n-localization property in iterations*

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Abstract

The n-localization property is preserved under the countable support iteration of suitably definable forcings. This solves a question of Rosłanowski and greatly simplifies the proofs in the area.

1 Introduction

Newelski and Rosłanowski [6] introduced the n-localization property of forcings.

Definition 1.1. A tree $T \subset \omega^{<\omega}$ is an n-tree if for every sequence $t \in \omega^{<\omega}$ the set $\{m \in \omega : t^\frown m \in T\}$ has size at most n.

Definition 1.2. A forcing P has an *n-localization property* if for every function $x \in \omega^\omega$ in the extension there is an n-tree T in the ground model such that $x \in [T]$.

This property can serve as a tool to discern between closely related forcings such as the usual Sacks forcing, which has 2-localization, and the 3-Sacks forcing, in which the nodes in trees split into three immediate successors and it does not have the 2-localization property. Several people [3, 4, 7, 8] wondered about the preservation of the n-localization property in countable support iteration and product. The existing approaches yield awkward proofs applicable only in very special situations. In this paper, I will prove

Theorem 1.3. *Assume that suitable large cardinals exist. Let $n \in \omega$ be a number. The n-localization property is preserved under the countable support iterations of suitably definable proper forcings.*

Here, a suitably definable forcing is one of the form P_I=Borel sets positive with respect to some σ-ideal I on a Polish space X such that, writing $A \subset 2^\omega \times X$ for a universal analytic set, the set $\{x \in 2^\omega : A_x \in I\}$ is universally Baire [2]. The large cardinal assumption sufficient to carry the proof is the existence of

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proper class many Woodin cardinals. Many definable proper forcings adding a single real are of this form [12, Section 2.1.3].

There is a ZFC version of the previous theorem that is sufficient in all practical cases that I know of. I will say that P is an *analytic CRN forcing* if P is an analytic set of finitely branching trees on ω ordered by inclusion, closed under restriction, and such that for every P-name $\dot{y}$ there is a condition $p \in P$ and a continuous function $f : [p] \to 2^\omega$ such that $p \Vdash \dot{y} = \dot{f}(\dot{x}_{gen})$ where $\dot{x}_{gen} \in \omega^\omega$ is the name for the intersection of all conditions in the generic filter. This class should be compared with the snep forcings of [11]. Analytic CRN forcings are bounding. Most definable proper bounding forcings adding a single real can be represented as such. There are some unpleasant exceptions to this rule, such as the posets of [9, Section 2.2], and the methods of this paper cannot handle them directly.

Theorem 1.4. *Let n be a number. The n-localization property is preserved under the countable support iterations of analytic CRN proper forcings.*

This solves some open questions of Rosłanowski [8]: for example, the countable support iteration of 2-Silver forcing does not add a 3-Silver generic. The theorem fails for arbitrary (undefinable) proper forcings already for iterations of length 2, as Theorem 5.4 shows.

The proof of the iteration theorems follows a pattern familiar from [12, Section 6.3.1], and uses the concept of Fubini properties of ideals [12, Section 3.2]. I will first identify some c.c.c. forcings, I will then show that their Fubini properties precisely characterize the n-localization property, and then use [12, Theorem 6.3.3] to show that these Fubini properties are preserved under the countable support iteration of suitably definable forcings.

The notation used in this paper follows the set theoretic standard of [5]. If $t \in 2^{<\omega}$ is a finite binary sequence then O_t denotes the clopen subset of 2^ω consisting of all infinite binary sequences containing t as an initial segment. If I is a σ-ideal on a Polish space X then P_I is the quotient poset of all Borel sets not in the ideal I ordered by inclusion. This forcing adds a single element of the Polish space X, namely the point contained in all sets in the generic filter; the name for this point will be denoted by $\dot{x}_{gen}$. For a tree $T \subset \omega^{<\omega}$ the symbol $[T]$ stands for the set of all infinite branches of T. A subset of a Polish space is *universally Baire* [2] if its continuous preimages in Hausdorff spaces have the property of Baire.

2 A c.c.c. forcing

The main tool of this paper is the n-localization forcing P_n:

Definition 2.1. Let $n \in \omega$ be a natural number. The n-localization forcing P_n consists of finite sets $a \subset \omega^\omega$ such that for every $t \in \omega^{<\omega}$ the set $\{m \in \omega : \exists x \in a\ t^\frown m \subset x\}$ has size at most n. The ordering is that of reverse inclusion.

It is not difficult to see that if $G \subset P_n$ is a generic filter then $y_{gen} = \{t \in \omega^{<\omega} : \exists a \in G\ \exists x \in a\ t \subset x\}$ is an n-ary tree, and the generic filter G can be recovered from y_{gen} as $G = \{a \in P_n \cap V : a \subset [y_{gen}]\}$. Thus the poset P_n can be viewed as adding a single point in the Polish space Y_n of all n-ary trees on ω, with topology inherited from $K(\omega^\omega)$. An obvious genericity argument shows that given a ground model function in the Baire space ω^ω, one can change finitely many values of it in such a way that the resulting function is a branch of the generic n-ary tree. A critical observation: the forcing P_n satisfies a certain strengthening of the countable chain condition.

Claim 2.2. *P_n is σ-n-centered.*

Proof. I must show that $P_n = \bigcup_m A_m$ where every n many elements of A_m have a common lower bound. For every condition $a \in P_n$ let $t(a) \subset 2^{<\omega}$ be the inclusion-smallest finite tree such that for every terminal node of $t(a)$ there is exactly one element of a extending it. Decompose the forcing P_n into countably many pieces according to the value of $t(a)$. It is not difficult to see that for any collection $\{a_i : i \in n\} \subset P_n$ with a common value of $t(a_i)$ the union $\bigcup_i a_i$ is a condition in P_n and a common lower bound. □

Let J_n be the σ-ideal associated with the forcing P_n. That is, J_n is the σ-ideal on the Polish space Y_n generated by those Borel sets $B \subset Y_n$ such that $P_n \Vdash \dot{y}_{gen} \notin \dot{B}$. Another elementary but critical observation: the forcing P_n is suitably homogeneous and therefore the ideal J_n is ergodic in the sense of [12, Section 3.7.1]: there is a countable Borel equivalence relation E such that for every Borel E-invariant set either it or its complement belongs in the ideal J_n.

Claim 2.3. *The ideal J_n is ergodic.*

Proof. Suppose that $k \in \omega$ is a number and π is an automorphism of the tree $k^{\leq k}$. Extend π to an automorphism $\hat{\pi}$ of the whole space Y_n by setting $\hat{\pi}(y) = \{\pi(s)^\frown t : s^\frown t \in y$ and s is the longest initial segment that belongs to $\mathrm{dom}(\pi)\}$. Note that the same definition also yields an automorphism of the forcing P_n. Let E be the countable Borel equivalence relation on the space Y_n generated by the graphs of all the countably many automorphisms obtained in this way. I claim that E has the required properties.

Indeed, suppose that $B \subset Y_n$ is a Borel E-invariant set and assume for contradiction that neither B nor its complement are in the ideal J_n. This means that there must be conditions $p, q \in P_n$ such that $p \Vdash \dot{y}_{gen} \in \dot{B}$ and $q \Vdash \dot{y}_{gen} \notin \dot{B}$. There is a sufficiently large number $k \in \omega$ and an automorphism π of $k^{\leq k}$ such that the conditions p and $\hat{\pi}(q)$ are compatible in P_n, with a lower bound r. Then r forces that $\hat{\pi}^{-1}$-image of the generic filter is a generic filter containing the condition q, and by the forcing theorem $\dot{y}_{gen} \in \dot{B}$ and $\hat{\pi}^{-1}(\dot{y}_{gen}) \notin \dot{B}$. Thus the set B is not E-invariant in the generic extension, and by an absoluteness argument, it is not invariant in the ground model either. Contradiction! □

To simplify several complexity computations and identify natural variations of the localization concept, I will use restricted versions of the above localization forcings. Suppose $f \in \omega^\omega$ is a function, and $n \in \omega$ is a number. The forcing $P_n \restriction f$ is defined in exactly the same way as P_n, except the conditions consist of functions dominated pointwise by f. The whole treatment transfers verbatim to the restricted versions. I will denote the space of all n-ary trees dominated by f by $Y_n \restriction f$, and the σ-ideal on it generated by the forcing $P_n \restriction f$ will be denoted by $J_n \restriction f$. The main difference between the original forcings P_n and their restricted versions is that the restricted σ-ideal $J_n \restriction f$ is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ [12, Section 3.8]: for every analytic set $A \subset 2^\omega \times Y_n \restriction f$ the set $\{x : A_x \in I_n \restriction f\}$ is coanalytic.

Claim 2.4. *Let $f \in \omega^\omega$ and $n \in \omega$. The ideal $J_n \restriction f$ is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$.*

Proof. By [12, Proposition 3.8.11], it is enough to show that the set of maximal antichains of $Q_n \restriction f$ is a Borel subset of $(Q_n \restriction f)^\omega$–in the language of [10], the poset is very Suslin. Fix a countable set $A \subset Q_n \restriction f$. Pairwise incompatibility of elements of A is certainly a Borel condition. The maximality of A is equivalent to the statement $\forall t \bigcap_{a \in A} B_{t,a} = 0$, where $B_{t,a} = \{b \in Q_n \restriction f : t = t(b) \wedge a \perp b\}$ and $t(b)$ is defined as in the proof of Claim 2.2. It is not difficult to check that the sets $B_{t,a}$ are closed subsets of the compact set $C_t \subset P_n \restriction f$ where $a \in C_t$ if and only if for every endnode of the tree t there is exactly one element of a extending it. Therefore they and their intersections are compact, and the statement that they are empty is Borel. □

While this definability property may seem mysterious, it has immediate forcing consequences.

Corollary 2.5. *The forcings $P_n \restriction f$ do not add dominating reals.*

This follows immediately from [12, Proposition 3.8.15]. Note that the unrestricted forcings P_n do add dominating reals and therefore the ideals J_n are not $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$.

3 Localization vs. Fubini property

This section is the heart of the paper. It contains just one key proposition connecting the n-localization property with the Fubini properties of the σ-ideal J_n. Such properties were introduced in [12, Section 3.2]: for σ-ideals K, L on respective Polish spaces X and Y, the symbol $K \perp L$ denotes the fact that there are a Borel K-positive set $B \subset X$, a Borel L-positive set $C \subset Y$, and a Borel set $D \subset B \times C$ such that the vertical sections of D are L-small, while the horizontal sections of its complement are K-small.

Proposition 3.1. *Let I be a σ-ideal on a Polish space X such that the quotient forcing P_I is proper, and every analytic I-positive set has a Borel I-positive subset. Let n be a natural number. The following are equivalent:*

1. *P_I has the n-localization property;*

2. *P_I is bounding and for every function $f \in \omega^\omega$, $I \not\leq J_n \restriction f$.*

Towards the proof of the proposition, first note that if the first item fails, then so does the other. If P_I does not have the n-localization property, then either it is not bounding or else it adds a function $\dot{g} \in \omega^\omega$ forced to be dominated by some ground model function $f \in \omega^\omega$, and not covered by any ground model n-tree. In the former case (2) fails immediately. In the latter case find a Borel I-positive set $B \subset X$ and a Borel function $h : B \to \omega^\omega$ such that $B \Vdash \dot{g} = \dot{h}(\dot{x}_{gen})$ and observe that the Borel set $D = \{\langle x, T\rangle \in B \times Y_n \restriction f : h(x)$ is not modulo finite equal to any branch of the tree $T\}$ has Borel $J_n \restriction f$-small vertical sections, and the horizontal sections of its complement are I-small, and (2) fails again.

For the reverse direction, let $n \in \omega$ be a natural number and suppose that the quotient forcing P_I does have the n-localization property. Clearly, it has the Sacks property and so is bounding. Let $f \in \omega^\omega$ be a function; I must show that $I \not\leq J_n \restriction f$. Suppose that $B \subset X$ is an I-positive Borel set, and $D \subset B \times Y_n \restriction f$ is a Borel set whose horizontal sections are $J_n \restriction f$-small. It will be enough to produce an I-positive horizontal section of the complement of the set D.

To simplify the notation, assume $X = 2^\omega$. Choose a countable elementary submodel M of a large enough structure, and use the properness and the bounding property of the poset P_I to find an I-positive compact set $C \subset B$ consisting of M-generic points, such that every subset of X in the model M has relatively clopen intersection with the set C. I will show that whenever $k \in \omega^\omega$ is a fast increasing function and $C' \subset C$ is a compact set such that $\forall j \in \omega\ |\{t \in 2^{k(j)} : O_t \cap C' \neq 0\}| \leq 2^j$, then there is a point $y \in Y_n \restriction f$ such that $C' \times \{y\} \cap D = 0$. Note that it is possible to find an I-positive set $C' \subset C$ like that simply by using the Sacks property of the forcing P_I to find a condition enclosing the sequence $\dot{x}_{gen}(k(j)) : j \in \omega$ into a tunnel of thickness 2^j. This will complete the proof.

The construction of the n-ary tree y is the key step, and the following notion will be instrumental. A *wall* is a Borel function $h \in M$ with Borel I-positive domain and range consisting of conditions in $P_n \restriction f$ which *cohere*: $\bigcup \mathrm{rng}(h)$ is covered by branches of some n-tree, or equivalently, subsets of $\mathrm{rng}(h)$ of size $n+1$ all have lower bounds. The walls are ordered by $h' \leq h$ if $\mathrm{dom}(h') \subset \mathrm{dom}(h)$ and $\forall x \in \mathrm{dom}(h')\ h'(x) \leq h(x)$. Finally, consider the poset Q of all walls h such that $C' \subset \mathrm{dom}(h)$. I will show

Claim 3.2. *Whenever $\dot{O} \in M$ is a P_I-name for an open dense subset of the poset $P_n \restriction f$, the collection of all walls h such that* $\mathrm{dom}(h) \Vdash \dot{h}(\dot{x}_{gen}) \in \dot{O}$ *is dense in Q.*

Once this claim is proved, the proposition follows: suppose that $g \subset Q$ is a filter meeting all the countably many open dense subsets of Q described in this claim. For every point $x \in C'$, the set $\{h(x) : h \in g\} \subset P_n \restriction f$ is then $M[x]$-generic. The resulting n-ary tree y does not depend on the choice of the point x, due to the coherence condition in the definition of a wall. Since the tree y is $M[x]$-generic, it cannot belong to the $J_n \restriction f$-small set $D_x \subset Y_n \restriction f$. Thus $C' \times \{y\} \cap D = 0$ as required.

To prove the claim, fix a wall $h \in M$ and a P_I-name $\dot{O} \in M$ for an open dense set. Choose a number $m \in \omega$. I will show that there is a number $l = l(m, h, \dot{O}) \in \omega$ such that for every m-tuple $\langle t_i : i \in m \rangle$ of binary sequences of length l,

- either for some index $i \in m$, $O_{t_i} \cap \mathrm{dom}(h) \cap C = 0$
- or there is a wall $h' \leq h$ such that $C \cap \bigcup_{i \in m} O_{t_i} \subset \mathrm{dom}(h')$ and $\mathrm{dom}(h') \Vdash h'(\dot{x}_{gen}) \in \dot{O}$.

This will immediately prove the claim. If $h \in Q$ is a wall and $\dot{O} \in M$ is a name for an open dense set, then the fast growth of the function k ensures that there will be a number $j \in \omega$ such that $k(j) > l(2^j, h, \dot{O})$. The set $\{t \in 2^{k(j)} : C' \cap O_t \neq 0\}$ has size $\leq 2^j$, and the second item above produces a wall $h' \in Q$, $h' \leq h$, and $\mathrm{dom}(h') \Vdash h'(\dot{x}_{gen}) \in \dot{O}$ as required.

To produce the number $l = l(m, h, \dot{O})$, first investigate generic extensions of the model M. Suppose $\vec{x}_i : i \in m$ are distinct points in the set $C \cap \mathrm{dom}(h)$. If they are not distinct just erase the repetitions. The set $p = \bigcup_{i \in m} h(x_i)$ is a condition in the poset $P_n \restriction f$ by the coherence condition in the definition of a wall. For every index $i \in m$, the point x_i is M-generic, so the expression $\dot{O}/x_i$ makes sense and denotes an open dense subset of the forcing $P_n \restriction f \cap M[x_i]$. An analytic absoluteness argument shows that this set is in fact predense in the whole poset $P_n \restriction f$, and there must be conditions $q_i \in \dot{O}/x_i, q_i \leq h(x_i)$ such that the whole collection $\{p, q_i : i \in m\}$ has a lower bound. Creatively use the n-localization property to find an n-tree $y \in M$ such that $\bigcup_{i \in m} q_i \subset [y]$.

By the forcing theorem, this situation must be reflected in the model M. That is, there are pairwise disjoint sets $B_i : i \in m$ in $P_I \cap M$ and Borel functions $h_i : B_i \to P_n \restriction f : i \in m$ in M such that for every index $i \in m$, $x_i \in B_i$, $B_i \Vdash \dot{h}_i(\dot{x}_{gen}) \in \dot{O}$, and for every point $x \in B_i$, $h_i(x) \leq h(x)$ and $h_i(x) \subset y'$.

The point now is that the sets $\mathrm{dom}(h), B_i : i \in m$ are relatively clopen in the set C. Thus the compact set $(C \cap \mathrm{dom}(h))^m$ is covered by relatively open sets with certain properties. A compactness argument yields a finite subcover and the required number l.

4 The cinch

The work in the previous sections leads to the proof of the theorems from the introduction via [12, Theorem 6.3.3].

I will first treat the ZFC case. Suppose that P is an analytic CRN forcing. Consider the ideal I on ω^ω generated by analytic sets A such that there is no tree $p \in P$ such that $[p] \subset A$. [12, Proposition 2.1.6, Theorem 3.8.9] shows that this is a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ ideal, every positive analytic set has a positive compact subset, and the forcing P is naturally isomorphic to a dense subset of the quotient P_I. Theorem 1.4 then immediately follows from the conjunction of Proposition 3.1, Theorem 6.3.3 of [12], and the fact that iterations of bounding forcings are bounding [1, Theorem 6.3.5].

The more general large cardinal case is almost identical. If I is suitably definable, then so is the ideal I^* generated by all universally Baire sets without an I-positive subset. The amended ideal I^* then satisfies the assumptions of Proposition 3.1, the quotients P_I and P_{I^*} are identical, and the last sentence of the previous paragraph applies again.

5 Variations and limitations

The n-localization property implies the Sacks property, and therefore very few forcings actually exhibit it. A number of partial orders adding unbounded reals nevertheless possess a *bounded* 2-localization property: every function $x \in \omega^\omega$ in the extension bounded by some ground model function is in fact a branch of a ground model binary tree. In some cases, a straightforward generalization of the above approach yields a nice iteration theorem.

Theorem 5.1. *The countable support iteration of Miller forcing has the bounded 2-localization property.*

Proof. Fix a function $f \in \omega^\omega$. The ideal $J_2 \restriction f$ is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$, and therefore the forcing $P_2 \restriction f$ does not add a dominating real. Thus $P_2 \restriction f \Vdash \omega^\omega \cap V \notin I$, where I is the σ-ideal associated with the Miller forcing: the ideal of σ-bounded sets. By [12, Proposition 3.2.2], this is equivalent to $I \not\perp J_2 \restriction f$. This Fubini property is preserved by the countable support iteration of Miller forcing by [12, Theorem 6.3.3], and therefore the countable support iteration of Miller forcing exhibits the bounded 2-localization property. □

I conjecture that even the countable support iterations of Mathias forcing have the bounded 2-localization property. However, the approach of this paper cannot lead to such a result. Mathias forcing adds a reaping real while every suitably definable c.c.c. forcing adds a splitting real, leading to a failure of the requisite Fubini property.

The iteration theorems from the introduction deal with suitably definable forcings only. This is no accident, as 2-localization property is not preserved even under iterations of undefinable forcings of length 2. I will show that the 4-Silver forcing Q_4 can be decomposed into a two step iteration $Q_2 * \dot{R}$ such that Q_2 is the 2-Silver forcing (and so has 2-localization) and $Q_2 \Vdash \dot{R}$ has the 2-localization property as well. It is not difficult to see that the 4-Silver forcing fails the 2-localization–the generic point is not a branch of any ground model 2-tree–and therefore the general iteration theorem fails. The point of course is that the remainder forcing $\dot{R}$ does not have a definition to which Theorems 1.3 or 1.4 can apply.

Definition 5.2. Let $n \in \omega$. The *n-Silver forcing* Q_n consists of partial functions $p : \omega \to n$ with coinfinite domain, ordered by reverse inclusion.

Theorem 5.3. *Let $n \in \omega$. The n-Silver forcing has the n-localization property.*

This result is optimal. Clearly, the n-Silver forcing fails the $n-1$-localization property, since the generic real cannot be enclosed by any ground model $n-1$-tree.

Proof. Suppose $p \Vdash \dot{y} \in \omega^\omega$ is a function; strengthening p if necessary we may find a continuous function $f : n^\omega \to \omega^\omega$ such that $p \Vdash \dot{y} = \dot{f}(\dot{x}_{gen})$. For a point $x \in n^\omega$ and a finite partial function $u : \omega \to n$ let $x\dot{\cup}u$ be the function obtained from x by replacing $x \restriction \mathrm{dom} u$ with u. By a standard fusion argument find a condition $q \leq p$ such that, enumerating the infinite set $\omega \setminus \mathrm{dom}(q)$ by $\{n_i : i \in \omega\}$ in increasing order, the following holds.

(*) For every $i \in \omega$ there is a number $m_i \geq n_i$ such that for every function $u : \{n_j : j \in i\} \to n$, for every $x \in n^\omega$ with $q \subset x$ the initial segment $f(x\dot{\cup}u) \restriction m_i$ is the same sequence $g(u)$, and for two such functions u, v, $g(u) = g(v) \leftrightarrow \forall x \in n^\omega\ q \subset x \to f(x\dot{\cup}u) = f(x\dot{\cup}v)$.

Now let $C = f''\{x \in n^\omega : q \subset x\}$. I will show that $C = [T]$ for some n-tree T; then clearly $q \Vdash \dot{y} \in [\check{T}]$ and the n-localization follows. Clearly C is a compact set and as such it consists of all branches of some tree T. Suppose for contradiction that the tree T branches into $n+1$ many immediate successors at some point, and let $\{x_l : l \in n+1\}$ be points in n^ω such that $q \subset x$ and such that the points $f(x_l) : l \in n+1$ split all at once at some natural number k.

Let $j \in \omega$ be the least number such that the set $a = \{x_l \restriction \{n_i : i \in j\} : l \in n+1\}$ has size greater than 1. Note that this set has size at most n. The key point: the sequences $\{g(u) : u \in a\}$ must be all the same. If two of them were different, then $m_j > k$, and since $\{f(x_l) \restriction m_j : l \in n+1\} = \{g(u) : u \in a\}$, this contradicts the fact that the set $\{f(x_l) \restriction k+1 : l \in n+1\}$ has size $n+1$.

This means that for every $l \in n+1$ and every $u \in a$, it is the case that $f(x_l) = f(x_l\dot{\cup}u)$, and it is possible to rewrite the sequences $\{x_l : l \in n+1\}$ in such a way that their restriction to the set $\{n_i : i \in j\}$ is any given single element $u \in a$, without changing the values $\{f(x_l) : l \in n+1\}$. One can repeat this procedure many times, pushing the first disagreement between the sequences $x_l : l \in n+1$ past the number n_k, but then the value $f(x_l)(k)$ will be the same for all numbers $l \in n+1$, contradiction. □

Theorem 5.4. *The 4-Silver forcing Q_4 can be decomposed as $Q_2 * \dot{R}$, where $Q_2 \Vdash \dot{R}$ has the 2-localization property.*

The remainder forcing $\dot{R}$ clearly preserves $\aleph_1$ since Q_4 does. If the Continuum Hypothesis holds then the remainder will be in fact proper; I will avoid the awkward argument.

Proof. The decomposition is simple. Let $4 = a_0 \cup a_1$ be a partition into two disjoint sets of size 2. Suppose x_4 is a 4-Silver generic point. Let $x_2 \in 2^\omega$ be the point defined by $x_2(n) = i \leftrightarrow x_4(n) \in a_i$. It is rather obvious that x_2 is a 2-Silver generic. The forcing decomposition then follows the chain $V \subset V[x_2] \subset V[x_4]$

of generic extensions. I just have to verify that the second step has the 2-localization property, in other words, every point $y \in V[x_4] \cap \omega^\omega$ is a branch of a 2-tree in the model $V[x_2]$.

Back to V. Suppose $p \in Q_4$ is a condition and $\dot{y}$ is a Q_4-name for a point in ω^ω. Strengthening the condition p if necessary find a continuous function $f : 4^\omega \to \omega^\omega$ such that $p \Vdash \dot{y} = \dot{f}(\dot{x}_{gen})$. Find a condition $q \leq p$ staisfying (*) in the proof of the previous theorem. Now move to the model $V[x_2]$ and consider the set $C = f''\{x \in 4^\omega : \forall i \in \omega\ x(i) \in a_{x_2(i)} \wedge q \subset x\}$. The same argument as in the previous theorem shows that $C = [T]$ for some 2-tree $T \subset \omega^{<\omega}$. Clearly, $T \in V[x_2]$ is a 2-tree such that $y \in [T]$, and the theorem follows. □

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