. □

□

3.2b Miller forcing and generalizations

The basic example not covered in the previous section is the Miller forcing. It is the poset of all superperfect trees in $\omega^{<\omega}$ ordered by inclusion is connected with the σ-ideal I on ω^ω σ-generated by compact sets:

Fact 3.2.8. *[26] Whenever $A \subset \omega^\omega$ is an analytic set, exactly one of the following is true: either $A \in I$ or A contains all branches of a superperfect tree. The ideal I is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$.*

Thus the map $\pi : T \to [T]$ is an isomorphism between Miller forcing and a dense subset of the poset P_I. The canonization properties of the ideal I have been thoroughly studied by Otmar Spinas.

Fact 3.2.9. *[43] The ideal I has the free set property. [45] The ideal I has the rectangular property. The ideal I has total canonization for Borel graphs.*

As every $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal σ-generated by closed sets, the ideal I has the transversal property, its free set property implies total canonization, and so in conjunction with Proposition 2.5.2 this yields

Corollary 3.2.10. *The ideal I has the Silver property.*

In another breakthrough, Otmar Spinas [44] computed the ideal associated with the product of Miller forcing with itself and canonized smooth equivalence relations on it. The ideal J on $\omega^\omega \times \omega^\omega$ consists of exactly those Borel sets which do not contain a product of two superperfect trees. Every smooth equivalence relation canonizes either to identity, or to equality on one of the two coordinates, or to EE on a superperfect rectangle. The product adds a dominating real and therefore, essentialy by the results of Section 3.10, E_{K_σ} belongs to the spectrum of the ideal J.

We will now attempt to generalize Spinas's results to partially ordered sets of infinitely branching trees with varying measures of the size of branching. For an ideal K on a countable set a let $P(K)$ be the poset of all trees $T \subset a^{<\omega}$ such that every node of T extends to a splitnode of T, and every splitnode $t \in T$ the set $\{i \in a : t^\frown i \in T\}$ is not in the ideal K. Thus for example the Miller forcing is P(Frechet ideal on ω). The computation of the ideal $I(K)$ associated with the forcing gives a complete information. Let $X = a^\omega$ with the product topology, where a is taken with the discrete topology. For every function $g : a^{<\omega} \to K$ consider the closed set $A_g = \{x \in a^\omega : \forall n\ x(n) \in g(x \restriction n)\} \subset X$, and the σ-ideal $I(K)$ on the space X σ-generated by all these closed sets. The following is obtained by a straightforward generalization of the proof of Fact 3.2.8.

Fact 3.2.11. *Let $A \subset X$ be an analytic set. Exactly one of the following happens:*

- *$A \in I(K)$;*
- *there is a tree $T \in P$ such that $[T] \subset A$.*

Moreover, if the ideal K is coanalytic then the σ-ideal $I(K)$ is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$.

Thus the map $T \mapsto [T]$ is a dense embedding of the poset $P(K)$ into $P_{I(K)}$. It is clear now that the poset $P(K)$ is proper and preserves Baire category since the corresponding ideal is generated by closed sets [49, Theorem 4.1.2]. If the ideal K is nonprincipal then the forcing adds an unbounded real. Other forcing properties depend very closely on the position of the ideal K in the Katětov ordering. The poset may have the Laver property (such as with K =the Frechet ideal), or it may add a Cohen real (such as in K =the nowhere dense ideal on $2^{<\omega}$). Similar issues are addressed in [36, Section 3]. We first look at a very well behaved special case, generalizing the result of [43].

Theorem 3.2.12. *Suppose that K is an F_σ ideal on a countable set. Then $I(K)$ has the free set property.*

Proof. Fix a F_σ-ideal K on ω and use Mazur's theorem [31] to find a lower continuous submeasure ϕ on $\mathcal{P}(\omega)$ such that $K = \{a \subset \omega : \phi(a) < \infty\}$. The argument now proceeds with a series of claims.

Claim 3.2.13. *The forcing $P(K)$ has the weak Laver property.*

Here, a forcing P has the *weak Laver property* if for every condition $p \in P$, every name $\dot{f}$ for a function in ω^ω forced to be dominated by a ground model function there is a condition $q \leq p$, an infinite set $b \subset \omega$, and sets $c_n : n \in b$ of the respective size n such that $q \Vdash \forall n \in \check{b}\ \dot{f}(n) \in c_n$. This is a forcing property appearing prominently in connection with P-point preservation [50].

Proof. Let $p \in P(K)$ be a condition, $g \in \omega^\omega$ a function and $\dot{f}$ a $P(K)$-name for a function in ω^ω pointwise dominated by g. By induction on $n \in \omega$ construct trees $U_n \in P(K)$ and their finite subsets $u_n \subset U_n$, numbers m_n and finite sets $c_n \subset \omega$ so that

- $T = U_0 \supset U_1 \supset \dots$ and $\{t\} = u_0 \subset u_1 \subset \dots$ where t is the shortest splitnode of T;
- u_n is an inclusion initial set of splitnodes of U_n and for every node $t \in u_n$, the set of its immediate successors that have some successor in u_{n+1} has ϕ-mass at least n;
- $U_{n+1} \Vdash \dot{f}(m_n) \in \check{c}_n$ and $|c_n| \leq m_n$.

The induction is not difficult to perform. Given U_n, u_n, let $m_n = |u_n|$ and for every splitnode $t \in u_n$ and every immediate successor s of t that has no successor in u_n, thin out the tree $U_n \restriction s$ to decide the value of $\dot{g}(m_n)$. Thinning out the set of immediate successors further if necessary, the decision can be assumed to be the same for all such immediate successors of t, yielding a number $k_t \in \omega$. Let $c_n = \{k_t : t \in u_n\}$ and let U_{n+1} be the thinned out tree. Finally, find $u_{n+1} \subset U_{n+1}$ satisfying the second item.

In the end, $U = \bigcap_n U_n$ is a tree in the poset $P(K)$ forcing $\forall n\ \dot{f}(m_n) \in \check{c}(n)$ as desired. ◻

Claim 3.2.14. *Whenever $a \subset \omega$ is a K-positive subset and $T \in P(K)$ forces $\dot{b} \subset \omega$ is an element of $\dot{K}$, then there is a condition $S \leq T$ and a K-positive ground model set $c \subset a$ such that $S \Vdash \check{c} \cap \dot{b} = 0$.*

Proof. This is in fact true for any forcing with weak Laver property in place of $P(K)$. Thinning out the condition T if necessary we may find a number $m \in \omega$ such that $T \Vdash \phi(\dot{b}) < \check{m}$. Find disjoint finite sets $a_n \subset a$ with $\phi(a_n) > 2mn$, and use the weak Laver property to find a condition $S \leq T$, an infinite set $b \subset \omega$ and sets $B_n \subset \mathcal{P}(a_n)$ for every $n \in b$ such that all sets in B_n have ϕ-mass $< m$, $|B_n| < n$, and $S \Vdash \dot{b} \cap \check{a}_n \in \check{B}_n$. Then for every number $n \in b$, the set $c_n = a_n \setminus \bigcup B_n$ has mass at least nm, and it is forced by S to be disjoint from $\dot{b}$. Thus the set $c = \bigcup_{n \in b} c_n$ works as desired. ◻

Now, suppose that $D \subset \omega^\omega \times \omega^\omega$ is a Borel set with $I(K)$-small vertical sections, and $T \in P(K)$ is a tree. We must find a tree $S \subset T$ such that $[S]\times[S]\cap D \subset \mathrm{ID}$. Thinning out the tree T if necessary we may assume that there is a continuous function $f : [T]\times\omega^{<\omega} \to K$ such that for every branch $x \in [T]$ the section D_x is included in the $I(K)$-small set $\{y \in \omega^\omega : \forall^\infty n\; y(n) \in f(x)(y \restriction n)\}$. Let $t_n : n \in \omega$ be an enumeration of $\omega^{<\omega}$ with infinite repetitions, and construct trees $U_n \in P(K)$ and their finite subsets $u_n \subset U_n$ so that

- $T = U_0 \supset U_1 \supset \dots$ and $\{t\} = u_0 \subset u_1 \subset \dots$ where t is the shortest splitnode of T;
- u_n is an inclusion initial set of splitnodes of U_n;
- if $t_n \in u_n$ then $u_{n+1} = u_n$ together with some set of splitnodes extending t_n. All the new splitnodes differ from all the old ones and among each other already at their $|t_n|$-th entry. Moreover the set $\{i \in \omega : \exists t \in u_{n+1}\; t_n^\frown i \subset t\}$ has ϕ-mass at least n. If $t_n \notin u_n$ then $u_{n+1} = u_n$;
- if $t_n \in u_n$ and $t \neq t_n$ is another node in u_n, then for every node $s \in u_{n+1} \setminus u$ and every $x \in [U_{n+1} \restriction s]$ it is the case that $f(x)(t)$ is disjoint from the set $\{i \in \omega : t^\frown i \in U_{n+1}$ but no node of u_n extends $t^\frown i\}$.

This is not difficult to do using the previous claim. In the end, the tree $S = \bigcap_n U_n$ belongs to $P(K)$, and its set of splitnodes is exactly $\bigcup_n u_n$. We must show that $[S] \times [S] \cap D \subset \mathrm{ID}$. To see this, suppose for contradiction that $x \neq y \in [S]$ are two branches and $y \in D_x$. This means that there is a number m_0 such that for every $m > m_0$, $y(m) \in f(x)(y \restriction m)$. Find a number $n \in \omega$ such that $t_n \in u_n$ is an initial segment of x, $u_{n+1} \setminus u_n$ contains still longer initial segment of x, and the longest node $t \in u_n$ which is an initial segment of y is of length greater than m_0. Then the last item shows that $y(|t|) \notin f(x)(t)$, contradicting the choice of m_0! □

Corollary 3.2.15. *For Borel ideals K as in the theorem, the ideal $I(K)$ has the Silver property.*

The previous result cannot be generalized to much more general ideals. A more or less canonical ideal on a countable set which is not a subset of an F_σ-ideal is $K = Fin \times Fin$, the ideal on $\omega \times \omega$ generated by vertical sections and sets with all vertical sections finite.

Example 3.2.16. The ideal $I(K)$ does not have the free set property.

Proof. Let $\pi : \omega\times\omega \to \omega$ be a bijection. For every point $x \in X = (\omega\times\omega)^\omega$ define the function $g(x) \in \omega^\omega$ by $g(x)(n) = \max\{\pi(x(m)) : m \in n\}$. Let $D \subset X \times X$ be defined by $\langle x, y\rangle \in D$ if for all but finitely many $n \in \omega$, writing $y(n) = (l, m)$, it is the case that $g(x)(l) > m$. It is not difficult to see that the vertical sections of the Borel set D are in the ideal $I(K)$.

To show that the set D contradicts the free set property, suppose that $T, U \in P(K)$ are two trees. We will find $x \in [T]$ and $y \in [U]$ such that $\langle x, y\rangle \in D$. By induction build splitnodes $t_j \in T$ and $u_j \in U$ so that

- $u_j : j \in \omega$ form an increasing sequence as well as $t_j : j \in \omega$;
- whenever $n \in \operatorname{dom}(u_{j+1}) \setminus \operatorname{dom}(u_j)$ and $u_{j+1} = (l, m)$ then $l > |t_j| + 1$ and $m < \pi(t_{j+1}(|t_j|)$.

The choice of the splitnodes t_0, u_0 is arbitrary. Suppose t_j, u_j have been found. Find a pair (l', m') such that $l' > |t_j|$ and $u_j^\frown(l', m') \in U$. Let u_{j+1} be an arbitrary splitnode of the tree U above the node $u_j^\frown(l', m')$. Find a pair $l, m \in \omega$ such that $t_j^\frown(l, m) \in T$ and $\pi(l, m)$ is larger than all numbers appearing in $\operatorname{rng}(u_{j+1})$. Let t_{j+1} be an arbitrary splitnode of the tree T above the node $t_j^\frown(l, m)$. The induction hypotheses continue to hold.

In the end, write $x = \bigcup_j t_j$ and $y = \bigcup_j u_j$. It is clear from the construction that for all numbers $n > |u_0|$, writing $(m, l) = y(n)$, it is the case that $l \in g(x)(m)$ and therefore $\langle x, y \rangle \in D$. Thus the free set property fails. $\square$

In view of the results of the next section, the ideal $I(Fin \times Fin)$ has the Silver property for all Borel equivalence relations reducible to E_{K_σ} or E_{c_0}. For this, it is just enough to show that the associated forcing preserves outer Lebesgue measure, and that in turn is equivalent to the Fubini property of the ideal $Fin \times Fin$ by [36]. The Fubini property is introduced in the next section. To prove the Fubini property of the ideal $Fin \times Fin$, work by the way of contradiction, and fix a putative offending Borel set $D \subset a \times [0, 1]$ of vertical sections of mass larger than some $\varepsilon > 0$. Thus, all horizontal sections of the set D are in the ideal $Fin \times Fin$. Use the continuity of Lebesgue measure in increasing unions to find a set $B \subset [0, 1]$ of mass larger than $1 - \varepsilon$ and numbers $\{n, m_i : i \in \omega\}$ such that for every $z \in B$ and every pair $\langle k, m \rangle \in a$ such that $\langle k, m, z \rangle \in D$ it is the case that either $k < n$ or $m < m_k$. use the positivity of the set a to find a pair $\langle k, m \rangle \in a$ such that $k > n$ and $m > m_k$. The vertical section $D_{\langle k,m \rangle \rangle}$ is of Lebesgue mass greater than ε, and therefore has nonempty intersection with the set B. However, for any point $z \in B \cap D_{\langle k,m \rangle}$, the triple $\langle k, m, z \rangle$ contradicts the properties of the set B!

3.2c The measure preserving case

Another partial result towards Conjecture 3.2.2 deals with σ-ideals whose quotient forcings preserve outer Lebesgue measure. The argument is very flexible and yields other canonization results as well.

Theorem 3.2.17. *If the σ-ideal I is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$, it is σ-generated by closed sets, and the quotient forcing preserves outer Lebesgue measure then I has the Silver property below E_{K_σ} and also below E_{c_0}.*

Proof. We will start with E_{K_σ}; the case of E_{c_0} is then only a minor variation of the whole argument.

Recall that E_{K_σ} is an equivalence relation on the space $Y = \omega^\omega$ below the identity, connecting two functions $y_0, y_1 \in Y$ if their distance, the maximum of

$|y_0(n) - y_1(n)| : n \in \omega$, is finite. Suppose now that B is an I-positive Borel set and E is a Borel equivalence relation on B Borel reducible to E_{K_σ} by a Borel function $f : B \to \omega^\omega$. We must prove that E has either an I-positive Borel equivalence class or an I-positive Borel set consisting of pairwise inequivalent elements. The argument is then completed as in Proposition 2.5.2. If there is an I-positive E-equivalence class, then we are certainly done. So assume that preimages of E_{K_σ}-equivalence classes are I-small, and work to produce a Borel I-positive set of pairwise inequivalent elements. Proceed in two stages: first get a Borel I-positive set $C \subset B$ on which the equivalence classes of E are countable, and then use the general Corollary 4.2.6.

The main tool in the argument is the *trace ideal* $tr(I)$ on the countable collection of basic open subsets of X introduced in [36]: a set a of basic open sets is in $tr(I)$ if the set the closed set $p(a) = \{x \in X :$ every open neighborhood of x contains a subset which is in $a\}$ is in the σ-ideal I. Obviously, $tr(I)$ is an ideal. The following is an easy abstract claim:

Claim 3.2.18. *If P_I preserves outer Lebesgue measure then the ideal $tr(I)$ has the Fubini property.*

Here, the Fubini property of ideals is the one introduced in [23]. An ideal J has the *Fubini property* if for every J-positive set a, every positive real $\varepsilon > 0$ and every Borel set $D \subset a \times 2^\omega$ with vertical sections of μ-mass $> \varepsilon$ for the standard Borel probability measure μ on 2^ω, for some point $y \in 2^\omega$ the horizontal section D^y is J-positive. Note that the definition does not depend on the choice of the probability measure space on the vertical coordinate by the standard measure isomorphism arguments.

Proof. Let $a \notin tr(I)$ be a set and $D \subset a \times 2^\omega$ be a Borel set with vertical sections of mass at least ε. Note that $p(a)$ as a condition in the forcing P_I forces the set $\{y \in 2^\omega :$ some open neighborhood of the generic point $\dot{x}_{gen}$ contains no subset in a containing $y\} \subset 2^\omega$ is of mass at most $1 - \varepsilon$, and as P_I preserves outer Lebesgue measure, there is a condition $C \subset p(a)$ forcing some definite point $y \in 2^\omega$ to not belong to this set. Then the vertical section $D^y \subset a$ is $tr(I)$-positive since $C \Vdash \dot{x}_{gen} \in p(D^y)$. $\square$

Thinning out the condition B if necessary we may assume that the function f is continuous on it, that B is G_δ, $B = \bigcap_n O_n$, and that the intersection of every open set with B is either empty or I-positive. By induction on $n \in \omega$, we will construct trees T_n of finite inclusion decreasing sequences of basic open subsets of X with nonempty intersection with the set B, T_n is of height n, T_{n+1} is an end-extension of T_n and writing $a(t) = \{O : t^O \in T_n\}$ for the set of immediate successors of a node $t \in T_n$, we have

- $a(t)$ consists of pairwise disjoint sets of radius $< 2^{-n}$;
- $\forall O \in a(t)\ O \subset O_{|t|}$;
- $a(t) \notin tr(I)$;

- whenever $O, P \in a(t)$ are distinct sets and $x \in O \cap B$ and $y \in P \cap B$ are points then $d(f(x), f(y)) \geq n$.

In the end, let $T = \bigcup_n T_n$ and let $C = \{x \in X : x$ belongs to infinitely many open sets on the tree $T\}$ be the set of points obtained by intersecting the open sets along the branches of the tree T. The first item shows that there is a one-to-one correspondence between points in C and infinite branches through the tree T. The second item shows that $C \subset B$.

The third item secures the I-positivity of the set C. Indeed, if $D_i : i \in \omega$ are closed sets in the ideal I, one can induce on i to pick open sets $\langle P_i : i \in \omega \rangle$ forming a branch through the tree T such that $D_i \cap P_{i+1} = 0$, and then the single point in the intersection of these sets will belong to the set $C \setminus \bigcup_i D_i$, witnessing the I-positivity of the set C. To pick the open set P_i once the sequence $t = \langle P_j : j \in i \rangle$ has been constructed, note that if every set $P \in a(t)$ had nonempty intersection with D_i, then $p(a(t)) \subset D_i$, contradicting the third item of the induction hypothesis above.

The fourth item shows that the E-equivalence classes on the set C are countable. If $x \in C$ and $n \in \omega$ then every open set at level n of the tree T can contain at most one point $y \in B$ such that $d(f(x), f(y)) < n/2$. There are only countably many numbers n and nodes in the tree T, and therefore the set $\{y \in C : x \; E \; y\}$ must be countable. The last item just makes the induction go through.

Thus, once the induction is complete, the theorem for E_{K_σ} will follow. The induction is simple except for the fourth item. Suppose that some endnode $t \in T_n$ has been constructed, labeled with an open set O. The collection $b(t) = \{P : P \subset O$ is a basic open set of radius $< 2^{-n}$, a subset of $O_{|t|}$, with nonempty intersection with $B\}$ is positive in the ideal $tr(I)$ since $B \cap O \subset tr(b(t))$. Enumerate the set $b(t)$ by $\{P_i : i \in \omega\}$ and by induction on $i \in \omega$ build sets $c_i \subset O \cap B$ and numbers $k_i \in \omega$ so that

- c_i is a set of size 2^i consisting of pairwise inequivalent elements of $P_i \cap B$;
- for each pair $\{x, y\}$ of distinct elements of c_i there is a number $k \in k_i$ such that $|f(x)(k) - f(y)(k)| > 2n$;
- for all elements $x \in c_{i+1}$, the finite sequence $f(x) \restriction k_i$ is the same.

This is easy to do. To obtain c_{i+1}, first pick an arbitrary point $x \in B \cap P_{i+1}$ and note that the set $P = \{y \in P_i : f(x) \restriction k_i = f(y) \restriction k_i\}$ is relatively open in B by the continuity of the function f, it has nonempty intersection with $B \cap P_{i+1}$ and therefore an I-positive intersection with this set, and since E-classes are I-small, there must be a collection of 2^{i+1} many inequivalent points in $B \cap P_{i+1} \cap P$. Then choose the number k_{i+1} to be sufficiently large to satisfy the second item.

In the end, consider the space $Z = \Pi_i c_i$ and the Borel probability measure μ on it which is the product of the normalized counting measures on the various c_i's. Consider the set $D \subset b(t) \times Z$ of those pairs $\langle P_i, z \rangle$ such that there is $j \in i$

such that $f(z(j)) \restriction k_j$ is n-close to $f(z(i)) \restriction k_i$. Observe that given numbers $j \in i \in \omega$, there can be at most one $x \in c_j$ such that $f(x) \restriction k_j$ is n-close to some (any) $f(y) \restriction k_i : y \in c_i$, and so the vertical sections of the set D have μ-mass at most $\Sigma_i 2^{-i-2} = 1/2$. By the Fubini property of the ideal $tr(I)$, there is $z \in Z$ such that the set $d(t) = \{P_i : \langle P_i, z\rangle \notin D\}$ is not in the trace ideal. For every open set $P_i \in d$ let $Q_i \subset P_i$ be a basic open set with nonempty intersection with B and such that $\forall x \in B \cap Q_i\ f(x) \restriction k_i = z(P_i) \restriction k_i$. The set $e(t) = \{Q_i : P_i \in d\}$ is not in the trace ideal since $p(d(t)) \subset p(e(t))$. If the immediate successors of the node t in T_{n+1} will be the basic open sets in the set $e(t)$, the induction hypotheses will be satisfied

Now, let us return to the case of equivalence relations below E_{c_0}. Recall that E_{c_0} is the equivalence relation on $\mathbb{R}^\omega$ connecting sequences x, y if $\lim |x(n) - y(n)| = 0$. Suppose that $B \in P_I$ is a Borel I-positive set and E a Borel equivalence relation on it reducible to E_{c_0} by a Borel function $f : B \to \mathbb{R}^\omega$. We must find a Borel I-positive subset of B such that E is equal to the identity or everything on it. As always, thinning out the set B if necessary we may assume that the function f is continuous on it, B is a G_δ set, and its intesection with any open set is either I-positive or empty. There are two distinct cases.

In the first case, for every relatively open I-positive subset of B and every $\varepsilon > 0$ there is a still smaller I-positive relatively open subset of B such that for every two points x, y in it, $\limsup |f(x)(n) - f(y)(n)| < \varepsilon$. In this case, a similar construction as in the E_{K_σ} case yields a tree T labeled by open sets such that for any open set at n-th level and any points $x, y \in B$ in this open set, $\limsup |f(x)(n) - f(y)(n)| < 2^{-n}$. Let $C \subset B$ be the Borel I-positive set associated with the tree T. It is immediate that the complement of E is relatively open in C^2, and therefore E is smooth on C^2 by [8, Proposition 5.4.7]. The forcing P_I adds a minimal forcing extension by [49, Theorem 4.1.7] and so has total canonization for smooth equivalences, which concludes the proof in this case.

In the second case, there is a relatively open I-positive subset $B' \subset B$ and a positive real $\varepsilon > 0$ such that every smaller relatively open set contains points x, y such that the $\limsup |f(x)(n) - f(y)(n)| > \varepsilon$. In this case, proceed just as in the treatment of E_{K_σ}. □

3.2d Cohen forcing

The Cohen forcing is the partial ordering of finite binary sequences with inclusion. The associated ideal I is the ideal of meager subsets of 2^ω as the following fact shows.

Fact 3.2.19. *Whenever $B \subset 2^\omega$ is an analytic set the there is an open set $O \subset 2^\omega$ such that $O \Delta B$ is meager.*

The spectrum of Cohen forcing is quite complicated; we only point out several fairly simple features.

Proposition 3.2.20. *E_0 is in the spectrum of Cohen forcing.*

No other countable Borel equivalence relations belong to the spectrum as by Theorem 4.2.7, essentially countable$\to_I$reducible to E_0.

Proof. It is enough to show that for every Borel nonmeager set $B \subset 2^\omega$, $E_0 \leq E_0 \restriction B$. By the Baire category theorem, there is a finite sequence $s \in 2^{<\omega}$ such that B is comeager in O_s, $B \subset O_s \cap \bigcap_n P_n$ where $P_n \subset 2^\omega$ is open dense. By induction on $n \in \omega$ build pairs $\{t_n^0, t_n^1\}$ of distinct finite binary sequences of the same length such that for every number $n \in \omega$, writing t for the concatenation $s^\frown(t_0^{i(0)})^\frown(t_1^{i(1)})^\frown \dots ^\frown (t_n^{i(n)})$, we have $O_t \subset P_n$ no matter what the choices of $i(0), i(1), \dots i(n)$. Now let $f : 2^\omega \to B$ be the continuous function for which $f(x)$ is defined as the concatenation of s with all $t_n^{x(n)} : n \in \omega$. It is not difficult to see that f is the required reduction of E_0 to $E_0 \restriction B$.

□

Proposition 3.2.21. *E_2 is in the spectrum of Cohen forcing.*

Proof. Consider the E_2 equivalence on 2^ω. The σ-ideal J σ-generated by Borel grainy subsets of 2^ω consists of meager sets only, as shown in Claim 3.5.7. Thus, whenever $B \subset 2^\omega$ is a Borel non-meager set then $B \notin J$ and therefore E_2 is Borel reducible to $E_2 \restriction B$ by Fact 3.5.1. The proposition follows. □

Proposition 3.2.22. *F_2 is in the spectrum of Cohen forcing.*

Proof. Consider the underlying space of the Cohen forcing to be $X = (2^\omega)^\omega$ with the product topology, and the ideal I to be the meager ideal in this topology. Consider the equivalence relation F_2 on this space. It will be enough to show that F_2 reduces to F_2 restricted to any Borel non-meager set.

we will need a preliminary observation. Let M be a countable elementary submodel of a large enough structure and b be a countable dense subset of 2^ω Cohen generic over M in finite tuples. It turns out that $M(b)$ is the $M[[\dot{x}_{gen}]]_{F_2}$ choiceless model; in order to prove the proposition, we need to look at it more closely. We will show that every Borel non-meager subset of X in the model M contains many one-to-one enumerations of the set b. Consider b with the discrete topology, the space b^ω with product topology, and the space Z_b of all bijections between b and ω as a G_δ subset of b^ω with the inherited (and therefore Polish) topology.

Claim 3.2.23. *Whenever $B \subset X$ is a Borel nonmeager set in the model M, the set $B \cap Z_b$ is nonmeager in Z_b.*

Proof. Let $P_n : n \in \omega$ be open dense subsets of the space Z_b; we must find a point in the intersection $B \cap \bigcap_n P_n$. Find a nonempty basic open set $O \subset X$ and countably many open dense sets $\{O_n : n \in \omega\}$ such that $B \supset O \cap \bigcap_n O_n$; by elementaricity these objects can be found in the model M. Use the density of the set b to find a finite injection $g_0 : \omega \to b$ such that any totalization of it will belong to the basic open set O. By induction on $n \in \omega$ build finite partial injections $g_n : \omega \to b$ so that

- $g_0 \subset g_1 \subset \dots$, $n \in \mathrm{dom}(g_{n+1})$ and the n-th element of b in some fixed enumeration belongs to $\mathrm{rng}(g_{n+1})$;
- every totalization of g_{n+1} in X belongs to O_n;
- the basic open subset of Z_b determined by g_{n+1} is a subset of P_n.

In the end, the function $\bigcup_n g_n$ will clearly be an element of Z_b in $\bigcap_n P_n \cap B$.

To perform the induction step, suppose that the function g_n has been found. First, find an extension g'_n such that all of its totalizations in X belong to O_n: the set $O_n \in M$ is open dense, its projection to the $\mathrm{dom}(g_n)$ coordinates is open dense as well, it is in the model M, and since g_n is Cohen-generic over the model M, it must be the case that $g_n \in A$. The vertical section A_{g_n} is nonempty an open, and as the set $b \setminus \mathrm{rng}(g_n)$ is dense, there must be an injective extension g'_n such that all of its totalizations belong to O_n. Now, just extend g'_n to g_{n+1} in an arbitrary way to satisfy the first and third item of the induction hypothesis. □

Now, let $B \subset X$ be a Borel nonmeager set. To find the reduction of F_2 to $F_2 \restriction B$, choose a countable elementary submodel M of a large structure. It is not difficult to find a Borel set $D \subset 2^\omega$ with uncountable intersection with every nonempty open set, consisting of points Cohen generic for the model M in finite tuples, and a Borel reduction $f : X \to D^\omega$ of F_2 to $F_2 \restriction D^\omega$ such that $\mathrm{rng}(f(\vec{x}))$ is dense for every $\vec{x} \in X$. Let S_∞ be the infinite permutation group with its natural Polish topology. Consider the Borel set $A \subset (2^\omega)^\omega \times S_\infty$ given by $\langle \vec{x}, \pi \rangle \in A$ if $\hat{f}(x) \circ \pi \in B$. The previous claim shows that the vertical sections of the set A are nonmeager in S_∞ and so by [27, Theorem 18.6], there is a Borel uniformization $h : (2^\omega)^\omega \to (2^\omega)^\omega$. It is immediate that the function h is a reduction of F_2 to $F_2 \restriction B$. □

Proposition 3.2.24. *The spectrum of Cohen forcing is cofinal in $\leq_B$ and it includes E_{K_σ}.*

Proof. In the Section 3.8 on the infinite countable support product of Sacks forcing, we prove that its associated ideal on $(2^\omega)^\omega$ consists of meager sets, and its spectrum is cofinal in $\leq_B$. Thus, the equivalence relations exhibited in that proof will also yield the same feature for Cohen forcing. These equivalence relations include E_{K_σ}, among others. □

3.3 Halpern-Läuchli forcing

The Halpern-Läuchli forcing P consists of those trees $T \subset 2^{<\omega}$ such that there is an infinite set $a_T \subset \omega$ such that $t \in T$ is a splitnode if and only if $|t| \in a_T$. The ordering is that of reverse inclusion. P is quite similar to the Sacks or Silver forcing notions, and the standard fusion arguments showthat it is proper and has the continuous reading of names. Consequently, the computation of the associated ideal can be found in [49, Proposition 2.1.6]. Let $X = 2^\omega$, and let I be the σ-ideal generated by those Borel sets A such that for no tree $T \in P$,

$[T] \subset A$. Then I is a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal, every positive analytic set contains a positive Borel subset, and the map $T \mapsto [T]$ is a dense embedding from P to P_I.

This forcing serves as a good example refuting several natural conjectures. It has the free set property, but not the mutual generics property. It has total canonization, but not the Silver dichotomy. There is a natural σ-closed regular subforcing, which then cannot be obtained as P^E_I from any Borel equivalence relation E on X.

Proposition 3.3.1. *I has the free set property.*

Proof. We will need a sort of a reduced product of the Halpern-Läuchli forcing. Let $P \times_r P$ be the poset of all pairs $\langle T, U\rangle \in P \times P$ such that $a_T = a_U$. Similarly to the usual product, the reduced product adds points $\dot{x}_{\mathrm{lgen}}, \dot{x}_{\mathrm{rgen}} \in X$ which are the intersections of all trees on the left (or right, respectively) side of conditions in the generic filter. Still, the difference between $P \times_r P$ and $P \times P$ should be immediately apparent. A more detailed analysis will show that P forces that the set $\{a_T : T \in G\}$ is a generic filter for the poset $\mathcal{P}(\omega)$ modulo finite, and the reduced product is equivalent to the two step iteration of $\mathcal{P}(\omega)$ modulo finite followed with the product of two copies of the remainder forcing $P/\mathcal{P}(\omega)$ modulo finite.

We will need a computation of the σ-ideal associated with the product $P \times_r P$. Let $I \times_r I$ be the collection of all Borel subsets $B \subset 2^\omega \times 2^\omega$ such that there is no pair $\langle T, U\rangle \in P \times_r P$ such that $[T] \times [U] \subset B$.

Claim 3.3.2. *$I \times_r I$ is a σ-ideal of Borel sets, and the map $T, U \mapsto [T] \times [U]$ is a dense embedding of $P \times_r P$ into $P_{I \times_r I}$.*

Proof. Suppose that $T, U \in P \times_r P$, and $[T] \times [U] = \bigcup_n B_n$ is a countable union of Borel sets. We must find a pair of trees $\bar{T}, \bar{U} \in P \times_r P$ such that the set $[\bar{T}] \times [\bar{U}]$ is a subset of one of the Borel sets in the union. Strengthen the condition $\langle T, U\rangle$ if necessary to find a specific number $n \in \omega$ such that $\langle T, U\rangle \Vdash \dot{x}_{\mathrm{lgen}}, \dot{x}_{\mathrm{rgen}} \in \dot{B}_n$. Let M be a countable elementary submodel of a large enough structure, and let $D_m : m \in \omega$ be an enumeration of all open dense subsets of $P \times_r P$ in the model M. By induction on $m \in \omega$ build conditions $\langle T_m, U_m\rangle \in M$ in the reduced product so that

- $T_0 = T$, the conditions $T_m \in M \cap P$ form a decreasing sequence, and the first $m+1$ splitting levels of T_m are equal to the first $m+1$ splitting levels of T_{m+1}, and the same on the U side;
- for every choice of nodes $t \in T_{m+1}$, $u \in U_{m+1}$ just past the m-th splitting level, the condition $\langle T_{m+1} \restriction t, U_{m+1} \restriction u\rangle \in P \times_r P$ is in the set D_m.

The induction is elementary, at each step making a pass through all pairs $\langle t, u\rangle$ as in the second item to handle them all. In the end, the pair $\langle \bar{T} = \bigcap_m T_m, \bar{U} = \bigcap_n U_m\rangle \in P \times_r P$ is a condition such that every pair $\langle x, y\rangle \in [\bar{T}] \times [\bar{U}]$ is M-generic for the reduced product below the condition $\langle T, U\rangle$. The forcing theorem implies that $M[x,y] \models \langle x,y\rangle \in B_n$, and by analytic absoluteness $\langle x,y\rangle \in B_n$. In other words, $[\bar{T}] \times [\bar{U}] \subset B_n$ as desired. $\square$

An essentially identical argument yields

Claim 3.3.3. *Suppose that M is a countable elementary submodel of a large enough structure and $T \in M \cap P$ is a tree. There is a tree $U \in P$, $U \subset T$, such that every two distinct elements of U are M-generic for the reduced product.*

The proposition immediately follows. Let $D \subset X \times X$ be a Borel set with all vertical sections in the ideal I; clearly, such a set belongs to the σ-ideal I. Let $T \in P$. Let M be a countable elementary submodel of a large enough structure containing both D and T. Let $U \subset T$ be a tree in P such that every two distinct points of $[U]$ are reduced product generic for M. An absoluteness argument immediately shows that $([U] \times [U]) \cap D \subset \mathrm{ID}$. $\square$

Proposition 3.3.4. *I does not have the mutual generics property.*

Proof. Suppose that M is a countable elementary submodel of a large structure, and suppose for contradiction that there is a tree $T \in P$ such that $[T]$ consists solely of points mutually generic for $P \times P$ over M. For every point $x \in [T]$ let $g_x \subset P \cap M$ be the M-generic filter generated by x. The key point: the set $B = \{x \in [T] : \exists S \in g_x\ a_T \subset a_S \text{ modulo finite}\}$ is relatively meager in $[T]$. If not, by the Baire category theorem there would be a tree $S \in P \cap M$ and a node $t \in T$ such that for comeagerly many points $x \in [T]$ passing through T, $S \in g_x$ and $a_T \setminus a_S$ contains some elements above $|t|$. Take two extensions $t_0, t_1 \in T$ of the tree T that split at some level in $a_T \setminus a_S$ and points $x_0, x_1 \in [T]$ extending them respectively such that $S \in g_{x_0}, g_{x_1}$. However, this means that $t_0 \cap t_1$ is a splitnode of S, contradicting the choice of the two nodes t_0, t_1.

Now, choose two distinct ponts $x_0, x_1 \in B$ and observe that the set $a_T \subset \omega$ diagonalizes both of the filters h_{x_0}, h_{x_1} on $\mathcal{P}(\omega)$ modulo finite in M given by g_{x_0}, g_{x_1}. In particular, h_{x_0} does not contain a set with finite intersection with some set in h_{x_1} as would be the case if the filters g_{x_0}, g_{x_1} were M-generic for the product $P \times P$! $\square$

Proposition 3.3.5. *I does not have the transversal property.*

Proof. Find a Borel injection $f : 2^\omega \to [\omega]^{\aleph_0}$ whose range consists of pairwise almost disjoint sets. Let $D \subset 2^\omega \times 2^\omega$ be the Borel set of all pairs $\langle x,y\rangle$ such that $y(n) = 0$ whenever $n \notin f(x)$. It is fairly obvious that the vertical sections of the set D are pairwise disjoint I-positive sets. It turns out that D is the sought counterexample to the transversal property.

Indeed, suppose that $T \in P$ is a tree such that the closed set $[T] \subset 2^\omega$ is covered by the sections of the set D, and visits each section in at most one point. There must be distinct points $y_0, y_1 \in [T]$ such that the set $a = \{n : y_0(n) = y_1(n) = 1\}$ is infinite. If $x \in 2^\omega$ is such that $y_0 \in D_x$, it must be the case that $a \subset f(x)$, and the same for y_1. However, since the sets in the range of f are pairwise almost disjoint, there can be only one such point $x \in 2^\omega$, and both y_0, y_1 must belong to D_x, contradicting the choice of the tree $T \in P$. $\square$

Proposition 3.3.6. *P_I regularly embeds a nontrivial σ-closed forcing.*

Proof. If $G \subset P_I$ is a generic filter, then $H \subset \mathcal{P}(\omega) \mod \text{finite}$, $H = \{a_T : T \in G\}$ is a generic filter as well. The function $T \mapsto a_T$ is the associated pseudoprojection from P to $\mathcal{P}(\omega) \mod \text{finite}$. $\square$

3.4 E_0 forcing

Let I be the σ-ideal on the space $X = 2^\omega$ generated by Borel partial E_0 selectors. The quotient forcing P_I is proper, bounding, preserves outer Lebesgue measure as well as Baire category [49, Section 4.7.1]. We will use a combinatorial description of the quotient forcing as creature forcing with gluing. A creature is a pair of distinct finite binary sequences of the same length; for notational simplicity we will require that the two sequences differ at their first entry. If $c(i) : i \in n$ is a finite sequence of creatures then $[c(i) : i \in n]$ is the set of all binary sequences obtained by choosing one sequence from each creature and concatenating them all. A composition of the finite sequence of creatures is another creature, a pair of distinct binary sequences in $[c_i : i \in n]$ which differ at their first entry–thus there are only finitely many different compositions. A partial order P consists of pairs $p = \langle t_p, \vec{c}_p \rangle$ where $\vec{c}_p$ is an ω-sequence of creatures and t_p is a finite binary string. The order is defined by $q \leq p$ if there is a decomposition of ω into finite consecutive intervals $i_n : n \in \omega$ such that $t_q = t_p^\frown s_0^\frown s_1^\frown \dots$ where $s_m \in \vec{c}_p(m) : m \in i_0$, and the creatures $\vec{c}_q(n)$ are obtained as a composition of creatures in $\vec{c}_p(m) : i_n \leq m < i_{n+1}$.

We will need some notation. If $p \in P$ is a condition, then $sp(p)$, the collection of *splitnodes* of p, is the collection of all binary sequences obtained as the concatenation of the *trunk* t_p with some binary sequence in $[c_p(i) : i \in n]$ for some $n \in \omega$; such a splitnode is said to be at n-th splitting level of p. If $t \in sp(p)$ is a splitnode at splitting level n and then $p \restriction t$ is the condition $\langle t, c_p(i) : i \geq n \rangle$; so $p \restriction t \leq p$. Two splitnodes $s, t \in s(p)$ at the same splitting level $n+1$ will be called *forked* if the sequences in $c_p(n)$ which are subsequences of s and t respectively are distinct. For a condition p, let $[p] \subset 2^\omega$ be the closed set of all binary sequences x such that $x = t_p^\frown s_0^\frown s_1^\frown \dots$ where $s_n \in \vec{c}_p(n)$ for all $n \in \omega$.

Theorem 3.4.1. *1. Every analytic subset of 2^ω is either in the ideal I, or it contains a subset of the form $[p]$ for some $p \in P$, and these two options are mutually exclusive;*

2. the σ-ideal I on 2^ω is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$;

3. the forcing P_I is proper, bounding, preserves Baire category and outer Lebesgue measure, and it adds no independent reals.

In particular, the map $p \mapsto [p]$ is an isomorphism of P and a dense subset of P_I.

Proof. The first two items were proved essentially in [48, Theorem 2.3.29] The other features come from [49, Section 4.7.1] except for adding no independent reals. This is as always more difficult than the other preservation properties, and it uses a partition claim of independent interest:

Claim 3.4.2. *If $p \in P$ is a condition and $\pi : sp(p) \to 2$ is an arbitrary function then there is a condition $q \leq p$ such that $\pi \restriction sp(q)$ is constant.*

Restated, the collection of those subsets of $2^{<\omega}$ which do not contain a subset of the form $sp[q]$ for some condition $q \in P$ is an ideal.

Proof. It is necessary to use a partition theorem here. Hindman theorem 1.3.22 is strong enough; in order to maintain the style of the exposition, we will use Fact 1.3.23. Choose subsequent intervals $i_m : m \in \omega$ of ω such that for every number $m \in \omega$, every coloring of pairs of $2^{|i_m|}$ with m colors contains a monochromatic triangle. Let $a_m = [[\vec{c}_p(j) : j \in i_m]]^2$ and let ϕ_m be the submeasure on a_m assigning a set b the minimum number of triangle-free sets covering b; thus $\phi_m(a_m) \geq m$. Consider the following partition $\bar{\pi}$ of the set $\Pi_m a_m \times \omega$ into two Borel pieces: $\pi(y, m) = \pi(s)$ where s is the splitnode of p obtained by concatenatingt_p, the lexicographically smaller sequences in the pairs $y(n) : n \in m$ and the longest common initial segment of the two sequences in $y(m)$. By Fact 1.3.23, there are nonempty sets $b_m : m \in \omega$ whose respective ϕ_m-masses tend to infinity and an infinite set $c \subset \omega$ such that $\bar{\pi} \restriction \Pi_m b_m \times c$ is constant.

Thininning out the set c if necessary, we may assume that the sets $b_m : m \in a$ all have ϕ_m-mass at least two, so they contain a triangle. For every number $m \notin a$, pick a pair in b_m and a lexicographically smaller sequence s_m in it, and for every number $m \in a$ find a triangle in b_m and let $s_m(0), s_m(1)$ be the two lexicographically smallest sequences in this triangle. It is not difficult to find the unique condition $q \in P$ such that all points in $[q]$ are obtained by concatenating sequences s_m and $s_m(j_m)$ for $m \in \omega$ and some choice of bits $j_m \in 2$ for $m \in a$. The definition of the partition $\bar{\pi}$ shows that $\pi \restriction sp(q)$ is constant as desired. □

To prove that no independent reals are added, suppose $p \in P$ is a condition and $\dot{x}$ a name for an element of 2^ω. We must find a condition $q \leq p$ and an infinite set $a \subset \omega$ such that $q \Vdash \dot{x} \restriction a$ is constant. First, use an obvious fusion argument to thin out the condition p so that for every splitnode $t \in sp(p)$ at n-th splitting level, $p \restriction t$ decides $\dot{x}(\check{n})$. Now let $\pi : sp(p) \to 2$ be the map recording this decision, and use the claim to find a condition $q \leq p$ such that $\pi \restriction sp(q)$ is constant, with the constant value equal to, say, 0. There are numbers $n_k : k \in \omega$ such that every splitnode of q at k-th splitting level is a splitnode of p at n_k-th splitting level. Clearly, $q \Vdash \dot{x} \restriction \{n_k : k \in \omega\}$ is constant with value 0, completing the proof. □

Clearly, the ideal I defined in such a way that the poset P_I has E_0 in its spectrum:

Theorem 3.4.3. *E_0 is in the spectrum of the ideal I. The poset $P_I^{E_0}$ is regularly embedded in P_I, it is $\aleph_0$-distributive, and it yields the $V[\dot{x}_{gen}]_{E_0}$ extension.*

Proof. This is just a particular case of the work of Section 4.2. $\square$

No equivalence relation beyond E_0 is in the spectrum, and in fact, every Borel equivalence relation on 2^ω simplifies to ID, E_0 or EE on a Borel I-positive set. This strong canonization result is the consequence of the fact that the reduced product $P_I \times_{E_0} P_I$ is exceptionally well behaved. Recall its general definition from Definition 2.2.8: it is the set of those pairs $\langle A, B\rangle \in P_I \times P_I$ such that some large collapse forces that there are V-generic points $x \in A$, $y \in B$ which are E_0 related. The reduced product adds a pair of points $\dot{x}_{\mathrm{lgen}}, \dot{x}_{\mathrm{rgen}}$, each of which is P_I-generic. We will denote the associated ideal by $I \times_{E_0} I$; it consists of those analytic subsets of $2^\omega \times 2^\omega$ such that the reduced-generic pair is outright forced not to belong to them. We will first define its combinatorial version and prove that it is equivalent to the general notion.

Definition 3.4.4. $P \times_{E_0} P$ is the set of all pairs $\langle p, q\rangle \in P \times P$ such that $|t_p| = |t_q|$ and $\vec{c}_p = \vec{c}_q$, ordered coordinatewise.

Theorem 3.4.5. *The ideal $I \times_{E_0} I$ is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$. For every analytic set $A \subset 2^\omega \times 2^\omega$, exactly one of the following happens:*

1. *$A \in I \times_{E_0} I$;*
2. *there is a condition $\langle p, q\rangle \in P \times_{E_0} P$ such that $[p] \times [q] \setminus E_0 \subset A$.*

Thus, the map $\langle p, q\rangle \mapsto [p] \times [q] \setminus E_0$ is an isomorphism of $P \times_{E_0} E$ with a dense subset of $P_{I \times_{E_0} I}$.

Proof. First observe that the map $\langle p, q\rangle \mapsto \langle [p], [q]\rangle$ is an isomorphism of $P \times_{E_0} P$ with $P_I \times_{E_0} P_I$. It is clear that $\langle [p], [q]\rangle$ is a condition of the general reduced product. We just must show that every condition $\langle B, C\rangle \in P_I \times_{E_0} P_I$ has a strengthening of this form. To see this, note that it must be the case that the analytic $[B]_{E_0} \cap [C]_{E_0}$ must be I-positive and therefore it must contain a subset of the form $[r]$ for some $r \in P$. Using the σ-additivity of the ideal I, it is possible to strengthen the condition r so that there will be finite sequences $u, v \in 2^{<\omega}$ shorter than the trunk of r such that $[t_r \oslash u, \vec{c}_r] \subset B$ and $[t_r \oslash v, \vec{c}_r] \subset C$. Thus, the conditions $p = \langle t_r \oslash u, \vec{c}_r\rangle$ and $q = \langle t_r \oslash v, \vec{c}_r\rangle$ will be as required.

To proceed, the following claim will be useful:

Claim 3.4.6. *Whenever M is a countable elementary submodel of a large enough structure, and $\langle p, q\rangle \in P \times_{E_0} P \cap M$ is a condition, then there is a condition $p', q' \leq p, q$ with the same trunks such that whenever $x \in [p]$ and $y \in [q]$ are non-E_0-equivalent sequences then the pair is $P \times_{E_0} P$-generic for the model M.*

The dichotomy immediately follows: if $A \subset 2^\omega \times 2^\omega$ is an analytic $I \times_{E_0} I$-positive set and $\langle p, q\rangle \Vdash \langle \dot{x}_{\mathrm{lgen}}, \dot{x}_{\mathrm{rgen}}\rangle \in \dot{A}$. M is a countable elementary submodel and $\langle p', q'\rangle$ is a condition as in the claim, then $([p'] \times [q']) \setminus E_0$ is a subset of A by the forcing theorem and analytic absoluteness between transitive models of ZFC.

Proof. To get the condition $\langle p', q'\rangle$, by induction on $n \in \omega$ build a decreasing sequence of conditions $\langle p_n, q_n\rangle \in P \times_{E_0} P \cap M$ so that

- $t_{p_n} = t_p, t_{q_n} = t_q$, $\vec{c}_{p_n} \restriction n = \vec{c}_{p_{n+1}} \restriction n$;
- whenever u, v are two forked sequences at $n+1$-st splitting level of p_n, q_n respectively then the condition $\langle p_n \restriction u, q_n \restriction v\rangle$ is in all the first n open dense subsets of $P \times_{E_0} P$ in the model M under some fixed enumeration.

The induction is elementary. In the end, consider $p' = \lim_n p_n$ and $q' = \lim_n q_n$. The first item of the induction shows that the pair $\langle p', q'\rangle$ is indeed a condition in $P \times_{E_0} P$. The second item shows that all elements of $[p', q']$ are M-generic. Clearly, $\langle p', q'\rangle \in P \times_{E_0} P$ is an M-master condition with the desired properties. □

The complexity of the ideal is a direct corollary of the previous method of proof. Let $A \subset 2^\omega \times 2^\omega \times 2^\omega$ be an analytic set. We must show that the set $B = \{x \in 2^\omega : A_x \notin I \times_{E_0} I\}$ is analytic. To this end, choose a closed set $C \subset 2^\omega \times 2^\omega \times 2^\omega \times \omega^\omega$ whose projection A is, and prove that $x \in B$ if and only the following formula $\phi(x)$ holds: there is a condition $\langle p, q\rangle \in P_I \times_{E_0} P_I$ and a function f mapping forked pairs $\langle u, v\rangle$ of nodes $u \in sp(p), v \in sp(q)$ to $\omega^{<\omega}$ such that f preserves extension, and whenever u, v are at the n-th splitting level then $f(u, v) \in \omega^n$ is a node such that the basic open set $O_{x\restriction n, u, v, f(u,v)} \subset 2^\omega \times 2^\omega \times 2^\omega \times \omega^\omega$ has nonempty intersection with the closed set C. $\phi(x)$ is clearly an analytic formula, and if we show the equivalence $\phi(x) \leftrightarrow x \in B$, we will have shown that $B \subset 2^\omega$ is an analytic set.

Now, $\phi(x)$ certainly implies that A_x is $I \times_{E_0} I$-positive and so $x \in B$: if $\langle p, q\rangle, f$ witness $\phi(x)$, then for every pair $y \in [p], z \in [q]$ of non E_0-equivalent points the functional values of $f(y \restriction n, z \restriction n)$ eventually converge to an infinite sequence $w \in \omega^\omega$, by the closedness of the set C it must be the case $\langle x, y, z, w\rangle$ must belong to the set C, and so $\langle y, z\rangle \in A_x$. Thus, $[p] \times [q] \setminus E_0 \subset A_x$ and the set A_x is indeed $I \times_{E_0} I$-positive.

For the opposite implication, suppose that $A_x \notin I \times_{E_0} I$, so there is a condition $\langle p, q\rangle \in P \times_{E_0} P$ such that $([p] \times [q] \setminus E_0) \subset A_x$ by the work in the previous paragraphs. This inclusion is a $\mathbf{\Pi^1_2}$ statement and therefore persists into the $P \times_{E_0} P$ extension by Shoenfield's absoluteness. So $\langle p, q\rangle \Vdash \langle \dot{x}_{\mathrm{lgen}}, \dot{x}_{\mathrm{rgen}}\rangle \in \dot{A}_x$, and there must be a name τ for an element of ω^ω such that $\langle p, q\rangle \Vdash \langle x, \dot{x}_{\mathrm{lgen}}, \dot{x}_{\mathrm{rgen}}, \tau\rangle \in \dot{C}$. The fusion arguments in the previous paragraphs then show that thinning out the condition $\langle p, q\rangle$, there is a function f on forked pairs of nodes that reads off the initial segments of τ from initial segments of $\dot{x}_{\mathrm{lgen}}, \dot{x}_{\mathrm{rgen}}$. The objects $\langle p, q\rangle, f$ then witness the formula $\phi(x)$! □

Theorem 3.4.7. *The reduced product forcing is proper. It adds an unbounded real and an independent real. It preserves outer Lebesgue measure and Baire category.*

Proof. The first sentence is proved in the previous argument. Simple genericity arguments will show that the function $\dot{f} : n \mapsto \min\{m : \dot{x}_{\mathrm{lgen}}(m) \neq \dot{x}_{\mathrm{rgen}}(m)\}$

is not bounded by any ground model function in ω^ω. The function $g : n \mapsto \dot{x}_{\mathrm{lgen}}(\dot{f}(n))$ in 2^ω cannot contain a ground model infinite subfunction, and so an independent real is added as well. The two preservation properties are harder.

For the preservation of Baire category, it is enough to verify that the ideal $I \times_{E_0} I$ is an intersection of a collection of meager ideals associated with Polish topologies generating the given Borel structure as in [49, Corollary 3.5.4]. Let $\langle p, q\rangle \in P \times_{E_0} P$ be a condition; it will be enough to produce a Polish topology on the set $([p] \times [q]) \setminus E_0$ such that I is a subset of its meager ideal. It is not difficult to verify that the product topology on $[p] \times [q]$ restricted to the set $[p, q]$ is exactly such. Note that $([p] \times [q]) \setminus E_0$, so it is a dense G_δ set in this topology and therefore Polish.

The preservation of outer measure seems to be a much more complicated deal. We need to use a probability version of a partition theorem of ??? to establish the following auxiliary claim. For a condition $p \in P$ let $sp(p)$, the set of *splitnodes* of p, be the collection of all binary sequences obtained as the concatenation of t_p and $u_i : i \in n$ where $u_i \in \vec{c}_p(i)$ are arbitrary sequences and $n \in \omega$ is a natural number. Clearly, $q \leq p \to sp(q) \subset sp(p)$

Claim 3.4.8. *For every condition $p \in P$, every positive real number $\varepsilon > 0$, and every assignment $t \mapsto B_t$ of Borel subsets of 2^ω of Lebesgue mass $> \varepsilon$ to splitnodes of p, there is $q \leq p$ and $y \in 2^\omega$ such that $y \in B_t$ for every splitnode t of q.*

Proof. Let $0 = i_0 < i_1 < i_2 < \dots$ be a sequence of natural numbers, increasing very fast. For every $n \in \omega$, let a_n be the collection of all distinct pairs of sequences obtained as concatenations of sequences $u_i : i \in [i_0, i_1)$, $u_i \in \vec{c}_p(i)$. Recall the quantitative Ramsey theorem ???: for every number $n \in \omega$ there is a positive real $\delta_n > 0$ such that every subset of a_n of normalized counting measure mass $> \delta_n$ contains a triangle, and the real δ_n can be made arbitrarily small by increasing the number i_{n+1} and therefore the size of the set a_n. Thus, the numbers $i_n : n \in \omega$ can be chosen increasing so fast that for some numbers $m_n : n \in \omega$, writing ϕ_n for the normalized counting measure on a_n multiplied by m_n, every set of ϕ_n-mass contains a triangle and the numbers m_n increase so fast as to satisfy [38, Theorem 1.5] for the given positive number ε.

Now, define a Borel set $D \subset \Pi_n a_n \times \omega \times 2^\omega$ in the following way: $\langle w_n : n \in \omega, k, y\rangle \in D$ if $y \in B_t$, where $t \in 2^{<\omega}$ is the splitnode of p obtained by the concatenation of t_p together with $u_n : n \in k$, (where $u_n \in w_n$ is the lexicographically smaller sequence in the pair), together with the longest common initial segment of the two sequences in the pair w_k. Note that the vertical sections of this set have Lebesgue mass at least ε. By [38, Theorem 1.5], there are sets $b_n : n \in \omega$ of respective ϕ_n-masses at least 1, an infinite set $c \subset \omega$ and a point $y \in 2^\omega$ such that $\Pi_n b_n \times c \times \{y\} \subset D$.

Now, for every number $n \in c$, the set b_n contains a triangle, and two vertices of the triangle will be lexicographically smaller than the third vertex, forming a pair w_k. For every $n \notin c$, just pick any pair $w_n \in b_n$, and consider the set $B = \{x \in 2^\omega : x$ is a concatenation of t_p and $\langle u_n : n \in \omega\rangle$, where for $n \in c$ the sequence $u_n \in w_n$ is arbitrary, and for $n \notin c$, $u_n \in w_n$ is the lexicographically smaller sequence$\}$. It is not difficult to see that there is exactly one condition $q \in P$ such that $[q] = B$, and this condition together with the point $y \in 2^\omega$ witness the statement of the claim!

$\square$

Now, suppose that some condition in the reduced product forces that $\dot{O} \subset 2^\omega$ is an open set of Lebesgue mass $< 1/4$. It will be enough to find a point $z \in 2^\omega$ and a stronger condition forcing $\check{z} \notin \dot{O}$. To simplify the considerations, assume that the original condition has empty stem part, that is, it is of the form $\langle p, p\rangle$ for some $p \in P$ with $t_p = 0$. Assume also that $\dot{O}$ is given as a union of $\{\dot{O}_n : n \in \omega\}$ where $\dot{O}_n$ are clopen sets such that the mass of $\bigcup_{n>m} \dot{O}_n$ is smaller than 2^{-4m} for every number $m \in \omega$. It is possible to strengthen the condition p by a straightforward fusion argument so that for every forked pair $s, t \in sp(p)$ of splitnodes at m-th level of p, the condition $\langle p \restriction s, p \restriction s\rangle$ decides the union $\bigcup_{n \leq m} \dot{O}_n$ to be equal to some specific clopen set $O_{s,t}$.

$\square$

The strong canonization properties of the ideal I are now very easy to prove.

Theorem 3.4.9. *Every Borel equivalence relation on an I-positive Borel set $B \subset 2^\omega$ simplifies to* ID *or to* EE *or to* E_0 *on I-positive Borel subset. Moreover, the ideal has total canonization of graphs disjoint from E_0.*

The total canonization of graphs is a result of Clinton Conley; we will prove a probabilistic version of it.

Proof. To argue for the canonization of Borel equivalences, fix a Borel I-positive set $B \subset 2^\omega$ and a Borel equivalence relation E on it. We will first prove that there is a Borel I-positive set $C \subset B$ such that on it, either $E \subseteq E_0$ or $E = C^2$.

Choose a countable elementary submodel M of a large structure and as in Claim 3.4.6 find an I-positive Borel set $C \subset B$ such that any two E_0-inequivalent points of C are M-generic for the reduced product. Thinning out the set C we may assume that its intersection with any Borel set in the model M is either I-positive or empty. If no two E_0-inequivalent points $x, y \in C$ are E-equivalent, then $E \subset E_0$ as desired. If there are two such points $x, y \in C$ such that $x\ E\ y$, then there must be a condition $\langle p, q\rangle \in P \times_{E_0} P$ in the model M forcing $\dot{x}_{\mathrm{lgen}}\ E\ \dot{x}_{\mathrm{rgen}}$, and such that $\langle x, y\rangle \in [p] \times [q] \setminus E_0$. The set $C \cap [q]$ is I-positive, and by the forcing theorem applied in the model M to $P \times_{E_0} P$, all of its points not E_0-related to x are M-generic with x for the reduced product, and hence E-related to x. Thus the I-positive set $(C \cap [q]) \setminus [x]_{E_0}$ is a subset of a single E-class $[x]_E$.

To complete the proof of the canonization of equivalence relations from here, it is just necessary to handle the special case $E \subseteq E_0$. So let $p \in P$ be a condition such that $E \restriction [p] \subseteq E_0$. By a standard fusion argument, find a condition $q \leq p$ such that for every splitnode $t \in q$, if there is a condition $r \leq q \restriction t$ with $t_r = t$ such that for all splitnodes $s_0 \neq s_1$ of r of the same length $r \restriction s_0 \Vdash \neg \dot{x}_{gen}\ E$

$\dot{x}_{gen} \oslash s_1$, then $q \restriction t$ is such a condition. Let $\pi : sp(q) \to 2$ be the map assigning the node t color 0 if this option occurred, and color 1 if it did not. Thin out the condition q if necessary so that the map q is homogeneous on $sp(q)$. If the homogeneous color is 0, it is not difficult to see that if M is a countable elementary submodel of a large structure containing q and $r \leq q$ is a condition such that the set $[r]$ consists solely of M-generic points, then $E \restriction [r] = \mathrm{ID}$: no two distinct E_0-equivalent elements of $[r]$ can be E-equivalent since there was no opportunity for them to be forced equivalent. If the homogeneous color is 1, then use a standard fusion argument to thin out the condition q so that for every number $n \in \omega$ there is a splitnode $t_n \in sp(q)$ at n-th splitting level so that for the two splitnodes $s_0 \neq s_1$ extending t on the next splitting level of q, it is the case that $q \restriction s_0 \Vdash \dot{x}_{gen}\ E\ \dot{x}_{gen} \oslash s_1$. Then use the transitivity of the equivalence relation E to argue by induction on n that for every pair u, v of splitnodes on n-th splitting level of q, $q \restriction u \Vdash \dot{x}_{gen}\ E\ \dot{x}_{gen} \oslash u$. Now let M be a countable elementary submodel of a large structure containing q, and let $r \leq q$ be a condition such that $[r]$ consists of M-generic points only. It is not difficult to see that $E = E_0 \restriction [r]$.

We will argue for the total canonization of graphs disjoint from E_0 in the following stronger form: if $B \subset 2^\omega$ is a Borel I-positive set and $D \subset [B]^2 \times 2^\omega$ is a Borel set whose vertical sections are of μ-mass bounded away from zero where μ is some fixed Borel probability measure on 2^ω, then there is a Borel I-positive set $C \subset B$ and a point $z \in 2^\omega$ such that for every two E-unrelated points $x, y \in C$, $z \in D_{\{x,y\}}$. The total canonization of graphs disjoint from E_0 then immediately follows: if $[B]^2 = G_0 \cup G_1$ is a Borel partition of $[B]^2 \setminus E_0$ into two pieces, then find two disjoint subsets A_0, A_1 of 2^ω of nonzero measure, let $D = \{\langle\{x,y\}, z\rangle : z \in A_0 \text{ if } \{x,y\} \in G_0 \text{ and } z \in A_1 \text{ if } \{x,y\} \in G_1\}$. If $C \subset B$ is a Borel I-positive set as above, either $[C]^2 \setminus E_0 \subset G_0$ or $[C]^2 \setminus E_0 \subset G_1$ depending on whether the homogeneous point z falls into A_0 or A_1.

Now, to produce the set $C \subset B$ and the point $z \in 2^\omega$, find a condition $p \in P$ with $[p] \subset B$ and consider the $P \times_{E_0} P$ name for the generic section $D_{\{\dot{x}_{\mathrm{lgen}}, \dot{x}_{\mathrm{rgen}}\}}$. This is a Borel set forced by the condition $\langle p, p\rangle$ to be of mass larger than zero. By the preservation of outer measure by the reduced product, there is a point $z \in 2^\omega$ and a condition $\langle r, s\rangle$ forcing z into the generic section. By Theorem ???, thinning out the conditions r, s if necessary we may arrange that $z \in D_{\{x,y\}}$ for all E_0-unrelated points $x \in [r], y \in [s]$. Now, since the set D is symmetric–the vertical section of D depends only on the points in the pair and not on their enumeration, this means that ??? □

3.5 E_2 forcing

E_2 is the equivalence relation on $X = 2^\omega$ defined by xE_2y if $\Sigma\{\frac{1}{n+1} : x(n) \neq y(n)\} < \infty$. It is a basic example of a turbulent equivalence relation, and therefore it is not classifiable by countable structures. There is an associated σ-ideal and a quotient proper notion of forcing. For xE_2y define $d(x,y) = \Sigma\{\frac{1}{n+1} : x(n) \neq y(n)\}$, otherwise let $d(x,y) = \infty$. Thus, d is a metric on each equivalence class. Kanovei [22, Definition 15.2.2] defined the collection of *grainy* subsets of X as those sets $A \subset 2^\omega$ such that there is a real number $\varepsilon > 0$ such that there is no finite sequence of elements of A, the successive elements of which have d-distance at most ε and the two endpoints are at a distance at least 1 from each other. Let I be the σ-ideal generated by the Borel grainy sets.

Fact 3.5.1. *[22, Section 15.2] The ideal I is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$, and every analytic I-positive set has a Borel I-positive subset. A Borel set $B \subset X$ is in the ideal I if and only if the equivalence $E_2 \restriction B$ is essentially countable if and only if E_2 is not reducible to $E_2 \restriction B$.*

The features of the quotient forcing were surveyed in [22, 49]. Here, we will provide a more thorough treatment:

Theorem 3.5.2. *The forcing P_I is proper, bounding, preserves Baire category and outer Lebesgue measure, and adds no independent reals.*

Proof. We will need some preliminary notation and observations. Abuse the notation a little and include I-positive analytic sets in it; by Fact 3.5.1, Borel sets will be dense in it anyway. For points $x, y \in 2^\omega$ and a number $k \in \omega$ write $d_k(x, y) = \sum\{\frac{1}{n+1} : n \geq k, x(n) \neq y(n)\} < \varepsilon$; this is a notion of pseudodistance between sets satisfying triangle inequality which can be infinite or zero for distinct points. For sets $A, B \subset 2^\omega$ let $d_k(A, B)$ be the associated Hausdorff distance: the infimum of all ε such that for every $x \in A$ there is $y \in B$ such that $d_k(x, y) < \varepsilon$ and vice versa, for every $x \in B$ there is $y \in A$ such that $d_k(x, y) < \varepsilon$. The following claims will be used repeatedly:

Claim 3.5.3. *If $A, B \subset 2^\omega$ are analytic sets such that $A \notin I$ and $\forall x \in A \exists y \in B\ x\ E_2\ y$ then $B \notin I$.*

Proof. If $B \in I$ then B can be enclosed in a countable union of Borel grainy sets, a Borel set $B' \in I$, and $E_2 \restriction B'$ can be reduced to some countable equivalence relation E on 2^ω via a Borel function $f : B' \to 2^\omega$. The set A can be thinned out to a Borel I-positive set $A' \subset A$. Consider the relation $R \subset A' \times 2^\omega$ where $\langle x, y\rangle \in R$ if $\exists z \in B'\ x\ E_2\ z \wedge f(z) = y$. This is an analytic relation such that non-E_2-equivalent points in A' have disjoint vertical sections, and the union of vertical sections of points belonging to the same fixed E_2 class is countable. These two are $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ properties of relations, and so by the first reflection theorem, R can be enclosed in a Borel relation R' with the same properties. Let $g : A' \to 2^\omega$ be a Borel uniformization of R'. This is a Borel function under which distinct E_2-classes have disjoint countable images. By a result of Kechris, [22, Lemma 7.6.1], $E_2 \restriction A'$ must then be essentially countable. □

Claim 3.5.4. *If $A_i : i \in n$ are I-positive analytic sets with $d_k(A_i, A_{i+1}) < \varepsilon$ for all $i \in n-1$ and $D \subset P_I$ is an open dense set, then there are analytic sets $A'_i : i \in n$ such that $A'_i \subset A_i$, $A'_i \in D$, and $d_k(A'_i, A'_{i+1}) < \varepsilon$ for all $i \in n-1$.*

Note that in the conclusion, the d_k-distances between the various sets $A'_i : i \in n$ must be smaller than $(n-1)\varepsilon$.

Proof. By induction on $i \in n$ find sets $A^i_j \subset A_j : j \in n$ such that at stage i, the d_k-distances are still $< \varepsilon$ and $A^i_i \in D$. The sets $A^{n-1}_j : j \in n$ will then certainly work as desired. The induction is easy. Given the sets $A^i_j : j \in n$, strengthen A^i_{i+1} to a set B in the dense set D. Then, for every $j \in n$, write $A^{i+1}_j = \{x \in 2^\omega : \exists \vec{y} \in (2^\omega)^n \ \forall l \in n-1 \ d_k(\vec{y}(l), \vec{y}(l+1)) < \varepsilon$ and $\forall l \in n \ \vec{y}(l) \in A^i_l$ and $\vec{y}(i+1) \in B$ and $\vec{y}(j) = x\}$. These are all analytic sets with $d_k(A^{i+1}_j, A^{i+1}_{j+1}) < \varepsilon$, $A^{i+1}_{i+1} = B$, and by the previous claim they must all be I-positive since the set B is. The induction can proceed. $\square$

Claim 3.5.5. *Suppose that $A_i : i \in n$ are I-positive analytic sets with d-distance $< \varepsilon$, and $1 > \delta > 0, \gamma > 0$ are real numbers. Then there are I-positive analytic sets $A^b_i : i \in n, b \in 2$, binary sequences $t^b_i \in 2^{<\omega}$ of the same length k such that*

1. *$|d(t^0_i, t^1_i) - \delta| < 2\varepsilon$ and $d(t^0_i, t^0_j), d(t^1_i, t^1_j)$ are both smaller than 2ε;*
2. *$A^b_i \subset A_i \cap O_{t^b_i}$, and the d_k-distances of the sets $A^b_i : i \in n, b \in 2$ are $< \gamma$.*

Proof. For each choice of k and $\vec{t} = t^b_i : i \in n, b \in 2$ as in the first item of the claim let $B_{\vec{t}} \subset (2^\omega)^{n \times 2}$ be the analytic set of all tuples $x^b_i : i \in n, b \in 2$ such that $t^b_i \subset x^b_i$, $x^b_i \in A_i$, and the d_k-distances between points on the tuple are $< \gamma$. If for some $\vec{t}$, the projection of $B_{\vec{t}}$ into the first coordinate is I-positive, then the projections into any coordinate are I-positive analytic sets by Claim 3.5.3, and they will work as $A^b_i : i \in n, b \in 2$ required in the claim.

So it will be enough to derive a contradiction from the assumption that the projections of the sets $B_{\vec{t}}$ into the first coordinate are always in the ideal I. These are countably many I-small analytic sets, so they can be enclosed in a Borel I-small set $C \subset 2^\omega$. The set $A_0 \setminus C$ is analytic and I-positive, and so it contains an ε-walk whose endpoints have d-distance > 1, so it has to contain two points x^0_0, x^1_0 with $|d(x^0_0, x^1_0) - \delta| < \varepsilon$. Choose points $x^b_i : i > 0, b \in 2$ in the sets A_i so that $d(x^0_0, x^0_i), d(x^1_0, x^1_i)$ are always smaller than ε, find a number k such that the d_k-distance of these points is pairwise smaller than γ, and let $\vec{t} = \langle x^b_i \restriction k : i \in n, b \in 2 \rangle$. This sequence should belong to the set $B_{\vec{t}}$, but its first coordinate should not belong to the projection of $B_{\vec{t}}$. This is of course a contradiction.

$\square$

This ends the preliminary part of the proof. For the properness of P_I, let M be a countable elementary submodel of a large enough structure and let $B \in P_I$ be a Borel I-positive set. We need to show that the set $C = \{x \in B : x$ is M-generic$\}$ is I-positive. To do that, we will produce a continuous injective embedding of the equivalence E_2 into $E_2 \restriction C$. Fix positive real numbers ε_n whose sum converges. Enumerate the open dense subsets of P_I in the model M by $D_n : n \in \omega$. By induction on $n \in \omega$ build functions $\pi_n : 2^n \to 2^{<\omega}$ as well as sets $A_t : t \in 2^n$ so that:

- if $m \in n$ then π_n extends π_m in the natural sense: for every $t \in 2^m$ and every $s \in 2^n$, $\pi_m(t) \subset \pi_n(s)$. Also $A_s \subset A_t$, and $A_s \in P_I \cap M \cap D_n$ is an I-positive analytic set;
- all sequences in the range of π_n have the same length $k_n \in \omega$, and π_n approximately preserves the d-distance: $|d(u, v) - d(\pi_n(u), \pi_n(v))| < \Sigma_{m\in n}\varepsilon_m$;
- the sets $A_u : u \in 2^n$ have d_{k_n}-distance smaller than $\varepsilon_n/2$.

The induction step is handled easily using the previous two claims. In the end, for every sequence $y \in 2^\omega$ the sets $A_t : t \subset y$ generate an M-generic filter, so they do have a nonempty intersection which must contain a single point $\pi(y) = \bigcup_n \pi_n(x \restriction n)$. It is immediate from the second item that the map $\pi : 2^\omega \to C$ is a continuous injective reduction of E_2 to $E_2 \restriction C$ and so $C \notin I$ as desired.

For the bounding part, we must show that for every I-positive Borel set $B \subset 2^\omega$ and every Borel function $f : B \to 2^\omega$ there is a compact I-positive subset of B on which the function f is continuous. This is obtained by a repetition of the previous argument. Just note that the image of the map π obtained there is compact, and if the open dense set D_n consists of sets B' on which the bit $f(x)(n) : x \in B'$ is constant, then $f \restriction \mathrm{rng}(\pi)$ is continuous.

For the preservation of outer Lebesgue measure, it is enough to argue that the ideal I is polar in the sense of [49, Section 3.6.1]; i.e., it is an intersection of null ideals for some collection of Borel probability measures. In other words, we need to show that every Borel I-positive set $B \subset 2^\omega$ carries a Borel probability measure that vanishes on the ideal I. This uses a simple claim of independent interest:

Claim 3.5.6. *The usual Borel probability measure μ on 2^ω vanishes on the ideal I, meaning that every set in I has μ-mass zero.*

Proof. First, an auxiliary statement: let $a \subset \omega$ be a finite set, and let $S \subset 2^a$ be a set containing more than half of all elements of 2^a. Then S contains two elements u, v with $d(u, v) = \sum_{n\in a} \frac{1}{n+1}$. To prove this, note that the map $\pi : 2^a \to 2^a$ flipping all bits of sequences in its domain is an involution, and so the set S must contain two points connected by π.

Now let $C \subset 2^\omega$ be a Borel set of positive mass; we must argue this set is not grainy. Let $\varepsilon > 0$ be a real number. We will construct a sequence of points $x_i : i \leq n$ in the set C such that $\varepsilon/4 \leq d(x_i, x_{i+1}) < \varepsilon$, and the sets $a_i = \{m \in \omega : x_i(m) \neq x_{i+1}(m)\}$ are finite and pairwise disjoint. If such a sequence has length at least $4/\varepsilon$, then its endpoints will have distance at least 1 as required to show that C is not grainy.

By induction on $i \in \omega$ build pairwise disjoint finite sets $a_i \subset \omega$, numbers $m_i \in \omega$, and sequences $u_i, v_i \in 2^{a_i}$, $w_i : m_i \setminus \bigcup_{j\in i} a_j$ so that

- $\varepsilon/2 \leq d(u_i, v_i) < \varepsilon$;

- the set $C_i = \{y \in 2^{\omega \setminus m_i}$: whenever $x \in 2^\omega$ is a point satisfying $w_i \cup y \subset x$ and for every $j \in i$, $x \restriction a_j$ is equal either to u_j or v_j, then $x \in C\}$ has mass greater than 3/4.

This is easy to do. To find m_0, w_0, use the Lebesgue density theorem to find a finite binary sequence w_0 of length greater than $4/\varepsilon$ such that C has relative mass in the basic open neighborhood defined by w_0 close to one, and let $m_0 = |w_0|$. Suppose m_i, w_i as well as $a_j, u_j, v_j : j \in i$ have been found. Let $k > m_i$ be a number such that the sum $\sum_{m_i \leq m < k} \frac{1}{m+1}$ is smaller than, but as close to ε as possible and let $a_i = [m_i, k)$. There must be a sequence $u_i \in 2^{a_i}$ such that $C_i \cap O_{u_i}$ has relative mass greater 3/4. The set $S \subset 2^{a_i}$ of those v such that the set $C_i \cap O_v$ has relative mass at least 1/4, contains more than half of all elements of 2^{a_i}. The set S contains two elements of maximal possible distance, so one of these elements, some $v_i \in S$, has distance at least $1/2 \cdot \Sigma_{m_i \leq m < k} \frac{1}{m+1} \geq \varepsilon/4$ from u_i. The set $D = \{y \in 2^{\omega \setminus k}$: whenever $x \in 2^\omega$ is a point satisfying $w_i \cup y \subset x$ and for every $j \leq i$, $x \restriction a_j$ is equal either to u_j or v_j, then $x \in C\}$ has nonzero mass, and so one can use the Lebesgue density theorem again to find a number $m_{i+1} \in \omega$ and a binary sequence $t \in 2^{m_i \setminus k}$ such that the set $D \cap O_t$ has relative mass close to 1. Let $w_{i+1} = w_i \cup t$ and continue the induction.

After some $n > 4/\varepsilon$ many steps of the induction, choose a point $y \in C_n$, and define points $x_i \in C$ for $i \in n$ by $w_n \cup y \subset x_i$ and $u_j \subset x_i$ if $j \in i$ and $v_j \subset x_i$ if $j \in n \setminus i$. These points constitute an ε-walk in C whose endpoints have distance at least 1. □

If $B \subset 2^\omega$ is a Borel I-positive set and $\pi : 2^\omega \to B$ is a continuous one-to-one reduction of E_2 to $E_2 \restriction B$ then the transported measure $\hat{\mu}$ on B must vanish on the sets in the ideal I as well.

For the preservation of Baire category, it is enough to argue that every Borel I-positive set $B \subset 2^\omega$ there is a Polish topology on B generating the same Borel structure as the usual one, such that all sets in the ideal I are meager in this topology. Then, a reference to Kuratowski-Ulam theorem and/or [49, Corollary 3.5.8] will finish the argument. Just as in the treatment of Lebesgue measure, it is enough to show

Claim 3.5.7. *The ideal I consists of meager sets.*

Proof. Let $B \subset 2^\omega$ be a Borel nonmeager set; we must show that it is not grainy. Let $\varepsilon > 0$ be a positive real number. Find a finite sequence $t \in 2^{<\omega}$ such that B is comeager in O_t. By a standard argument, find a point $x \in 2^\omega$ such that $t \subset x$ and all points obtained by rewriting t on finitely many positions above $\mathrm{dom}(t)$ belong to B. Find finite pairwise disjoint sets $u_i : i \in j$ of natural numbers larger than $\mathrm{dom}(t)$ such that $\sum_{n \in u_i} \frac{1}{n+1} < \varepsilon$ for every $i \in j$, and $\sum\{\frac{1}{n+1} : n \in \bigcup_{i \in j} u_i\} > 1$. Let $x_k : k \leq j+1$ be points obtained from x by rewriting x to its opposite on the respective sets $\bigcup_{i \in k} u_i$. It is clear that these points form an ε-walk in the set B with endpoints of distance at least 1; thus B is not grainy for the constant ε. □

The proof that no independent reals are added is as always more complicated. One can use an infinitary partition theorem to prove this, such as a theorem of Henle [12]. We take a different route which implicitly uses concentration of measure. Theorem 3.12.4 produces a fat tree forcing with E_2 in its spectrum. The fat tree forcings do not add independent reals by [49, Theorem 4.4.8]. It is now not difficult to transfer this feature to the E_2 forcing.

Suppose that $B \subset 2^\omega$ is an I-positive Borel set and $f : B \to 2^\omega$ is a Borel function. We must find an infinite partial function $y : \omega \to 2$ such that the set $\{x \in B : y \subset f(x)\}$ is still I-positive. In order to do that, let T_{ini} be a finitely branching tree and J be a σ-ideal on $[T_{ini}]$ corresponding to some fat tree forcing with E_2 in its spectrum, as witnessed by some equivalence E on $[T_{ini}]$ and a Borel function $g : [T_{ini}] \to B$ reducing E to $E_2 \restriction B$. Since the fat tree forcing P_J does not add an independent real, there must be an infinite partial function $y : \omega \to 2$ such that the set $C = \{z \in [T_{ini}] : y \subset f(g(z))\}$ is I-positive. Since E showed that E_2 is in the spectrum of the ideal J, it must be the case that the range $g''C$ is an I-positive analytic set; otherwise E_2 restricted to it would be essentially countable and so $E \restriction C$ would be essentially countable. We conclude that the set $\{x \in B : y \subset f(x)\} \supset g''C$ is I-positive as desired! □

Theorem 3.5.8. *I has total canonization for equivalences classifiable by countable structures.*

In particular, this shows that the forcing P_I adds a minimal real, improving [22, Theorem 15.6.3].

Proof. One can argue directly by a demanding fusion argument via the separation property of the poset P_I, which was our original way of dealing with the challenge. There is a simple proof that uses the existence of a forcing with E_2 in the spectrum and total canonization for equivalences classifiable by countable structures.

Let $B \subset 2^\omega$ be a Borel I-positive set and E an equivalence on B that is classifiable by countable structures. Fix a Borel function f with domain B reducing E to isomorphism of countable structures. Fix a function $g : 2^\omega \to B$ reducing E_2 to $E_2 \restriction B$. Fix a σ-ideal J on a compact space Y such that J has total canonization for equivalences classifiable by countable structures and E_2 is in the spectrum of J as witnessed by a Borel equivalence F on Y, reduced to E_2 by a Borel function $h : Y \to 2^\omega$–see theorem 3.12.4.

Consider the equivalence relation G on Y defined by $y_0 \, G \, y_1$ if the structures $fgh(y_0)$ and $fgh(y_1)$ are isomorphic. Use the total canonization feature of the σ-ideal J to find a Borel J-positive set $D \subset Y$ such that the function $g \circ h$ is one-to-one and the equivalence relation $G \restriction D$ is either equal to identity or to D^2. The set $C = gh''D \subset B$ is a one-to-one image of a Borel set and therefore Borel. It is also I-positive, since E_2 Borel reduces to $F \restriction D$ which Borel reduces to $E_2 \restriction C$ via $g \circ h$. The equivalence relation $E \restriction C$ is either equal to identity or to C^2 depending on the behavior of $G \restriction D$. The theorem follows. □

Theorem 3.5.9. *E_2 is in the spectrum of I. Moreover,*

1. *$P_I^{E_2}$ is a regular $\aleph_0$-distributive subposet of P_I, and it generates the model $V[\dot{x}_{gen}]_{E_2}$ of Definition 2.1.1;*

2. *$V[[\dot{x}_{gen}]]_{E_2} = V[\dot{x}_{gen}]_{E_2}$.*

Proof. The first sentence follows from the analysis of the ideal I immediately. The Borel sets in I are exactly those sets B for which $E_2 \restriction B$ is essentially countable, and so by the E_2 dichotomy 3.5.1, E_2 is Borel reducible to $E_2 \restriction B$ for every Borel I-positive set B.

Suppose that $B \in P_I$ is an analytic I-positive set. Its saturation, $[B]_{E_2}$, is a pseudoprojection of B to $P_I^{E_2}$: if $C \subset [B]_{E_2}$ is an analytic E_2-saturated I-positive set, the set $D = B \cap C$ is analytic and must be I-positive by Claim 3.5.3. The $\aleph_0$-distributivity follows from Proposition 2.2.6. Note that the fusion process from the previous proofs starts with an arbitrary analytic I-positive set B and inscribes into it a compact I-positive set $C \subset B$ such that the equivalence class of any point in C is dense in C.

To show that the poset P_I^E generates the $V[\dot{x}_{gen}]_{E_2}$ model, first observe that for every I-positive Borel set B and every Borel one-to-one map $f : B \to 2^\omega$ whose graph is a subset of E_2, the function f induces an isomorphism between P_I below B and P_I below $\mathrm{rng}(f)$, mapping any analytic set $C \subset B$ to its f-image. Note that $C \in I \leftrightarrow f''C \in I$ by Claim 3.5.3.

????? $\square$

Question 3.5.10. Find another equivalence relation in the spectrum of I.

3.6 Silver forcing

The well-known Silver forcing is just the partial order P of all partial functions $f : \omega \to 2$ with coinfinite domain, ordered by reverse inclusion. The calculation of the associated σ-ideal gives complete information.

Let X be the Cantor space and let G be the graph on X given by xGy if and only if there is exactly one n such that $x(n) \neq y(n)$. A set $B \subset X$ is G independent if no two elements of B are G-connected. Let I be the σ-ideal generated by Borel G-independent sets. The following is true:

Theorem 3.6.1. *1. Whenever $A \subset X$ is an analytic set, either $A \in I$ or there is $f \in P$ such that $\{x \in X : f \subset x\} \subset A$, and these two options are mutually exclusive;*

2. the ideal I is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$;

3. the ideal I does not have the transversal property;

4. the forcing P is proper, bounding, preserves outer Lebesgue measure and Baire category, and adds an independent real.

Proof. (1) comes from [48, Lemma 2.3.37]. (2) follows from (1) and [49, Theorem 3.8.9], or one can compute directly. (4) is well-known; the independent real $\dot{a} \subset \omega$ is read off the generic real $\dot{x}_{gen} \in 2^\omega$ as $a = \{n \in \omega : \{m \in n : \dot{x}_{gen}(m) = 1\}$ has even size$\}$. Finally, the failure of the transversal property is witnessed by the set $D \subset 2^\omega \times 2^\omega$ given by $\langle y, x\rangle \in D$ if $x(3n) = x(3n+1) = y(n)$. □

From now on, we will write $B_f = \{g \in X : f \subset g\}$ and call sets of this form *Silver cubes*. The theorem shows that the map $f \mapsto B_f$ is a dense embedding of the Silver forcing P into the quotient forcing P_I. The failure of the Silver property can be exhibited by any perfect collection of functions $\{f_y : y \in 2^\omega\}$ from ω to 2 with coinfinite domain such that any two distinct functions in the collection disagree on more than one entry, and the set $D \subset 2^\omega \times 2^\omega$ given by $\langle y, x\rangle \in D \leftrightarrow f_y \subset x$. Any transversal for this set would be G-independent, and if Borel then it would be in the ideal I.

Several features of the spectrum can be identified without much effort:

Theorem 3.6.2. *The ideal I has total canonization for smooth equivalences. In other words, Silver forcing adds a minimal real degree.*

This was originally proved by Grigorieff [9, Corollary 5.5] in a rather indirect fashion. We provide a direct argument.

Proof. The argument uses a claim of independent interest.

Claim 3.6.3. *Whenever $f_i : i \in n$ is a finite collection of continuous functions from X to X such that preimages of singletons are I small, and $A \subset X$ is an analytic I-positive set, there is a point $x \in X$ and a number $m \in \omega$ such that $x \in A$, $x \odot m \in A$ and for every $i \in n$, $f_i(x) \neq f_i(x \odot m)$.*

Proof. This is proved by induction on n. The case $n = 0$ is trivial. Suppose that we know it for a given n, and $f_i : i \in n+1$ is a collection of continuous functions of size $n+1$, and the conclusion fails for some I-positive analytic set $A \subset X$. By induction on $j \in \omega$ build numbers m_j and functions $g_j \in P$ so that

- $B_{g_0} \subset A$ and $g_0 \subset g_1 \subset \dots$
- $m_0 \in m_1 \in \dots$ and $m_j \notin \mathrm{dom}(g_k)$ for all k;
- for every point $x \in B_{g_{j+1}}$, the value of $f_n(x)$ does not depend on the values of x at $m_k : k \leq j$.

Once this induction is finished, find a function $h \in P$ with $\bigcup_j g_j \subset h$ and $\omega \setminus \mathrm{dom}(h) = \{m_j : j \in \omega\}$. The continuity of the function f_n together with the last item of the induction hypothesis shows that $f \restriction B_h$ is constant, contradicting the assumptions on the function f_n.

To perform the induction step, suppose that the function g_j and the numbers $m_k : k \in j$ have been found. Use the induction hypothesis for the n induction to conclude that there is a number $m \in \omega$ greater than all the $m_k : k \in j$ and an I-positive Borel set $B \subset B_{g_j}$ such that for every $x \in B$ it is the case that

$x \odot m \in B_j$ and for every $i \in n$, $f_i(x \odot m) \neq f_i(x)$. (If for every number m the set B_m of all such points x were I-small, consider the I-positive Borel set $B \setminus \bigcup_m B_m$ and apply the n induction hypothesis to it to reach a contradiction.) Now the assumption that the n-induction step cannot be performed implies that for every $x \in B$, $f_n(x) = f_n(x \odot m)$. Let $m = m_j$ and find a function $g_{j+1} \in P$ such that $B_{g_{j+1}} \subset B$ and erase the numbers $m_k : k \leq j$ from its domain if necessary. This completes the j-induction step and the proof of the claim.

□

Now suppose that $B \subset X$ is an I-positive Borel set and $f : B \to 2^\omega$ is a Borel function with I-small preimages of singletons. We must produce a Borel I-positive set $C \subset B$ such that $f \restriction C$ is an injection. By standard homogeneity and bounding arguments, we may assume that the function f is continuous and the set B in fact equals to the whole space 2^ω. By induction on $n \in \omega$ build functions $g_n \in P$ and numbers $m_n, j_n \in \omega$ such that

- $g_0 \subset g_1 \subset \dots$, $m_0 \in j_0 \in m_1 \in j_1 \in \dots$, $m_n \notin \mathrm{dom}(g_k)$ for every $n, k \in \omega$;
- Whenever $x \in B_{g_n}$ then $f(x \odot m_n) \neq f(x)$ and the least point of difference between these two infinite binary sequences as well as their values at this point depend only on $x \restriction j_n$.

To perform the induction, suppose that the function g_n and the numbers m_n and j_n have been found. Let $a = \{m_i : i \leq n\}$ and for each element $h \in 2^a$ define the continuous function $f_h : B_{g_n} \to 2^\omega$ by setting $f_h(x) = f(x \oslash h)$. Use the claim to find a number m_{n+1} and an I-positive Borel set $D \subset B_{g_n}$ such that for every point $x \in B$ and every function $h \in 2^a$ it is the case that $f_h(x \odot m) \neq f_h(x)$. Let j_{n+1} be a number large enough so that the least points of difference at all these sequences as well as the values at those points depend only on $x \restriction j_{n+1}$, Let $g_{n+1} \in P$ be a function such that $B_{g_{n+1}} \subset D$, erase the numbers $m_k : k \leq n+1$ from the domain of g_{n+1} if necessary and continue the induction.

After the induction process is complete, choose a function $h \in P$ such that $\bigcup_n g_n \subset h$ and $\omega \setminus \mathrm{dom}(h) = \{m_n : n \in \omega\}$. It is clear that the function f is an injection on the set B_h! □

Theorem 3.6.4. *E_0 belongs to the spectrum of the Silver forcing. The poset $P_I^{E_0}$ is regularly embedded in P_I, it is $\aleph_0$-distributive, and it yields the model $V[\dot{x}_{gen}]_{E_0}$ of Definition 2.1.1.*

Proof. Look at the equivalence relation E_0 on X. Whenever $B_f \notin I$ is a Silver cube and $\pi : \omega \to \omega \setminus \mathrm{dom}(f)$ is a bijection then the map $g : X \to B_f$ defined by $g(x) = f \cup (x \circ \pi^{-1})$ is a continuous reduction of E_0 to $E_0 \restriction B_f$, and therefore E_0 is in the spectrum. The remainder of the theorem is a consequence of the work in Section 4.2 as soon as we prove that E_0-saturations of sets in I are still in I. The easiest way to see that is to consider the action of the countable group of finite subsets of ω with symmetric difference on the space X where a finite set acts on an infinite binary sequence by flipping all entries of the sequence on the set. This action generates E_0 as its orbit equivalence relation, and it also preserves the graph G and with it the σ-ideal I. Thus, E_0-saturations of I-small sets are still I-small.

As an interesting extra bit of information, the equivalence class $[x_{\mathrm{gen}}]_{E_0}$ is the only one in the extension $V[G]$ which is a subset of all sets in the filter K, where $G \subset P_I$ is a generic filter and $K = G \cap P_I^{E_0}$. In the ground model, this is immediately translated to the following: If $B \subset X$ is an I-positive Borel set and $f : B \to X$ is a Borel function such that $\forall x \in B\ \neg f(x)\ E_0\ f(y)$, then there is a Borel I-positive set $C \subset B$ such that $f''C \cap [C]_{E_0} = 0$.

Since the forcing P_I is bounding, there must be a partition of ω into intervals $I_n : n \in \omega$ and an I-positive compact set $B_g \subset B$ such that for every $x \in C$ and every $n \in \omega$ there is a number $m \in I_n$ with $f(x)(m) \neq x(m)$, and the function $f \restriction B_g$ is continuous. Strengthen the function g if necessary so that infinitely many of the intervals do not intersect the set $\omega \setminus g$. The condition $C = B_g$ works as desired. □

Theorem 3.6.5. *E_{K_σ} is in the spectrum of I as witnessed by a certain Borel equivalence relation E on X. Moreover,*

1. *the poset P_I^E is regular in P_I, it is $\aleph_0$-distributive and it yields the $V[\dot{x}_{gen}]_E$ extension of Definition 2.1.1;*
2. *the reduced product $P_I \times_E P_I$ is not proper.*

Proof. Let $c : 2^\omega \to \omega^\omega$ be the continuous function defined by $c(x)(n) = |\{m \in n : x(m) = 1\}|$ and let E be the Borel equivalence relation on 2^ω connecting $x_0, x_1 \in 2^\omega$ if the function $|c(x_0) - c(x_1)| \in \omega^\omega$ is bounded. Clearly, the function h reduces the equivalence relation E to E_{K_σ}. To show that $E_{K_\sigma} \leq_B E$, choose successive finite intervals $I_n \subset \omega$ of length $2n + 1$ and for every function $y \in \omega^\omega$ below the identity consider the binary sequence $g(y) \in 2^\omega$ specified by the demand that $g(y) \restriction I_n$ begins with $f(n)$ many 1's, ends with $n - f(n)$ many 1's and has zeroes in all other positions. It is not difficult to see that if $y_0, y_1 \in \omega^\omega$ below the identity are two functions then $|c(g(y_0)) - c(g(y_1)) \restriction I_n|$ is bounded by $|y_0(n) - y_1(n)|$ and the bound is actually attained at the midpoint of the interval I_n. Thus, the function g reduces E_{K_σ} to E. To show that $E \leq E \restriction B$ for every Borel I-positive set B, find a Silver cube $B_g \subset B$, let $\pi : \omega \to \omega \setminus \mathrm{dom}(g)$ be the increasing bijection, and observe that the map $\bar{\pi} : 2^\omega \to B_g$ defined by $\bar{\pi}(x) = g \cup (x \circ \pi^{-1})$ reduces E to $E \restriction B$. Thus, the equivalence relation E shows that E_{K_σ} is in the spectrum.

The following definition will be helpful for the investigation of the poset P_I^E. Let $B_g, B_h \subset 2^\omega$ be two Silver cubes. We will say that they are *n-dovetailed* if between any two successive elements of $\omega \setminus \mathrm{dom}(g)$ is exactly one element of $\omega \setminus \mathrm{dom}(h)$ and $\min(\omega \setminus \mathrm{dom}(g)) \in \min(\omega \setminus \mathrm{dom}(h))$, and moreover, for every number $m \in \omega$, the difference $|\{k \in m : g(k) = 1\}| - |\{k \in m : h(k) = 1|$ is bounded in absolute value by n. The two Silver cubes are *dovetailed* if they are n-dovetailed for some n. It is clear that if B_g, B_h are dovetailed and $f : \omega \setminus \mathrm{dom}(g) \to \omega \setminus \mathrm{dom}(h)$ is the unique increasing bijection, then the continuous function $\pi : x \mapsto h \cup x \circ f^{-1}$ from B_g to B_h preserves the σ-ideal I, and its graph is a subset of E. Moreover, if B_g, B_h are arbitrary Silver cubes and $x \in B_g$ and $y \in B_h$ are E-equivalent points such that both $x \setminus g, y \setminus h$ take both 0 and 1 value infinitely many times, then there are dovetailed cubes $B_{g'} \subset B_g$ and $B_{h'} \subset B_h$: just let $a, b \subset \omega$ be dovetailed infinite subsets of $\omega \setminus \mathrm{dom}(g)$ and $\omega \setminus \mathrm{dom}(h)$ respectively such that $x \restriction a$ and $y \restriction b$ both return constantly zero value, and let $g' = x \restriction (\omega \setminus a)$ and $h' = y \restriction (\omega \setminus b)$.

For the regularity of P^E_I, let $B = B_g$ be a Silver cube and let $B_h \subset [B_g]_E$ be another one. We must show that $B_g \cap [B_h]_E$ is an I-positive set. Choose a point $y \in 2^\omega$ such that $h \subset y$ and on $\omega \setminus \mathrm{dom}(h)$, y attains both values 0 and 1 infinitely many times. Since $y \in B_h \subset [B_g]_E$, there must be a point $x \in B_g$ such that $x \mathrel{E} y$. Note that x must attain both values infinitely many times on $\omega \setminus g$: if it attained say 1 only finitely many times, there would be no E equivalent point in B_g to the point $y' \in B_h$ which extends h with only zeroes. As in the previous paragraph, find dovetailed Silver cubes $B_{g'} \subset B_g$ and $B_{h'} \subset B_h$, and conclude that $B_{g'} \subset B_g \cap [B_h]_E$ as desired.

For the distributivity just note that E is I-dense and apply Proposition 2.2.6. To show that P^E_I yields the $V[x_{\mathrm{gen}}]_E$ extension, it is enough to prove that for all I-positive analytic sets $B, C \subset X$, either there is a Borel I-positive subset $B' \subset B$ such that $[B']_E \cap C \in I$, or there is a Borel I-positive subset $B' \subset B$ and an injective Borel map $f : B' \to C$ such that $f \subset E$ and f preserves the ideal I; and then apply Theorem 2.2.7. So fix the sets B, C. First, inscribe a Silver cube B_g into B. If $[B_g]_E \cap C \in I$ then we are done; otherwise inscribe a Silver cube B_h into $[B_g]_E \cap C$. As in the previous paragraph, there will be dovetailed Silver cubes $B_{g'} \subset B_g$ and $B_{h'} \subset B_h$ and a homeomorphism $\pi : B_{g'} \to B_{h'}$ whose graph is a subset of E and which preserves the ideal I. Thus, the assumptions of Theorem 2.2.7 are satisfied and P^E_I yields the $V[x_{\mathrm{gen}}]_E$ extension.

As an additional piece of information, in the P_I-generic extension $V[G]$ the equivalence class $[x_{\mathrm{gen}}]_E$ is the only one contained in every set in the filter $H = G \cap P^E_I$. In the ground model, this translates to the statement that whenever $B \in P_I$ is an I-positive Borel set and $g : B \to 2^\omega$ is a Borel function such that $\forall x \in B \ \neg g(x) \mathrel{E} x$ then there is a Borel I-positive set $C \subset B$ such that $C \cap [g''C]_E = 0$. Since Silver forcing is bounding, thinning out the set B if necessary we may assume that $g \restriction B$ is continuous and there are successive intervals $I_n : n \in \omega$ in ω such that for every $x \in B$ and every $n \in \omega$ there is $m \in I_n$ such that $|f(x)(m) - f(g(x))(m)| > 2n$. Find a function $h \in P$ such that $B_h \subset B$ and moreover the set $\omega \setminus \mathrm{dom}(h)$ visits every interval $I_n : n \in \omega$ in at most one point. It is immediate that $C = B_g$ works as required.

For the nonproperness of the reduced product forcing, we will show that it forces the following sentence Ψ: for every $k \in \omega$ there are numbers $l_0 < l_1$ larger than k and a ground model partition of ω into finite intervals such that for every interval J in it, the set $\{c(\dot{x}_{\mathrm{lgen}})(n) - c(\dot{x}_{\mathrm{rgen}})(n) : n \in J\}$ has cardinality between l_0 and l_1. However, no condition in the reduced product can identify a countable ground model set of partitions such that one of them is guaranteed to work; this will contradict properness.

To proceed with this plan, first use the general definition of the reduced product forcing to see that the dovetailed pairs of Silver cubes are dense in it. On one hand, if B_g, B_h are dovetailed Silver cubes and $\pi : B_g \to B_h$ is the natural homeomorphism, then $B_g \Vdash \pi(\dot{x}_{gen}) \in B_h$ is a Silver generic real E-equivalent to $\dot{x}_{gen}$, and therefore the pair $\langle B_g, B_h \rangle$ is in the reduced product. On the other hand, if $\langle B, C \rangle \in P_I \times_E P_I$ is a pair of analytic sets, there are Silver cubes $B_g \subset B$ and $B_h \subset C$ such that the pair $\langle B_g, B_h \rangle$ is still in the reduced product. The large collapse then forces E-equivalent Silver generic reals $x \in B_g$ and $y \in B_h$; these points extend g, h respectively with infinitely many zeroes and ones, and by analytic absoluteness there must be E-equivalent points in B_g and B_h extending g, h in this fashion already in the ground model. It follows that there are dovetailed cubes $B_{g'} \subset B_g$ and $B_{h'} \subset B_h$ as required.

To see why Ψ is forced, let $\langle B_g, B_h \rangle \in P_I \times_E P_I$ be a pair of n-dovetailed cubes for some n, and let $k \in \omega$. Let $J_i : i \in \omega$ be successive finite intervals of natural numbers, each of them containing exactly $nk + 1$ many elements of $\omega \setminus \mathrm{dom}(g)$ and $\omega \setminus \mathrm{dom}(h)$. Extend g to g' by assigning value 1 to the first nk many elements of $J_i \setminus \mathrm{dom}(g)$ if i is even, and value 0 if i is even. Extend h to h' similarly, exchanging 0's for 1's. It is not difficult to see that the cubes $B_{g'}, B_{h'}$ are $n(k+1)$-dovetailed, and as a condition in the reduced product they force that for every $i \in \omega$, the set $\{c(\dot{x}_{\mathrm{lgen}})(m) - c(\dot{x}_{\mathrm{rgen}})(m) : m \in J_i\}$ has size between k and $n(k+1)$.

To see why the partitions cannot be enclosed into a ground model countable set, suppose that $\langle B_g, B_h \rangle$ is a pair of n-dovetailed Silver cubes, and suppose that A is a countable set of partitions. Find a partition $\vec{J} = \langle J_i : i \in \omega \rangle$ of ω into finite intervals such that for every partition in A, all but finitely many intervals in it meet at most two intervals in $\vec{J}$, and moreover, every interval in $\vec{J}$ contains numbers $m_0 \in m_1$, ??? □

There are also complementary canonization results regarding the spectrum of I:

Theorem 3.6.6. *If $B \in P_I$ is a Borel I-positive set and E is an equivalence relation on it classifiable by countable structures, then there is a Borel I-positive set $C \subset B$ such that either $E \restriction C = \mathrm{EE}$, or $E \restriction C \subset E_0$.*

Note that every equivalence relation which is a subset of E_0 is hyperfinite and as such reducible to E_0. The relation E on 2^ω defined by $x \mathrel{E} y$ if and only if the set $\{n \in \omega : x(n) \neq y(n)\}$ is finite and of even cardinality, happens to be an equivalence relation reducible to E_0 which is not equal to E_0 on any I-positive Borel set.

Proof. Suppose that $B \subset 2^\omega$ is a Borel I-positive set and E is an equivalence on it classifiable by countable structures. Since the Silver forcing extension contains a single minimal real degree, Corollary 4.3.10 implies that E simplifies to an essentially countable equivalence relation on a positive Borel set. Since the Silver forcing extension preserves Baire category, Theorem 4.2.7 shows that E restricted to a further I-positive subset is reducible to E_0. Now we can deal with this fairly simple situation separately.

Suppose that $B \in P_I$ is a Borel I-positive set, and E is an equivalence relation on B reducible to E_0 by a Borel function $f : B \to 2^\omega$. By standard homogeneity and bounding arguments, we may assume that $B = X$ and f is continuous. Suppose that E has no I-positive equivalence classes. We must produce a Borel set such that on it, $E \subset E_0$.

By induction on $n \in \omega$ build functions $g_n \in P$ and numbers m_n such that

- $g_0 \subset g_1 \subset \dots$, $m_0 \in m_1 \in \dots$ and $\forall n, k\ m_k \notin \mathrm{dom}(g_n)$;
- writing $a = \{m_k : k \in n\}$, for every $x \in B_{g_n}$ and functions $h_0, h_1 \in 2^a$, if the functions h_0, h_1 differ in at least l many values then the functions $f(x \oslash h_0)$, $f(x \oslash h_1)$ differ in at least l many values, and these differences are visible already when looking at $x \restriction m_n$.

If the induction has been performed, then choose a function $h \in P$, $\bigcup_n g_n \subset h$ and $\omega \setminus \mathrm{dom}(h) = \{m_n : n \in \omega\}$. The second item immediately implies that $E \restriction B_h \subset E_0$ as required.

To perform the induction, suppose that $a = \{m_k : k \in n\}$ as well as g_{n-1} have been found. Thin out the condition $B_{g_{n-1}}$ to some set C such that there is an equivalence relation F on 2^a such that for every $x \in C$, $h_0\ F\ h_1 \to f(x \oslash h_0) \restriction (m_{n-1}, \omega) = f(x \oslash h_1) \restriction (m_{n-1}, \omega)$ and $\neg h_0\ F\ h_1 \to f(x \oslash h_0) \restriction (m_{n-1}, m) \neq f(x \oslash h_1) \restriction (m_{n-1}, m)$. Choose representatives $h_j : j \in J$ for each F-equivalence class, and consider the functions $g_j : x \mapsto f(x \oslash h_j) \restriction m_{n-1}, \omega)$ for every number $j \in J$. The preimage of any singleton under any of these maps must be an I-small set–otherwise there would be an I-positive E-equivalence class. Use Claim 3.6.3 to find a function g_n and a number $m_n = \min(\omega \setminus \mathrm{dom}(g_n) > m$ such that for every index $j \in J$ and every point $x \in B_{g_n}$, $g_j(x) \neq g_j(x \odot m_n)$. The induction can proceed.

□

Theorem 3.6.7. *(Michal Doucha) Every equivalence relation reducible to E_2 canonizes to a subset of E_0 or* EE *on a Borel I-positive set.*

Proof. Let $f : B \to 2^\omega$ be a Borel function from a Borel I-positive set, and let E be the pullback equivalence relation $f^{-1}E_2$. we will canonize E on a Borel I-positive subset of B. By the usual homogeneity and fusion arguments we may assume that $2^\omega = B$ and f is a 1-Lipschitz function with the usual minimum difference metric on 2^ω.

We will need a little bit of notation. For distinct finite binary sequences $s, t \in 2^{<\omega}$ of the same length and a positive real $\varepsilon > 0$ we will write $sE_\varepsilon t$ if for every infinite sequence $x \in 2^\omega$, $d(f(s^\frown x), f(t^\frown x)) < \varepsilon$. We will also write $sE_2 t$ if for every infinite binary sequence $x \in 2^\omega$, $f(s^\frown x) E_2 f(t^\frown x)$. There are several cases.

In the first case, for every real $\varepsilon > 0$ the set $S_\varepsilon = \{s : \exists t \in 2^{<\omega}\ (s^\frown 0^\frown t) E_\varepsilon (s^\frown 1^\frown t)\} \subset 2^{<\omega}$ is dense. In this case we will find an I-positive Borel set of pairwise equivalent elements. By induction on $n \in \omega$ construct binary sequences s_n and sets $a_n \subset \mathrm{dom}(s_n)$ so that

- $s_0 \subset s_1 \subset \dots$, $a_0 \subset a_1 \subset \dots$, $|a_n| = n$;
- any two sequences u, v of the same length as s_n such that for every $i \in \mathrm{dom}(s_n) \setminus a_n$, $s_n(i) = u(i) = v(i)$, are $E_{1-2^{-n}}$ related.

If this succeeds then in the end let $g : \omega \to 2$ be the partial function with domain $\omega \setminus \bigcup_n a_n$ equal to s_n on its domain and note that every two points in B_g are E-related since their f-images have distance at most one. To perform the induction step, given a_n and s_n, extend s_n into an element $s \in S_{2^{-n-2}}$ with witness $t \in 2^{<\omega}$, let $s_{n+1} = s^\frown 1^\frown t$ and $a_{n+1} = a_n \cup \{|s|\}$. The triangle inequality applied to d will show that the induction hypothesis continues to hold.

In the second case, suppose that the first case fails and find a positive number $\varepsilon > 0$ and a sequence $\bar{s} \in 2^{<\omega}$ such that no extension of s is in S_ε, and let $S = \{s \supset \bar{s} : \exists t\ s^\frown 0^\frown t) E_2 (s^\frown 1^\frown t)\}$. The second case will occur when the set S is dense below $\bar{s}$. Here, we will find an I-positive Borel set such that on it $E = E_0$. By induction on $n \in \omega$ construct binary sequences s_n and sets $a_n \subset \mathrm{dom}(s_n)$ so that

- $s_0 \subset s_1 \subset \dots$, $a_0 \subset a_1 \subset \dots$, $|a_n| = n$;
- for any two sequences u, v of the same length as s_n such that for every $i \in \mathrm{dom}(s_n) \setminus a_n$, $s_n(i) = u(i) = v(i)$, and for every $x \in 2^\omega$, $d(f(u^\frown x) \restriction \mathrm{dom}(s_n), f(v^\frown x) \restriction \mathrm{dom}(s_n)) > \varepsilon/2 \cdot |\{i : u(i) \neq v(i)\}|$ and $d(f(u^\frown x) \setminus \mathrm{dom}(s_n), f(v^\frown x) \setminus \mathrm{dom}(s_n) < \varepsilon/4 \cdot (1 - 2^{-n})$.

If this succeds then let $g : \omega \to 2$ be the partial function with domain $\omega \setminus \bigcup_n a_n$ equal to s_n on its domain and observe that $E \restriction B_g = E_0$ by the second item. The induction step uses the Lipschitz condition on the function f. To perform the induction step, assume that s_n, a_n have been found. Find an extension $s \in S$ of s_n and a witness $t \in 2^{<\omega}$. Since $s \notin S_\varepsilon$, we can extend t if necessary so that for every point $x \in 2^\omega$, $d(f(s^\frown 0^\frown t^\frown x) \restriction |s| + |t| + 1, f(s^\frown 1^\frown t^\frown x) \restriction |s| + |t| + 1 > \varepsilon$. Extending t further if necessary we may arrange that for every point $x \in 2^\omega$, $d(f(s^\frown 0^\frown t^\frown x) \setminus |s| + |t| + 1, f(s^\frown 1^\frown t^\frown x) \setminus |s| + |t| + 1)) < \varepsilon/4 \cdot 2^{-n-3}$. It is not difficult to use the triangle inequality to show that $s_{n+1} = s^\frown 1^\frown t$ together with $a_{n+1} = a_n \cup \{|s|\}$ satisfies the induction hypothesis.

In the last case, the first and second case both fail, and so there is an extension $\bar{\bar{s}}$ such that none of its extensions belong to the set S. In this case, we will find a Borel I-positive set B such that on it the E-equivalence classes are countable and finish by the result of the previous section. To find the Borel set B, note that for every number $n \in \omega$ and every finite set a of sequences extending $\bar{\bar{s}}$ there is a sequence $t \in 2^{<\omega}$ such that for every $s \in a$, $\neg(s^\frown 0^\frown t) E_n (s^\frown 1^\frown t)$. This allows a standard fusion process to build a partial function $g : \omega \to 2$ such that for points $x_0, x_1 \in B_g$, $d(f(x_0), f(x_1)) >$minimal n such that $x_0(n) \neq x_1(n)$. Clearly, the equivalence classes of the relation $E \restriction B_g$ are at most countable, and the results of the previous subsection apply to yield the desired conclusion.

□

3.7 Milliken forcing

Let $(\omega)^\omega$ be the space of all infinite sequences a of finite sets of natural numbers with the property that $\min(a_{n+1}) > \max(a_n)$ for every number $n \in \omega$. Define the partial ordering P to consist of all pairs $p = \langle t_p, a_p \rangle$ where $t_p \subset \omega$ is a finite set and $a_p \in (\omega)^\omega$, ordered by $q \leq p$ if $t_p \subset t_q$ and t_q is the union of t_p with some sets on the sequence a_p, and all sets on a_q are unions of some sets on a_p. This partial ordering is folklorically well-known to resemble Miller forcing in many aspects. In this section, we will establish its main forcing and canonization properties.

Note that the poset P adds an infinite set $\dot{x}_{gen} \subset \omega$, the union of the first coordinates of the conditions in the generic filter. As always, the most important issue from the strategic point of view is the identification of the associated σ-ideal I on the space $[\omega]^{\aleph_0}$. For every condition $p \in P$ define $[p] \subset [\omega]^{\aleph_0}$ to be the collection of all sets which are the union of t_p with infinitely many sets on the sequence a_p, and let I be the collection of all analytic sets which do not contain a subset of the form $[p]$ where $p \in P$.

Theorem 3.7.1. *1. The collection I is a σ-ideal and the map $p \mapsto [p]$ is an isomorphism of P with a dense subset of P_I;*

2. *I is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$;*

3. *I fails to have the transversal property;*

4. *the forcing P is proper, preserves Baire category and outer Lebesgue measure, and it adds no independent reals. It adds an unbounded real.*

Proof. The proof starts with the identification of basic forcing properties of the poset P. The following notation and terminology will be helpful: $q \leq p$ is a *direct extension* of p if $t_q = t_p$. If $a \in (\omega)^\omega$ is a sequence and $m \in \omega$ then $a \setminus m$ is the sequence in $(\omega)^\omega$ obtained by erasing the first m entries on the sequence a. We will start with basic two claims.

Claim 3.7.2. *Let $O \subset P$ be an open dense set and $p \in P$ a condition. There is a direct extension $q \leq p$ such that for every $n \in \omega$, the condition $\langle t_q \cup a_q(n), a_q \setminus n \rangle$ belongs to the set O.*

Proof. Set up a simple fusion process to find a sequence $b \leq a$ such that for every finite set $u \subset \omega$, either $\langle t_p \cup \bigcup_{n \in u} a(n), b \setminus \max(u) \rangle \in O$ or else none of direct extensions of this condition belong to O. Applying Fact 1.3.22 to the partition $\pi : (\omega)^\omega \to 2$ given by $\pi(d) = 0$ if the former case holds for $d(0)$, we get a homogeneous sequence $c \subset b$. The homogeneous color cannot be 1 since the condition $\langle t_p, c \rangle$ has an extension in the open dense set O. If c is homogeneous in color 0 then it confirms the statement of the claim with $q = \langle t_p, c \rangle$. $\square$

Claim 3.7.3. *If $p \in P$ is a condition and $[p] = B_0 \cup B_1$ is a partition into two Borel sets then there is a direct extension $q \leq p$ such that $[q]$ is contained in one of the pieces of the partition.*

Proof. This is an immediate consequence of Fact 1.3.22. □

For the proof of properness of the poset P, let M be a countable elementary submodel of a large enough structure, let $p \in P \cap M$ be a condition. We must produce a master condition $q \leq p$. Enumerate the open dense sets of P in the model M by $\langle O_n : n \in \omega \rangle$ and use the claim and the elementarity of the model M repeatedly to build a sequence $p = p_0, p_1, p_2, \dots$ of direct extensions of p in the model M such that for every $n \in \omega$, $a_{p_n} \restriction n = a_{p_{n+1}} \restriction n$ and for every finite set $u \subset \omega$ with $\max(u) > n$ it is the case that the condition $\langle t_p \cup \bigcup_{m \in u} a_{p_{n+1}}(m), a_{p_{n+1}} \setminus \max(u) \rangle \in M$ belongs to the open dense set O_n. The limit $q \leq p$ of the sequence of conditions $\langle p_n : n \in \omega \rangle$ will be the required master condition.

In order to show that I is a σ-ideal, suppose that $\{B_n : n \in \omega\}$ are Borel sets in the collection I and suppose for contradiction that there is a condition $p \in P$ such that $[p] \subset \bigcup_n B_n$. Clearly $p \Vdash \dot{x}_{gen} \in \bigcup_n B_n$ and so, strengthening p if necessary, we may find a single natural number $n \in \omega$ such that $p \Vdash \dot{x}_{gen} \in \dot{B}_n$. Let M be a countable elementary submodel of a large enough structure, and let $q \leq p$ be an M-generic condition described in the previous proof. It is immediate that for every point $x \in [q]$ the conditions $\langle t_q \cup (q \cap \bigcup_{i \in n} a_q(i), a_q \setminus n \rangle : n \in \omega$ generate an M-generic filter $g_x \subset P \cap M$. By the forcing theorem applied in the model M, $M[x] \models x \in B_n$, and by Borel absoluteness, $x \in B_n$. Thus $[q] \subset B_n$, contradicting the assumption that $B_n \in I$.

In order to show that $p \mapsto [p]$ is an isomorphism between P and a dense subset of P_I, it is necessary to show that if $p, q \in P$ are conditions such that $[q] \subset [p]$ then in fact $q \leq p$. To simplify the discussion a bit, assume that $t_p = t_q = 0$. Suppose that $q \leq p$ fails, and find a number $n \in \omega$ such that $a_q(n)$ is not a union of sets on the sequence a_p. In such a case, either there is an element $i \in a_q(n)$ such that $i \notin \bigcup \mathrm{rng}(a)$, in which case no set $x \in [q]$ containing i can be in $[p]$; or else there is a number m such that $a_p(m) \cap a_q(n) \neq 0$ and $a_p(m) \setminus a_q(n) \neq 0$. In the latter case find a point $x \in [q]$ containing $a_q(n)$ as a subset and disjoint from the set $a_p(m) \setminus a_q(n)$; such a point cannot belong to $p]$ either. In both cases we have a contradiction with the assumption that $[q] \subset [p]$!

To show that the ideal I is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$, let $A \subset 2^\omega \times [\omega]^{\aleph_0}$ be an analytic set, a projection of a closed set $C \subset 2^\omega \times [\omega]^{\aleph_0} \times \omega^\omega$. We must show that the set $\{y \in 2^\omega : A_y \in I\}$ is coanalytic. Fix a point $y \in 2^\omega$ and suppose that $A_y \notin I$. This means that there is a condition $p \in P$ such that $[p] \subset A_y$, by Shoenfield absoluteness this will be true in the P-forcing extension as well, and so there must be a P-name τ for a point in ω^ω such that $p \Vdash \langle \check{y}, \dot{x}_{gen}, \tau \rangle \in \dot{C}$. By a fusion argument, there is a condition $r \leq p$ and a function $g : [\omega]^{<\aleph_0} \to \omega^{<\omega}$ such that the following holds. For every $b \in [\omega]^{<\aleph_0}$, write $r_b \leq r$ to be that condition whose trunk consists of t_r and the union of i-th elements of the sequence a_r for all $i \in b$, and $a_{r_b} = a_r \setminus \max(b) + 1$; we will have that for every $b \in [\omega]^{<\aleph_0}$, $\mathrm{dom}(g(b)) = \max(b) + 1$ and $r_b \Vdash g(b) \subset \tau$. In particular, the following formula $\phi(y, r, g)$ holds: the function g respects inclusion, $\mathrm{dom}(g(b)) = \max(b) + 1$ and the basic open neighborhood of the space $2^\omega \times [\omega]^{\aleph_0} \times \omega^\omega$ given by the first $\max(b)$

elements of y in the first coordinate, the trunk t_{r_b} in the second coordinate, and the sequence $g(b)$ in the third coordinate has nonempty intersection with the closed set C. It is also clear that $\psi(y, r, g)$ implies that $[r] \subset A_y$ since the function g yields the requisite points in the set C witnessing that elements of the set $[r]$ belong to A_y. We conclude that $A_y \notin I$ if and only if there is $r \in P$ and a function $g : [\omega]^{<\aleph_0} \to \omega^{<\omega}$ such that $\psi(y, r, g)$ holds, and this is an analytic statement.

For the continuous reading of names, let $p \in P$ and let $f : [p] \to \omega^\omega$ be a Borel function. Use Claim 3.7.2 to find a direct extension $q \leq p$ such that for every number $n \in \omega$ and every set $u \subset n + 1$ with $n \in u$ there is a number m such that the condition $\langle t_p \cup \bigcup_{i \in u} a_q(i), a_q \setminus n + 1\rangle$ forces $\dot{f}(\dot{x}_{gen})(\check{n}) = \check{m}$. Let M be a countable elementary submodel of a large structure and find a direct extension $r \leq q$ as in the previous argument. The forcing theorem will then show that $f \restriction [r]$ is a continuous function.

For the preservation of Baire category, there is a simple fusion-type proof. A more elegant argument will show that the ideal I is the intersection of a collection of meager ideals corresponding to Polish topologies on the Polish space $[\omega]^{\aleph_0}$ that yield the same Borel structure, and use the Kuratowski-Ulam theorem as in [49, Corollary 3.5.5]. This is the same as to show that for every condition $p \in P$ there is a Polish topology on $[p]$ such that all sets in the ideal I are meager with respect to it.

We will deal with the case of the largest condition $p \in P$; there, $t_p = 0$ and $a_p(n) = \{n\}$, so $[p] = [\omega]^{\aleph_0}$. We will show that every Borel nonmeager set in the usual topology on the space $[\omega]^{\aleph_0}$ is I-positive. Suppose that $B \subset [\omega]^{\aleph_0}$ is such a nonmeager set, containing the intersection $O \cap \bigcap O_n : n \in \omega$, where O is a nonempty basic open set and $O_n : n \in \omega$ are open dense sets. Let $r_0 \in 2^{<\omega}$ be a finite sequence such that whenever the characteristic function of a point $x \in [\omega]^{\aleph_0}$ contains r_0 then $x \in O$. By a simple induction on $n \in \omega$ build finite sequences $r_0 \subset r_1 \subset r_2 \subset \dots$ such that for every point $x \in [\omega]^{\aleph_0}$ whose characteristic function contains $r_{n+1} \setminus r_n$, $x \in \bigcap_{m \in n} O_m$ holds. In the end, let $t = \{i \in \mathrm{dom}(r_0) : r_0(i) = 1\}$ and let $a \in (\omega)^\omega$ be the sequence defined by $a(n) = \{i \in \mathrm{dom}(r_{n+1} \setminus r_n) : r_{n+1}(i) = 1\}$, and observe that $[t, a] \subset B$ and so $B \notin I$ as required.

For the preservation of outer Lebesgue measure, suppose that $A \subset 2^\omega$ is a set of outer Lebesgue mass $> \varepsilon + \delta$ and $p \in P$ is a condition forcing $\dot{O}$ to be a subset of 2^ω of mass $< \varepsilon$. We must find a condition $q \leq p$ and a point $z \in A$ such that $q \Vdash \check{z} \notin \dot{O}$.

Find positive real numbers $\delta_u : u \in [\omega]^{<\aleph_0}$ that sum up to less than δ, and find names $\dot{O}_j : j \in \omega$ for basic open sets whose union $\dot{O}$ is. Using Claim 3.7.2 and a simple fusion process, find a direct extension $q \leq p$ such that for every nonempty set $u \in [\omega]^{<\aleph_0}$, there is an open set P_u such that the condition $t_p \cup \bigcup_{i \in u} a_q(i) \Vdash \bigcup_{j \in \max(u)} \dot{O}_j \subset P_u \subset \dot{O}$ and the mass of $\dot{O} \setminus P_u$ is less than δ_u. Clearly, the set $R_0 = \bigcup\{P_u \setminus P_{u \cap \max(u)} : u \in [\omega]^{<\aleph_0}, |u| > 1\}$ has mass at most δ, and the set $R_1 = \limsup\{P_{\{n\}} : n \in \omega\}$ has size at most ε. Find a point $z \in A \setminus (R_0 \cup R_1)$, thin out the sequence q if necessary to make sure that

$z \notin P_{\{n\}}$ for any $n \in \omega$ and conclude that $q \Vdash \check{z} \notin \dot{O}$ as desired.

Not adding independent reals is always a somewhat more delicate issue. Suppose that $p \in P$ is a condition and $\dot{y}$ a name for an infinite binary sequence. A simple fusion process using Corollary 3.7.3 repeatedly at each stage yields a direct extension $\langle t_p, b\rangle \leq p$ such that for every $n \in \omega$ and every set $u \subset n$ the condition $\langle t_p \cup \bigcup_{i\in u} b(i), b \setminus n+1\rangle$ decides the value $\dot{y}(\min(b(n)))$. Define a partition $\pi : ([\omega]^{<\aleph_0})^2 \to 2$ by setting $\pi(u, v) = 0$ if the decision for u and $\min(b_{\min(v)})$ was 0. This is clearly a clopen partition and therefore there is a homogeneous sequence $d \in (\omega)^\omega$ by Fact 1.3.22. Let us say that the homogeneous color is 0. Consider the condition $q \leq p$ given by $t_q = t_p \cup \bigcup_{i\in d(0)} b(i)$, and $a_q(j) = \bigcup_{i\in d_{2j+1}} b(i)$. It is not difficult to use the homogeneity of the sequence d to show that the condition q forces $\dot{y} \restriction \{\min(b_{\min(d(2j))}) : j > 0\} = 0$.

Finally, P certainly adds an unbounded real: the increasing enumeration of the generic subset of ω cannot be bounded by any ground model function. The failure of the transversal property of the ideal I is also simple. Let $C \subset \mathcal{P}(\omega)$ be a perfect set consisting of pairwise almost disjoint sets, and consider the set $D \subset C \times [\omega]^{\aleph_0}$ consisting of all pairs $\langle c, x\rangle$ such that $c \in C$ and $x \subset c$. It is clear that D is a Borel set with I-positive vertical sections. If $B \subset [\omega]^{\aleph_0}$ is a Borel set covered by the vertical sections of the set D, visiting each vertical section in at most one point, it must be the case that the elements of B are pairwise almost disjoint, therefore B cannot contain a subset of the form $[p]$ for any $p \in P$, and so $B \in I$. □

The investigation of the spectrum of the ideal I begins with identifying the only obvious feature:

Theorem 3.7.4. *E_0 is in the spectrum of the σ-ideal I. The poset $P_I^{E_0}$ is regularly embedded in P_I, it is $\aleph_0$-distributive, and it yields the model $V[\dot{x}_{gen}]_{E_0}$.*

Proof. This is an immediate corollary of the work in Theorem 4.2.1 and the simple observation that E_0-saturations of I-small sets are still I-small.

As an additional piece of information, the equivalence class $[x_{\text{gen}}]_{E_0}$ is forced by P_I to be the only E_0 equivalence class in the intersection of all E_0-invariant sets in the generic filter. Suppose that p is a condition and $p \Vdash \dot{y} \subset \omega$ is an infinite set not E_0 related to $\dot{x}_{gen}$. Strengthening the condition p if necessary we may find a continuous function $f : [p] \to \mathcal{P}(\omega)$ such that $p \Vdash \dot{y} = \dot{f}(\dot{x}_{gen})$. We must find a condition $q \leq p$ such that no element of $[q]$ is E_0-equivalent to $f''[q]$; then clearly $q \Vdash \dot{y} \notin [q]_{E_0}$ and so the class $[\dot{y}]_{E_0}$ is not a subset of all E_0-invariant sets in the P_I-generic filter.

To produce the condition q, first use a standard fusion process to strengthen p so that for every finite set $u \in \omega$, the condition $\langle t_p \cup \bigcup_{n\in u} a_p(n), a_p \setminus \max(u)\rangle$ identifies some number $m_u > \max(a_p(\max(u) - 1))$ such that $m_u \in \dot{x}_{gen} \Delta \dot{y}$, and some number $k_u > \max(a_p(\max(u) - 1))$ such that $k_u \in \dot{y}$. Thinning out the sequence a_p if necessary, assume that the numbers $\{k_u, m_u : u \subset n\}$ are all smaller than $\min(a_p(n))$. It is now not difficult to verify that the thinned out condition q works as desired; that is, whenever $u, v \subset \omega$ are infinite sets, writing

$x_u = t_p \cup \bigcup_{n\in u} a_q(n))$ and $x_v = t_q \cup \bigcup_{n\in v} a_q(n)$ we must show $\neg f(x_u)\ E_0\ x_v$. For every $n \in u$, we must produce a number $k > \max(a_q(n-1))$ which belongs to the symmetric difference of $f(x_u)$ and x_v. There are two cases:

- either $n \notin v$. Then $k = k_{u\cap n+1}$ will work: as $n \in u$, $k \in f(x_u)$, while x_v contains no numbers in the interval $(\max(a_q(n-1)), \min(a_q(n+1)))$, where k belongs;
- or $n \in v$. In this case, $k = m_{u\cap n+1}$ will work. Either it is in the set $a_q(n)$, in which case it belongs to both x_u, x_v and does not belong to $f(x_u)$, or it is not in the set $a_q(n)$, in which case it does not belong to either x_u, x_v and does belong to $f(x_u)$.

The proof is complete. □

In the absence of any other tangible features of the spectrum, it is difficult to resist making a sweeping general conjecture.

Conjecture 3.7.5. Borel$\to_I \{\mathrm{ID}, \mathrm{EE}, E_0\}$.

This would strengthen the canonization results of Section **??**, since the characteristic functions of sets in any given set $[p]$ form a positive set with respect to the E_0-ideal investigated there. However, the treatment of the reduced product forcing seems to yield much less information in the present case, and we are limited to affirmative results in several particular cases only.

Theorem 3.7.6. $\leq E_{K_\sigma} \to_I \{\mathrm{ID}, \mathrm{EE}, E_0\}$.

In particular, I has total canonization for smooth equivalence relations and so the forcing P adds a minimal real degree.

Proof. Let $p \in P$ be a condition and $E \leq E_{K_\sigma}$ be a Borel equivalence relation on $[p]$, and let $f : [p] \to \omega^\omega$ below the identity be a Borel function witnessing the reducibility. To simplify the notation assume that $t_p = 0$ and the function f is continuous. Assume that E has I-small equivalence classes and proceed to find an I-positive Borel set on which the equivalence E is either equal to ID or to E_0. We will first show that there is an I-positive Borel set on which $E \subset E_0$ and then handle that fairly simple case separately.

We will start with a sequence of simple considerations. If $t \in [\omega]^{<\aleph_0}$, $n \in \omega$ and $a \in (\omega)^\omega$, we will say that a is t, n-separated if there is a map $g : \omega \to \omega^{<\omega}$ such that for every $i < j$ it is the case that $\mathrm{dom}(g(i)) \subset \mathrm{dom}(g(j))$ and there is $k \in \omega$ such that $|g(i)(k) - g(j)(k)| > n$, and moreover, for every $x \in [t_q^\frown a(i), a \setminus i+1]$ it is the case that $g(i) \subset f(x)$.

Claim 3.7.7. *Whenever $t \in [\omega]^{<\aleph_0}$, $n \in \omega$ and $a \leq a_p$ then there is $b \leq a$ such that b is t, n-separated.*

Proof. Note that in every Borel positive set there are two inequivalent elements. This observation means that there are finite sets $u(m, j) : m \in \omega, j \in 2$ and finite sequences of natural numbers $g(m, j) : m \in \omega, j \in 2$ so that

- all numbers in $u(m,j)$ are smaller than all numbers in $u(m+1,k)$ no matter what $j,k \in 2$ are;
- the two sequences $g(m+1,j) : j \in 2$ have the same domain larger than the common domain of $g(m,j)$, they are equal on $\mathrm{dom}(g(m,j))$ and there is i such that $|g(m+1,0) - g(m+1,1)| > 2n$;
- for all $x \in [t_p \cup \bigcup_{i \in u(m,j)} a(i), a \setminus \max(u(m,j))+1]$, $g(m,j) \subset f(x)$.

Find an infinite set $v \subset \omega$ such that the sequence $g(m+1,j) \restriction \mathrm{dom}(m,j) : m \in v$ converges to some $z \in \omega^\omega$ below the identity. For every number $m \in v$ find $j_m \in 2$ such that there is $i \in \mathrm{dom}(g(m+1,j_m)$ such that $|g(m+1,j_m)(i) - z(i)| > n$. Thinning out v even further, we can achieve now that for every $m < m' \in v$ there is $i \in \mathrm{dom}(g(m+1,j_m))$ such that $|g(m+1,j_m)(i) - g(m'+1,j_{m'})(i)| > n$. The sequence $b \in (\omega)^\omega$ given by $b(m) = \bigcup_{i \in u(m+1,j_m)} a(i)$ is n separated as required. $\square$

If $t,s \in [\omega]^{<\aleph_0}$, $n \in \omega$ and $a \in (\omega)^\omega$, say that a is t,s,n-separated if there are maps $g_0, g_1 : \omega \to \omega^{<\omega}$ such that for every $i < j$ it is the case that $\mathrm{dom}(g_0(i)) \subset \mathrm{dom}(g_1(j))$ and there is $k \in \mathrm{dom}(g_0)$ such that $|g_0(i)(k) - g_1(j)(k)| > n$, and moreover, for every $x \in [t^\frown a(i), b \setminus i+1]$ it is the case that $g_0(i) \subset f(x)$, and for every $x \in [s^\frown a(j), b \setminus j+1]$ it is the case that $g_0(i) \subset f(x)$.

Claim 3.7.8. *Whenever $t,s \in [\omega]^{<\aleph_0}$, $n \in \omega$ and $a \leq a_p$, there is $b \leq a$ which is s,t,n-separated.*

Proof. Use the previous claim to find b which is both s,n-separated and t,n-separated and then thin out if necessary. $\square$

Now, by a simple fusion process using the claim repeatedly at each step we can find a sequence $b \leq a_p$ such that for every number n and sets $u,v \subset n$, the sequence $b \setminus n+1$ is s,t,n-separated where $s = t_p \cup \bigcup_{i \in u} b(i)$ and $t = t_p \cup \bigcup_{i \in v} b(i)$. It is immediate then that $E \restriction [t_p, b] \subset E_0$.

Now suppose that there is no I-positive Borel subset of $[t_p, b]$ such that $E = E_0$ on it and work to find a Borel positive set of pairwise inequivalent elements. If $t,s \in 2^{<\omega}$ and $a \in (\omega)^\omega$, say that a is s,t-discrete if for every finite set $v \subset \omega$ and every point $x \in [0, a \setminus \max(v)+1]$ the points $s \cup x$, $t \cup \bigcup_{i \in v} a(i) \cup x$ are inequivalent.

Claim 3.7.9. *Whenever $t,s \in [\omega]^{<\aleph_0}$ and $a \in (\omega)^\omega$ then there is $c \leq a$ which is s,t-discrete.*

Proof. By a simple fusion process one can strengthen the sequence a such that for every finite set $u \subset \omega$, one of the following happens:

- for every $x \in [0, a \setminus \max(u)+1]$, $s \cup x$ is equivalent with $t \cup \bigcup_{i \in u} a(i) \cup x$;
- for every such x, the two points are inequivalent.

Let $\pi : [\omega]^{<\aleph_0} \to 2$ be defined by $\pi(u) = 0$ if the first item above holds. Fact 1.3.22 yields a homogeneous sequence c. Observe that the homogeneous color cannot be 0. In such a case $E \restriction [t,d] = E_0$: whenever $y, z \in [t,d]$ are E_0-equivalent points, we can find a number m such that $y \setminus m = z \setminus m$ and then both y, z are equivalent to $s \cup (y \setminus m)$ by the homogeneity of the set d. This contradicts our assumption that there is no positive Borel set on which $E = E_0$. If the homogeneous color is 1, then c is easily seen to be s,t-discrete. $\square$

By a simple fusion process (inducing on n and using the previous claim at each step of the induction repeatedly), build a sequence $c \le b$ such that for every $n \in \omega$ and sets $u, v \subset n$, $c \setminus n$ is s,t-discrete where $s = t_p \cup \bigcup_{i\in u} c(i)$ and $t = t_p \cup \bigcup_{i \in v} c(i)$. We claim that in the end $E \restriction [q] = \mathrm{ID}$. Indeed, if $x \neq y \in [q]$ are two distinct E_0 equivalent points, then there is the largest number m such that $a_q(m) \subset x$ and $a_q(m) \cap y = 0$ or vice versa. The definition of discreteness then immediately shows that $\neg x \mathrel{E} y$ as required. The claim follows!

$\square$

Theorem 3.7.10. *Classifiable by countable structures*$\to_I \{\mathrm{ID}, \mathrm{EE}, E_0\}$.

Proof. This follows by modus ponens from previous results. The forcing P_I adds a minimal real degree as per the previous theorem. Corollary 4.3.10 then shows that every equivalence relation calssifiable by countable structures simplifies to an essentially countable one on a Borel I-positive set, which then is reducible to E_{K_σ}, and by the previous theorem simplifies to either ID, EE, or E_0 on a further Borel I-positive subset. $\square$

3.8 Infinite product of Sacks forcing

The infinite product of Sacks forcing is the poset P of all ω-sequences $\vec{T}$ of perfect binary trees, ordered by coordinatewise inclusion. The computation of the associated ideal yields a complete information. Let I be the collection of those Borel subsets of $X = (2^\omega)^\omega$ which do not contain a product $\Pi_n C_n$ of nonempty perfect subsets of 2^ω. The following Fact is a conjunction of the rectangular property of Sacks forcing [49, Theorem 5.2.6] and [49, Proposition 2.1.6].

Fact 3.8.1. *I is a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal of Borel sets. Every positive analytic set contains a Borel positive subset.*

Thus, the function $\pi : P \to P_I$ defined by $\pi : \vec{T} \mapsto \Pi\vec{T}$ is a dense embedding of P to P_I, where $\Pi\vec{T}$ denotes the product $\Pi_n[\vec{T}(n)]$.

The spectrum of the ideal I is very complicated. Already the understanding of smooth equivalence relations (i.e. names for reals) seems to be out of reach. We will include several basic results. First, consider the obvious equivalence E_1 on X.

Theorem 3.8.2. *E_1 is in the spectrum of I. Moreover,*

1. *the poset $P_I^{E_1}$ is regular in P_I, it is $\aleph_0$-distributive, and its extension is equal to $V[\vec{x}_{gen}]_{E_1}$;*

2. *the reduced product $P_I \times_{E_1} P_I$ is proper and adds a dominating real and a Cohen real;*

3. *$V[[\vec{x}_{gen}]]_{E_1} = V[\vec{x}_{gen}]_{E_1}$.*

Proof. To see that E_1 is indeed in the spectrum, if $B \subset X$ is an I-positive set, use Fact 3.8.1 to find a sequence $\vec{T} \in P$ of perfect trees such that $\Pi\vec{T} \subset B$ and let $\pi_n : 2^\omega \to [\vec{T}(n)]$ be a homeomorphism for every number n. Then $\pi : \vec{x} \mapsto \langle \pi_n \vec{x}(n) : n \in \omega\rangle$ is a Borel reduction of E_1 to $E_1 \restriction B$.

To prove that $P_I^{E_1}$ is a regular $\aleph_0$-distributive subposet of P_I, suppose that B is a Borel I-positive set, and thinning it out if necessary assume that it is a product of countably many perfect sets, $B = \Pi_n C_n$. We will show that $[B]_{E_1}$ is a pseudoprojection of P_I into $P_I^{E_1}$. For that, suppose that $\Pi_n D_n \subset [B]_{E_1}$ is a positive set below the saturation of B. We must show that the set $B \cap [\Pi_n D_n]_{E_1}$ is I-positive. Indeed, since $\Pi_n D_n \subset [B]_{E_1}$, for all but finitely many numbers $n \in \omega$, say for all $n > n_0$, it must be the case that $D_n \subset C_n$. Then the product $\Pi_{n \leq n_0} C_n \times \Pi_{n > n_0} D_n$ is the required I-positive subset of $B \cap [\Pi_n D_n]_{E_1}$. The distributivity of the poset $P_I^{E_1}$ follows immediately from the I-density of the equivalence E_1 and Proposition 2.2.6.

To show that the poset $P_I^{E_1}$ yields the model $V[\dot{x}_{gen}]_{E_1}$, we must show that for analytic I-positive sets $B, C \subset X$, either there is I-positive Borel $B' \subset B$ such that $[B']_{E_1} \cap C \in I$, or there is an I-positive Borel $B' \subset B$ and an I-preserving Borel injection $f : B' \to C$ with $f \subset E_1$. After that, Theorem 2.2.7 completes the proof.

Thus, fix the sets B, C and find a product $\Pi\vec{T} \subset B$. If $[\Pi\vec{T}]_{E_1} \cap C \in I$, then we are done, otherwise find a product $\Pi\vec{S} \subset [\Pi\vec{T}]_{E_1} \cap C$. There must be a number $n_0 \in \omega$ such that $\forall n \geq n_0\ \vec{S}(n) \subset \vec{T}(n)$. Consider the set $B' = \Pi_{n \in n_0} \vec{T}(n) \times \Pi_{n \geq n_0} \vec{S}(n)$, fix homeomorphisms $\pi_n : [\vec{T}(n) \to \vec{S}(n)$ for every $n \in n_0$, and consider the map $f : B' \to C$ defined by $f(\vec{x})(n) = \pi_n(\vec{x})(n)$ if $n \in n_0$ and $f(\vec{x})(n) = \vec{x}(n)$ for $n \geq n_0$. It is not difficult to see that the function f is a Borel injection with the required properties. As an interesting aside, note that in $V[G]$, $[\vec{x}_{gen}]_{E_1}$ is the only equivalence class contained in all sets in the filter K and so K and $[\vec{x}_{gen}]_{E_1}$ are interdefinable elements. This is to say, in the ground model, if $B \subset X$ is an I-positive Borel set and $f : B \to X$ is a Borel function such that $\forall x \in B\ \neg f(x)\ E_1\ x$, then there is an I-positive Borel set $C \subset B$ such that $[C]_{E_1} \cap f''C = 0$. Since the poset P_I is bounding, there are increasing functions $g, h \in \omega^\omega$ and an I-positive Borel set $C \subset B$ such that for every point $x \in C$ and every number $i \in \omega$ there are numbers $n \in [g(i), g(i+1))$ and $k \in h(i)$ such that $f(x)(n)(k) \neq x(n)(k)$. Thinning the set C further if necessary we may assume that $C = \Pi_n [T_n]$ for some perfect binary trees T_n such that whenever $n \in [g(i), g(i+1))$ then the first splitnode of the tree T_n is past the level $h(i)$. It is not difficult to see that such a set $C \subset B$ works as desired.

For the computation of the reduced product forcing, suppose that $\vec{S}, \vec{T} \in P$ are sequences of trees such that in some generic extension, $\Pi\vec{S}$ and $\Pi\vec{T}$ contain E_1-related generic sequences of Sacks reals. It is clear then that for all but finitely many $n \in \omega$, the intersection $\vec{T}(n) \cap \vec{S}(n)$ must contain a perfect tree: for all but finitely many $n \in \omega$, the n-th entry on the E_1-equivalent generic sequences must be a branch through both $\vec{T}(n)$ and $\vec{S}(n)$ and so the set $[\vec{T}(n)] \cap [\vec{S}(n)] \in V$ must not be countable. It follows that the collection of all pairs $\langle \vec{T}, \vec{S} \rangle$ such that for all but finitely many $n \in \omega$, $\vec{T}(n) = \vec{S}(n)$ holds, is dense in the reduced product $P_I \times_{E_1} P_I$.

To prove the properness of the reduced product, let M be a countable elementary submodel of a large structure, let $\langle \vec{T}, \vec{S} \rangle \in M$ be a condition in the reduced product. By the previous paragraph we may assume that $\vec{T} = \vec{S}$ on all but finitely many entries, and to simplify the notation, we will assume that in fact $\vec{S} = \vec{T}$. To construct the master condition for the model M, enumerate the open dense subsets of the reduced product in the model M with repetitions by $D_n : n \in \omega$, and by a simple induction produce sequences $\vec{T}_n \in P \cap M$ such that

- the first n splitnodes of the first n trees on the sequence $\vec{T}_n$ also belong to the trees on the sequence $\vec{T}_{n+1}$;
- for every two sequences $\vec{u}, \vec{v} \in (2^{<\omega})^n$ picking one node at $n+1$-th splitting level for all trees $\vec{T}_{n+1}(i) : i \in n$, if there is a condition $\langle \vec{U}, \vec{V} \rangle \leq \langle \vec{T}_{n+1} \restriction \vec{u}, \langle \vec{T}_{n+1} \restriction \vec{v} \rangle \in D_n$ such that for every $i \in n$, $\vec{U}(i) \subset \vec{T}_{n+1}(i) \restriction \vec{u}(i)$, $\vec{V}(i) \subset \vec{T}_{n+1}(i) \restriction \vec{v}(i) \wedge \vec{u}(i) =$ and if $\vec{u}(i) = \vec{v}(i)$ then $\vec{U}(i) = \vec{V}(i)$, and for all $i \geq n$, $\vec{U}(i) = \vec{V}(i)$, then $\langle \vec{U}, \vec{V} \rangle \leq \langle \vec{T}_{n+1} \restriction \vec{u}, \langle \vec{T}_{n+1} \restriction \vec{v} \rangle$ is such a condition.

In the end, the sequence $\vec{W}$ which is in the natural sense the limit of $\langle \vec{T}_n : n \in \omega$ will be a condition in the poset P by the first item, and the condition $\langle \vec{W}, \vec{W} \rangle$ will be the desired master condition for the model M by the second item. For if $n \in \omega$ is a number and $\langle \vec{U}, \vec{V} \rangle \leq \langle \vec{W}, \vec{W} \rangle$ is a condition, then thin it out if necessary to fall into the open dense set D_n, and for some $m > n$ such that $D_m = D_n$ it is the case that for all $i \geq m$, $\vec{U}(i) = \vec{V}(i)$ holds, and for all $i \in m$, there is a unique node $\vec{u}(i)$ at $m+1$-th splitting level of $\vec{W}(i)$ such that $\vec{U}(i) \subset \vec{W}(i) \restriction \vec{u}(i)$, also a unique node $\vec{v}(i)$ at that level such that $\vec{V}(i) \subset \vec{W}(i) \restriction \vec{v}(i)$, and if $\vec{u}(i) = \vec{v}(i)$ then $\vec{U}(i) = \vec{V}(i)$. The second item of the induction hypothesis then implies that the condition $\langle \vec{T}_{m+1} \restriction \vec{u}, \vec{T}_{m+1} \restriction \vec{v} \rangle$ is a condition in $D_n \cap M$ compatible with $\langle \vec{U}, \vec{V} \rangle$ as desired.

The reduced product adds a dominating real $\dot{f} \in \omega^\omega$ defined as $\dot{f}(n) =$the minimal number k such that $\vec{x}_{lgen}(n)(k) \neq \vec{x}_{rgen}(n)(k)$. The Cohen real is then read off as $\vec{x}_{lgen} \circ \dot{f}$.

To show that $V[\vec{x}_{gen}]_{E_1} = V[[\vec{x}_{gen}]]_{E_1}$, suppose that $\vec{T} \in P$ is a condition forcing $\sigma \in V[[\vec{x}_{gen}]]_{E_1}$ and σ is a set of ordinals; it will be enough to find a stronger condition $\vec{S}$ forcing $\sigma \in V[\vec{x}_{gen}]_{E_1}$. By induction on n build a decreasing sequence of trees $\vec{T}_n \leq \vec{T}$ and names condition $\vec{S} \leq \vec{T}$ and names σ_n for the product of the Sacks reals indexed by numbers larger than n such that

- $\vec{T}_0 = \vec{T}$, $\sigma_0 = \sigma$ and $\vec{T}_n \restriction n = \vec{T}_{n+1} \restriction n$;
- $\vec{T}_{n+1} \Vdash \sigma/\vec{x}_{gen} = \sigma_{n+1}/\vec{x}_{gen} \restriction (n, \omega)$.

To see how this is done, suppose that $\vec{T}_n, \sigma_n$ are known. Now $\vec{T}_n \restriction [n, \omega)$ forces in the tail of the product that $\tau_n \in V[[\vec{x}_{gen} \restriction [n, \omega)]]_{E_1}$ and therefore there must be a tree $S \subset \vec{T}_n(n+1)$ and a sequence $\vec{U} \leq \vec{T}_n \restriction [n+1, \omega)$ and a name σ_{n+1} such that in the tail of the product, $(S, \vec{U}) \Vdash \sigma_n/\vec{x}_{gen} \restriction [n, \omega) = \sigma_{n+1}\vec{x}_{gen} \restriction [n+1, \omega)$. The name σ_{n+1} together with the sequence $\vec{T}_{n+1} = \vec{T}_n \restriction n^\frown S^\frown \vec{U}$ will work.

In the end, the natural limit of the sequences $\vec{T}_n$ forces that σ is definable from parameters in V and the equivalence class $[\vec{x}_{gen}]_{E_1}$ as the eventual stable value of the sequence $\langle \sigma_n/\vec{y} \restriction [n, \omega) : n \in \omega \rangle$ for any $\vec{y} \in [\vec{x}_{gen}]_{E_1}$. $\square$

Theorem 3.8.3. *The spectrum of I is cofinal among the Borel equivalence relations under $\leq_B$. It contains E_{K_σ}.*

Proof. Let J be a Borel ideal on ω, then the equivalence relation E_J, the modulo J equality of sequences in $(2^\omega)^\omega$ is a Borel equivalence, it is in the spectrum of the ideal I by the same reason as E_1, and the equalities of this type are cofinal in $\leq_B$ by Fact 1.3.15. There is an F_σ ideal J such that equality modulo J is bireducible with E_{K_σ}, and so E_{K_σ} is in the spectrum of I as well. $\square$

The equivalences from the previous proof are never reducible to orbit equivalences of Polish group actions, which leaves open the possibility of a strong canonization theorem for I and orbit equivalences. We will prove something weaker. The proofs transfer to infinite products of other forcings in place of Sacks–for example the bounding forcings generated by $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideals of closed sets.

In order to avoid the nightmarish and repetitive fusion arguments, the following definitions will be useful.

Definition 3.8.4. if $a \subset \omega$ is a finite set then ID_a is the equivalence relation on $(2^\omega)^\omega$ connecting two sequences if they agree on all entries outside of the set a.

Definition 3.8.5. if $\vec{S} \leq \vec{T}$ are two sequences of trees in P and k is a number, we write $\vec{S} \leq_k \vec{T}$ if every node at k-th splitting level of a tree $\vec{T}(i)$ for some $i \in k$ still belongs to the tree $\vec{S}(i)$.

It is not difficult to see that if $\vec{T}_0 \geq_0 \vec{T}_1 \geq_1 \vec{T}_2 \geq_2 \dots$ is a decreasing sequence of trees then its coordinatewise intersection is still a sequence of trees in the poset P.

Definition 3.8.6. Let M be a countable elementary submodel of a large structure. A condition $\vec{T} \in P$ is *strong master condition* for M if

1. the set $\Pi\vec{T}$ consists of M-generic sequences, and every subset of the space $(2^\omega)^\omega$ in the model M is relatively open in it;
2. whenever $a \subset \omega$ is a finite set and t_i^0, t_i^1 are incomparable nodes of $\vec{T}(i)$ for every number $i \in a$, then the set $\Pi_{i\in a}[\vec{T} \restriction t_0^i] \times \Pi_{i\in a}[\vec{T} \restriction t_1^i] \times \Pi_{i\notin a}[\vec{T}(i)]$ consists of M-generic sequences of Sacks product reals, and every subset of the product space in the model M is relatively open in it.

Every condition in the model M can be thinned out to a strong master condition by a straightforward fusion argument, and further strengthening of a strong master condition is still strong master.

Theorem 3.8.7. *Equivalences classifiable by countable structures*$\to_I$*smooth.*

Proof. We will show that the product of Sacks forcing has the separation property of Definition 4.3.4. The proof is then concluded by a reference to Theorem 4.3.5.

So suppose that $\vec{T} \in P$ is a condition forcing $\dot{y} \in 2^\omega$, $\dot{a} \subset 2^\omega$ is a countable set not containing $\dot{y}$. We must produce a ground model coded Borel set A and a condition $\vec{S}$ such that $\vec{S} \Vdash A$ separates $\dot{y}$ from $\dot{a}$. Thinning out the trees on the sequence $\vec{T}$ if necessary, find continuous functions $f, g_n : \Pi\vec{T} \to 2^\omega$ for all $n \in \omega$ such that $\vec{T} \Vdash \dot{y} = \dot{f}(\vec{x}_{gen})$ and $\dot{a} = \{\dot{g}_n(\vec{x}_{gen}) : n \in \omega\}$ and for all $\vec{x} \in \Pi\vec{T}$, $f(\vec{x}) \notin \{g_n(\vec{x}) : n \in \omega\}$. Let M be a countable elementary submodel of a large enough structure containing all the above information, and thin out $\vec{T}$ to a strong master condition for M.

By induction on $k \in \omega$, build conditions $\vec{T}_k \in P$ and clopen sets $U_k \subset 2^\omega$ so that

- $\vec{T}_k$ form a decreasing fusion sequence in P. That is, $\vec{T}_{k+1} \leq_k \vec{T}_k$;
- $f''\Pi\vec{T}_{k+1} \subset U_{k+1}$ and $g_k''\Pi\vec{T}_{k+1} \cap U_{k+1} = 0$;
- for every number $n \in \omega$ and all pairs $\vec{x}, \vec{y} \in \Pi\vec{T}_k$, if $\vec{x}\ \mathrm{ID}_k\ \vec{y}$ then $f(\vec{x}) \neq g_n(\vec{y})$.

Once this construction is complete, the coordinatewise intersection $\vec{S}$ of the conditions $\vec{T}_k$ will be a condition in the poset P by the first item. Writing $U = \bigcap_k U_k$, $\vec{S}$ forces that the set $\dot{U}$ separates $\dot{y}$ from the set $\dot{a}$ by the second item. The third item is there just to keep the induction going.

Start the induction with $\vec{T} = \vec{T}_0$ and $U_0 = \{0\}$. Suppose that $\vec{T}_k, U_k$ have been constructed. Thin out the sequence $\vec{T}_k$ to $\vec{T}'_k \leq_k \vec{T}_k$ so that for every choice $\vec{t}$ of splitnodes on k-th splitting level from the trees $\vec{T}'_k(j) : j \in k$, if there is some condition below $\vec{T}'_k \restriction \vec{t}$ on which the value of f does not depend on the k-th coordinate of the input, then already $\vec{T}'_k \restriction \vec{t}$ is such a condition. Note that this thinned out sequence $\vec{T}'_k$ already satisfies the third item for $k+1$. For suppose that $\vec{x}, \vec{y} \in \Pi\vec{T}'_k$ are such that $\vec{x}\ \mathrm{ID}_{k+1}\ \vec{y}$ and $f(\vec{x}) = g_n(\vec{y})$. Note that $\vec{x}(k) \neq \vec{y}(k)$ by the third item of the induction hypothesis. Use the second item of strong master property of $\vec{T}'_k$ to conclude that there is a condition below $\vec{T}'_k$ containing $\vec{x}$ such that the values of the function f on this condition do not depend on the k-th coordinate of the input. The choice of $\vec{T}'_k$ then implies that $f(\vec{x}) = f(\vec{x}')$ where $\vec{x}'$ is obtained from $\vec{x}(k)$ by replacing $\vec{x}(k)$ with $\vec{y}(k)$. But then, $\vec{x}', \vec{y}$ contradict the third item of the induction hypothesis at k.

Now, pick a sequence $\vec{y} \in \Pi_{l>k}[\vec{T}'_k(l)]$. Let $R = \Pi_{l\leq k}\vec{T}'_k(l) \times \{y\}$. The induction assumption at k implies that $f''R \cap g''_k R = 0$. These two sets are compact, disjoint, and therefore separated by a clopen set U_{k+1}. By the first item of the strong master property, there is a sequence $\vec{T}_{k+1} \leq_k \vec{T}'_k$ which satisfies the first item of the induction hypothesis and such that $f''\Pi\vec{T}_{k+1} \subset U_{k+1}$ and $g''_k\Pi\vec{T}_{k+1} \cap U_{k+1} = 0$ as required in the second item. This completes the inductive step. ☐

Theorem 3.8.8. $\leq_B E_2 \to_I$ *smooth.*

The proof transfers without change to other equivalence relations E on 2^ω defined by $x\ E\ y \leftrightarrow \{n \in \omega : x(n) \neq y(n)\} \in K$, where K is an F_σ P-ideal, using a theorem of Solecki [42] to find a suitable submeasure.

Proof. Suppose that $\vec{T} \in P$ is a sequence of perfect trees and F on $\Pi\vec{T}$ is an equivalence relation reducible to E_2 by a Borel function $f : \Pi\vec{T} \to 2^\omega$. We will construct a condition $\vec{S} \leq \vec{T}$ such that on it, F is reducible to E_0. A reference to the previous theorem will complete the proof. The reduction to E_0 will take a quite specific form. There will be numbers $\{m_k : k \in \omega\}$ such that for every $\vec{x}, \vec{y} \in \Pi\vec{S}$ the real number $d_k(\vec{x}, \vec{y}) = \Sigma\{1/m+1 : m_k \leq m < m_{k+1}, f(\vec{x})(m) \neq f(\vec{y})(m)\}$ is either smaller than 2^{-k-2} or else larger than 1. This means that the relation G_k connecting two sequences $\vec{x}, \vec{y} \in \Pi\vec{S}$ if $d_k(\vec{x}, \vec{y}) \leq 2^{-k}$ is an equivalence relation with finitely many classes. Moreover, $\vec{x}, \vec{y}$ are F-connected if and only if they are G_k connected for all but finitely many numbers k. Thus the function assigning a sequence $\vec{x}$ its sequence of G_k-equivalence classes for $k \in \omega$ reduces the equivalence $F \restriction \Pi\vec{S}$ to a hyperfinite equivalence relation.

Let M be a countable elementary submodel of a large enough structure containing the equivalence F, and thin out $\vec{T}$ to a strong master condition for M.

Claim 3.8.9. *On the product $\Pi\vec{T}$, for every number $n \in \omega$, the equivalence relation $F \cap \mathrm{ID}_n$ has a Borel selector.*

Proof. First show that the complement of the equivalence relation $F \cap \mathrm{ID}_n$ is F_σ by the second item of the strong master definition, and for the same reason the equivalence relation $F \cap \mathrm{ID}_n$ is F_σ itself. So it is a G_δ equivalence relation, therefore smooth, and has K_σ equivalence classes, therefore a Borel selector exists. ☐

Let $\{g_n : n \in \omega\}$ be the Borel selectors indicated by this claim. By induction on $k \in \omega$ build sequences $\vec{T}_k$ of trees and numbers $m_k \in \omega$ so that

- $\vec{T}_k$ form a descending fusion sequence of trees below $\vec{T}$. That is, $\vec{T}_{k+1} \leq \vec{T}_k$ and the k-th splitting level of all trees $\vec{T}_k(i) : i \in k$ is included in the corresponding tree $\vec{T}_{k+1}(i)$;
- the numbers $d_k(\vec{x}, \vec{y})$ are either smaller than 2^{-k-2} or else larger than 1 for all sequences $\vec{x}, \vec{y} \in \Pi\vec{T}_k$;
- for all $\vec{x} \in \Pi\vec{T}_k$, the number $\Sigma\{1/m{+}1 : m \geq m_k, f(\vec{x})(m) \neq f(g_k(\vec{x}))(m)\}$ is smaller than 2^{-k-3}.

The first two items show that the fusion $\vec{S}$ of the sequence $\vec{T}_k : k \in \omega$ will have the required properties. The third item is present just to keep the induction going.

Now suppose that $\vec{T}_k, m_k$ have been found so that the first and third item are satisfied. There is a strengthening $\vec{T}'_k \leq \vec{T}_k$ and a number $m'_k > m_k$ such that for every $\vec{x} \in \Pi\vec{T}'_k$, the number $\Sigma\{1/m{+}1 : m \geq m'_k, f(\vec{x})(m) \neq f(g_{k+1}(\vec{x}))(m)\}$ is smaller than 2^{-k-4}. Note that no matter what $\vec{T}_{k+1} \leq_k \vec{T}'_k$ and $m_{k+1} > m'_k$ we produce, the first and third item of the induction hypothesis will be satisfied, so it is just enough to concentrate on the second item.

For every $i \in k$ and every node t on the k-th splitting level of the tree $\vec{T}'_k$, choose a branch x^t_i through the tree $\vec{T}'_k(i) \restriction t$. Choose also a sequence $\vec{y} \in \Pi_{i \geq k}\vec{T}'_k(i)$. Let $Z = \{\vec{t} : \vec{t}$ is a sequence of length k picking a node at k-th splitting level of the tree $\vec{T}'_k(i)$ for every $i \in k\}$. For every sequence $\vec{x} \in Z$ let $\vec{x}_{\vec{t}}$ be the concatenation $\langle x_i^{\vec{t}(i)} : i \in k\rangle^\frown\vec{y}$, and for sequences $\vec{t}, \vec{s} \in Z$ let $e(\vec{s}, \vec{t})$ be the sum $\Sigma\{1/m + 1 : m_k \geq m, f(\vec{x}_{\vec{t}})(m) \neq f(\vec{x}_{\vec{s}})(m)\}$. The critical observation: if the sequences $\vec{x}_{\vec{t}}$ and $\vec{x}_{\vec{s}}$ are F-connected, then the number $e(\vec{t}, \vec{s})$ cannot be larger than 2^{-k-2} by the third item of the induction hypothesis, since in this case $g_k(\vec{x}_{\vec{s}}) = g_k(\vec{x}_{\vec{t}})$. On the other hand, if the sequences are not F-connected then the sum $e(\vec{t}, \vec{s})$ is infinite. Thus, there is a number $m_{k+1} > m_k$ such that all pairs of sequences among $\{\vec{x}_{\vec{t}} : \vec{t} \in Z\}$ satisfy the second item of the induction hypothesis. Use the first item of the strong master definition to find a relatively open tree $\vec{T}_{k+1} \leq_k \vec{T}'_k$ such that $\{\vec{x}_{\vec{t}} : \vec{t} \in Z\} \subset \Pi\vec{T}_{k+1}$ and moreover $\{f(\vec{x}) \restriction [m_k, m_{k+1}) : \vec{x} \in \Pi\vec{T}_{k+1}\} = \{f(\vec{x}_{\vec{t}}) \restriction [m_k, m_{k+1}) : \vec{t} \in Z\}$. The induction hypothesis continues to hold. □

This leaves a basic question begging for an answer:

Question 3.8.10. Can every equivalence relation on $(2^\omega)^\omega$, Borel reducible to an orbit equivalence, be simplified to smooth on a Borel I-positive subset?

3.9 Illfounded iteration of Sacks forcing

The illfounded iterations of forcings are significantly more difficult to handle than products or wellfounded iterations. A general treatment is provided in [49, Section 5.4] In this book, we will only mention the relatively well-understood case of iteration of Sacks forcing along the ordertype inverted ω.

Start with a combinatorial description of the poset. Let $X = (2^\omega)^\omega$. A continuous injective function $h : X \to X$ is called a *projection-keeping homeomorphism* if the functional value $h(x) \restriction (n, \omega)$ depends only on $x \restriction (n, \omega)$ for every natural number $n \in \omega$. Let P be the poset of the ranges of projection-keeping homeomorphisms ordered by inclusion. The associated σ-ideal I on X is the σ-ideal on σ-generated by sets of the form A_y, where $y \in 2^\omega$ and $A_y = \{\vec{x} \in X :$ for some number $n \in \omega$, $\vec{x}(n)$ is hyperarithmetic in $\vec{x} \restriction (n, \omega)$ and $y\}$. The fundamental theorem of the inverse ω Sacks iteration:

Theorem 3.9.1. *The ideal I is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$. For every analytic set $A \subset X$, exactly one of the following holds:*

1. *$A \in I$;*
2. *A contains the range of a projection-keeping homeomorphism.*

Theorem 3.9.2. *The forcing P_I is proper, bounding, and for every number $n \in \omega$, the point $\vec{x}_{gen}(n)$ is forced to be Sacks-generic over the model $V[\vec{x}_{gen} \restriction (n, \omega)]$.*

The canonization of the smooth equivalence relations reveals an interesting structure. For every number $n \in \omega$, let F_n be the smooth equivalence relation on X defined by $\vec{x} \; F_n \; \vec{y}$ iff $\vec{x} \restriction [n, \omega) = \vec{y} \restriction [n, \omega)$.

Theorem 3.9.3. *Let $B \subset X$ be a Borel I-positive set and E a smooth equivalence relation on it. Then there is a Borel I-positive set $C \subset B$ and a number $n \in \omega$ such that $E \restriction C = F_n \restriction C$.*

Theorem 3.9.4. *E_1 is in the spectrum of I.*

Proof. If $B \subset X$ is an I-positive set then we can inscribe the range of a projection-keeping homeomorphism into it. It is immediate that projection-keeping homeomorphisms preserve the equivalence E_1, and so $E_1 \leq_B E_1 \restriction B$ as desired. □

3.10 Laver forcing

The Laver forcing P is the partial order of those infinite *Laver trees* $T \subset \omega^{<\omega}$ such that there is a finite trunk and all nodes past the trunk split into infinitely many immediate successors. The associated *Laver ideal* I on ω^ω is generated by all sets A_g where $g : \omega^{<\omega} \to \omega$ is a function and $A_g = \{f \in \omega^\omega : \exists^\infty n \; f(n) \in g(f \restriction n)\}$.

Theorem 3.10.1. 1. *Every analytic subset of ω^ω is either in I or it contains all branches of a Laver tree, and these two options are mutually exclusive;*

2. *the ideal I is $\mathbf{\Delta^1_2}$ on $\mathbf{\Pi^1_1}$;*
3. *the ideal I has the transversal property;*

4. *the forcing P is $< \omega_1$-proper, preserves outer Lebesgue measure, and adjoins a minimal generic extension.*

Proof. The first item comes from [4], [49, Proposition 4.5.14]. The second item comes from [49, Proposition ??]. The minimality of Laver extension is due to Groszek [10, Theorem 7]; it can be also derived from the general treatment in [49, Theorem 4.5.13]. This leaves us with the third item. The proof of the transversal property is very flexible, it will be recalled in other places in this book, and we therefore treat it in some detail. It uses an integer game associated with the ideal. The Laver game was in its simplest form presented in [49, Section 4.5]. If $A \subset \omega^\omega$ is a set, Players I and II can play in the following fashion: Player II indicates an initial finite sequence $t_{ini} \in \omega^{<\omega}$ and then in every round n Player I indicates a number $m_n \in \omega$ and Player II responds with $k_n > m_n$. Player II wins if the result of the play, the sequence $x = t_{ini}^\frown \langle k_0, k_1, \dots \rangle \in \omega^{<\omega}$ belongs to the set A. There is also the appropriate unraveled version of this game if the set A is analytic. We have

Fact 3.10.2. *Player II has a winning strategy (in the original or unraveled version) if and only if the set A contains all branches of some Laver tree. Player I has a winning strategy if and only if $A \in I$. If the set A is analytic then the game is determined.*

Now let $D \subset 2^\omega \times \omega^\omega$ be a Borel I-positive set with pairwise disjoint I-positive sections. First argue that there is a perfect set $C \subset 2^\omega$ and a continuous function f on C assigning to each point $y \in C$ a winning strategy for player I in the unraveled game associated with the vertical section D_y. To do this, consider the space Z of all strategies for Player I in this game with a suitable Polish topology on it, fix a lightface Σ^1_1 set $A \subset 2^\omega \times 2^\omega \times \omega^\omega$ universal for analytic subsets of $2^\omega \times \omega^\omega$ and the set $A' \subset 2^\omega \times 2^\omega \times Z$ defined by $\langle x, y, z \rangle \in A'$ if z is a winning strategy for Player I in the unraveled game associated with the section $A_{x,y}$. This is a Π^1_1 set whose projection into the first two coordinates is, by the above fact, provably Π^1_2: it is the set of those pairs $\langle x, y \rangle$ such that Player II has no winning strategy in the game. Now find $x \in 2^\omega$ such that $D = A_x$; thus the set $\{\langle x, y \rangle : y \in 2^\omega\}$ is a subset of the projection of A'. Now use Theorem 1.3.5 to find a perfect set C and a map f as desired.

Let $\tau_y : y \in C$ be a Borel list of all strategies for Player I in the unraveled game, and let B be the set of all results of plays of the strategies $f(y)$ against τ_y as y ranges over all elements of the set C. The set B is Borel, since it is the image of the Borel set A under the Borel function $y \mapsto$the play of $f(y)$ against τ_y, that function is injective as the vertical sections of the set D are pairwise disjoint, and injective Borel images of Borel sets are Borel. The set B also I-positive, since if it was I-small, one of the strategies $\tau_y : y \in A$ should be winning in the game associated with it, but the play of σ_y against that τ_y would defeat it. Thus, the set B is the required transversal.

□

There are a standard set of corollaries and remarks. The function $\pi : P \to P_I$ defined by $\pi(T) = [T]$, is a dense embedding of the poset P into P_I. The study of the ideal I is complicated by the fact that it is not $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$. Laver forcing adds a dominating real and therefore its ideal cannot have such a simple definition by [49, Proposition 3.8.15]. Theorem 1.3.21 remains true for the Laver ideal, as a manual fusion construction immediately shows. We also have The computation of the spectrum of the Laver ideal is then facilitated by the general results such as Corollary 4.3.8 and Theorem 4.3.5 to yield

Theorem 3.10.3. *The σ-ideal I has total canonization for equivalences classifiable by countable structures.*

The application of Proposition 2.5.2 to get the Silver dichotomy for Borel equivalences classifiable by countable structures hits a snag: the Laver ideal is not $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ and the complexity calculations seem not to work out to yield a ZFC theorem. Instead, we have

Theorem 3.10.4. *Suppose that $\mathbf{\Delta^1_2}$ determinacy holds. Then the Laver ideal has the Silver property for Borel equivalences classifiable by countable structures.*

Proof. Follow the argument for Proposition 2.5.2. Let E be a Borel equivalence classifiable by countable structures on a Borel I-positive set $B \subset \omega^\omega$. Let $C = \{x \in B : [x]_E \in I\}$. The set C is $\mathbf{\Delta^1_2}$: it is the set of those points $x \in B$ for which Player I has a winning strategy in the game associated with $[x]_E$, and also the set of those points $x \in B$ for which Player II does not have such a winning strategy. By $\mathbf{\Delta^1_2}$ determinacy applied to the Laver game, the set C contains all branches of some Laver tree T or else is in the ideal I. In the former case, use the total canonization below the tree T, where it has to yield a Borel I-positive set of pairwise inequivalent elements and we are done. In the latter case, enclose C by a Borel set D in the ideal I and consider the Borel set $B \setminus D$. By the Silver dichotomy applied to E and this set, either there are only countably many classes of E meeting this set, or there is a perfect set of mutually E-inequivalent points in $B \setminus D$. In the former case, we decomposed B into a set in the ideal and countably many E-equivalence classes and we are done. In the latter case, note that each point in the perfect set has an I-positive equivalence class, and the transversal property of the ideal I yields the desired Borel I-positive set of pairwise inequivalent elements. The theorem follows. $\square$

Question 3.10.5. Is the determinacy assumption necessary in the above theorem?

Further inquiry into the spectrum identifies a nontrivial feature:

Theorem 3.10.6. *E_{K_σ} is in the spectrum of the Laver forcing.*

Proof. Consider the equivalence relation E on the set of increasing functions in ω^ω defined by $f E g$ if and only if there is a number $n \in \omega$ such that for all m, $f(m) < g(m+n)$ and $g(m) < f(m+n)$. In other words, a finite shift of the graph of f to the right is below the graph of g, and vice versa, a finite shift of the graph of g is below the graph of f.

It is not difficult to show that $E \leq_B E \restriction B$ for every Borel E-positive set. Such positive set must contain all branches of some Laver tree T; to simplify the notation assume that T has empty trunk. Find a level-preserving injection π from the set of all finite increasing sequences of natural numbers into T so that if the last number on s is less than the last number on t then also the last number on $\pi(s)$ is less than the last number on t. It is immediate that π extends to a continuous map which embeds E to $E \restriction B$.

The last point is to show that E_{K_σ} is bireducible with E. To embed E_{K_σ} in E, look at the two partitions $\omega = \bigcup_n K_n = \bigcup_n L_n$ into consecutive intervals such that $|K_n| = n+1$, $|L_n| = 2(n+1)$ and for a point $y \in \mathrm{dom} E_{K_\sigma}$ let $f(y)$ be the function in ω^ω specified by $f(y)(\min(K_n)+i) = \min(L_n)+i+y(n)$ for every n and $i \in n$. It is not difficult to see that f reduces E_{K_σ} to E. To embed E into E_{K_σ}, note that $E = \bigcup_n F_n$ is a countable union of closed sets with compact sections. Viewing ω^ω as a G_δ subset of a compact metric space Y, it is clear that the sets $\bar{F}_n$ =the closures of F_n in $Y \times Y$ have the same sections corresponding to the points in ω^ω as the sets F_n. Thus, $\bar{E}$, the relation on the space Y defined by $y \mathrel{\bar{E}} z$ if either $z, y \notin \omega^\omega$ or there is n such that $\langle y, z\rangle \in \bar{F}_n$, is a K_σ equivalence relation equal to E on ω^ω. By a result of Rosendal [35], it is Borel reducible to E_{K_σ} and the restriction of that Borel reduction reduces E to E_{K_σ} as desired.

□

Corollary 3.10.7. *Whenever J is a σ-ideal on a Polish space X such that the quotient forcing P_J is proper and adds a dominating real, then E_{K_σ} is in the spectrum of the ideal J.*

Question 3.10.8. Is E_2 in the spectrum of Laver forcing?

3.11 Mathias forcing

Adrian Mathias introduced the Mathias forcing P as the poset of all pairs $p = \langle a_p, b_p\rangle$ where $a_p \subset \omega$ is finite and $b_p \subset \omega$ is infinite, ordered by $q \leq p$ if $a_p \subset a_q$, $b_q \subset b_p$, and $a_q \setminus a_p \subset b_p$. The computation of the ideal gives full information. Let I be the σ-ideal on $X = [\omega]^{\aleph_0}$ consisting of those sets $A \subset X$ which are nowhere dense in the quotient partial order $\mathcal{P}(\omega)$ modulo the ideal of finite sets.

Fact 3.11.1. *[30] Let $A \subset X$ be an analytic set. Exactly one of the following is true:*

1. *$A \in I$;*
2. *there is a condition $p \in P$ such that $B_p \subset A$.*

Here, for a condition $p \in P$ the set $B_p \subset X$ consists of those infinite sets $c \subset \omega$ such that $a_p \subset c$ and $c \subset a_p \cup b_p$. Thus, the map $p \mapsto B_p$ is a dense embedding of the poset P into the quotient P_I.

The structure of real degrees in the Mathias extension is fairly complicated, and therefore the canonization of smooth equivalences with respect to the σ-ideal I cannot be simple. Prömel and Voigt gave an explicit canonization formula which can be also derived from earlier work of Mathias [30, Theorem 6.1]. For a function $\gamma : [\omega]^{<\aleph_0} \to 2$ let $\Gamma_\gamma : [\omega]^{\aleph_0} \to [\omega]^{\leq\aleph_0}$ be the function defined by $\Gamma_\gamma(a) = \{k \in a : \gamma(a \cap k) = 0\}$ and E_γ the smooth equivalence relation on $[\omega]^{\aleph_0}$ induced by the function Γ_γ.

Fact 3.11.2. *[30, 34] For every smooth equivalence relation E on $[\omega]^{\aleph_0}$ there is an infinite set $a \subset \omega$ and a function γ such that on the set $[a]^\omega$, $E = E_\gamma$.*

Theorem 3.11.3. *E_0 is in the spectrum of Mathias forcing. The poset $P_I^{E_0}$ is regularly embedded in P_I, it is $\aleph_0$-distributive and it yields the $V[\dot{x}_{gen}]_{E_0}$ model.*

Proof. Identify subsets of ω with their characteristic functions and use this identification to transfer E_0 to the space X. The theorem follows from Theorem 4.2.1 once we prove that E_0 saturations of Borel I-small sets are I-small. However, it is clear that E_0-saturation of a set nowhere dense in the algebra $\mathcal{P}(\omega)$ modulo finite is again nowhere dense, because the saturation does not change the place of the elements of the set in this algebra. $\square$

In fact, it is not difficult to see that the decomposition of Mathias forcing associated with the E_0 equivalence is just the standard decomposition into $\mathcal{P}(\omega)$ modulo finite followed with a poset shooting a set through a Ramsey ultrafilter. Note that in the Mathias extension, there are many E_0-classes which are subset of every E_0-invariant set in the generic filter; every infinite subset of the generic real will serve as an example.

Theorem 3.11.4. *The spectrum of the Mathias forcing is cofinal among Borel equivalence relations in $\leq_B$. It includes E_{K_σ} and E_0^ω.*

Proof. Let K be a Borel ideal on ω containing all finite sets and let E_K be the equivalence relation on the space Y of increasing functions in ω^ω defined by $y \; E_K \; z$ if $\{n \in \omega : y(n) \neq z(n)\} \in K$. For the cofinality of the spectrum, by Fact 1.3.15, it is enough to show that E_K is in the spectrum of Mathias forcing. Look at E, the relation on $[\omega]^{\aleph_0}$ connecting two points if their increasing enumerations are E_K-equivalent. Clearly, this is an equivalence relation Borel reducible to E_K. On the other hand, if $B \subset X$ is a Borel I-positive set, then E_K is Borel reducible to $E \restriction B$. Just find a condition $p \in P$ such that $B_p \subset B$ and let $\pi : \omega \to a_p \cup b_p$ be the increasing enumeration. To each point $y \in Y$ associate the set $b_y = a_p \cup \mathrm{rng}((\pi \circ y) \restriction \omega \setminus |a_p|)$ and observe that the map $y \mapsto b_y$ reduces E_K to $E \restriction B$.

If K is an F_σ-ideal on ω such that $=_K$ is bireducible with E_{K_σ}, then also E_K is bireducible with E_{K_σ}, showing that E_{K_σ} is in the spectrum of I. To see that E_0^ω is in the spectrum, choose a partition $\omega = \bigcup_n a_n$ of ω into infinitely many infinite pairwise disjoint sets, for $x \in [\omega]^{\aleph_0}$ let $f(x) \subset \omega \times \omega$ be the set of all pairs $\langle n, m\rangle$ such that m is i-th element of x in the increasing enumeration for some $i \in a_n$, and let $x \; E \; y$ if $f(x) \; E_0^\omega \; f(y)$. This is clearly a Borel equivalence relation reducible to E_0^ω. To show that E_0^ω is Borel reducible to $E \restriction B_p$ for every condition $p \in P$, just find ??? $\square$

Theorem 3.11.5. *(Mathias) Essentially countable*$\to_I \leq E_0$.

Question 3.11.6. Is F_2 in the spectrum of Mathias forcing?

3.12 Fat tree forcings

Let T_{ini} be a finitely branching tree without terminal nodes, and for every node $t \in T_{ini}$ fix a submeasure ϕ_t on the set a_t of immediate successors of t in T_{ini}, such that $\lim_t \phi_t(a_t) = \infty$. Consider the *fat tree forcing* P of those trees $T \subset T_{ini}$ such that $\lim_{t\in T} \phi_t(b_t) = \infty$, where $b_t \subset a_t$ is the set of all immediate successors of the node t in the tree T; the ordering is that of inclusion.

This is a special case of the generalized Hausdorff measure forcings of [49, Section 4.4]. The computation of the associated ideal gives a complete information. The underlying space is $X = [T_{ini}]$ and the σ-ideal I on X is generated by all sets A_f, where f is a function with domain T_{ini} associating with each node $t \in T_{ini}$ a set of its immediate successors of ϕ_t-mass $\leq n_f$ for some number $n_f \in \omega$. The set A_f is defined as $\{x \in X : \exists^\infty n\ x(n) \in f(x \restriction n)\}$. The following sums up the work in [49]:

Theorem 3.12.1. *Let $\phi_t : t \in T_{ini}$ be submeasures such that $\lim_t \phi_t(a_t) = \infty$. Then*

1. *Every analytic subset of X either is in the ideal I or contains all branches of some tree $T \in P$, and these two options are mutually exclusive;*
2. *The ideal I is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$;*
3. *the ideal I has the transversal property;*
4. *The forcing P is $< \omega_1$-proper, bounding, adds no independent reals, and adds a minimal forcing extension.*

Proof. The first item comes from [49, Proposition 4.4.14], the second item comes from [49, Theorem 3.8.9]. For the transversal property, use the integer game associated with the ideal I as described in [49, Proposition 4.4.14], and then follow the proof of Theorem 3.10.1(3). The first three assertions in the last item follow from [49, Theorem 4.4.2, Theorem 4.4.8]. The minimality of the extension can be proved either by an awkward manual argument, or by an analysis of the possible intermediate extensions parallel to the proof of Theorem 2.4.5. First, the transversal property and $< \omega_1$-properness imply that every intermediate extension is c.c.c. and potential c.c.c. intermediate extensions without new reals are ruled out just as in the proof of that theorem. Thus, it is enough to rule out the possibility that the forcing P adds c.c.c. reals. Since the ideal I is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$, these c.c.c. reals would have to be obtained from a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal which then is guaranteed to either add a Cohen real or else be generated by a Maharam submeasure. However, all of these options would add independent reals [1, Lemma 6.1], while P does not add such a real! □

The usual corollaries apply. The map $\pi : T \mapsto [T]$ is an isomorphism between the poset P and a dense subset of the quotient poset P_I, and Corollary 4.3.8 and Proposition 2.5.2 yield

Corollary 3.12.2. *The ideal I has the Silver property for Borel equivalence relations classifiable by countable structures.*

Further forcing and/or canonization properties very much depend on the choice of the submeasures ϕ_t.

Theorem 3.12.3. *There is a choice of submeasures such that the resulting ideal I has the Silver property for all Borel equivalence relations.*

Proof. For a submeasure ϕ on a finite set a let $\mathtt{add}(\phi)$ be the minimum size of a set $B \subset \mathcal{P}(a)$ such that $\phi(\bigcup B) \geq \max\{\phi(b) : b \in B\} + 1$. Submeasures with large $\mathtt{add}$ can be obtained for example by starting with the counting measure and then inflicting on it the operation $\phi \mapsto \log(1 + \phi)$ repeatedly many times. Now choose a tree T_{ini} with the submeasures $\phi_t : t \in \omega$ so that there is a linear ordering of T_{ini} in type ω in which $\mathtt{add}(\phi_t)$ is much bigger than $\Pi_s |a_s|$ where s ranges over all nodes of T_{ini} preceding t in the linear order. We claim that this choice of submeasures works.

Let $T \subset T_{ini}$ be a fat tree and E be a Borel equivalence relation on $[T]$. For simplicity of notation assume that the masses of sets of immediate successors of nodes in T are always greater than 1. We must find a fat subtree $S \subset T$ such that $E \restriction [S]$ is equal either to the identity or to $[S]^2$. The Silver property of the associated σ-ideal then follows from Proposition 2.5.2.

For a node $t \in T$ let b_t be the set of immediate successors of t in T, equipped with the pavement submeasure ϕ_t obtained from the pavers $b \subset b_t$ for which $\phi_t(b) < \phi_t(b_t) - 1$ and each paver is assigned weight 1. The additivity properties of the submeasures ϕ_t imply that the numbers $\psi_t(b_t)$ increase to infinity very fast. Consider the space $Y = \Pi_{t \in T} b_t$. For every node $u \in T$ there is a continuous map $f_u : Y \to X$ which assigns to every point $y \in Y$ the unique path $x \in X$ such that $u \subset x$ and for all $n \geq |u|$, $u(n) = y(u \restriction n)$. Consider the sets $B_{u,v} = \{y \in Y : f_u(y) E f_v(y)\}$ for all pairwise incompatible nodes $u, v \in T$. The creature forcing arguments of [38] yield nonempty sets $c_t \subset b_t : t \in T$ such that the numbers $\psi_t(c_t)$ are all greater than 1 and tend to infinity, and all the Borel sets $B_{u,v}$ are relatively clopen in $\Pi_t c_t$. Note that $\phi_t(c_t) > \phi_t(b_t) - 1$. Find a tree $S \subset T$ in the forcing P such that for every node $t \in S$, the set of the immediate successors of t in S is exactly c_t. It is not difficult to see that the equivalence relation E on the product $[S] \times [S]$ is now a union of an open set together with the diagonal. Thus, $E \restriction [S]$ is a G_δ equivalence relation and as such it must be smooth [8, Theorem 6.4.4]. Now, the ideal I has total canonization of smooth equivalence relations, and so it canonizes E!

□

Theorem 3.12.4. *There is a choice of submeasures such that E_2 and E_{K_σ} are in the spectrum of the associated σ-ideal.*

In fact, it will be obvious from the construction that there are very many complicated equivalence relations on the underlying space X. For every Borel equivalence relation E there will be a Borel equivalence relation F on X such that on every Borel I-positive set $B \subset X$, $E \leq F \restriction B$ holds. In order to see this, just note that the equivalence relations $=_K$, where K is a Borel ideal on ω and $x =_K y$ if $\{n : x(n) \neq y(n)\} \in K$ for $x, y \in 2^\omega$, are cofinal in the reducibility ordering of Borel equivalence relations by Fact 1.3.15.

Proof. Recall that the *Hamming cubes* 2^n with the normalized counting measure μ_n and the normalized Hamming distance $d_n(x,y) = |\{m \in n : x(m) \neq y(m)\}|/n$ form a *Levy family* with concentration of measure as n varies over all natural numbers, see for example [33, Theorem 4.3.19]. This means that for all real numbers $\varepsilon, \delta, \gamma > 0$, for every sufficiently large number n, every set $A \subset 2^n$ of μ_n-mass at least ε the set $A_\delta = \{x \in 2^n : d_n(x, A) < \delta\}$ has μ-mass at least $1 - \gamma$. We will need the following corollary of this fact:

Claim 3.12.5. *For every $m \in \omega$ and positive reals $\varepsilon, \delta > 0$, there is a number $n \in \omega$ such that whenever $A_i : i \in m$ are subsets of 2^n of μ_n-mass at least ε, there is a ball of d_n-radius δ intersecting them all.*

Now, choose a fast increasing sequence of numbers $n_i : i \in \omega$ so that for every subset of $\mathcal{P}(2^{n_i})$ of cardinality $2\Pi_{j \in i} 2^{n_j}$, consisting of sets of μ_{n_i}-mass at least $1/i$, there is a ball of radius 2^{-i} intersecting them all. Consider the tree T_{ini} consisting of all finite sequences t such that $\forall i \in \mathrm{dom}(t)\ t(i) \in 2^{n_i}$, and for every $t \in T$ of length i let ϕ_t be the normalized counting measure on 2^{n_i} multiplied by i. We claim that the fat tree forcing given by this choice of submeasures works.

To get E_{K_σ} in the spectrum, let J be an F_σ-ideal on 2^ω such that $=_J$ is bireducible with E_{K_σ} as in the construction after fact 1.3.15. Define an equivalence E on $[T_{ini}]$ by setting $x\ E\ y$ if there is a set $a \in J$ and a number $k \in \omega$ such that for every $i \in \omega \setminus a$, $2^i d_{n_i}(x(i), y(i)) < k$. This is clearly K_σ, so reducible to E_{K_σ}. We will show that $=_J$ is Borel reducible to $E \restriction B$ for every Borel I-positive set $B \subset [T_{ini}]$.

So suppose B is such a Borel I-positive set, and inscribe a fat tree T into it. Find a node $t \in T$ such that past t, the nodes of the tree T split into a set of immediate successors of mass at least 1. Let $j = \mathrm{dom}(t)$. The choice of the numbers n_i shows that for every $i > j$ there is a point $p_i \in 2^{n_i}$ such that its neigborhood of d_{n_i}-radius 2^{-i} intersects all sets of immediate successors of nodes in T extending the node t of length i, and also their flips (a *flip* of a subset of 2^{n_i} is obtained by interchanging 0's and 1's in all of its elements). Now it is easy to find a continuous function $h : 2^\omega \to [T]$ such that for every $x \in 2^\omega$ and every $i > j$, $h(x)(i)$ is some point within d_{n_i}-distance 2^{-i} from p_i if $x(i) = 0$, and $h(x)(i)$ is some point within d_{n_i}-distance 2^{-i} from the flip of p_i if $x(i) = 1$. Note that the d_{n_i}-distance of p_i from its flip is equal to 1. This clearly means that for $x, y \in 2^\omega$, $x =_J y$ if and only if $h(x)\ E\ h(y)$, and therefore E_{K_σ} is in the spectrum.

To get E_2 in the spectrum, consider the equivalence relation F on T_{ini} defined by $x\ F\ y$ if $\Sigma\{1/j+1 :$ for some $i \in \omega$, $n_i \leq j < 2n_i$ and $x(i)(j-n_i) \neq y(i)(j-n_i)\} < \infty$. This is clearly reducible to E_2; we must show that E_2 is Borel reducible to $F \restriction B$ for every I-positive Borel set $B \subset [T_{ini}]$.

A direct construction of the embedding is possible by a trick similar to the previous paragraph. We note that if E_2 was not reducible to $F \restriction B$, then $F \restriction B$ would be essentially countable by the E_2 dichotomy, the ideal I has total canonization for essentially countable equivalences, and so there would be a Borel I-positive subset consisting of either pairwise inequivalent or pairwise equivalent elements. So it is enough to show that every Borel I-positive set B contains a pair of equivalent elements, and a pair of inequivalent elements. Inscribe a fat tree T into B, choose the node t and the points $p_i \in 2^{n_i} : i > \mathrm{dom}(t)$ as in the E_{K_σ} case. Note that $u, v \in 2^{n_i}$ are sequences such that u is 2^{-i}-close to p_i and v is 2^{-i}-close to the flip of p_i, then $\Sigma\{1/j+1 : n_i \leq j < 2n_i$ and $u(i)(j-n_i) \neq v(i)(j-n_i)\} > 1/2 - 4 \cdot 2^{-i}$, and if u, v are both close to p_i or both close to the flip of p_i, then this sum is smaller than $4 \cdot 2^{-i}$. Thus, with the continuous function h constructed in the E_{K_σ} argument, $h(x)\ F\ h(y)$ if and only if $x\ E_0\ y$; in particular, there are both F-related and F-unrelated pairs in $[T] \subset B$.

$\square$

The theorem (and its proof) has an interesting corollary, complementary to the treatment of turbulent equivalence relations in [22, Chapter 13]. If E and F are equivalence relations on respective Polish spaces X and Y, follow Hjorth and Kechris and say that E is *generically ergodic* with respect to F if for every partial Borel map $f : X \to Y$ defined on a comeager set, such that $x_0\ E\ x_1 \to f(x_0)\ F\ f(x_1)$, there is a comeager set $B \subset X$ such that $f''B$ is a subset of a single F-equivalence class. Hjorth and Kechris showed that every generically turbulent equivalence relations (such as E_2) are generically ergodic with respect to every equivalence relation classifiable by countable structures. The previous theorem allows a similar treatment for measure. If μ is a Borel probability measure on the space X, say that E is *μ-ergodic* with respect to E if for every partial Borel map $f : X \to Y$ defined on a co-μ-null set, such that $x_0\ E\ x_1 \to f(x_0)\ F\ f(x_1)$, there is a co-$\mu$-null set $B \subset X$ such that $f''B$ is a subset of a single F-equivalence class. Of course, this can be satisfied trivially if the measure μ concentrates on a single E-equivalence class. The interesting cases occur when the μ-mass of every E-equivalence class is zero.

Corollary 3.12.6. *E_2 is μ-ergodic with respect to every equivalence relation classifiable by countable structures.*

Here, μ is the usual Haar probability measure on 2^ω invariant under coordinatewise binary addition.

Proof. The argument uses a similar fat tree forcing as the previous theorem, with an improved computation of measure concentration. Let $\omega = \bigcup_n J_n$ be a partition of ω into successive finite intervals so long that for every $n \in \omega$, $1/n^2 \geq e^{-2^{-n-1}/8k_{n+1}}$. Here, $k_n = \Sigma_{j\in J_n} 1/(j+1)^2$. Such a partition exists since the sum $\Sigma_{j\in\omega} 1/(j+1)^2$ converges. [33, Theorem 4.3.19] now shows that whenever $A \subset 2^{J_n}$ is a set of normalized counting measure $\geq 1/n^2$ then $A_{2^{-n}} \subset 2^{J_n}$ has normalized counting measure greater than $1/2$. Here, A_ε is the set of all points with d_n-distance at most ε from the set A, and d_n is the metric on 2^{J_n} defined by $d_n(x, y) = \Sigma_{x(j)\neq y(j)} 1/j+1$. Just as in the previous proof, if $A, B \subset 2^{J_n}$ are two sets of normalized counting measure at least $1/n^2$, then they contain points $x \in A, y \in B$ of d_n-distance at most 2^{-n}: the 2^{-n-1}-neighborhoods of the sets A and B have normalized counting measure greater than $1/2$ and therefore they intersect. Replacing the set A with the flips of its points and repeating this argument, we also find other points $\bar{x} \in A, \bar{y} \in B$ of d_n-distance at least $(\Sigma_{j\in J_n} 1/j+1) - 2^{-n}$.

Consider the fat tree forcing on the initial tree $T_{ini} = \Pi_n 2^{J_n}$, and the submeasure ϕ_n is simply the normalized counting measure on 2^{J_n} multiplied by n^2. Clearly, every branch through the tree T_{ini} can be identified with the union of the labels on its nodes, which is an element of 2^ω. With this identification, the generating sets of the associated σ-ideal I have μ-mass 0 by the Borel-Cantelli lemma, since the sum of the numbers $1/n^2$ converges. And just as in the previous proof, one can show that every two trees $T, S \in P$ contain branches $x \in [T], y \in [S]$ which are E_2-related, and branches $\bar{x} \in [T]$ and $\bar{y} \in [S]$ that are E_2-unrelated.

Now, suppose that $f : 2^\omega \to Y$ is a partial Borel map defined on a co-μ-null set, where Y is some Polish space with an equivalence relation F generated by a Polish action of a closed subgroup of S_∞, and f is such that $x_0 \; E_2 \; x_1 \to f(x_0) \; F \; f(x_1)$. Let E be the pullback of F, so $E_2 \subset E$. Thinning out to a co-μ-null set $B \subset 2^\omega$, we may assume that E is Borel by Theorem 4.4.1. Now, the ideal I has the Silver property for Borel equivalence relations classifiable by countable structures by Corollary 3.12.2. This means that either B decomposes into an I-small set and countably many E-classes, or B contains a Borel I-positive subset consisting of pairwise E-unrelated elements. The latter case cannot occur, since every Borel I-positive set contains distinct E_2-related elements which then would have to be E-related. In the former case, only one of the E-classes can be I-positive, since any two Borel I-positive sets contain E_2-related elements that then have to be E-related. Thus, the whole space2^ω decomposes into the μ-null complement of B, an I-small set which must be μ-null, countably many I-small E-classes, which then must be μ-null as well, and a single I-positive E-class, which then has to have full μ-mass. $\square$

Note that as the fat tree forcings allow total canonization of equivalence relations classifiable by countable structures and the one exhibited in the previous theorem does not allow canonization of an equivalence relation reducible to E_2, we have a new proof of the fact that E_2 is not classifiable by countable structures which does not use Hjorth's concept of turbulence [15], [8, Chapter 10], [22, Chapter 13]. Note also that the fat tree forcings have the separation property of Definition 4.3.4 by Theorem 4.3.5, and so the collection of Borel equivalence relations for which they have total canonization and the Silver property is closed under the operation of Friedman-Stanley jump and modulo finite countable product, which is reminiscent of the treatment of turbulence and generic ergodicity in [22, Chapter 13].

3.13 The random forcing

Let λ be the usual Borel probability measure on 2^ω, and let $I = \{B \subset 2^\omega : \lambda(B) = 0\}$. The quotient forcing P_I is the random forcing.

Proposition 3.13.1. *E_0 is in the spectrum of I.*

Proof. The classical Steinhaus theorem [2, Theorem 3.2.10] shows that in every Borel set of positive mass there are two distinct E_0-related points. Thus, Borel partial E_0-selectors must have Lebesgue mass zero, and since every Borel set on which E_0 is smooth is a countable union of partial Borel E_0 selectors, all such sets must have Lebesgue mass zero. It follows that E_0 is non-smooth on every Borel set $B \subset 2^\omega$ of positive mass, and by the Glimm-Effros dichotomy, $E_0 \leq E_0 \restriction B$ as desired. □

Proposition 3.13.2. *E_2 is in the spectrum of I.*

Proof. this follows from the treatment of the ideal J associated with E_2 of Section 3.5. As Claim 3.5.6 shows, $J \subset I$, and so E_2 is Borel reducible to $E_2 \restriction B$ for every Borel non-null set $B \subset 2^\omega$. □

Proposition 3.13.3. *The spectrum of I is cofinal in the Borel equivalence relations under the Borel reducibility order, and it includes E_{K_σ}.*

Proof. Look at the space $X = (2^\omega)^\omega$ with the product Borel probability measure instead. Consider the ideal J associated with the product of Sacks forcing, as in Section 3.8. Then $J \subset I$, the ideal J is suitably homogeneous, and so the spectrum of J is included in the spectrum of I. This proves the proposition. □

Chapter 4

Particular equivalence relations

4.1 Smooth equivalence relations

In the case that the equivalence relation E is smooth, the model $V[x_{\text{gen}}]_E$ as described in Theorem 2.1.3 takes on a particularly simple form.

Proposition 4.1.1. *Let I be a σ-ideal on a Polish space X such that the quotient forcing P_I is proper, and suppose that E is a smooth equivalence relation on the space X. Then $V[x_{\text{gen}}]_E = V[f(\dot{x}_{gen})]$ for every ground model Borel function f reducing E to the identity.*

Proof. Choose a Borel function $f : X \to 2^\omega$ reducing the equivalence E to the identity. Let $G \subset P_I$ and $H \subset \text{Coll}(\omega, \kappa)$ be mutually generic filters, and let $x_{\text{gen}} \in X$ be a point associated with the filter G. First of all, $f(x_{\text{gen}}) \in 2^\omega$ is definable from the equivalence class $[x_{\text{gen}}]_E$: it is the unique value of $f(x)$ for all $x \in [x_{\text{gen}}]_E$. Thus, $V[f(x_{\text{gen}})] \subset V[x_{\text{gen}}]_E$. On the other hand, the equivalence class $[x_{\text{gen}}]_E$ is also definable from $f(x_{\text{gen}})$, and the model $V[G, H]$ is an extension of $V[f(x_{\text{gen}})]$ via a homogeneous notion of forcing $\text{Coll}(\omega, \kappa)$, and therefore $V[x_{\text{gen}}]_E \subset V[f(x_{\text{gen}})]$. □

The total canonization of smooth equivalence relations has an equivalent restatement with the quotient forcing adding a minimal real degree. It is necessary though to discern between the forcing adding a minimal real degree, and the forcing producing a minimal extension. In the former case, we only ascertain that $V[y] = V$ or $V[y] = V[x_{\text{gen}}]$ for every set $y \subset \omega$ in the generic extension. In the latter case, this dichotomy in fact holds for every set y of ordinals in the extension. For example, the Silver forcing adds a minimal real degree, while it produces many intermediate σ-closed extensions.

Proposition 4.1.2. *Let X be a Polish space and I be a σ-ideal on X such that the quotient forcing P_I is proper. The following are equivalent:*

1. *P_I adds a minimal real degree;*

2. *I has total canonization of smooth equivalence relations.*

Proof. First, assume that the canonization property holds. Suppose that $B \in P_I$ forces that τ is a new real; we must show that some stronger condition forces $V[\tau] = V[\dot{x}_{gen}]$. Thinning out the set B if necessary we may find a Borel function $f : B \to 2^\omega$ such that $B \Vdash \tau = \dot{f}(\dot{x}_{gen})$. Note that singletons must have I-small f-preimages, because a large preimage of a singleton would be a condition forcing τ to be that ground model singleton. Consider the smooth equivalence relation E on B, $E = f^{-1}\mathrm{ID}$. Since it is impossible to find a Borel I-positive set $C \subset B$ such that $E \restriction B = \mathrm{EE}$, there must be a Borel I-positive set $C \subset B$ such that $E \restriction C = \mathrm{ID}$, in other words $f \restriction C$ is an injection. But then, $C \Vdash V[\tau] = V[\dot{x}_{gen}]$ since $\dot{x}_{gen}$ is the unique point $x \in C$ such that $f(x) = \tau$.

Now suppose that P_I adds a minimal real degree, and let $B \in P_I$ be a Borel I-positive set and E a smooth equivalence relation on B with I-small classes. We need to produce a Borel I-positive set on which E is equal to the identity. Let $f : B \to 2^\omega$ be a Borel function reducing E to the identity. Note that since f-preimages of singletons are in the ideal I, $\dot{f}(\dot{x}_{gen})$ is a name for a new real and therefore it is forced that $V[\dot{x}_{gen}] = V[\dot{f}(\dot{x}_{gen})]$. Let M be a countable elementary submodel of a large structure, and let $C \subset B$ be the Borel I-positive set of all M-generic points in B for the poset P_I. The set $f''C$ is Borel. Consider the Borel set $D \subset f''C \times C$ defined by $\langle y, x\rangle \in D$ if and only if $f(x) = y$. The set D has countable vertical sections: If $f(x) = y$ then by the forcing theorem applied in the model M it must be the case that $M[x] = M[y]$ and thus the vertical section D_y is a subset of the countable model $M[y]$. Use the uniformization therem to find countably many graphs of Borel functions $g_n : n \in \omega$ whose union is D. The sets $\mathrm{rng}(g_n) : n \in \omega$ are Borel since the functions g_n are Borel injections, and they cover the whole set C. Thus there is a number $n \in \omega$ such that the set $\mathrm{rng}(g_n)$ is I-positive. On this set, the function f is an injection as desired. $\square$

4.2 Countable equivalence relations

Borel countable equivalence relations are a frequent guest in mathematical practice. Their treatment in this book is greatly facilitated by essentially complete analysis of the associated forcing extensions.

Theorem 4.2.1. *Suppose that I is a σ-ideal on a Polish space X such that the quotient forcing is proper. Suppose that E is a countable Borel equivalence relation on X. Then for every Borel I-positive set $B \subset X$ there is a Borel I-positive set $C \subset B$ such that $P_I^{E,C}$ is a regular subposet of P_I below C. Moreover, C forces the model $V[\dot{x}_{gen}]_E$ to be equal to the $P_I^{E,C}$ extension. The remainder forcing $P_I \restriction C/P_I^{E,C}$ is homogeneous and c.c.c.*

Here, $P_I^{E,C}$ is the poset of I-positive Borel subsets of C which are relatively E-saturated inside C. The set $C \subset X$ may not be the whole space, as Claim 2.2.4 shows.

Proof. Use the Feldman–Moore theorem to find a Borel action of a countable group G on B whose orbit equivalence relation is exactly E.

Claim 4.2.2. *There is a Borel I-positive set $C \subset B$ such that the E-saturation of any I-small Borel subset of C has I-small intersection with C.*

Proof. Let M be a countable elementary submodel of a large structure and let C be the set of all M-generic elements of the set B for the poset P_I. We claim that this set works.

Suppose for contradiction that there is an I-small Borel subset D of C whose E-saturation has an I-positive intersection with C. Note that $[D]_E$ is a Borel set again as E is a countable Borel equivalence relation. Thinning out the set $C \cap [D]_E$ we may find a Borel I-positive subset C' and a group element $g \in G$ such that $gC' \subset D$.

Now consider the set $O = \{A \in P_I :$ if there is a Borel I-positive subset of A whose shift by g is I-small, then A is such a set$\}$. Clearly, this is an open dense subset of P_I below B in the model M, and therefore there is $A \in M \cap O$ such that $A \cap C' \notin I$. Now $g(A \cap C') \subset D$ and therefore this set is in the ideal I, and by definition of the set O it must be the case that $gA \in I$. But as $gA \in I \cap M$, it must be the case that $C \cap gA = 0$, which contradicts the fact that $g(A \cap C')$ is a nonempty subset of C. □

Fix a set $C \subset B$ as in the claim.

Claim 4.2.3. *$P_I^{E,C}$ is a regular subposet of P_I below C.*

Proof. We will show that whenever $D \subset C$ is a Borel I-positive set, then the E-saturation of D is a pseudoprojection of P_I to $P_I^{E,C}$. Indeed, if $D \subset C$ is a Borel I-positive set, then every Borel I-positive set $A \subset [D]_E \cap C$ which is E-saturated relatively in C, must have a Borel I-positive intersection with D. Otherwise, $A \subset C$ would be a Borel I-positive subset of the E-saturation of the I-small set $A \cap D$, which contradicts the choice of the set C. □

Claim 4.2.4. *The remainder forcing $P_I \restriction C/P_I^{E,C}$ is homogeneous and c.c.c.*

Proof. Let $G \subset P_I$ be a generic filter containing the condition C with its associated generic point x_{gen}, and let $H = G \cap P_I^{E,C}$. Note first that the remainder forcing consists of exactly those Borel sets $D \subset C$ such that their E saturation belongs to the filter H, or restated, those Borel sets containing some element of the equivalence class of x_{gen}.

For the c.c.c. suppose that $D_\alpha : \alpha \in \omega_1$ is a putative antichain in the remainder poset in the model $V[H]$. Since the equivalence class of x_{gen} is countable and $\aleph_1$ is preserved in $V[G]$, there must be $\alpha \neq \beta \in \omega_1$ such that D_α, D_β contain the same element of the equivalence class. It follows that $D_\alpha \cap D_\beta$ is a lower bound of D_α, D_β in the remainder forcing.

For the homogeneity, return to the ground model. Suppose that $D_0, D_1, D_2 \subset C$ are Borel I-positive sets, D_2 is E-saturated and forces in $P_I^{E,C}$ both D_0, D_1 to the remainder forcing. By the countable additivity of the ideal I there must be an element $g \in G$ and a positive set $\bar{D}_0 \subset D_0 \cap D_2$ such that $g\bar{D}_0 \subset D_1$. The map π defined by $\pi(A) = gA$ preserves the ideal I, preserves the E-saturations and therefore the condition $[\bar{D}_0]_E \in P_I^{E,C}$ forces that it induces an automorphism of the remainder poset moving a subset of D_0 to a subset of D_1. The homogeneity of the quotient forcing follows.

□

The fact that $P_I^{E,C}$ generates the $V[\dot{x}_{gen}]_E$ model now follows directly from Theorem 2.2.7; the group action provides the necessary automorphisms of the poset $P_I^{E,C}$. □

Inspecting the proofs in the previous chapter, the fact that the model $V[\dot{x}_{gen}]_E$ is obtained by the forcing P_I^E are invariably proved in a significantly tighter fashion: it is proved that in the P_I extension, the equivalence class of the generic real is the only one which is a subset of all E-invariant Borel sets in the generic filter. This means that, in the notation of the previous proof, the filter H is interdefinable with $[x_{\mathrm{gen}}]_E$ in the P_I-extension $V[G]$ and they therefore have to define the same HOD model. This, however, does not have to be the case in general even in the very restrictive case that the forcing P_I adds a minimal real degree. Consider the case where P_I is the Silver forcing and E is the equivalence relation on 2^ω defined by $x \mathrel{E} y$ if x, y differ on finite set of entries, and moreover, the set of their differences has even cardinality. Comparing E with E_0, we see that $E \subset E_0$ and every E_0 equivalence class decomposes into exactly two E classes. Note that the E_0-saturation of any Silver cube is equal to its E-saturation, and therefore the posets $P_I^{E_0}$ and P_I^E are co-dense. If G is the Silver generic filter and $H = G \cap P_I^E$ then the E_0-class of the Silver real belongs to all sets in H, and it splits into two E-classes, each of which has the property that it is a subset of all sets in the P_I^E-generic filter. Moreover, neither of these classes is ordinally definable from H or parameters in V.

Let us now return back to the general treatment of Borel countable equivalence relations. In the common case that the forcing P_I adds a minimal real degree, there is much more information available about the $V[\dot{x}_{gen}]_E$ model.

Theorem 4.2.5. *Suppose that I is a σ-ideal on a Polish space X such that the quotient forcing is proper and adds a minimal real degree. Suppose that E is an essentially countable equivalence relation on X which does not simplify to smooth on any Borel I-positive set. Then the model $V[\dot{x}_{gen}]_E$ is forced to contain no new countable sequences of ordinals, and it is the largest intermediate extension in the P_I-extension.*

Proof. Let $f : X \to Y$ be a Borel reduction of E to a Borel countable equivalence relation F on a Polish space Y. Let $B \subset X$ be an I-positive Borel set. By the total canonization of smooth equivalences we may pass to a Borel I-positive set C on which the function is injective and therefore E itself has countable classes, and by the previous theorem we may assume that the poset $P_I^{E,C}$ is a regular subposet of P_I below C.

We will first argue that the poset $P_I^{E,C}$ is $\aleph_0$-distributive. The forcing is a regular suposet of a proper forcing, therefore proper, and so it is enough to show that it does not add reals. Suppose that τ is a $P_I^{E,C}$-name for a real, and $D \subset C$ is a Borel I-positive set. We must produce a Borel I-positive subset of D forcing a definite value to τ. Let M be a countable elementary submodel of a large enough structure, and let $A \subset D$ be the Borel I-positive subset consisting of all M-generic points. The map $x \mapsto K_x$ assigning to each generic point the corresponding $P_I^{E,C}$ generic filter for M, is E-invariant: K_x is the set of all relatively E-saturated Borel subsets containing x, which depends only on the E-equivalence class of a point $x \in C$. Thus the map $x \mapsto \tau/K_x$ is Borel and also E-invariant. Since the forcing P_I adds a minimal real degree, there is a Borel I-positive set $A' \subset A$ such that this map is either one-to-one or constant on A'. The former possibility cannot occur–since the function is E-invariant, it would have to be the case that $E \restriction A' = \mathrm{ID}$, contradicting the assumption that E cannot be reduced to a smooth equivalence on a positive Borel set. The latter possibility forces τ to be equal to the constant value of the function on A', as desired.

Now we need to argue that the model $V[x_{\mathrm{gen}}]_E$ is the largest intermediate model between V and $V[G]$, in other words, if U is a model of ZFC such that $V \subseteq U \subseteq V[G]$ then either $U = V[G]$ or $U \subseteq V[x_{\mathrm{gen}}]_E$. Suppose that $U = V[Z]$ is an intermediate model of ZFC, for a set $Z \subset \kappa$ for some ordinal κ. Fix a name $\dot{Z}$ for the set Z. If there is a countable set $a \subset \kappa$ in V such that $Z \cap a \notin V$ then $V[G] = V[Z]$ by the minimal real degree assumption. Suppose then that for every countable set $a \in V$, it is forced that $\dot{Z} \cap \check{a} \in V$. Consider the names for countably many sets $\dot{Z}/y : y \in [\dot{x}_{gen}]_E$ and y is V-generic. There must be a countable set $b \subset \kappa$ such that if any two of these sets differ at some ordinal, then they differ at an ordinal in b. By properness, this set b has to be covered by a ground model countable set $a \subset \kappa$. Suppose that $D \subset C$ in P_I is a condition which forces this property of the set a, and moreover decides the value of $\dot{Z} \cap \check{a}$ to be some definite set $c \in V$. Then D forces that Z is definable from $[x_{\mathrm{gen}}]_E, D, c$, and $\dot{Z}$ in the model $V[G]$ by the formula "Z is the unique value of $\dot{Z}/y$ for all V-generic points $y \in [x_{\mathrm{gen}}]_E \cap D$ such that $\dot{Z}/y \cap a = c$". Thus, $V[Z] \subset V[x_{\mathrm{gen}}]_E$!

$\square$

An important heuristic consequence of the last sentence of this theorem: if the poset P_I adds a minimal real degree, we cannot use the model $V[\dot{x}_{gen}]_E$ to discern between the various nonsmooth countable equivalence relations on the underlying Polish space X, because they all happen to generate the same model. This is apparently not to say that every two such countable equivalence relations must be similar or of the same complexity at some I-positive Borel set.

Corollary 4.2.6. *If the forcing P_I is proper, nowhere c.c.c. and adds a minimal* *forcing extension, then I has total canonization for essentially countable Borel equivalences.*

Proof. Looking at the previous two theorems, if the poset P_I adds a minimal real degree and E is an essentially countable equivalence relation on the underlying Polish space, then P_I decomposes into an iteration of $\aleph_0$-distibutive and c.c.c. forcings. If P_I is nowhere c.c.c. then the first extension of the iteration must be nontrivial, contradicting the assumptions. $\square$

In the next section, we will be able to extend the conclusion of the corollary to total canonization of equivalences classifiable by countable structures, but the treatment of the essentially countable equivalences is a necessary preliminary step towards that goal.

The final observation in this section concerns the simplification within the class of countable equivalence relations. The class of Borel countable equivalence relations is very complex and extensively studied. It is possible that many forcings have highly nontrivial intersection of their spectrum with this class. In the fairly common case that the forcing preserves Baire category, all that complexity boils down to E_0 though.

Theorem 4.2.7. *Whenever I is a σ-ideal on a Polish space X such that the forcing P_I is proper and preserves Baire category, then every essentially countable Borel equivalence relation is reducible to E_0 on a positive Borel set.*

Proof. Suppose $B \in P_I$ is an I-positive Borel set and $f : B \to Y$ is a Borel map into a Polish space with a countable equivalence relation E on it. The argument of Miller and Kechris [25, Theorem 12.1] shows that there is a Borel set $D \subset \omega^\omega \times Y \times Y$ such that for every $z \in \omega^\omega$ the section D_z is a Borel hyperfinite equivalence relation included in E, and for every $y \in Y$ the set $A_y = \{z \in \omega^\omega : [y]_E = [y]_{D_z}\}$ is comeager in ω^ω. Now the set $D' \subset B \times \omega^\omega$ of all pairs x, z such that $z \in A_{f(x)}$, has comeager vertical sections. Since the forcing P_I preserves the Baire property, it must be the case that there is a point $z \in \omega^\omega$ such that the corresponding horizontal section $C \subset B$ is I-positive. But then, $D_z = E$ on $f''C$ and therefore the pullback $f^{-1}E$ is on C reducible to a hyperfinite equivalence relation D_z and therefore to E_0. $\square$

Question 4.2.8. Is there a σ-ideal I such that the quotient P_I is proper, E_0 is not in the spectrum of I and still there is some countable Borel equivalence relation in the spectrum of I?

4.3 Equivalences classifiable by countable structures

The main purpose of this section is to bootstrap Corollary 4.2.6 to include all equivalences classifiable by countable structures.

Definition 4.3.1. An equivalence relation E on a Polish space X is *classifiable by countable structures, cbcs* if there is a Borel map $f : X \to \Pi_n \mathcal{P}(\omega^n)$ such that elements $x, y \in X$ are E-related if and only if $f(x), f(y)$ are isomorphic as relational structures on ω.

An isomorphism of structures on ω is naturally induced by a permutation of ω, and therefore every cbcs equivalence relation is Borel reducible to an orbit equivalence generated by a Polish action of the infinite permutation group S_∞. The converse implication also holds by a result of Becker and Kechris [22, Theorem 12.3.3]: every orbit equivalence relation generated by a Polish action of the infinite permutation group or its closed subgroups is cbcs.

Cbcs equivalence relations are in general analytic; however, the results of Section 4.4 show that for canonization purposes we only need to look at the Borel case. The analysis of the structure of such Borel equivalence relations uses the Friedman-Stanley jump:

Definition 4.3.2. Let E be an equivalence relation on a Polish space X. E^+ is the equivalence relation on X^ω defined by $\vec{x}\ E^+\ \vec{y}$ if the E-saturations of $\mathrm{rng}(\vec{x})$ and $\mathrm{rng}(\vec{y})$ are the same.

If E is a Borel equivalence relation then E^+ is again Borel and it is strictly higher than E in the Borel reducibility hierarchy. By induction on $\alpha \in \omega_1$ define Borel equivalence relations F_α by setting $F_1 = \mathrm{ID}$ on the Cantor space, $F_{\alpha+1} = F_\alpha^+$ and F_α =the disjoint union of $F_\beta : \beta \in \alpha$ for limit ordinals α.

Fact 4.3.3. *[22, Theorem 12.5.2] Every cbcs Borel equivalence relation is Borel reducible to some F_α for some $\alpha \in \omega_1$.*

This fact allows one to prove general facts about Borel cbcs equivalences by showing that the property in question is downward directed in the Borel reducibility hierarchy and persists under the Friedman-Stanley jump and countable disjoint union. This is the path taken in [22, Chapter 13], and we will take it here. The following purely forcing property will be instrumental:

Definition 4.3.4. A poset P has the *separation property* if it forces that for every countable set $a \subset 2^\omega$ and every point $y \in 2^\omega \setminus a$ there is a ground model coded Borel set $B \subset 2^\omega$ separating y from a: $y \in B$ and $a \cap B = 0$.

Theorem 4.3.5. *Suppose that I is a σ-ideal on a Polish space X such that P_I is proper.*

1. *If P_I has the separation property then classifiable by countable structures$\to_I$smooth;*
2. *if I has total canonization for essentially countable Borel equivalence relations, then P_I has the separation property.*

Proof. The first item follows abstractly from two claims of independent interest.

Claim 4.3.6. *Suppose that I is a σ-ideal on a Polish space X such that the quotient P_I is proper and has the separation property. Let F be a Borel equivalence relation on some Polish space. If $\leq_B F \to_I$smooth then $\leq F^+ \to_I$smooth.*

Proof. Let Y be a Polish space and F a Borel equivalence relation on it. Let E be an equivalence relation on some Borel I-positive set $B \subset X$ that is Borel reducible to F^+ by a Borel function $f : B \to Y^\omega$. We need to find a Borel I-positive set $C \subset B$ on which E is smooth.

Let M be a countable elementary submodel of a large structure and let $C \subset B$ be the Borel I-positive set of generic points. Let $A = \{f(x)(n) : n \in \omega, x \in C\} \subset Y$. We will show that $F \restriction A$ is smooth by decomposing A into countably many Borel sets on which F is smooth and applying [22, Corollary 7.3.2]. For every number $n \in \omega$, consider the equivalence relation E_n on B given by $x_0\ E_n\ x_1 \leftrightarrow f(x_0)(n)\ F\ f(x_1)(n)$, the Borel function $f_n : B \to X$, $f_n(x) = f(x)(n)$ reducing E_n to F and the set $D_n = \{B' \subset B : B' \in P_I$ and $E_n \restriction B'$ is smooth and $f_n'' B'$ is Borel$\}$. The set $D_n \in M$ is dense in P_I by the assumptions. Moreover, for every set $B' \in D_n$, the equivalence $F \restriction f_n'' B'$ is smooth: the f_n-image of a countable analytic separating family for $E_n \restriction B'$ will be a countable analytic separating family for $F \restriction f_n'' B'$. Finally, since the set $C \subset B$ consists of M-generic points only, it must be the case that $C \subset \bigcap_n \bigcup (D_n \cap M)$ and so $A = \bigcup_n f_n'' C \subset \bigcup \{f_n'' B' : n \in \omega, B' \in D_n \cap M\}$ decomposes into countably many sets on which F is smooth.

Let $g : A \to 2^\omega$ be a Borel function reducing $F \restriction A$ to identity. Consider the Borel functions $g_n : C \to 2^\omega$ defined by $g_n(x) = g(f(x)(n))$. The separation property implies that C forces every element of this set to be separated from the rest of the set by a ground model coded Borel set. Let N be another countable elementary submodel of a large structure and $D \subset C$ a set of its generic points. Let $A_m : m \in \omega$ be a list of the Borel sets in the model N. By the forcing theorem, for every $x \in D$ and every $n \in \omega$ there is $m \in \omega$ such that $g_n(x)$ is the only point of $\{g_k(x) : k \in \omega\}$ in the set A_m. Now, let $h : D \to (2^\omega)^\omega$ be the function defined by $h(x)(m) = y$ if the intersection $\{g_k(x) : k \in \omega\} \cap A_m$ contains exactly one point y, and $h(x)(m) =$trash otherwise. The function h is Borel: $h(x)(m) = g_k(x)$ if and only if there is a Borel set $B' \in N$ such that $x \in B'$ and $B' \Vdash \dot{A}_m$ separates $g_k(\dot{x}_{gen})$ from the set $\{g_l(\dot{x}_{gen}) : l \neq k\}$ bz the forcing theorem applied in N and analytic absoluteness between V and the generic extensions of N. It is not difficult to see that h reduces the equivalence $E \restriction D$ to the identity.

□

Claim 4.3.7. *Suppose that I is a σ-ideal on a Polish space X and $G_n : n \in \omega$ are equivalence relations on some Polish spaces. If $\leq_B G_n \to_I$ smooth holds for every number $n \in \omega$ then $\leq_B G \to_I$ smooth holds as well, where G is the disjoint union of $\{G_n : n \in \omega\}$*

Proof. Suppose that $B \subset X$ is a Borel I-positive set and E an equivalence relation on it reducible to G by a Borel function f. Since I is a σ-ideal, for some number $n \in \omega$ the set $C = \{x \in B : f(x) \in \mathrm{dom}(G_n)\}$ is Borel and I-positive. By $\leq_B G_n \to_I$smooth, it is possible to thin out C so that $E \restriction C$ is smooth, and the claim follows. □

Now by induction on $\alpha \in \omega_1$ prove that $\leq F_\alpha \to_I$smooth: the successor and limit stage are handled by the two separate claims. Finally, every cbcs equivalence relation is reducible to an orbit equivalence relation induced by a Polish action of S_∞, and therefore it simplifies to a Borel equivalence relation on a Borel I-positive set by Theorem 4.4.1. The first item follows by Fact 4.3.3.

For the second item, suppose that the assumptions hold and $B \Vdash \dot{a} \subset 2^\omega$ is a countable set. Without loss of generality we may assume that $V \cap \dot{a} = 0$ is forced as well. (If not, use properness to enclose the countable set $\dot{a} \cap V$ into a ground model countable set, and separate it from the rest by a countable collection of singletons.) Thinning out the set B we may assume that there are Borel functions $f_n : n \in \omega$ on the set B such that $B \Vdash \dot{a} = \{\dot{f}_n(\dot{x}_{gen}) : n \in \omega\}$. Let M be a countable elementary submodel of a large enough structure, and let $C \subset B$ be the I-positive Borel set of all M-generic points for P_I in the set B.

Observe that each of the functions $f_n : n \in \omega$ is countable-to-one on the set C: since the poset P_I adds a minimal real degree, the set $D_n \subset P_I$ consisting of those conditions on which the function f_n is one-to-one is dense in P_I, and $C \subset \bigcup (M \cap D_n)$. This means that the equivalence relation E generated by the relation $x_0 E x_1 \leftrightarrow \exists n_0 \exists n_1\ f_{n_0}(x_0) = f_{n_1}(x_1)$ has countable equivalence classes; it is easily verified that it is Borel as well. Since (1) holds, this equivalence relation can be reduced to smooth, and since the minimal real degree is added, there must be then a Borel I-positive set $D \subset C$ such that $E \restriction D = \mathrm{ID}$. This means that for distinct points $x_0, x_1 \in D$ the countable sets $\{f_n(x_0) : n \in \omega\}$ and $\{f_n(x_1) : n \in \omega\}$ are disjoint; in particular, the functions $f_n : n \in \omega$ are all injections on D. Let $\bar{f}_n = (f_n \restriction D) \setminus \bigcup_{m \in n} f_m$ and let $B_n = \mathrm{rng} \bar{f}_n$. The sets B_n are ranges of Borel injections and as such they are Borel, and for every point $x \in D$ their collection separates the set $\{f_n(x) : n \in \omega\}$. It follows that $D \Vdash$ the sets in $\{B_n : n \in \omega\}$ separate each point of the set a from the rest of the set a. □

Corollary 4.3.8. *If the quotient forcing P_I is proper, nowhere c.c.c. and generates a minimal forcing extension, then the ideal I has total canonization for equivalences classifiable by countable structures.*

Proof. This follows by modus ponens from the previous results. I has total canonization for smooth equivalences since it adds a minimal real degree. Once that is known, Corollary 4.2.6 shows that it has total canonization for essentially countable equivalence relations. Theorem 4.3.5(2) then yields the separation property of P_I, and (1) together with the minimal real degree yields the total canonization for equivalences classifiable by countable structures. □

To conclude this section, we will show that the theory of the choiceless model $V[[x]]_E$ corelates with the canonization properties of the equivalence relations classifiable by countable structures.

Theorem 4.3.9. *Suppose that X is a Polish space and I is a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal on it such that the forcing P_I is proper. Let E be an equivalence on X classifiable by countable structures. The following are equivalent:*

1. *E is not reducible to a countable equivalence relation on any I-positive Borel set;*
2. *it is forced that $V[[\dot{x}_{gen}]]_E$ fails the axiom of choice.*

If E is a more complicated Borel equivalence relation, then AC can probably fail in the model $V[[x]]_E$ for more complicated reasons, even though we do not see any good examples at this point.

Proof. First, $\neg(1) \to \neg(2)$. If $B \subset X$ is a Borel I-positive set on which the equivalence E is reducible to a countable equivalence F on a Polish space Y, as witnessed by a Borel function $f : B \to Y$, then f attains only countably many values at each equivalence class and so $B \Vdash \dot{f}(\dot{x}_{gen}) \in V[[x_{\text{gen}}]]_E$ by Theorem 2.1.9. For absoluteness reasons, every generic extension containing $\dot{f}(\dot{x}_{gen})$ will also contain one of its f-preimages in the set B, which then must be E-equivalent to $\dot{x}_{gen}$; thus $V[[\dot{x}_{gen}]]_E = V[\dot{f}(\dot{x}_{gen})]$ by the definition of the model $V[[\dot{x}_{gen}]]_E$ and this model satisfies the axiom of choice.

For the opposite implication, we will first inductively define functions h_α on $\text{dom}(F_\alpha)$ which assign each F_α-equivalence class a countable transitive set in a fairly canonical fashion, this for every countable ordinal $\alpha \in \omega_1$. For $\alpha = 1$, identify $\text{dom}(F_1)$ with the space of all infinite co-infinite subsets of ω and set $h_1(x) = \text{trcl}(\{1, x\})$. For the induction, let $h_{\alpha+1}(x) = \text{trcl}(\{\alpha + 1, h_\alpha(x(n))\})$ and $h_\alpha(x) = \text{trcl}(\{\alpha, h_\beta(x(\beta))\})$ for a limit ordinal $\alpha \in \omega_1$. A simple transfinite induction argument will show that two points in $\text{dom}(F_\alpha)$ are F_α-related if and only if their h_α-images are the same.

So now assume that some condition B forces $V[[\dot{x}_{gen}]]_E$ to satisfy the axiom of choice. Thinning out the condition B if necessary, we may assume that there is an ordinal α such that $E \restriction B$ is Borel reducible to F_α via some Borel function f–Theorem 4.4.1. Let $a = h_\alpha(\dot{f}(\dot{x}_{gen}))$.

First argue that B forces $V[[\dot{x}_{gen}]]_E = V(\dot{a})$. To see this, note that every generic extension containing an E-equivalent of x_{gen} will contain one of its E-equivalents in the Borel set B by an absoluteness argument, so it will contain an F_α-equivalent of the point $f(x_{\text{gen}})$ and so the set a; thus $a \in V[[x_{\text{gen}}]]_E$. On the other hand, any two mutually generic extensions $V(a)[y], V(a)[z]$ satisfying ZFC that contain an enumeration of the set a in type ω will contain points in the set B that are mapped by $h_\alpha \circ f$ to a, and these are E-equivalent to x_{gen}. By mutual genericity, $V[[x_{\text{gen}}]]_E \subset V(a) = V(a) \cap V(a)[y] \cap V(a)[z]$, and so $V[[x_{\text{gen}}]]_E = V(a)$.

Now, if $V[[x_{\text{gen}}]]_E$ satisfies the axiom of choice, then it must be equal to $V[z]$ for a certain point $z \in 2^\omega$ which in some simple fixed way codes the set a. By Theorem 2.1.9, it must be the case that there is a Borel I-positive set $C \subset B$ and a Borel function $g : C \to 2^\omega$ representing $\dot{z}$ such that g attains only countably many values on each E-equivalence class, and $g(x)$ codes $h_\alpha(f(x))$ for every $x \in C$; in particular, the g images of distinct E-equivalence classes are disjoint. By a result of Kechris, [8, Lemma 12.5.6], the existence of such a function guarantees that $E \restriction C$ is essentially countable.

□

Corollary 4.3.10. *If I is a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal such that P_I is proper and adds a minimal real degree then classifiable by countable structures$\to_I$essentially countable.*

Proof. To see this, let B be an I-positive Borel set and E an equivalence on B that is classifiable by countable structures, and look at the set a in the extension defined in the previous proof; so $V[[x]]_E = V(a)$. Look at the $\in$-minimal element $z \in a \cup \{a\} \setminus V$. If there is no such an element, then $a \in V$, $V(a) = V[[\dot{x}_{gen}]]_E = V$ satisfies the axiom of choice and we are finished by the theorem. If z is well-defined, then by properness z is covered by a ground model countable set, so $V[z]$ contains a real not in V. Since P_I adds a minimal real degree, $V[x_{\text{gen}}] = V[z] \subset V[[x_{\text{gen}}]]_E = V[x_{\text{gen}}]$, the model $V[[x_{\text{gen}}]]_E$ satisfies choice and the theorem seals the deal again. □

A brief discussion is in order here. The analysis of the failure of choice in the model $V[[x]]_E$ may provide further information about the Borel reducibility properties of the equivalence relation E. We provide three examples:

Consider the countable support product of infinitely many copies of E_0-forcing, with the equivalence relation E_0^ω on it. Thus, the underlying space X is $(2^\omega)^\omega$, and the equivalence relation connects $x, y \in X$ if $\forall n\ x(n)\ E_0\ y(n)$. The model $V[[x]]_{E_0^\omega}$ contains the sequence of E_0-orbits of the points $x_{\text{gen}}(n) : n \in \omega$. Each of these sets is countable there, since it is just an E_0-orbit, the sets form a countable sequence, but their union is not countable in $V[[x_{\text{gen}}]]_{E_0^\omega}$.

A more significant violation of choice occurs in the model $V[[\vec{x}_{gen}]]_{E_0^\omega}$ where $\vec{x}_{gen}$ is the countable sequence of points in 2^ω obtained by the countable support product of Silver forcing. Again, the model contains the sequence $\langle[\vec{x}_{gen}(n)]_{E_0} : n \in \omega\rangle$ of countable sets whose union is not countable. Each of the equivalence classes $[\vec{x}_{gen}(n)]_{E_0}$ splits into two equivalence classes of the equivalence E_s connecting two binary sequences if they differ on a finite set of even size. The model $V[[\vec{x}_{gen}]]_{E_0^\omega}$ does not contain a choice function on this countable system of pairs. This is in contradistinction to the previous case. In the countable support product of E_0-forcings, there is a dense set consisting of conditions p such that for every $n \in \omega$, $E_0 \restriction p(n) = E_s \restriction p(n)$ by the canonization theorem ???. Thus, in that model one could simply choose for every number $n \in \omega$ that unique E_s-equivalence class included in $[\vec{x}_{gen}(n)]_{E_0}$ which has nonempty intersection with $p(n)$.

In contrast, look at the finite support product of infinitely many copies of Cohen forcing, with the equivalence relation F_2 on it. That is, the underlying space is $X = (2^\omega)^\omega$, with the equivalence relation connecting points x, y if $\{x(n) : n \in \omega\} = \{y(n) : n \in \omega\}$. The model $V[[x_{\text{gen}}]]_{F_2}$ contains the set $\{x_{\text{gen}}(n) : n \in \omega\}$ and not its enumeration. In this model, countable unions of countable sets are countable.

If the equivalence E is not classifiable by countable structures, the axiom of choice may probably fail in the model $V[[x]]_E$ in much more complex fashion. The possibility that $V[[x]]_E$ has the same reals as V and still cannot well-order its $\mathcal{P}(\omega_1)$ is left open in this book.

4.4 Orbit equivalence relations

Orbit equivalence relations of Polish group actions form a quite special subclass of analytic equivalence relations. In our context, they can be simplified to Borel equivalence relations by passing to a Borel I-positive Borel set:

Theorem 4.4.1. *Suppose that X is a Polish space, I a σ-ideal on it such that the quotient poset P_I is proper, $B \in P_I$ an I-positive Borel set, and E an equivalence relation on the set B reducible to an orbit equivalence relation of a Polish action. Then there is a Borel I-positive set $C \subset B$ such that $E \restriction C$ is Borel. If the ideal I is c.c.c. then in fact the set C can be found so that $B \setminus C \in I$.*

Note that the equivalence E may have been analytic non-Borel in the beginning.

Proof. Suppose that a Polish group G acts on a Polish space Y, and $f : B \to Y$ is a Borel reduction of E to the orbit equivalence relation E_G of the action. By [3, Theorem 7.3.1], the space Y can be decomposed into a union of a collection $A_\alpha : \alpha \in \omega_1$ of pairwise disjoint invariant Borel sets such that $E_G \restriction A_\alpha$ is Borel for every ordinal α; moreover, this decomposition is absolute between various transitive models of ZFC with the same $\aleph_1$.

Let M be a countable elementary submodel of a large enough structure and let $C \subset B$ be the Borel I-positive subset of all M-generic points for P_I in the set B. This is a Borel I-positive set by the properness of the quotient poset P_I, and if I is c.c.c. then in fact $B \setminus C \in I$. For every point $x \in C$, $M[x] \models f(x) \in \bigcup\{A_\alpha : \alpha \in M \cap \omega_1\}$. Absoluteness considerations imply that $f''C \subset \bigcup_{\alpha \in \omega_1 \cap M} A_\alpha$. The equivalence relation E_G restricted on the latter set is Borel, and so $E \restriction C$ is Borel. $\square$

This feature is quite special to orbit equivalence relations, and elsewhere it may not hold. Consider the following example:

Example 4.4.2. Let K be an analytic non-Borel ideal on ω and the equivalence relation E_K on $X = (2^\omega)^\omega$ defined by $\vec{x} E_K \vec{y}$ if $\{n \in \omega : \vec{x}(n) \neq \vec{y}(n)\} \in K$. Consider the ideal I on X associated with the countable support product of Sacks forcing. Then The equivalence E_K is non-Borel, and since $E_K \leq E_K \restriction B$ for every Borel I-positive set $B \subset X$, it is also the case that $E_K \restriction B$ is non-Borel.

4.5 Hypersmooth equivalence relations

Equivalence relations reducible to E_1 are connected to decreasing sequences of real degrees in the generic extension.

Theorem 4.5.1. *Suppose that X is a Polish space and I is a σ-ideal such that the forcing P_I is proper. The following are equivalent:*

1. *E_1 is in the spectrum of I;*

2. *some condition forces that in the generic extension there is an infinite decreasing sequence of V-degrees.*

Proof. Suppose first that E_1 is in the spectrum of I, as witnessed by a Borel I-positive set $B \subset X$ an equivalence relation E on B, and a Borel reduction $f : B \to (2^\omega)^\omega$ of E to E_1. We claim that B forces that in the generic extension, for every number $n \in \omega$ there is $m \in \omega$ such that $f(x_{\mathrm{gen}}) \restriction [n, \omega) \notin V[f(x_{\mathrm{gen}}) \restriction [m, \omega)]$.

Suppose for contradiction that some condition $C \subset B$ forces the opposite for some definite number $n \in \omega$. Let M be a countable elementary submodel of a large enough structure containing C and f, and let $D \subset C$ be the Borel I-positive set of all M-generic points in the set C. Consider the Borel set $Z \subset (2^\omega)^{\omega \setminus n}$ defined by $z \in Z$ if and only if for every number $m > n$ there is a Borel function $g \in M$ such that $g(z \restriction [m, \omega)) = z$. It is not difficult to see that E_1 restricted to Z has countable classes, since there are only countably many functions in the model M. Now, whenever $x \in D$ then $f(x) \restriction [n, \omega) \in Z$ by the forcing theorem, and for $x, y \in D$, $f(x) E_1 f(y)$ if and only if $f(x) \restriction [n, \omega) E_1 f(y) \restriction [n, \omega)$. This reduces $E \restriction D$ to a Borel equivalence relation with countable classes, which contradicts the assumption that E_1 is embeddable into it.

Now, in the generic extension $V[x]$ define a function $h \in \omega^\omega$ by induction: $h(0) = 0$, and $h(n+1) =$the least number $m \in \omega$ such that $f(x) \restriction [h(n), \omega) \notin V[f(x) \restriction [m, \omega)]$. The function h clearly belongs to all models $V[f(x) \restriction [m, \omega)]$, and therefore the sequence $f(x) \restriction [h(n), \omega) : n \in \omega$ forms the required strictly decreasing sequence of degrees.

For the other direction, suppose that some condition forces a decreasing sequence of V-degrees into the forcing extension. Thus, some condition forces $\dot{x}_n : n \in \omega$ to be the decreasing sequence in 2^ω. We can adjust the sequence to attain the following properties:

- for every $n \in \omega$, $x_n \notin V[\langle x_m : m > n \rangle]$
- $\langle x_m : m > n \rangle \in V[x_n]$.

The second item is necessary to the construction below. To secure it, replace $x_n : n \in \omega$ by a decreasing sequence $\bar{x}_n : n \in \omega$ such that $V[\bar{x}_n] \models \bar{x}_{n+1}$ is the $\leq_n$-least point below which there is an infinite decreasing sequence of degrees and $V[\bar{x}_{n+1}] \neq V[x_n]$. Here, $\leq_n$ is any well-ordering of $2^\omega \cap V[\bar{x}_n]$ simply definable from a wellordering of V and $\bar{x}_n$; for example, let Q be the least poset of size $\leq \mathfrak{c}$ in V for which $\bar{x}_n$ is generic, and then turn any well-order of Q-names for reals into a well-order of $V[\bar{x}_n]$.

Moving to a stronger condition $B \subset P_I$, we may represent each of the degrees as a Borel function $f_n : B \to 2^\omega$, for each $n \in \omega$. We claim that the product of these functions, $g : B \to (2^\omega)^\omega$ defined by $g(x)(n) = f_n(x)$, has the property that the pullback $E = g^{-1} E_1$ cannot be reduced to a countable Borel equivalence relation on any Borel I-positive set. This will prove the theorem. Suppose for contradiction that there is a Borel I-positive set set $C \subset B$ such that $E_1 \not\leq E \restriction$

B; by the dichotomy 1.3.13 it must be the case that $E \restriction C \leq E_0$ via some Borel map $g : C \to 2^\omega$. Move to the forcing extension $V[x_{\text{gen}}]$ for some P_I-generic point $x_{\text{gen}} \in C$.

Claim 4.5.2. *For every number $n \in \omega$, $g(x_{\text{gen}}) \in V[f_n(x_{\text{gen}})]$.*

Proof. Working in $V[f_n(x_{\text{gen}})]$, we see that there is a point $x \in C$ such that $\langle f_m(x) : m > n\rangle E_1 \langle f_m(x_{\text{gen}}) : m > n\rangle$, since such an x exists in the model $V[x_{\text{gen}}]$ and analytic absoluteness holds between the two models. Note that the sequence $\langle f_m(x_{\text{gen}}) : m > n\rangle$ is in the model $V[f_n(\dot{x}_{gen})]$. Now all such numbers x must be mapped by g into the same E_0-equivalence class. This class is countable and contans $g(x_{\text{gen}})$. The claim follows. $\square$

Now in the model $V[g(x_{\text{gen}})]$, there is a point $x \in C$ such that $g(x) = g(x_{\text{gen}})$, since there is such a point x in the model $V[x_{\text{gen}}]$ and analytic absoluteness holds between the two models. The sequence $\langle f_n(x) : n \in \omega\rangle$ must be E_1-equivalent to the sequence $\langle f_n(x_{\text{gen}}) : n \in \omega\rangle$; let us say that they are equal from some m on. But then, $f_m(x_{\text{gen}}) \in V[g(x_{\text{gen}})] \subset V[f_{m+1}(x_{\text{gen}})]$, contradiction! $\square$

Corollary 4.5.3. *If the forcing P_I adds a minimal real degree then E_1 does not belong to the spectrum of I.*

Chapter 5

Cardinals in choiceless models

The study of Borel reducibility of Borel equivalences is reminiscent of the study of the cardinalities of the sets X/E, where E is a Borel equivalence on a Polish space X and X/E is the set of all E-equivalence classes. This is of course not at all true in the context of the axiom of choice, since these sets are either countable or of size at least continuum by Silver dichotomy 1.3.10, and since they are all surjective images of the continuum, there are only two possibilities for their cardinality: $\aleph_0$ or $\mathfrak{c}$. Without the axiom of choice, a surjective image of a set may have cardinality strictly larger than the domain set itself; thus, the previous argument breaks down and the cardinalities of the quotient spaces take a life on their own.

The study of cardinal inequalities in the choiceless context is much more complicated and meaningful than the corresponding subject under the axiom of choice. In order to delineate the subject, we will only study the cardinalities of surjective images of the reals. Note that for every surjection $f : \mathbb{R} \to A$ there is the correspoding equivalence E on $\mathbb{R}$ defined by $xEy \leftrightarrow f(x) = f(y)$, for which $|\mathbb{R}/E| = |A|$; thus these cardinals can be called *equivalence cardinals.* It is clear that if E is an equivalence on a Polish space X Borel reducible to an equivalence F on a Polish space Y, then $|X/E| \leq |Y/F|$ since the map $[x]_E \mapsto [f(x)]_F$ is an injection whenever $f : X \to Y$ is a Borel reduction. On the other hand, there does not seem to be any clear way of using a cardinal inequality to find a Borel (or even more complicated) reduction between the underlying Polish spaces. The connection between these two notions of comparison was investigated in print by [13] and [3, Chapter 8].

One should point out the connection with so-called Turing thesis for analysis ??? Many results in the theory of Borel equivalence relations are worded in the following way: a certain equivalence on a certain class of structures is not Borel reducible to another equivalence on another class of structures. On the face of it, the validity of this statement depends on the way how the structures in question are represented as points in Polish spaces, so that the statement can be formalized properly. A question arises whether the validity of this statement may change if a different presentation appears. The Turing thesis for analysis asserts that this never occurs:

Whenever C is a natural class of mathematical structures with a natural equivalence on it, then for every representation X of objects in this class as points in a Polish space, with a corresponding equivalence relation E, and for every other representation Y, F of this class, it is the case that $E \leq_B F$ and vice versa, $F \leq_B E$.

This is never an issue when we compare only cardinalities of the sets of equivalence classes; for two representations as above, there is the obvious bijection between the sets X/E and Y/F that maps any E-class to that unique F-class that represents the objects equivalent to those in the E-class. Thus, obtaining a Borel reduction from a mere cardinal inequality may be viewed as a confirmation of Turing thesis for the equivalences concerned, in the particular model of set theory without choice.

In order to simplify the expressions, for a Borel equivalence relation E on a Polish space X we will denote by $|E|$ the cardinality of the set X/E.

5.1 Completely regular models

We will first turn our attention to the study of completely regular models. By this we understand either the choiceless Solovay extension derived from a strongly inaccessible cardinal, or any of the models satisfying the axiom of determinacy and containing all the reals, as obtained in the presence of suitable large cardinals. The Solovay model is easier to study and requires weaker initial assumtpions; on the other hand, the axiom of determinacy or even the theory ZF+AD$\mathbb{R}$+θ regular offers a nearly complete resolution of questions regarding the cardinalities of surjective images of the reals as well as other issues. However, the answers to the questions we pose here are the same in both contexts.

What is the cardinal structure of the surjective images of reals under AD or AD$\mathbb{R}$+θ regular? Well, first there are the wellorderable equivalence cardinal, whose size is capped by the cardinal θ. The study of wellordered cardinals under θ was the main concern of the Cabal school of set theory. There is the smallest non-well-orderable equivalence cardinal, which happens to be the size of the powerset of ω. By the celebrated coding theorem of Moschovakis, the powersets of the wellorderable cardinals below θ are also surjective images of the reals; these are the linearly orderable equivalence cardinals. Their structure was studied by Woodin in [47]. There is the smallest non-linearly-orderable equivalence cardinal, which happens to be E_0 [5]. Beyond that, there is the rich realm of Borel equivalences and a whole zoo of other things that no one has ever seen.

5.1a Turning cardinal inequalities into Borel reductions

As indicated in the introduction to this chapter, if E is an equivalence on a Polish space reducible to another equivalence F, then $|E| \leq |F|$, but the reverse implication does not quite seem to hold. In this section we will show that the metods of this book can be used to turn cadinal inequalities into Borel and even more regular reductions between equivalence relations.

Proposition 5.1.1. *Suppose that E is a Borel equivalence relation on a Polish space X and I is a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal on X such that ZFC proves that the quotient forcing is proper and has E in its spectrum. Then in the Solovay model, whenever F is an equivalence relation on a Polish space and $|E| \leq |F|$, then E is Borel reducible to F.*

Proof. Suppose that some condition in $\mathrm{Coll}(\omega, <\kappa)$ forces $|E| \leq |F|$ as witnessed by an injection $\hat{f}$. Since the Solovay model has uniformization, there will be also a name $\dot{f}$ for a lifting from $\mathrm{dom}(E)$ to $\mathrm{dom}(F)$ of $\hat{f}$. By standard homogeneity arguments as in Fact 1.3.16 we may assume that both F and $\dot{f}$ are definable from parameters in the ground model. Let I be the σ-ideal from the assumptions. Note that $P_I \Vdash$there is y such that $\mathrm{Coll}(\omega, <\kappa) \Vdash \dot{f}(\dot{x}_{gen}) = \check{y}$, since in the P_I-extension, the homogeneity of the collapse implies that the largest condition in $\mathrm{Coll}(\omega, <\kappa)$ decides whether $\dot{f}(\dot{x}_{gen})$ belongs to any given basic open set, and so there will be only one possibility left. Let $\dot{y}$ be the P_I-name for the value of $\dot{f}(\dot{x}_{gen})$. There is a condition $B \in P_I$ and a Borel function $g : B \to \mathrm{dom}(F)$ such that $B \Vdash \dot{y} = \dot{g}(\dot{x}_{gen})$. Now in the Solovay model $V(\mathbb{R})$, the set $C \subset B$ consisting only of P_I-generic points for the model V is Borel and I-positive by Theorem 1.3.21–this is the only place where we use the definability of the ideal I. The forcing theorem then implies that for every point $x \in C$ it is the case that $g(x) = f(x)$ and so g is a Borel reduction of $E \restriction C$ to F. However, the assumptions imply that E is reducible to $E \restriction C$, and the proposition follows. $\square$

In fact, the statement of the proposition can be strengthened in many circumstances. Theorem 1.3.21 holds true in ZFC for many ideals that are not $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$, for example for the Laver ideal. If the forcing P_I is bounding, then the Borel function g in the proof will be continuous on a Borel I-positive set. If one wants to get a continuous reduction of E to F in such a case, it still has to be proved that E reduces continuously to $E \restriction B$ for every Borel I-positive set. This is satisfied in most examples discussed in this book, but it always requires a nontrivial argument. A similar degree of caution applies to the search for injective reduction from the assumption that the forcing P_I adds a minimal real degree.

Corollary 5.1.2. *In $V(\mathbb{R})$, if F is an equivalence relation and $|E_0| \leq |F|$ then E_0 reduces to F in a continuous injective fashion.*

Proof. Consider the ideal I on 2^ω of Section 3.4. It has E_0 in its spectrum, and the quotient forcing is bounding and adds a minimal real degree. In view of the proposition, it is enough to verify that for every I-positive Borel set $B \subset 2^\omega$, there is a continuous injection $f : 2^\omega \to B$ reducing E_0 to $E_0 \restriction B$. To see that, let P be the creature forcing in that section, choose $p \in P$ such that $[p] \subset B$ by Fact 3.4.1; $p = \langle t_p, \vec{c}_p \rangle$ where $\vec{c}_p$ is a sequence of pairs of finite binary sequences and the sequences in any given pair have the same length. Let f be the continuous function from 2^ω to $[p]$ for which $f(x)$ is the concatenation of t_p with $\vec{c}_p(n)(x(n)) : n \in \omega$. This is the desired injective continuous reduction. $\square$

Corollary 5.1.3. *In $V(\mathbb{R})$, if F is an equivalence relation and $|E_1| \leq |F|$ then E_1 reduces to F in a continuous injective fashion.*

Proof. There are a number of bounding forcings with E_1 in the spectrum, but none of them add a minimal real degree, as Theorem 4.5.1 shows. How does one manufacture an injective reduction then?

Look at the illfounded iteration of Sacks forcing of length inverted ω. ??? $\square$

Corollary 5.1.4. *In $V(\mathbb{R})$, if F is an equivalence relation and $|E_2| \leq |F|$ then E_2 reduces to F in a continuous injective fashion.*

Proof. Consider the ideal I on 2^ω of Section 3.5. It has E_2 in its spectrum and the quotient forcing P_I is proper, bounding, and adds a minimal real degree. If $B \subset 2^\omega$ is an I-positive Borel set, the properness proof of Theorem 3.5.9 provides a continuous injective reduction of E_2 to $E_2 \restriction B$. Together with the proposition, this completes the proof of the corollary. $\square$

Corollary 5.1.5. *In $V(\mathbb{R})$, if F is an equivalence relation and $|E_3| \leq |F|$ then E_3 reduces to F in a continuous injective fashion.*

Corollary 5.1.6. *In $V(\mathbb{R})$, if F is an equivalence relation and $|E_{K_\sigma}| \leq |F|$ then E_{K_σ} reduces to F in a continuous injective fashion.*

Proof. Consider the Silver forcing of Section 3.6 and its associated σ-ideal I on 2^ω. E_{K_σ} is in the spectrum of I as witnessed by some Borel equivalence relation E on 2^ω to which E_{K_σ} continuously injectively reduces. Also, for every I-positive Borel set $B \subset 2^\omega$, E reduces to $E \restriction B$ by a continuous injective function. Moreover, the Silver forcing is bounding and adds a minimal real degree, Theorem 3.6.2. The corollary follows. $\square$

5.1b Regular and measurable cardinals

The commonly discussed equivalence cardinals have features that resemble regular and measurable cardinals in ZFC. We will make several perhaps ad hoc definitions and show how they relate to the common ZFC notions.

Definition 5.1.7. A set A is a *regular cardinality* if for every equivalence relation E on A, either one of the classes of E has the same cardinality as A, or there is a set $B \subset A$ of the same cardinality as A consisting of pairwise inequivalent elements.

Definition 5.1.8. An ideal I on a set A is *weakly normal* if for every equivalence relation E on A, either one of the classes of E is I-positive, or there is an I-positive set $B \subset A$ consisting of pairwise inequivalent elements.

It is not difficult to see that if κ is a wellordered cardinal regular in the usual sense, it is also regular in the sense of Definition 5.1.7. Moreover, if I is a normal ideal on κ in the usual sense, it is also weakly normal in the sense of Definition 5.1.8. If $S \subset \kappa$ is I-positive and E is an equivalence relation on S, let $f : S \to S$ be defined as $f(\alpha)$ =the least ordinal equivalent to α. Either this function is equal to identity on an I-positive set, leading to an I-positive set of pairwise inequivalent elements, or it is regressive on an I-positive set, which after stabilization leads to an I-positive equivalence class. However, the ideal of bounded subsets of κ is not normal, and still it is weakly normal in the above sense.

From now on we will work in the Solovay model $V(\mathbb{R})$, derived from a $\mathrm{Coll}(\omega, < \kappa)$ extension for some inaccessible cardinal κ.

Proposition 5.1.9. *$\mathbb{R}$ is a regular cardinality.*

Proposition 5.1.10. *E_0 is a regular cardinality.*

The notion of ergodicity as introduced in the wording of Theorem 2.1.3 gives rise to ultrafilters on the equivalence cardinals with a great degree of completeness.

Definition 5.1.11. An ideal on a set A is B-complete, where B is a set, if for every function f from an I-positive set to B there is an I-positive set on which f is constant.

This should be compared with the notion of F-ergodicity of Hjorth and Kechris [22, Lemma 13.3.4].

Proposition 5.1.12. *There is no nonprincipal ω-complete ultrafilter on 2^ω.*

Proof. Suppose that U is a ω-complete ultrafilter on 2^ω. For every number $n \in \omega$, there will be an element $i_n \in 2$ such that $A_n = \{x \in 2^\omega : x(n) = i_n\} \in U$. The set $\bigcap_n A_n$ is in the ultrafilter U, and it contains only one point, so U is a principal ultrafilter. $\square$

Theorem 5.1.13. *In $V(\mathbb{R})$, there are at least two pairwise Rudin-Keisler incomparable $\mathbb{R}$-complete ultrafilters on E_0.*

Proof. Let J_0 be the ideal on $2^\omega/E_0$ defined by $B \in J_0 \leftrightarrow \bigcup B$ is meager. It is not difficult to argue that J_0 is a maximal ideal. Suppose for contradiction that $B \subset 2^\omega/E_0$ is a set such that neither $\bigcup B$ nor its complement are meager subsets of 2^ω. Since all sets in $V(\mathbb{R})$ have the Baire property, both $\bigcup B$ and its complement are equal to a nonempty open set modulo the meager ideal, and it is well known that any two such sets contain E_0-connected elements, which is of course impossible. To see that the ideal J_0 is $\mathbb{R}$-complete, let $f : B \to \mathbb{R}$ be an arbitrary function. The function $\hat{f} : \bigcup B \to \mathbb{R}$ defined by $\hat{f}(x) = f([x]_{E_0})$ is continuous on a comeager set by [27, Theorem 8.38], since all sets in $V(\mathbb{R})$ have the Baire property. A continuous E_0-invariant function on a comeager set is constant, proving the $\mathbb{R}$-completeness.

Let J_1 be the ideal on $2^\omega/E_0$ defined by $B \in J_1 \leftrightarrow \bigcup B$ is null in the usual Borel probability measure on 2^ω. The proof in the previous paragraph transfers literally using the Steinhaus theorem [2, Theorem 3.2.10] and the fact that all sets in $V(\mathbb{R})$ are measurable.

To show that the two ideals from the previous paragraphs are not Rudin-Keisler reducible to each other, we will argue that J_0 is Rudin-Keisler above the meager ideal but not above the null ideal and J_1 is above the null ideal and not above the meager ideal. Obviously, J_0 is above the meager ideal. Suppose for contradiction that it is above the null ideal as witnessed by a function $\hat{f} : 2^\omega \to 2^\omega/E_0$. $V(\mathbb{R})$ satisfies uniformization, so there is a function $f : 2^\omega \to 2^\omega$ such that $[f(x)]_{E_0} = \hat{f}(x)$. Since all sets are measurable, this function is Borel when restricted to a measure one set; to simplify the notation assume that this measure one set is the whole space 2^ω. Since "random real does not add a Cohen real", there is a Borel meager set $B \subset 2^\omega$ whose f-preimage is not null. Since the standard countable group action inducing the equivalence E_0 preserves the meager ideal, the E_0 saturation $[B]_{E_0}$ is meager as well. Thus, the set $\{x \in 2^\omega/E_0 : x \cap B \neq 0\}$ is in the ideal J_0 while its $\hat{f}$-preimage is not null, contradicting the properties of a Rudin-Keisler reduction. The treatment of J_1 is similar. □

Theorem 5.1.14. *In $V(\mathbb{R})$, there are at least two Rudin-Keisler E_{S_∞}-complete ultrafilters on E_2.*

Proof. By a result of Hjorth and Kechris, E_2 is generically E_{S_∞}-ergodic. This means that for every comeager set $B \subset 2^\omega$ and every Borel function $f : B \to (2^\omega)^2$ such that xE_2y implies $f(x)$ is as a graph isomorphic to $f(y)$, there is a comeager set C such that the range of $f \restriction C$ is contained in a single E_{S_∞} class. This can be fairly easily turned into the proof that the σ-ideal J_0 on $2^\omega/E_2$ defined by $B \in J_0 \leftrightarrow \bigcup B$ is meager, is an E_{S_∞}-complete nonprincipal maximal ideal on E_2.

By Corollary 3.12.6, E_2 is μ-ergodic for some Borel probability measure μ on $\mathcal{P}(\omega)$ that assigns mass zero to E_2 equivalence classes. As in the previous paragraph, this means that the σ-ideal J_1 on $2^\omega/E_2$ defined by $B \in J_1 \leftrightarrow \bigcup B$ has μ-mass zero, is an E_{S_∞}-complete nonprincipal maximal ideal on E_2.

It is now necessary to prove that the two maximal ideals are not Rudin-Keisler reducible to each other. We will show that J_0 is Rudin-Keisler above the meager ideal but not above the null ideal, while J_1 is above the null ideal but not above the meager ideal. ???? □

Theorem 5.1.15. *In $V(\mathbb{R})$, there is a E_{S_∞}-complete ultrafilter on E_{K_σ}, which is not Rudin-Keisler above the meager or the Lebesgue null ideal.*

Proof. Let E be the equivalence relation on ω^ω introduced in Section 3.10; that is, xEy if there is a number $m \in \omega$ such that for all $n \in \omega$, $x(n) < y(m+n)$ and $y(n) < x(m+n)$. The equivalence E is bireducible with E_{K_σ} and so we can treat the space ω^ω/E instead of E_{K_σ}.

Let J be the σ-ideal on ω^ω/E defined by the formula $B \in J \leftrightarrow \bigcup B \in I$ where I is the Laver ideal. To show that J is a maximal ideal, assume for contradiction that $B \subset \omega^\omega/E$ is a set such that both $\bigcup B$ and its complement are I-positive. In the model $V(\mathbb{R})$, every subset of ω^ω is either in the Laver ideal or else it contains all branches of a Laver tree [49, Section 4.5.2, Theorem 4.5.6 (2)]. Thus both $\bigcup B$ and its complement contain all branches of some Laver tree, but two Laver trees always contain a pair of E-connected branches by the proof of Theorem 3.10.6. This is of course a contradiction. The completeness of the ideal J follows from the Silver property of the Laver ideal for equivalences classifiable by countable structures.

We must now show that the ideal J is neither Rudin-Keisler above the meager ideal nor above the Lebesgue null ideal. The proofs are similar, we will treat the meager ideal. Suppose that $\hat{f} : 2^\omega \to \omega^\omega/E$ is a function. By the uniformization in the model $V(\mathbb{R})$, there is a function $f : 2^\omega \to \omega^\omega$ such that for every $x \in 2^\omega$, $[f(x)]_E = \hat{f}(x)$. Since all sets in $V(\mathbb{R})$ are Baire measurable, the function f is Borel if restricted to a comeager set; without loss of generality assume that this set is the whole space 2^ω. Since "Cohen real does not add a dominating real", there is a function $y \in \omega^\omega$ such that the set $\{x \in 2^\omega : y$ is modulo finite dominated by $f(x)\}$ is meager. The E-saturation of the set $B = \{z \in \omega^\omega : y$ is not modulo finite dominated by $z\}$ is in the Laver ideal, since none of its elements modulo finite dominate the function $n \mapsto \max\{y(i) : i \in 2n\}$. Thus, the set $\{x \in \omega^\omega/E : x \cap B \neq 0\}$ is in the maximal ideal J, and its preimage is comeager in 2^ω, contradicting the properties of Rudin-Keisler reduction. □

5.2 Ultrafilter models

The completely regular models of the previous section are long studied and fairly well understood. New insights are possible if one inserts fractions of the axiom of choice into them. One can ask for example what is the possible structure of cardinal inequalities between surjective images of the reals in various forcing extensions of models of the Axiom of Determinacy. A sample question:

Question 5.2.1. Is there a generic extension of $L(\mathbb{R})$ in which $|\omega_1| \leq |E_1|$ and $\neg|\omega_1| \leq |E_0|$?

Very little is known in this direction. There one exceptionally well understood generic extension of $L(\mathbb{R})$, and that is the model $L(\mathbb{R})[U]$ obtained from $L(\mathbb{R})$ by adjoining a single Ramsey ultrafilter U to it. Recall that an ultrafilter is *Ramsey* if it contains a homogeneous set for every partition $\pi : [\omega]^2 \to 2$. Such ultrafilters may fail to exist. Forcing with the σ-closed poset $\mathcal{P}(\omega)$ modulo finite adds such an ultrafilter as the generic set, and under suitable assumptions this is the only way to obtain it:

Fact 5.2.2. *[6] If suitable large cardinals exist, every Ramsey ultrafilter U is $\mathcal{P}(\omega)$ modulo finite generic set for the model $L(\mathbb{R})$.*

This fact generalizes to all standard models of the Axiom of Determinacy containing all the reals. Since the poset $\mathcal{P}(\omega)$ modulo finite is homogeneous, the theory of the model $L(\mathbb{R})[U]$ can be identified within $L(\mathbb{R})$, as such it cannot be changed by set forcing under suitable large cardinal assumptions, and therefore it is an interesting, canonical object to study. What is true in $L(\mathbb{R})[U]$? On one hand, this model contains an ultrafilter on ω, which is a non-measurable set without the Baire property. On the other hand, many consequences of determinacy survive, such as the perfect set property [6]. In this section, we will study the reducibility of Borel equivalence relations in this model and show how the work of the previous chapters can be applied to this end.

5.2a Basic features

We will study the model $V(\mathbb{R})[U]$, where κ is a strongly inaccessible cardinal, $G \subset \mathrm{Coll}(\omega, <\kappa)$ is a V-generic filter, $\mathbb{R}$ is the set of reals in the model $V[G]$, and $U \subset \mathcal{P}(\omega)$ modulo finite is a $V[G]$-generic filter. While under large cardinal assumptions, the theory of this model is not equal to that of $L(\mathbb{R})[U]$ (for example there are stationary costationary subsets of ω_1 in $V(\mathbb{R})[U]$ while in $L(\mathbb{R})[U]$ the closed unbounded filter on ω_1 is an ultrafilter), still all theorems we prove transfer directly to the model $L(\mathbb{R})[U]$ by the following proposition. At the same time, the model $V(\mathbb{R})[U]$ is easier to study and its analysis does not require any large cardinal assumptions beyond an inaccessible, in contradistinction to $L(\mathbb{R})[U]$.

Proposition 5.2.3. *Suppose ψ is a Π^2_1 sentence with real parameters. If $V(\mathbb{R})[U] \models \psi$ then also $L(\mathbb{R})[U] \models \psi$.*

This is an immediate consequence of the fact that $L(\mathbb{R})[U] \subset V(\mathbb{R})[U]$ and that the two models share the same reals. A careful look at the sentences proved in this section will show that all of their interesting instances are of Π^2_1 form.

The basic tool for the study of the model $V(\mathbb{R})[U]$ is the diagonalization forcing $P_{\hat{U}}$ for a Ramsey ultrafilter $\hat{U}$ on ω. It is the set of all pairs $p = \langle a_p, b_p \rangle$ such that $a_p \subset \omega$ is finite and $b_p \in \hat{U}$, with the ordering defined by $q \leq p$ if $a_p \subset a_q$, $b_q \subset b_p$, and $a_q \setminus a_p \subset b_p$. The forcing adds an infinite subset of ω, the union of the first coordinates of the conditions in the generic filter, that we will denote $\dot{a}_{gen}$. The following is a basic fact:

Fact 5.2.4. *Suppose that V is a transitive model of ZFC and $\hat{U}$ is a Ramsey ultrafilter in it. The following are equivalent for an infinite set $a \subset \omega$:*

1. *a is a $P_{\hat{U}}$-generic set for the model V;*

2. a diagonalizes the filter $\hat{U}$: for every $b \in \hat{U}$, $a \setminus b$ is finite.

Proposition 5.2.5. *Let $\hat{U}$ be a Ramsey ultrafilter and let κ be an inaccessible cardinal. Let ϕ be a formula with ground model parameters, another parameter $\dot{U}$, and one free variable x such that $P_{\hat{U}} * \mathrm{Coll}(\omega, < \kappa) * \mathcal{P}(\omega)$ forces that there is a unique $x \in V(\mathbb{R})$ satisfying $\phi(x, U)$. Then the first two steps of the iteration force that $\dot{a}_{gen}$ decides which x it is.*

Here, $V(\mathbb{R})$ is the model obtained from the reals after the first two stages of the iteration. Note that $P_{\hat{U}} * \mathrm{Coll}(\omega, < \kappa)$ is isomorphic to $\mathrm{Coll}(\omega, < \kappa)$ by the basic homogeneity Fact 1.3.16.

Proof. Suppose for contradiction that p_0, p_1 forces that $\dot{a}_{gen}$ does not decide the value of x. Find generic filters $G_0 \subset P_{\hat{U}}$ and $G_1 \subset \mathrm{Coll}(\omega, < \kappa)$ such that $p_0 \in G_0$, $p_1 \in G_1$, and work in the model $V[G_0, G_1]$. Find $a \subset \dot{a}_{gen}/G_0$ and x such that $a \Vdash \phi(\check{x}, \dot{U})$. By virtue of Fact 5.2.4, $a \subset \dot{a}_{gen}/G_0$ is V-generic for the poset $P_{\hat{U}}$. Find a V-generic filter $H_0 \subset P_{\hat{U}}$ containing the condition p_0 such that $\dot{a}_{gen}/H_0 = a$ modulo finite, use the collapse homogeneity 1.3.16 to find a filter $H_1 \subset \mathrm{Coll}(\omega, < \kappa)$ so that $p_1 \in H_1$ and $V[G_0, G_1] = V[H_0, H_1]$, and observe that a forces $\phi(\check{x}, \dot{U})$ by its choice, and at the same time a should not force $\phi(y, \dot{U})$ for any $y \in V(\mathbb{R})$ by the assumption and the forcing theorem applied to the ground model and the filters H_0, H_1! □

Proposition 5.2.6. *Suppose that ϕ is a formula with parameters in the ground model and perhaps another parameter U, and no free variables. Let $\hat{U}$ be a Ramsey ultrafilter and let κ be an inaccessible cardinal. In the iteration $P_{\hat{U}} *$ $\mathrm{Coll}(\omega, < \kappa) * \mathcal{P}(\omega)$, the condition $1, 1, \dot{a}_{gen}$ decides the truth value of $\phi(\dot{U})$.*

Proof. Assume for contradiction that $\langle p_0, p_1, p_2 \rangle$, $\langle q_0, q_1, q_2 \rangle$ are two conditions below $\langle 1, 1, \dot{a}_{gen} \rangle$ that decide ϕ in different ways. Then $\langle p_0, p_1 \rangle$ forces p_2 to be a $P_{\hat{U}}$-generic set over the ground model as per Fact 5.2.4. Let $G_0 * G_1$ be a V-generic filter on the iteration $P_{\hat{U}} * \mathrm{Coll}(\omega, < \kappa)$ containing the condition $\langle p_0, p_1 \rangle$. In the model $V[G_0 * G_1]$, make a finite change to the set $a = \dot{p}_2/G_0 * G_1$ to get a V-generic filter H_0 on $P_{\hat{U}}$ containing the condition q_0 producing this set. Use the homogeneity of the collapse 1.3.16 in the model $V[H_0]$ to find a $V[H_0]$-generic filter $H_1 \subset \mathrm{Coll}(\omega, < \kappa)$ containing the condition q_1 such that $V[G_0, G_1] = V[H_0, H_1]$. Thus, the set $q_2/H_0 * H_1$ is modulo finite included in the set $\dot{a}_{gen}/H_0$, which is modulo finite equal to $p_2/G_0 * G_1$. Looking into the model $V[G_0, G_1] = V[H_0, H_1]$, these two conditions should decide ϕ in a different way in the poset $\mathcal{P}(\omega)$ modulo finite, which is of course a contradiction. □

5.2b Preservation of cardinals

A basic question regarding any forcing extension is whether cardinals are preserved. In the context of the axiom of choice, this question may be difficult in exceptional cases. However, in the choiceless context, the question takes on a whole new dimension of complexity. Suppose that A and B are two (non-well-orderable) sets such that $\neg|A| \leq |B|$. Is this still true in our favorite generic extension? The basic question remains open:

Question 5.2.7. (In the context of the Solovay model or AD+) Does the forcing with $\mathcal{P}(\omega)$ modulo finite add new cardinal inequalities between old sets?

Of course, the most intriguing case is that of cardinal inequalities between sets that are surjective images of the real line. There, the question can be rephrased in a more intuitive and attractive way: can a (Ramsey) ultrafilter be used to construct new reductions between equivalences, where there are no such *Borel* reductions? A little bit of experimentation will show that it is difficult to even conceive of a way to use an ultrafilter in this way. Correspondingly, we conjecture that the answer to the question is negative.

The question can be motivated from another angle as well. Nonexistence of Borel reductions between various equivalences is often a difficult problem, and its resolution always leads to significant progress in understanding the equivalence relations concerned. Nearly all known results in this direction use Baire category or measurability methods in one way or another. It may be of interest to find diametrally different arguments. One way to generate such arguments will attempt to prove that the cardinal inequality between the sets of equivalence classes concerned fails to hold even in the presence of an ultrafilter. Such a proof must use a trick different from Baire category or measurability, since the ultrafilter in question is a nonmeasurable set without Baire property and therefore is likely to kill any such trick.

While the general negative answer to the above question seems to be out of reach, the methods of this book allow us to prove that no cardinal inequalities are added for many of the equivalence cardinals. The results of this form will essentially always use the following proposition.

Proposition 5.2.8. *Suppose that E is a Borel equivalence relation on a Polish space X. If ZFC proves that there is a σ-ideal I on the space X which has E in its spectrum and such that the forcing P_I preserves Ramsey ultrafilters, then whenever $A \in V(\mathbb{R})[U]$ is a set then $V(\mathbb{R}) \models |E| \leq |A| \leftrightarrow V(\mathbb{R})[U] \models |E| \leq |A|$.*

Proof. Suppose that $\mathrm{Coll}(\omega, <\kappa) * \mathcal{P}(\omega)$ forces $|E| \leq |\dot{A}|$ as witnessed by an injection $\dot{f}$. The standard homogeneity arguments as in Fact 1.3.16 show that we may assume withut loss of generality that the set A is definable from parameters in the ground model, the injection $\dot{f}$ is definable from the parameters in the ground model and $\dot{U}$, and the ground model contains a Ramsey ultrafilter u. Let P_I be the forcing with E in its spectrum, and observe that $P_I \Vdash P_u \Vdash \mathrm{Coll}(\omega, <\kappa) \Vdash \dot{a}_{gen}$ decides the value of $\dot{f}([\dot{x}_{gen}]_E)$ by Proposition 5.2.5. Pass to the Solovay model $V(\mathbb{R})$, and in this model find an infinite set $a \subset \omega$ diagonalizing the ultrafilter u, and use Claim 1.3.21 to find an I-positive Borel set $B \subset X$ consisting of V-generic reals only. Note that a is P_u generic over all the models $V[x] : x \in B$ since P_I preserves the ultrafilter u. Since E is in the spectrum of the ideal I, $|E| \leq |E \restriction B|$, and $|E \restriction B|$ can be injectively mapped to A by the map $[x]_E \mapsto$the unique element of the set A such that a forces it to be equal to $\dot{f}([x]_E)$. The proposition follows. $\square$

Designing quotient forcings that preserve Ramsey ultrafilters and contain a given equivalence relation in its spectrum is not an entirely trivial endeavor. The notion of spectrum is the subject of this book. Suitably definable forcings preserve Ramsey ultrafilters if they are proper, bounding, and add no independent reals by [49, Theorem 3.4.1]. The bounding condition is usually fairly easy to verify, but not adding independent reals is more difficult, and it typically requires some nontrivial applications of Ramsey theory on Polish spaces.

Theorem 5.2.9. *Suppose that $A \in V(\mathbb{R})$ is a set such that in $V(\mathbb{R})$, $\neg|E_0| \leq |A|$. Then in $V(\mathbb{R})[U]$, $\neg|E_0| \leq |A|$ as well.*

Proof. Consider the E_0 forcing P and the corresponding σ-ideal I on 2^ω as defined in Section 3.4. We know that the forcing P is proper and bounding, and E_0 is in the spectrum. We must prove that independent reals are not added.

Suppose that $B \Vdash \dot{a} \subset \omega$ is a set. Thinning out the condition B if necessary we may find a Borel function $f : B \to \mathcal{P}(\omega)$ representing the name $\dot{a}$. Using the fact that $E_0 \leq E_0 \restriction B$, it is not difficult to find a Borel injection $g : \omega^\omega \to B$ preserving the equivalence E_0, which on the space ω^ω is defined by $xE_0y \leftrightarrow \{n : x(n) \neq y(n)\}$ is finite. Consider the partition $\omega^\omega \times \omega = D_0 \cup D_1$ into two Borel pieces defined by $\langle x, n\rangle \in D_0 \leftrightarrow n \in g(f(x))$. Use a result of Henle [12] to find two element sets $a_i : i \in \omega$ as well as an infinite set $c \subset \omega$ such that the product $\Pi_i a_i \times c$ is a subset of one piece of the partition, say of D_0. The equivalence $E_0 \restriction \Pi_i a_i$ is still not smooth, and so the Borel set $C = g''\Pi_i a_i \subset B$ is still I-positive. At the same time, for every point $x \in C$ it is the case that $c \subset f(x)$, and by the usual absoluteness argument it follows that $C \Vdash \check{c} \subset \dot{a}$! $\square$

Theorem 5.2.10. *Suppose that $A \in V(R)$ is a set such that in $V(\mathbb{R})$, $\neg|E_1| \leq |A|$. Then in $V(\mathbb{R})[U]$, $\neg|E_1| \leq |A|$ as well.*

Proof. Consider the poset P, the product of countably many copies of Sacks forcing. The poset is proper, bounding, and E_1 is in its spectrum by the results in Section 3.8. The poset adds no independent reals by a theorem of Laver, and therefore it preserves Ramsey ultrafilters. Perhaps a tighter argument would use the illfounded iteration of Sacks forcing of length inverted ω as developed for example in [21] or [49, Section 5.4]. $\square$

Theorem 5.2.11. *Suppose that $A \in V(\mathbb{R})$ is a set such that in $V(\mathbb{R})$, $\neg|E_2| \leq |A|$. Then in $V(\mathbb{R})[U]$, $\neg|E_2| \leq |A|$ as well.*

Proof. One approach is to recall Theorem 3.12.4. It yields a fat tree forcing with E_2 in the spectrum. The fat tree forcings are proper, bounding, and do not add independent reals by [49, Theorem 4.4.8]. Together with the definability it implies that the forcing preserves Ramsey ultrafilters by [49, Theorem 3.4.1]. The theorem follows!

Another option is to use the E_2 forcing of Section 3.5 and use a complex Ramsey theoretic argument to argue that it does not add independent reals. $\square$

Theorem 5.2.12. *Suppose that $A \in V(\mathbb{R})$ is a set such that in $V(\mathbb{R})$, $\neg|E_{K_\sigma}| \leq |A|$. Then in $V(\mathbb{R})[U]$, $\neg|E_{K_\sigma}| \leq |A|$ as well.*

Proof. Theorem 3.12.4 provides for a fat tree forcing such that E_{K_σ} is in the spectrum of the associated ideal. Then continue as in the previous argument. $\square$

Theorem 5.2.13. *Suppose that $A \in V(\mathbb{R})$ is a set such that in $V(\mathbb{R})$, $\neg|F_2| \leq |A|$. Then in $V(\mathbb{R})[U]$, $\neg|F_2| \leq |A|$ as well.*

Proof. The previous proof scheme cannot be used since there is no proper forcing that adds an interesting new F_2 class. Instead, consider the finite support product P of infinitely many Sacks forcings with finite support, adding Sacks reals $\{x_n : n \in \omega\}$. This poset of course collapses the continuum to $\aleph_0$ and therefore does not preserve any Ramsey ultrafilters, but we will instead consider the intermediate choiceless model $V(\{x_n : n \in \omega\})$, with the ground model enriched by the *symmetric Sacks set* of the Sacks generics without the function that enumerates them. This is the *symmetric Sacks extension.* The usual symmetricity arguments yield the following.

Claim 5.2.14. *The symmetric Sacks extension contains only those sets of ordinals that belong to $V[x_n : n \in m]$ for some finite m.*

Claim 5.2.15. *Every infinite set of reals in finite tuples product generic for Sacks forcing, which intersects every ground model perfect set, is a symmetric Sacks set.*

If u is a Ramsey ultrafilter in the ground model, then it generates a Ramsey ultrafilter in finite Sacks product extensions, and so even in the symmetric Sacks extension. Every infinite set diagonalizing u is P_u-generic for the symmetric Sacks extension.

Now suppose that $\mathrm{Coll}(\omega, < \kappa) \Vdash \dot{A}$ is a set such that $\mathcal{P}(\omega) \Vdash |F_2| \leq |A|$ as witnessed by an injection $\dot{f}$. We must argue that $\mathrm{Coll}(\omega, < \kappa) \Vdash |F_2| \leq |A|$. By the usual symmetricity arguments we can assume that the set A is definable from parameters in the ground model, and the function f is definable from parameters in the ground model and the ultrafilter $\dot{U}$, and the ground model contains a Ramsey ultrafilter u. Pass to the extension $V[G]$ where $G \subset \mathrm{Coll}(\omega, < \kappa)$ is a generic filter. Find an F_σ-set $C \subset 2^\omega$ that meets every ground model perfect set in a perfect set and consists of reals that are Sacks generic in finite tuples over V. For example, a finite support product of infinitely many copies of the amoeba for Sacks forcing will yield such a set. Find a Borel function $f : 2^\omega \to C^\omega$ such that for every $x \in 2^\omega$, $f(x)$ enumerates a set which intersects every perfect set in the ground model, and for distinct $x, y \in 2^\omega$, $\mathrm{rng}(f(x)) \cap \mathrm{rng}(f(y)) = 0$. Note that for every countable set $b \subset 2^\omega$, the set $c_b = \bigcup_{x \in b} \mathrm{rng}(f(x)) \subset 2^\omega$ is a countable symmetric Sacks set over the ground model. Fix an infinite set $a \subset \omega$ diagonalizing the filter u. By Proposition 5.2.5 applied in the model $V(c_b)$, a

decides the value of $\dot{f}(c_b)$ for every countable set $b \subset 2^\omega$. It is now easy to check that the map $b \mapsto$that element of the set A that a forces $f(c_b)$ to be equal to, is the required injection of $[\mathbb{R}]^{\aleph_0}$ to the set A in the model $V(\mathbb{R})$!

□

5.2c Linear orderability

Without the axiom of choice, certain sets may fail to carry a linear ordering. Clearly, if $|A| \leq |B|$ and the set B is linearly orderable then so is A, and so the concept of linear orderability may serve as a tool for disproving inequalities between sizes of various sets. Under hypotheses such as AD$\mathbb{R}$, this a priori interesting tool proves to be fairly blunt: the linearly ordered sets are exactly those whose size is smaller than some $\mathcal{P}(\kappa)$ for an ordinal κ, and there is a smallest cardinal which is not linearly ordered, namely the E_0 cardinal [5]. However, if we insert an ultrafilter into our universe, the situation changes and linear orderability turns into a much more intriguing concept. Start with a seminal observation:

Theorem 5.2.16. *(Paul Larson, personal communication) (ZF) If there is an ultrafilter, the E_1 and E_3 cardinals are linearly orderable.*

Proof. For E_1, choose linear orderings $\leq_n$ on $(2^\omega)^{\omega \setminus n}$ and let $\leq$ be the pre-ordering on $(2^\omega)^\omega$ defined by $\vec{x} \leq \vec{y}$ if the set $\{n : x \restriction \omega \setminus n \leq_n y \restriction \omega \setminus n\}$ belongs to the fixed ultrafilter. It is immediate that $\leq$ linearly orders the set ofE_1 equivalence classes. For E_3, first observe that since $E_0 \leq E_1$, the set of all E_0 classes is linearly orderable from an ultrafilter and fix such a linear order $\prec_0$. The linear order $\prec_3$ on $E_3 = E_0^\omega$ is then defined by $\vec{x} \prec_3 \vec{y}$ if for the least n such that $\vec{x}(n) \neq \vec{y}(n)$ it is the case that $\vec{x}(n) \prec_0 \vec{y}(n)$. □

The proof clearly shows that the class of linearly ordered sets is closed under power ω (or any ordinal power) and, if an ultrafilter is present, under power ω modulo finite. How does one argue that a set is not linearly orderable? We know of exactly one trick:

Definition 5.2.17. Let P be a partial ordering and E a Borel equivalence relation on a Polish space X. We say that P adds *interchangeable E equivalence classes* if there are P-names τ, σ for non-E-equivalent elements of X such that for every condition $p \in P$ there are, in some further generic extension, V-generic filter $G, H \subset P$ containing the condition p so that $(\sigma/G)E(\tau/H)$ and $(\sigma/H)E(\tau/G)$.

The most basic example is the Cohen forcing and the E_0-equivalence. If one considers the name σ for the Cohen generic and τ for its flip, then their E_0-classes are interchangeable: whenever $p \in 2^{<\omega}$ is a Cohen condition and $x \in 2^\omega$ is a Cohen real extending p, then the flip of x rewritten with p at the appropriate initial segment is also a Cohen real, E_0-equivalent to the flip of x. This feature can be used to argue that in the Solovay model, the set of E_0 classes is not linearly orderable. If one wants to argue that other sets are not linearly orderable even if an ultrafilter is present, it is necessary to produce forcings adding interchangeable equivalence classes with additional preservation properties:

Proposition 5.2.18. *Suppose that E is a Borel equivalence relation. If ZFC proves that there is a Ramsey ultrafilter preserving forcing adding interchangeable equivalence classes, then in $V(\mathbb{R})[U]$, the set of E equivalence classes is not linearly orderable.*

Proof. Suppose for contradiction that some condition in $\mathrm{Coll}(\omega, <\kappa) * \mathcal{P}(\omega) \Vdash V(\mathbb{R})[U] \models \leq$ linearly orders E-equivalence classes. Then $\leq$ will be definable from parameters which may be elements of the ground model, reals, or the ultrafilter U itself. The homogeneity arguments show that we can assume that all of the parameters except for the Ramsey ultrafilter lie in the ground model, and the condition is the largest condition. We may also assume that the ground model contains a Ramsey ultrafilter u.

By the assumption, there is a forcing P adding interchangeable E-equivalence classes which preserves the Ramsey ultrafilter u. By Proposition 5.2.6, $P \Vdash$ the condition $1, 1, \dot{a}_{gen}$ in the iteration $P_{\dot{U}} * \mathrm{Coll}(\omega, <\kappa) * \mathcal{P}(\omega)$ decides $[\tau]_E \leq [\sigma]_E$ or vice versa. Let $p \in P$ be a condition which decides which way this decision will go, thus for example $p, 1, 1, \dot{a}_{gen} \Vdash [\tau]_E \leq [\sigma]_E$. The interchangeability feature of the names σ and τ together with standard homogeneity arguments shows that there are filters $G_0 \subset P, G_1 \subset P_u, G_2 \subset \mathrm{Coll}(\omega, <\kappa)$ and $H_0 \subset P, H_1 \subset P_u, H_2 \subset \mathrm{Coll}(\omega, <\kappa)$ such that the two triples are generic for the iteration indicated, $p \in G_0 \cap H_0$, $V[G_0, G_1, G_2] = V[H_0, H_1, H_2]$, $(\sigma/G_0)E(\tau/H_0)$, $(\sigma/H_0)E(\tau/G_0)$, and $\dot{a}_{gen}/G_1 = \dot{a}_{gen}/H_1$. Let U be a Ramsey ultrafilter in $V[G_0, G_1, G_2]$ containing the set $\dot{a}_{gen}/G_1$. The forcing theorem then shows that both $[\sigma/G_0]_E \leq [\tau/G_0]$ and $[\tau/G_0]_E \leq [\alpha/G_0]_E$ should hold, which is impossible. $\square$

Designing posets adding interchangeable equivalence classes is an interesting discipline in its own right. We mastered the discipline well enough to be able to prove two theorems:

Theorem 5.2.19. *In $V(\mathbb{R})[U]$, the set of E_2 equivalence classes is not linearly orderable.*

Proof. We will design a fat tree forcing adding two interchangeable E_2 classes. These forcings are simply definable, bounding, add no independent reals, and therefore preserve Ramsey ultrafilters. The construction uses concentration of measure on Hamming cubes, similarly to Theorem 3.12.4. The following will come handy:

Claim 5.2.20. *For arbitrary positive reals $\varepsilon, \delta > 0$ there is a natural number $n \in \omega$ such that for any sets $A_0, A_1 \subset 2^n$ of normalized counting measure mass $\geq \varepsilon$ there are points $x_0 \in A_0, x_1 \in A_1$ such that x_0 is within normalized Hamming distance $\leq \delta$ to the flip of x_1.*

To prove this, use the concentration of measure to find a number n such that for every set $A \subset 2^n$ of normalized counting measure mass $\geq \varepsilon$, the set $A_\delta = \{y \in 2^n : \exists x \in A\ d(x, y) \leq \delta\}$ has mass greater than $1 - \varepsilon$. This number must work, since the set $(A_0)_\delta$ must intersect with the set of flips of all points in A_1; their masses added give a number greater than 1!

Now consider the finitely branching tree T_{ini} such that for every node $t \in T_{ini}$ at level m there is a number $n_m \in \omega$ such that the set of immediate successors of t is the set $a_t = 2^{2n_m \setminus n_m}$, and such that n_m satisfies the previous claim with $\varepsilon = 1/m$ and $\delta = 2^{-m}$. Moreover, we require that $n_{m+1} > 2n_m$. The submeasure ϕ_t will be defined by $\phi_t(b) = m$ times the normalized counting measure mass of the set $b \subset a_t$.

It is immediately clear that the numbers $\phi_t(a_t) : t \in T_{ini}$ tend to infinity and so the forcing P of fat trees associated with this system of submeasures is well defined and nonatomic. To describe the interchangeable names for E_2 classes, note that P adds a cofinal branch $\dot{x}_{gen} \in [T_{ini}]$. Define $\sigma = \{i \in \omega : \exists m\ i \in [n_m, 2n_m) \wedge \dot{x}_{gen}(m)(i) = 1\}$ and $\tau = \{i \in \omega : \exists m\ i \in [n_m, 2n_m) \wedge \dot{x}_{gen}(m)(i) = 0\}$. These two subsets of ω are not E_2-equivalent since $\sigma \Delta \tau = \bigcup_m [n_m, 2n_m)$; this is not a summable set since $\Sigma\{1/i : i \in [n_m, 2n_m)\} > 1/2$ for every $m \in \omega$. How does one interchange their equivalence classes though?

Suppose that $p \in P$ is a condition, and move to a generic extension $V[K]$ collapsing the ground model powerset of the continuum to $\aleph_0$. Using standard fusion arguments or Theorem 1.3.21, find a tree $q \leq p$ in the poset $P^{V[K]}$ consisting exclusively of V-generic branches through the tree T_{ini}. There is a node $t \in q$ such that all nodes above it split into a set of immediate successors of mass at least 1. Use the property of numbers $n_m : m \geq |t|$ described in Claim 5.2.20 to find tow distinct branches $x, y \in [q]$ such that $t \subset x, y$ and for all numbers $m \geq |t|$ the sequence $x(m)$ is 2^{-m} close in the normalized Hamming distance of the cube $2^{2n_m \setminus n_m}$ to the flip of $y(m)$. Let $G_x, G_y \subset P$ be the associated V-generic filters. It is clear that $(\sigma/G_x) E_2 (\tau/G_y)$, since for every number $m \geq |t|$, the symmetric difference of the two sets intersected with $[n_m, 2n_m)$ has cardinality at most $2^{-m} n_m$, its summable mass is $\leq 2^{-m}$, and the numbers $2^{-m} : m \geq |t|$ have a finite sum! Similarly, $(\sigma/G_y) E_2 (\tau/G_x)$, and the theorem follows. □

Theorem 5.2.21. *The class of all c_0 equivalence classes is not linearly orderable in $L(\mathbb{R})[U]$.*

Theorem 5.2.22. *In $V(\mathbb{R})[U]$, the set of F_2 equivalence classes is not linearly orderable.*

Proof. Consider the product of two copies of Sacks forcing, $P \times P$. For every condition $\langle p, q \rangle \in P \times P$ there is a automorphism π of $P \times P \restriction \langle p, q \rangle$ such that, when we naturally extend the automorphism to the space of all $P \times P$ names, $\pi(\mathbb{R} \cap V[\dot{x}_{\mathrm{lgen}}]) = \mathbb{R} \cap V[\dot{x}_{\mathrm{rgen}}]$ and vice versa, $\pi(\mathbb{R} \cap V[\dot{x}_{\mathrm{rgen}}]) = \mathbb{R} \cap V[\dot{x}_{\mathrm{lgen}}]$. Here $\dot{x}_{\mathrm{rgen}}, \dot{x}_{\mathrm{lgen}}$ denote the right and left Sacks reals obtained from the corresponding copies of the Sacks forcing P. It is easy to describe the automorphism π: note that p, q are perfect binary trees, choose a homeomorphism between $[p]$ and

$[q]$, abuse the notation by calling it π again, and let $\pi(p_0, q_0) = (p_1, q_1)$ where q_1 is a perfect tree such that $[q_1] = \pi''[p_0]$ and p_1 is a perfect tree such that $[p_1] = \pi''[q_0]$.

Now, the forcing $P \times P$ does not really add an interchangeable pair of F_2 degrees in the sense of Definition 5.2.17, because the sets $\mathbb{R} \cap V[\dot{x}_{\mathrm{lgen}}]$ and $\mathbb{R} \cap V[\dot{x}_{\mathrm{rgen}}]$ are forced to be uncountable in the $P \times P$ extension. However, they will be countable in the larger Solovay extension and define interchangeable F_2-classes, which is all that is needed in the proof of Proposition 5.2.18. The proof of the theorem is completed by noting that the product of Sacks reals preserves Ramsey ultrafilters. $\square$

As a final remark, in all of the nonorderability theorems it is really necessary to specify that U is a Ramsey ultrafilter, as the following result shows:

Theorem 5.2.23. *(Shelah) There is a Borel ideal J on ω such that for every ultrafilter U disjoint from it, the set of all E_{K_σ} classes is linearly orderable in the model $V(\mathbb{R})[U]$.*

In fact, the set of all E_{K_σ}-equivalence classes is Boolean linearly orderable in ZF+DC, in the sense that there is a Borel map $f : \mathrm{dom}(E_{K_\sigma})^2 \to \mathcal{P}(\omega)$ satisfying the following demands:

1. $x_0 E_{K_\sigma} x_1$ and $y_0 E_{K_\sigma} y_1$ implies $f(x_0, y_0) = f(x_1, y_1)$ modulo J;
2. $f(x, y) = \omega \setminus f(y, x)$ modulo J when $\neg x E_{K_\sigma} y$. Moreover $f(x, x) = \omega$ modulo J;
3. $f(x, y) \cap f(y, z) \subset f(x, z)$ modulo J.

Question 5.2.24. Is the set of E_∞ equivalence classes linearly orderable in $V(\mathbb{R})[U]$?

Question 5.2.25. (ZF+DC) Suppose that the set of all E_0 equivalence classes is linearly orderable. Does it follow that there is a nonprincipal ultrafilter on ω?

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