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Diploma Thesis

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On higher dimensional Kerr-Schild spacetimes

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Declaration.

I declare that I have written my diploma thesis on my own with a help of literature listed in Bibliography. I agree with lending of my work.

In Prague, 5th of August 2009 Bc. Martin Kršš́ák

Abstrakt.

Název práce: O vícerozměrných Kerr-Schildových časo-prostorech

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Abstrakt: V tejto diplomovej práci sa venujeme Kerr-Schildovým (KS) časopriestorom vo vyšších ($n > 4$) dimenziách s geodetickým nulovým vektorom ℓ. S použitím zovšeobecnenia Newman-Penrosového formalizmu do vyšších dimenzii, nájdeme, pre metriku v tvare KS ansatzu, zodpovedajúce Einsteinové rovnice a zameriame sa na vákuové riešenia.

Spomenieme najnovšie výsledky v prípade neexpandujúcich riešení a my sa zameriame na expandujúce. Zistíme, že jedna z Einsteinových rovníc, ktorú nazývame *optická väzba*, plne určuje ℓ. Zvyšok Einsteinových rovníc určuje KS funkciu $\mathcal{H}$.

Nájdené rovnice analyzujeme a zistíme, že sa jedná o systém nelineárnych PDR s nelineárnymi väzbami, čo nám zabraňuje nájsť analytické riešenie vo všeobecnom prípade. Preto sa musíme uspokojiť s čiastočnými výsledkami.

Zameriame sa na riešenia bez "twistu". V rámci tejto triedy riešení nájdeme vhodný ansatz splňujúci optickú väzbu s pomocou ktorého sme schopní nájsť explicitné riešenie všetkých Einsteinových rovníc. Ukážeme, že takéto riešenie zodpovedá čiernej strune a ukážeme ako sa dá toto riešenie zovšeobecniť, tak aby odpovedalo čiernej "p-bráne".

Druhým prístupom je skúmanie r-závislosti KS časo-priestorov. Za r si zvolíme afinný parameter pozdĺž vektora ℓ. Nájdeme r-závislosť všetkých Ricciho koeficientov, KS funkcie $\mathcal{H}$ a operátorov smerových derivácii do ľubovolného rádu v r. S týmito vedomosťami sme schopní nájsť r-závislosť všetkých komponent Riemannovho tenzora a analyzujeme jeho asymptotické vlastnosti.

Klíčová slova: Kerr-Schildové časo-prostory, vyšší dimenze, černé díry, Newman-Penrose formalizmus.

Abstract.

Title: On higher dimensional Kerr-Schild spacetimes

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Abstract: In this thesis we investigate Kerr-Schild (KS) metrics in higher ($n > 4$) dimensions with geodetic null-congruence ℓ, using a generalization of Newman-Penrose formalism to higher dimensions. We use KS ansatz for metric and find corresponding Einstein's equations. We focus on vacuum solutions of Einstein's equations.

We give some remarks on recent results about non-expanding solution and we focus on expanding solutions. One of Einstein's equation, to which we refer as the *optical constraint*, determines completely the null-congruence. The remaining determine KS function $\mathcal{H}$.

We analyze these equations and find that they are a systems of nonlinear PDEs with non-linear couplings, what prohibit us from finding analytical solution in general case. Therefore, we have to satisfy ourselves with partial results.

We thus focus on non-twisting solutions. Within this class, we find a suitable ansatz which satisfies the optical constraint and we solve all the corresponding Einstein equations explicitly. We further demonstrate that this solution corresponds to (static) black strings and we give instruction how to generalize it to the black p-branes.

Second approach is that we fix r-depependence choosing affine parameter along null vector ℓ. We derive r-dependance of all Ricci coefficients, KS function $\mathcal{H}$ and operators of directional derivative to arbitrary term in r. Using this knowledge we find r-dependance of all components of Riemann tensor and analyze their asymptotic properties.

Keywords: Kerr-Schild solutions, higher dimensions, black holes, Newman-Penrose formalism.

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Chapter 1

Introduction

One of the main tasks of theoretical physics during the last 5-6 decades was, and still remains, to find a theory which would unify all four fundamental forces of Nature. During 70s, the quantum description of three of these forces was successfully unified to one theory, presently known as the *Standard Model* . However, despite of a huge effort of physicists world-wide it was not possible to incorporate gravitation into the Standard Model or to find a consistent quantum theory of gravitation. It turned out that gravitation at sub-atomic scales is very different from all other interactions and that there is a need for a new theory.

Nowadays, we have basically three main theories which are aspiring to be correct quantum theory of gravitation. The main one, which dominates today in theoretical physics is *string theory*, which suggests that problems of quantization of gravitation can be resolved by considering fundamental particles as one-dimensional objects, i.e. we can think of them as a tiny strings [1]. Second approach, a wide family of approaches called *canonical quantum gravitation*, is trying to develop non-perturbative methods to quantize gravitation [2]. The most famous of these theories is the *loop quantum gravity* in which space is represented by quantized loops called spin networks [3]. Other approach, gaining popularity recently, is *non-commutative geometry* [4].

Our work presented in this thesis is directly in neither of these theories, but in a field of classical gravitation in spacetimes with more than 4 dimensions. In order to understand motivation for such work, we will mention main features of string theory and how it is connected with matter of our research.

One of the main flaws of Standard Model, beside its lack of description of gravitation, is that it has 19 free parameters (masses of 3 leptons, 6 quarks, coupling constants,...) which are needed to fully describe dynamics of the theory [5]. These parameters in Standard model can take arbitrary values and have to be find experimentally. However, this is a very inelegant feature of Standard Model, because it seems that if these free-parameters would have just slightly different values, our Universe would be very different from what we are presently experiencing and would not allow creation of galaxies, star systems, stable atoms and support creation of life.

String theory solves this problem elegantly, because the only free parameter of this theory is string tension and all particles are considered to be just different vibrational modes of fundamental string. Beside claiming of explanation of masses of fundamental particles, there was found spin-two field representing gravitation. Thus string theory is claiming to be a consistent quantum theory of gravitation.

However, to make this theory consistent we need to introduce supersymmetry and 6 extra dimensions, i.e. spacetime is required to be 10 dimensional. The two most popular ways how to deal with extra-dimensions are *compactification* of extra-dimensions and *brane models*. The last one suggests that our universe lives on 4 dimensional brane floating in 10 dimensional space, where 3 fundamental interactions of Standard Model are confined to this brane, while gravitation acts in full 10 dimensional space. This solves so-called *hierarchy problem*, i.e. explains relative weakness of gravitation against other interactions [6].

Idea of compactification of extra-dimensions suggests that the reason why we cannot see these extra dimensions is that they are such small that they are undetectable by ordinary experiments. Scale of these extra-dimensions is not determined by string theory and theoretically can be as small as the Planck length, what would make any laboratory test of string theory unrealistic in any foreseeable future, or, as it suggests so-called *large dimensions scenarios*, can be sufficiently large to make string theory testable at LHC which should start to produce the first measurements soon.

If the scale of extra-dimensions is large enough we could be able to observe microscopic black-holes at the LHC, see [7], [8], [9], [10] or recent review [11]. This would not allow us to test just string theory, but also to make laboratory observation of black holes for first time[1].

Beside important role in possible experiments, higher dimensional black holes (more specific their generalizations black p-branes) are important in investigation of non-perturbative effect of string theory [1], [13]. Also in gauge/string duality known as AdS/CFT [14], [15], solutions of higher dimensional gravitation can be useful.

But, first we need to better understand classical black hole solutions in higher dimensions , in order to understand which effects are specific to string theory and which are purely classical gravitational effects. Thus, string theory is an important source of motivation to understand solutions of classical gravitation in higher dimensions.

Beside, interesting results for string theory or possible experiments, classical gravitation in higher dimensions is interesting on its own. For example, famous topology theorem by Hawking [16], [17] that a stationary black hole horizon has always the spherical topology does not hold in higher dimensions. Solutions as black strings, or recently found black rings [18] are well-known counter-examples. Thus, question of uniqueness of black holes in higher dimensions is much more complicated, see [21] and [20] for recent progress.

Thus, research in higher dimensional gravitation is interesting from many perspectives. In this thesis we will focus on class of solutions called *Kerr-Schild solutions*, which played important role in four dimensional gravitation and we will try to analyze some of their properties in higher dimensions.

Let us first give very brief outlook of the most important results about black holes in higher dimensions before we start to analyze Kerr-Schild spacetimes.

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[1] Note, that we would not observe directly black holes, but analogue of Hawking radiation from these black holes. Hawking radiation was already experimentally observed in solid states [12], so we may expect to occur in a gravitational case and thus saving us from growing black hole to dangerous size [11].

1.1 Basic black hole solutions

General relativity describes dynamics of space-time. Fundamental equations of this theory are Einstein's equations, which in a natural units $\hbar = G = c = 1$ and without cosmological constant have a form

$$R_{ab} - \frac{1}{2} g_{ab} R = 8\pi T_{ab}, \tag{1.1}$$

i.e. they say how geometry of spacetime (left-hand side of these equations) is coupled to matter (described by right-hand side of these equations). Dynamical variable of Einstein's equations is metric tensor g_{ab}, which is a symmetric rank-two tensor, i.e. it has generally $\frac{n(n+1)}{2}$ independent components. Thus, to solve Einstein's equations (1.1) means to find all independent components of the metric tensor. Our work in whole thesis will be restricted to the case of vacuum space-times[2], where geometry of space-time is given by *vacuum Einstein's equations*

$$R_{ab} = 0. \tag{1.2}$$

Even vacuum Einstein's equations are a system of non-linear partial differential equations and there exists no general analytical solution. However, it is possible to find some solutions if one assumes that metric tensor possess symmetry. In the case of *spherical symmetry* we are able to find that the most general spherically symmetric metrical tensor should have a following form [17]

$$ds^2 = -f(t,r)dt^2 + g(t,r)dr^2 + r^2 d\Omega^2, \tag{1.3}$$

where

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2. \tag{1.4}$$

If we insert this metric into the vacuum Einstein's equations we find well-known *Schwarzschild solution*

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} + r^2 d\Omega^2. \tag{1.5}$$

[2]However, we will occasionally mention generalizations containing simple form of energy-momentum, like electromagnetic field, or cosmological constant, and we will give corresponding references.

Generalization of this solution to arbitrary dimension is quite straightforward and was found by Tangherlini in 1963 [22]

$$ds^2 = -\left(1 - \frac{2M}{r^{n-3}}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r^{n-3}}\right)} + r^2 d\Omega_{n-2}^2, \tag{1.6}$$

this solution is usually referred as a *Scharzschild-Tangherlini solution.*

If we impose less-restrictive symmetry we are able to derive solution representing rotating black hole. In four dimensions black hole can rotate just in a single plane, however in higher dimensions situation is different and we can have $\lfloor \frac{n-1}{2} \rfloor$ planes of rotation [23], where $\lfloor \cdots \rfloor$ represents integer part of a given number.

Solution with a single plane of rotation is given by [23], [24]

$$\begin{aligned} ds^2 &= -dt^2 + \frac{M}{r^{n-5}\Sigma}(dt - a\sin^2\theta d\phi)^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 \\ &+ (r^2 + a^2)\sin^2\theta d\phi^2 + r^2\cos^2\theta d\Omega_{n-4}^2, \end{aligned} \tag{1.7}$$

where

$$\Sigma = r^2 + a^2\cos^2\theta, \qquad \Delta = r^2 + a^2 - \frac{M}{r^{n-5}}. \tag{1.8}$$

In the case $n = 4$ this is well-known *Kerr solution* [25], which is one of most important known solution in general relativity. Generalization of this solution with more spins are *Myers-Perry solutions* [23], which will be briefly described in section 1.3. Other black hole solution in higher dimensional gravitation are black strings and generally black p-branes [13], black rings [18] and many other solutions, for example solutions with supersymmetry, or cosmological constant. We refer reader to reviews [24], [19] for more details.

1.2 Original Kerr-Schild solutions

In 1960s Kerr, *et. al.* [26], [27], [28], [29] found, in four dimensions, general class of solutions which contains Kerr solutions as special case. This class of solutions is presently known as *Kerr-Schild spacetimes.* In this approach we do not demand spherical symmetry or axisymmetry of metric tensor, but we use following ansatz for a metric

$$g_{ab} = \eta_{ab} + 2\mathcal{H}k_a k_b, \tag{1.9}$$

where function $\mathcal{H}$ is some analytic function and η_{ab} is metric of a four-dimensional Minkowski space-time and k^a is a congruence of null vectors with respect to both metrics g_{ab} and η_{ab}

$$g_{ab}k^a k^b = \eta_{ab}k^a k^b = 0. \tag{1.10}$$

It is usefull to use null coordinates

$$u = 2^{-\frac{1}{2}}(z+t), \qquad \zeta = 2^{-\frac{1}{2}}(x+iy),$$
$$v = 2^{-\frac{1}{2}}(z-t), \qquad \bar{\zeta} = 2^{-\frac{1}{2}}(x-iy).$$

Most general null direction in such coordinates is given by

$$k = du + \bar{Y}d\zeta + Yd\bar{\zeta} - Y\bar{Y}dv, \tag{1.11}$$

where Y is some analytic function of coordinates and $\bar{Y}$ its complex conjugate. Advantage of using complex function and coordinates is that then we have just half of necessary equation, because other half can be obtained by complex conjugation. General solution of Einstein's equations in such case exists and is given by

$$\begin{aligned} ds^2 &= 2d\zeta d\bar{\zeta} + 2dudv \\ &+ P^{-3}\left[m(Z+\bar{Z}) - \psi\bar{\psi}Z\bar{Z}\right]\left(du + \bar{Y}d\zeta + Yd\bar{\zeta} - Y\bar{Y}dv\right)^2, \end{aligned} \tag{1.12}$$

where

$$P = pY\bar{Y} + qY + \bar{q}\bar{Y} + c, \tag{1.13}$$
$$Z = -PF_Y^{-1}, \tag{1.14}$$

and Y is given implicitly by

$$F = 0, \tag{1.15}$$

and F is defined as

$$F = \phi + (qY + c)(\zeta - Yv) - (pY + \bar{q})(u + Y\bar{\zeta}), \tag{1.16}$$

where ϕ and ψ are arbitrary analytic functions of the complex variable Y and m, p, c are real constants and q is complex constant.

Let us note, that equations (1.15) and (1.16) fully determine Y and consequently fully determine the null-congruence k^a. This is known as the Kerr theorem and in four dimensions its solution is the most general shear-free, geodetic congruence in flat space [27], [30]. This is an important result not only in investigations of Kerr-Schild spacetimes, but have applications also in other field of physics, for example *twistor theory* [30].

1.3 Myers-Perry solutions

In 1986 Myers and Perry [23] used generalization of the Kerr metric [25] and thus obtained one of very few known Kerr-Schild solutions in higher dimensions representing black hole rotating in many planes of rotation.

As it was mentioned earlier, black holes in higher dimensions can rotate in $\lfloor \frac{n-1}{2} \rfloor$ plane of rotation and thus it is convenient to pair the coordinates as $x^a = \{x^\alpha, y^\alpha\}$, where $\alpha = 1, \cdots \lfloor \frac{n-1}{2} \rfloor$. Consequence of this is that Myers-Perry black holes have separate solutions for odd and even dimensional spacetimes. Let us start with **odd** dimensions. In this case Kerr-Schild metric (1.9) is fully determined by

$$k = k_a dx^a = dt + \sum_{\alpha=1}^{\lfloor \frac{n-2}{2} \rfloor} \frac{r(x^\alpha dx^\alpha + y^\alpha dy^\alpha) + a_\alpha (x^\alpha dy^\alpha - y^\alpha dx^\alpha)}{r^2 + {a_\alpha}^2}, \tag{1.17}$$

and

$$\mathcal{H} = \frac{\mu r^2}{\Pi F}, \tag{1.18}$$

where a_α are some constants and r is defined implicitly by

$$\sum_{\alpha=1}^{\lfloor \frac{n-2}{2} \rfloor} \frac{x^{\alpha 2} + y^{\alpha 2}}{r^2 + {a_\alpha}^2} = 1, \tag{1.19}$$

and

$$F = 1 - \sum_{\alpha=1}^{\lfloor \frac{n-2}{2} \rfloor} \frac{{a_\alpha}^2 (x^{\alpha 2} + y^{\alpha 2})}{(r^2 + {a_\alpha}^2)^2}, \tag{1.20}$$

$$\Pi = \prod_{\alpha=1}^{\lfloor \frac{n-2}{2} \rfloor} (r^2 + {a_\alpha}^2). \tag{1.21}$$

In **even** dimensions the solution is given by

$$k = k_a dx^a = dt + \sum_{\alpha=1}^{\lfloor \frac{n-2}{2} \rfloor} \frac{r(x^\alpha dx^\alpha + y^\alpha dy^\alpha) + a_\alpha (x^\alpha dy^\alpha - y^\alpha dx^\alpha)}{r^2 + {a_\alpha}^2} + \frac{z dz}{r}, \tag{1.22}$$

and r is defined implicitly by

$$\sum_{\alpha=1}^{\lfloor\frac{n-2}{2}\rfloor} \frac{x^{\alpha 2} + y^{\alpha 2}}{r^2 + {a_\alpha}^2} + \frac{z^2}{r^2} = 1, \tag{1.23}$$

with F and Π as in previous case. We can easily check that in 4 dimensions this is equivalent to the Kerr metric in coordinate basis [25]. There exist generalizations of Myers-Perry solution to the case of dS and AdS spacetimes [31].

1.4 Outline

Generalization of these results to the higher dimensions will be our main interest in this thesis. We will see that this generalization is not straightforward at all and that we will encounter many problems. Situation indeed is very different from four dimensional case because we have many degrees of freedom.

As a illustration take the Kerr theorem, mentioned at the end of section 1.2. In four dimensions introducing complex function Y allowed us to specify null congruence k uniquely by this single function. Consequence of this was that null congruence k was fully determined by simple partial differential equation of first order, which we know how to solve analytically.

As we will see in following chapters, in higher dimension our null congruence will be specified by $n-2$ functions and resulting equations will be system of non-linearly coupled partial differential equations. However, we will try to analyze these equations and try to find their solutions at least in some special cases.

We will start in chapter 2 with introduction of Newman-Penrose formalism in higher dimensions and we will summarize most important results which we will need in following calculations.

We will continue in chapter 3 with adopting Kerr-Schild ansatz metric (1.9) in Newman-Penrose formalism and we will find Einstein's equations for it.

In chapter 4 we will analyze Einstein's equations, we will give reasons why we have to give up hope to solve them analytically in general case and we solve them at least in some special cases.

Chapter 2

Newman-Penrose formalism in higher dimensions

In investigation of Kerr-Schild spacetimes we will use generalization of Newman-Penrose (NP) formalism [32] in higher dimensions. In this chapter we will give introduction to tetrad formalism and to Newman-Penrose formalism.

2.1 Frame formalism

At each point of space-time it is possible to introduce n-tuple of n independent vectors $m_{\hat{a}}{}^{b}$, where normal letters are *tensor indices* and indices with hat are *frame indices* and label the different vectors of the n-tuple. Both kinds of indices run from 0 to $n-1$, but we always have to keep in mind that they are two different kinds of indices. Einstein summation convention holds for both kinds of indices.

This n-tuple of vectors in four dimensions is usually called *tetrad* or, what is more appropriate in higher dimensions, *frame*, from where we have name for the whole formalism- *frame formalism*.

Dual frame is defined by either of the equivalent relations

$$m_{\hat{a}}{}^{c} m^{\hat{b}}{}_{c} = \delta^{\hat{b}}{}_{\hat{a}}, \qquad m_{\hat{a}}{}^{b} m^{\hat{a}}{}_{c} = \delta^{b}{}_{c}. \tag{2.1}$$

Using this we can find the frame components of any tensor $T^{b\cdots}_{a\cdots}$ from relation

$$T_{\hat{a}\cdots}{}^{\hat{b}\cdots} = m_{\hat{a}}{}^{c} m^{\hat{b}}{}_{d} \cdots T_{c\cdots}{}^{d\cdots}, \tag{2.2}$$

and inverse relation is given by

$$T_{a\cdots}^{\ b\cdots} = m^{\hat{c}}{}_{a} m_{\hat{d}}{}^{b} \cdots T_{\hat{c}\cdots}^{\ \hat{d}\cdots}. \tag{2.3}$$

The frame vectors determine differential forms

$$m^{\hat{a}} \ = m^{\hat{a}}{}_{b} dx^{b}, \tag{2.4}$$

and their duals

$$m_{\hat{a}} = m_{\hat{a}}{}^{b} \partial_{b}. \tag{2.5}$$

So, the metric form will be given by

$$ds^2 = m_{\hat{a}} m^{\hat{a}} \ = g_{ab} dx^a dx^b. \tag{2.6}$$

The directional derivative of function along a frame vector will be denoted by $_{|\hat{a}}$ or by $\delta_{\hat{a}}$

$$f_{|\hat{a}} = \delta_{\hat{a}} f = m_{\hat{a}}{}^{b} \frac{\partial f}{\partial x^b}. \tag{2.7}$$

The frame components of a covariant derivative are projections of coordinate covariant derivatives

$$T_{\hat{a}\cdots\ ;\hat{c}}^{\ \hat{b}\cdots} = T_{d\cdots\ ;g}^{\ e\cdots} m_{\hat{c}}{}^{g} m_{\hat{a}}{}^{d} m^{\hat{b}}{}_{e} \cdots , \tag{2.8}$$

and they are given by

$$T_{\hat{a}\cdots;\hat{c}}^{\hat{b}\cdots} \equiv T_{\hat{a}\cdots|\hat{c}}^{\hat{b}\cdots} - \Gamma^{\hat{d}}{}_{\hat{a}\hat{c}} T_{\hat{d}\cdots}^{\hat{b}\cdots} - \cdots + \Gamma^{\hat{b}}{}_{\hat{d}\hat{c}} T_{\hat{a}\cdots}^{\hat{d}\cdots} + \cdots , \tag{2.9}$$

where $\Gamma^{\hat{a}}{}_{\hat{b}\hat{c}}$ are Ricci rotation coefficients

$$\Gamma^{\hat{a}}{}_{\hat{b}\hat{c}} = -m^{\hat{a}}{}_{d;e} m_{\hat{b}}{}^{d} m_{\hat{c}}{}^{e}, \tag{2.10}$$

we can lower first index to obtain

$$\Gamma_{\hat{a}\hat{b}\hat{c}} = g_{\hat{a}\hat{d}} \Gamma^{\hat{d}}{}_{\hat{b}\hat{c}}. \tag{2.11}$$

We can define new 1-forms called *connection forms*, which are projections of Ricci rotation coefficients on frame vectors

$$\Gamma^{\hat{a}}{}_{\hat{b}} = \Gamma^{\hat{a}}{}_{\hat{b}\hat{c}} m^{\hat{c}} \ , \tag{2.12}$$

and *curvature form*, which are projections of frame components of curvature tensor on frame vectors

$$\mathcal{R}_{\hat{a}\hat{b}} = R_{\hat{a}\hat{b}\hat{c}\hat{d}} m^{\hat{c}} \wedge m^{\hat{d}} \; . \tag{2.13}$$

Both of them can be obtained from frame forms using *Cartan equations*

$$dm^{\hat{a}} = m^{\hat{b}} \wedge \Gamma^{\hat{a}}{}_{\hat{b}} = \Gamma^{\hat{a}}{}_{\hat{b}\hat{c}} m^{\hat{b}} \wedge m^{\hat{c}} \; , \tag{2.14}$$

$$\frac{1}{2}\mathcal{R}^{\hat{a}}{}_{\hat{b}} = d\Gamma^{\hat{a}}{}_{\hat{b}} + \Gamma^{\hat{a}}{}_{\hat{c}} \wedge \Gamma^{\hat{c}}{}_{\hat{b}}. \tag{2.15}$$

Equation (2.14) determines just skew-symmetric part of the Ricci coefficients $\Gamma_{\hat{a}[\hat{b}\hat{c}]}$. The symmetric part $\Gamma_{(\hat{a}\hat{b})\hat{c}}$ of the Ricci coefficients is given by

$$2\Gamma_{(\hat{a}\hat{b})\hat{c}} = g_{\hat{a}\hat{b}|\hat{c}}, \tag{2.16}$$

since

$$0 = g_{\hat{a}\hat{b};\hat{c}} = g_{\hat{a}\hat{b}|\hat{c}} - 2\Gamma_{(\hat{a}\hat{b})\hat{c}}. \tag{2.17}$$

In all calculations in this thesis we will restrict just to the case of rigid frames, where $g_{\hat{a}\hat{b}}$ are constants and thus from (2.16) we can find that Ricci rotation coefficients are skew-symmetric in the first two indices. Thus they can be determined from (2.14) by

$$\Gamma_{\hat{a}\hat{b}\hat{c}} = -\Gamma_{\hat{b}\hat{a}\hat{c}} = \Gamma_{\hat{a}[\hat{b}\hat{c}]} + \Gamma_{\hat{b}[\hat{c}\hat{a}]} - \Gamma_{\hat{c}[\hat{a}\hat{b}]}. \tag{2.18}$$

For more details and derivations see [27], [29] or [33].

2.2 Newman-Penrose formalism

Special form of tetrad formalism was developed in 1960s by E. Newman and R. Penrose [32], who used a special form of the frame metric $g_{\hat{a}\hat{b}}$. While in tetrad formalism Minkowski metric, with orthonormal frame vectors, was used, the idea of Newman and Penrose was to use null metric, which turned out to be very useful, mainly in the investigation of algebraically special spacetimes (according to Petrov classification) [17], [29]. One feature of this approach is that we obtain large amount of equations, from which many are redundant, however advantage of this approach is that all obtained differential equations are of first order.

2.2.1 Null frames

NP formalism which we will use in the analyzes of Kerr-Schild spacetimes has to be generalized to higher dimensions. This was done in papers [34], [35] and [36].

We choose our frame metric to be

$$g_{\hat{a}\hat{b}} = \begin{pmatrix} 0 & 1 & . & \cdots & . \\ 1 & 0 & . & \cdots & . \\ . & . & 1 & \cdots & . \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ . & . & . & \cdots & 1 \end{pmatrix}, \tag{2.19}$$

where $\cdots$ always represents zero, except diagonal where it represents 1. So, this means that we are choosing null frame with two null vectors $m^{\hat{0}} = n$ and $m^{\hat{1}} = \ell$ and $n-2$ orthonormal space-like vectors $m^{\hat{i}}$ for which

$$\begin{aligned} l^a l_a = n^a n_a = n^a m^{\hat{i}}{}_a = l^a m^{\hat{i}}{}_a &= 0, \\ l^a n_a &= 1, \\ m^{\hat{i}a} m^{\hat{j}}_a &= \delta_{\hat{i}\hat{j}}. \end{aligned} \tag{2.20}$$

From now on, indices taken from start of alphabet runs $a, b, \cdots = 0, \ldots, n-1$ and indices from middle of alphabet $i, j, \cdots = 2, \ldots, n-1$.

Lorentz transformations of frame vectors $m^{\hat{a}}$ can be described by null rotations (where ℓ or n are fixed), spatial rotations of the vectors $m^{\hat{i}}$ (spins) and boosts.

Null rotations:

$$\tilde{\ell} = \ell, \tag{2.21}$$

$$\tilde{n} = n + z_{\hat{i}} m^{\hat{i}} - \frac{1}{2} z^2 \ell, \tag{2.22}$$

$$\tilde{m}^{\hat{i}} = m^{\hat{i}} - z_{\hat{i}} \ell, \tag{2.23}$$

where $z_{\hat{i}}$ are some real functions and $z^2 = z^{\hat{i}} z_{\hat{i}}$.

Spins:

$$\tilde{\ell} = \ell, \tag{2.24}$$

$$\tilde{n} = n, \tag{2.25}$$

$$\tilde{m}^{\hat{i}} = X^{\hat{i}}_{\hat{j}} m^{\hat{j}}, \tag{2.26}$$

where $X \in SO(n-2)$.

Boosts:

$$\tilde{\ell} = \lambda \ell, \tag{2.27}$$

$$\tilde{n} = \lambda^{-1} n, \tag{2.28}$$

$$\tilde{m}^{\hat{i}} = m^{\hat{i}}\ , \tag{2.29}$$

where λ is some function $\lambda \in \mathcal{R}$.

2.2.2 Ricci rotation coefficients

We can find Ricci rotation coefficients (2.10) in this null frame. They will give us information about covariant derivatives of frame vectors. We will define them to be

$$\begin{aligned} l_{a;b} &= L_{\hat{c}\hat{d}} m^{\hat{c}}_a m^{\hat{d}}_b, \\ n_{a;b} &= N_{\hat{c}\hat{d}} m^{\hat{c}}_a m^{\hat{d}}_b, \\ m^{\hat{i}}_{a;b} &= \overset{\hat{i}}{M}_{\hat{c}\hat{d}} m^{\hat{c}}_a m^{\hat{d}}_b. \end{aligned} \tag{2.30}$$

Note, that in notation used in (2.10) these coefficients are

$$N_{\hat{b}\hat{c}} = -\Gamma^{\hat{0}}{}_{\hat{b}\hat{c}} = -\Gamma_{\hat{1}\hat{b}\hat{c}}, \tag{2.31}$$

$$L_{\hat{b}\hat{c}} = -\Gamma^{\hat{1}}{}_{\hat{b}\hat{c}} = -\Gamma_{\hat{0}\hat{b}\hat{c}}, \tag{2.32}$$

$$\overset{\hat{i}}{M}_{\hat{b}\hat{c}} = -\Gamma^{\hat{i}}{}_{\hat{b}\hat{c}} = -\Gamma_{\hat{i}\hat{b}\hat{c}}. \tag{2.33}$$

If we take first derivatives of scalar products of basis vectors from previous section (2.20) we find

$$L_{\hat{0}\hat{a}} = N_{\hat{1}\hat{a}} = \overset{\hat{i}}{M}_{\hat{i}\hat{a}} = 0, \tag{2.34}$$

$$N_{\hat{0}\hat{a}} + L_{\hat{1}\hat{a}} = \overset{\hat{i}}{M}_{\hat{0}\hat{a}} + L_{\hat{i}\hat{a}} = \overset{\hat{i}}{M}_{\hat{1}\hat{a}} + N_{\hat{i}\hat{a}} = \overset{\hat{i}}{M}_{\hat{j}\hat{a}} + \overset{\hat{j}}{M}_{\hat{i}\hat{a}} = 0, \tag{2.35}$$

which reduces a number of independent rotation coefficients to $n^2(n-1)/2$.

We define covariant derivatives along the frame vectors in analogy with 4 dimensional NP formalism, which acts on scalar as

$$Df = f_{|\hat{0}} \equiv l^a \nabla_a f, \qquad \Delta f = f_{|\hat{1}} \equiv n^a \nabla_a f, \qquad \delta_i f = f_{|\hat{i}} \equiv m^{\hat{i}a} \nabla_a f. \tag{2.36}$$

Expansion, shear and twist

In four dimensions for complex null frame we can define well-known *optical scalars*, namely *expansion*, *shear* and *twist*. In higher dimensions the situation is more complicated. We will be able to obtain similar quantities as in 4 dimensions, but with difference that they will be not scalars anymore, they have to be generalized to vectors or matrices.

Physically interesting case is when the vector ℓ is geodetic. From (2.30) we can see that

$$l_{a;b}l^b = L_{\hat{1}\hat{0}}l_a + L_{\hat{i}\hat{0}}m_a^{(i)}, \tag{2.37}$$

so vector ℓ is geodetic iff $L_{\hat{i}\hat{0}} = 0$. In a such case, according to transformation properties of rotation coefficients given in [36], we can find that $L_{\hat{i}\hat{j}}$ is invariant under null rotations and transforms under boosts simply as

$$\tilde{L}_{\hat{i}\hat{j}} = \lambda L_{\hat{i}\hat{j}}, \tag{2.38}$$

and thus it has special geometric meaning, because it characterizes null congruence in an invariant way. The matrix $L_{\hat{i}\hat{j}}$ can be decomposed into

$$L_{\hat{i}\hat{j}} = S_{\hat{i}\hat{j}} + A_{\hat{i}\hat{j}}, \tag{2.39}$$

where

$$S_{\hat{i}\hat{j}} \equiv L_{(\hat{i}\hat{j})} = \sigma_{\hat{i}\hat{j}} + \theta\delta_{\hat{i}\hat{j}}, \tag{2.40}$$

$$A_{\hat{i}\hat{j}} \equiv L_{[\hat{i}\hat{j}]}, \tag{2.41}$$

where $\sigma_{\hat{i}\hat{j}}$ and $A_{\hat{i}\hat{j}}$ are *shear* and *twist* matrices, respectively. From these we can define three scalars under space and null rotations θ, σ, ω, which we call *expansion*, *shear* and *twist* scalars, respectively.

They are given by

$$\theta \equiv \frac{1}{n-2}S_{\hat{i}\hat{i}}, \qquad \sigma^2 \equiv \sigma_{\hat{i}\hat{j}}\sigma_{\hat{i}\hat{j}}, \qquad \omega^2 \equiv A_{\hat{i}\hat{j}}A_{\hat{i}\hat{j}}, \tag{2.42}$$

where Einstein summation convention applies.

2.2.3 Curvature tensor and Ricci tensor

In order to solve Einstein's equations we have to find components of Ricci tensor in Newman-Penrose formalism. Second Cartan equation (2.15) will be used

$$d\Gamma_{\hat{a}\hat{b}} + \Gamma_{\hat{a}\hat{c}} \wedge \Gamma_{\hat{d}\hat{b}} g^{\hat{c}\hat{d}} = \frac{1}{2}\mathcal{R}_{\hat{a}\hat{b}}. \tag{2.43}$$

Ricci rotation coefficients are determined from connection forms which are skew-symmetric, so we have just these independent forms $\Gamma_{\hat{0}\hat{1}}, \Gamma_{\hat{0}\hat{i}}, \Gamma_{\hat{1}\hat{i}}, \Gamma_{\hat{i}\hat{j}}$:

$$d\Gamma_{\hat{0}\hat{1}} + \Gamma_{\hat{0}\hat{i}} \wedge \Gamma_{\hat{i}\hat{1}} = \frac{1}{2}\mathcal{R}_{\hat{0}\hat{1}} = \frac{1}{2}R_{\hat{0}\hat{1}\hat{a}\hat{b}} m^{\hat{a}} \wedge m^{\hat{b}} , \tag{2.44}$$

$$d\Gamma_{\hat{0}\hat{i}} + \Gamma_{\hat{0}\hat{1}} \wedge \Gamma_{\hat{0}\hat{i}} + \Gamma_{\hat{0}\hat{j}} \wedge \Gamma_{\hat{j}\hat{i}} = \frac{1}{2}\mathcal{R}_{\hat{0}\hat{i}} = \frac{1}{2}R_{\hat{0}\hat{i}\hat{a}\hat{b}} m^{\hat{a}} \wedge m^{\hat{b}} , \tag{2.45}$$

$$d\Gamma_{\hat{1}\hat{i}} - \Gamma_{\hat{0}\hat{1}} \wedge \Gamma_{\hat{1}\hat{i}} + \Gamma_{\hat{1}\hat{j}} \wedge \Gamma_{\hat{j}\hat{i}} = \frac{1}{2}\mathcal{R}_{\hat{1}\hat{i}} = \frac{1}{2}R_{\hat{1}\hat{i}\hat{a}\hat{b}} m^{\hat{a}} \wedge m^{\hat{b}} , \tag{2.46}$$

$$d\Gamma_{\hat{i}\hat{j}} + \Gamma_{\hat{i}\hat{0}} \wedge \Gamma_{\hat{1}\hat{j}} - \Gamma_{\hat{j}\hat{0}} \wedge \Gamma_{\hat{1}\hat{i}} + \Gamma_{\hat{i}\hat{k}} \wedge \Gamma_{\hat{k}\hat{j}} = \frac{1}{2}\mathcal{R}_{\hat{i}\hat{j}} = \frac{1}{2}R_{\hat{i}\hat{j}\hat{a}\hat{b}} m^{\hat{a}} \wedge m^{\hat{b}} . \tag{2.47}$$

The Ricci tensor is defined to be

$$R_{\hat{a}\hat{b}} = R_{\hat{b}\hat{a}} = R^{\hat{c}}{}_{\hat{a}\hat{c}\hat{b}}. \tag{2.48}$$

It has the following independent components obtained from curvature tensor

$$R_{\hat{0}\hat{0}} = R_{\hat{i}\hat{0}\hat{i}\hat{0}}, \tag{2.49}$$

$$R_{\hat{0}\hat{1}} = R_{\hat{1}\hat{0}\hat{0}\hat{1}} + R_{\hat{i}\hat{0}\hat{i}\hat{1}}, \tag{2.50}$$

$$R_{\hat{0}\hat{i}} = R_{\hat{1}\hat{0}\hat{0}\hat{i}} + R_{\hat{j}\hat{0}\hat{j}\hat{i}}, \tag{2.51}$$

$$R_{\hat{1}\hat{1}} = R_{\hat{i}\hat{1}\hat{i}\hat{1}}, \tag{2.52}$$

$$R_{\hat{1}\hat{i}} = R_{\hat{0}\hat{1}\hat{1}\hat{i}} + R_{\hat{j}\hat{1}\hat{j}\hat{i}}, \tag{2.53}$$

$$R_{\hat{i}\hat{j}} = R_{\hat{1}\hat{i}\hat{0}\hat{j}} + R_{\hat{0}\hat{i}\hat{1}\hat{j}} + R_{\hat{k}\hat{i}\hat{k}\hat{j}}, \tag{2.54}$$

where we have used basic symmetries of curvature tensor

$$R_{\hat{a}\hat{b}\hat{c}\hat{d}} = -R_{\hat{b}\hat{a}\hat{c}\hat{d}}, \tag{2.55}$$

$$R_{\hat{a}\hat{b}\hat{c}\hat{d}} = -R_{\hat{a}\hat{b}\hat{d}\hat{c}}, \tag{2.56}$$

$$R_{\hat{a}\hat{b}\hat{c}\hat{d}} = R_{\hat{c}\hat{d}\hat{a}\hat{b}} = R_{\hat{d}\hat{c}\hat{b}\hat{a}}. \tag{2.57}$$

2.2.4 Other issues in NP formalism

Expressions given above are the most important equations of NP formalism in higher dimensions, which we will use directly in our calculations in following chapters. However, if we talk about generalization of NP formalism to higher dimensions we should mention that this contains many other issues than we just mentioned here. We found here just rotation coefficients, connection and curvatures form for null congruence. To give complete generalization of NP formalism to higher dimensions it means to find:

1. Transformation properties of rotation coefficients under transformations (2.21)-(2.29), see [36].
2. Commutators of directional derivatives (2.36), see [35].
3. Ricci indetities, see [36].
4. Bianchi identities, see [34].
5. Classification of the Weyl tensor [37].

We refer reader for more details to given references and [38].

Chapter 3

Kerr-Schild spacetimes

Kerr-Schild (KS) spacetimes are very important in four dimensional general relativity. They include Kerr metric, which is one of the most important exact solutions of Einstein's equations in vacuum. Beside Kerr metric, KS spacetimes contain also pp-waves and some Kundt spacetimes. Moreover electromagnetic field can be included and so contain Kerr-Newman metric and pp-waves coupled to a null Maxwell field. For references, see original paper [27] or the reference book [29]. Kerr-Schild spacetimes also played a role in discovery of *Mayer-Perry black holes* in higher dimensions. Here we will try to find the generalization of the method [27] to higher dimensions.

3.1 Kerr-Schild ansatz

We will study spacetime with metric given by *Kerr-Schild ansatz*

$$g_{ab} = \eta_{ab} + 2\mathcal{H}k_a k_b, \tag{3.1}$$

where η_{ab} is usual Minkowski metric in n dimensions, $\mathcal{H}$ is a scalar function and k_a is a *null* vector field

$$g^{ab}k_a k_b = 0. \tag{3.2}$$

From this we can see that $-g = 1$ and that k_a is also null with respect to Minkowski metric.

Contravariant metric tensor is given by

$$g^{ab} = \eta^{ab} - 2\mathcal{H}k^a k^b. \tag{3.3}$$

We choose null coordinates (u, v) in Minkowski spacetime which are related to Cartesian coordinates $t, x^1, ..x^{n-1}$ by

$$\begin{aligned} u &= \frac{1}{\sqrt{2}}(x^1 - t), \\ v &= \frac{1}{\sqrt{2}}(x^1 + t). \end{aligned}$$

So then (3.1) can be expressed as

$$ds^2 = 2dvdu + dx_i dx^i + 2\mathcal{H}k^2. \tag{3.4}$$

Now, we have to find the general field of real null directions in Minkowski spacetime. We will find it to be [1] [35]

$$\ell \;=\; m^{\hat{1}} = du - z_{\hat{i}} dx^i - \frac{1}{2}z^2 dv, \tag{3.5}$$

and the frame will be completed to be

$$n \;=\; m^{\hat{0}} = dv + \mathcal{H}\ell, \tag{3.6}$$

$$m^{\hat{i}} \;=\; dx^i + z^{\hat{i}} dv. \tag{3.7}$$

We will make the standard identification [29], [39]

$$\ell = k \tag{3.8}$$

Frame directional derivatives (2.36) are acting on scalar function as

$$f_{|\hat{0}} = Df \;=\; (\partial_v - z_{\hat{i}}\partial_i - \frac{1}{2}z^2\partial_u)f, \tag{3.9}$$

$$f_{|\hat{1}} = \Delta f \;=\; (\partial_u - \mathcal{H}D)f, \tag{3.10}$$

$$f_{|\hat{i}} = \delta_{\hat{i}} f \;=\; (\partial_i + z_{\hat{i}}\partial_u)f. \tag{3.11}$$

It will turn out to be useful to have commutators of directional derivatives. In order to find these commutators we first have to find commutators of D with all space-time

[1] Note, that $z_{\hat{i}}$ and z_i represents same quantity.

directional derivatives.

$$
\begin{aligned}
\partial_v D - D\partial_v &= -(\partial_v z_{\hat{i}})\delta_{\hat{i}}, & (3.12)\\
\partial_u D - D\partial_u &= -(\partial_u z_{\hat{i}})\delta_{\hat{i}}, & (3.13)\\
\partial_j D - D\partial_j &= -(\partial_j z_{\hat{i}})\delta_{\hat{i}}. & (3.14)
\end{aligned}
$$

Now, we can find commutators of all frame directional derivatives, which are given by

$$
\begin{aligned}
D\Delta - \Delta D &= -\mathcal{H}_{|\hat{0}}D + (\partial_u z_{\hat{i}})\delta_{\hat{i}} = -\mathcal{H}_{|\hat{0}}D + (z_{\hat{i}|\hat{1}} + \mathcal{H}z_{\hat{i}|\hat{0}})\delta_{\hat{i}} & (3.15)\\
D\delta_{\hat{i}} - \delta_{\hat{i}}D &= z_{\hat{j}|\hat{i}}\delta_{\hat{j}} + z_{\hat{i}|\hat{0}}\partial_u = z_{\hat{j}|\hat{i}}\delta_{\hat{j}} + z_{\hat{i}|\hat{0}}(\Delta + \mathcal{H}D) & (3.16)\\
\Delta\delta_{\hat{i}} - \delta_{\hat{i}}\Delta &= (\partial_u z_{\hat{i}})\partial_u + \mathcal{H}_{|\hat{i}}D - \mathcal{H}(D\delta_{\hat{i}} - \delta_{\hat{i}}D) & \\
&= z_{\hat{i}|\hat{1}}(\Delta + \mathcal{H}D) - \mathcal{H}z_{\hat{j}|\hat{i}}\delta_{\hat{j}} + \mathcal{H}_{|\hat{i}}D & (3.17)\\
\delta_{\hat{i}}\delta_{\hat{j}} - \delta_{\hat{j}}\delta_{\hat{i}} &= -(z_{\hat{i}}z_{\hat{j}} - z_{\hat{j}}z_{\hat{i}})\partial_u\partial_u - (z_{\hat{j}|\hat{i}} - z_{\hat{i}|\hat{j}})\partial_u & \\
&= 2z_{[\hat{j}|\hat{i}]}(\Delta + \mathcal{H}D). & (3.18)
\end{aligned}
$$

These commutators can be simplified, as we will see later by assuming null congruence to be geodetic.

3.2 Rotation coefficients

In order to find the rotation coefficients we have to first find exterior derivatives of frame vectors (2.14)

$$
\begin{aligned}
dn &= \mathcal{H}_{|\hat{a}}m^{\hat{a}} \wedge \ell + \mathcal{H}d\ell = \mathcal{H}_{|\hat{a}}m^{\hat{a}} \wedge \ell + \mathcal{H}z_{\hat{i}|\hat{a}}m^{\hat{a}} \wedge m^{\hat{i}}\ , & (3.19)\\
d\ell &= -z_{\hat{i}|\hat{a}}m^{\hat{a}} \wedge dx^i - z^{\hat{i}}z_{\hat{i}|\hat{a}}m^{\hat{a}} \wedge dv = -z_{\hat{i}|\hat{a}}m^{\hat{a}} \wedge m^{\hat{i}}\ , & (3.20)\\
dm^{\hat{i}} &= z_{\hat{i}|\hat{a}}m^{\hat{a}} \wedge dv = z_{\hat{i}|\hat{a}}m^{\hat{a}} \wedge (n - \mathcal{H}\ell). & (3.21)
\end{aligned}
$$

From this we can calculate the connection coefficients by using the first Cartan equation (2.14)

$$
\begin{aligned}
d\ell &= -L_{\hat{a}\hat{b}}m^{\hat{a}} \wedge m^{\hat{b}}\ , & (3.22)\\
dn &= -N_{\hat{a}\hat{b}}m^{\hat{a}} \wedge m^{\hat{b}}\ , & (3.23)\\
dm^{\hat{i}} &= -\overset{\hat{i}}{M}_{\hat{a}\hat{b}}m^{\hat{a}} \wedge m^{\hat{b}}\ . & (3.24)
\end{aligned}
$$

Here we should not forget that from the Cartan structure equation we can only obtain antisymmetric part $\Gamma^{\hat{a}}{}_{[\hat{b}\hat{c}]}$. Full expression for $\Gamma^{\hat{a}}{}_{\hat{b}\hat{c}}$ is given by equation (2.18).

In the following three subsections we will show detailed calculations of rotation coefficients. For a reader who is not interested in these calculations we suggest to skip to the last subsection 3.2.6, where all results will be listed.

3.2.1 Antisymmetric part of rotation coefficients

Let us first compute all antisymmetric parts of rotation coefficients from which we will then compute all coefficients.

$\mathbf{L}_{[\hat{\mathbf{a}}\hat{\mathbf{b}}]}$: are found from equation (3.22) to be

$$z_{\hat{i}|\hat{a}} m^{\hat{a}} \wedge m^{\hat{i}} = L_{\hat{a}\hat{b}} m^{\hat{a}} \wedge m^{\hat{b}} , \tag{3.25}$$

$$\rightarrow \quad L_{[\hat{0}\hat{1}]} = 0, \tag{3.26}$$

$$L_{[\hat{0}\hat{i}]} = z_{\hat{i}|\hat{0}}, \tag{3.27}$$

$$L_{[\hat{1}\hat{i}]} = z_{\hat{i}|\hat{1}}, \tag{3.28}$$

$$L_{[\hat{i}\hat{j}]} = z_{\hat{j}|\hat{i}} - z_{\hat{i}|\hat{j}}. \tag{3.29}$$

$\mathbf{N}_{[\hat{\mathbf{a}}\hat{\mathbf{b}}]}$: are found from equation (3.23) to be

$$-\mathcal{H}_{|\hat{a}} m^{\hat{a}} \wedge \ell + \mathcal{H} z_{\hat{i}|\hat{a}} m^{\hat{a}} \wedge m^{\hat{i}} = N_{\hat{a}\hat{b}} m^{\hat{a}} \wedge m^{\hat{b}}, \tag{3.30}$$

$$\rightarrow \quad N_{[\hat{0}\hat{1}]} = -\mathcal{H}_{|\hat{0}}, \tag{3.31}$$

$$N_{[\hat{0}\hat{i}]} = \mathcal{H} z_{\hat{i}|\hat{0}}, \tag{3.32}$$

$$N_{[\hat{1}\hat{i}]} = \mathcal{H} z_{\hat{i}|\hat{1}} + \mathcal{H}_{|\hat{i}}, \tag{3.33}$$

$$N_{[\hat{i}\hat{j}]} = \mathcal{H} z_{\hat{j}|\hat{i}} - \mathcal{H} z_{\hat{i}|\hat{j}}. \tag{3.34}$$

$\overset{\hat{\mathbf{i}}}{\mathbf{M}}_{[\hat{\mathbf{a}}\hat{\mathbf{b}}]}$: are found from equation (3.24) to be

$$-z_{\hat{i}|\hat{a}} m^{\hat{a}} \wedge (n - \mathcal{H}\ell) = \overset{\hat{i}}{M}_{\hat{a}\hat{b}} m^{\hat{a}} \wedge m^{\hat{b}}, \tag{3.35}$$

$$\rightarrow \quad \overset{\hat{i}}{M}_{[\hat{0}\hat{1}]} = z_{\hat{i}|\hat{1}} + \mathcal{H} z_{\hat{i}|\hat{0}}, \tag{3.36}$$

$$\overset{\hat{i}}{M}_{[\hat{0}\hat{j}]} = z_{\hat{i}|\hat{j}}, \tag{3.37}$$

$$\overset{\hat{i}}{M}_{[\hat{1}\hat{j}]} = -\mathcal{H} z_{\hat{i}|\hat{j}}, \tag{3.38}$$

$$\overset{\hat{i}}{M}_{[\hat{j}\hat{k}]} = 0. \tag{3.39}$$

3.2.2 $L_{\hat{a}\hat{b}}$ coefficients

Here we will determine all $L_{\hat{a}\hat{b}}$ using the knowledge of antisymmetric parts of rotation coefficients and equation (2.18). Some of the rotation coefficients will be found o be zero by constraints (2.34)

$$\begin{aligned}
L_{\hat{0}\hat{0}} &= L_{\hat{0}\hat{1}} = L_{\hat{0}\hat{i}} = 0, && (3.40)\\
L_{\hat{1}\hat{0}} &= \frac{1}{2}(L_{[\hat{1}\hat{0}]} - L_{[\hat{0}\hat{1}]}) = 0, && (3.41)\\
L_{\hat{1}\hat{1}} &= \frac{1}{2}(N_{[\hat{1}\hat{0}]} - N_{[\hat{0}\hat{1}]}) = \mathcal{H}_{|\hat{0}}, && (3.42)\\
L_{\hat{1}\hat{i}} &= \frac{1}{2}(L_{[\hat{1}\hat{i}]} + N_{[\hat{i}\hat{0}]} - \overset{\hat{i}}{M}_{[\hat{0}\hat{1}]}) = 0, && (3.43)\\
L_{\hat{i}\hat{0}} &= \frac{1}{2}(L_{[\hat{i}\hat{0}]} - L_{[\hat{0}\hat{i}]}) = -z_{\hat{i}|\hat{0}}, && (3.44)\\
L_{\hat{i}\hat{1}} &= \frac{1}{2}(L_{[\hat{i}\hat{1}]} + \overset{\hat{i}}{M}_{[\hat{1}\hat{0}]} - N_{[\hat{0}\hat{i}]}) = -z_{\hat{i}|\hat{1}} - \mathcal{H}z_{\hat{i}|\hat{0}} = \partial_u z_{\hat{i}}, && (3.45)\\
L_{\hat{i}\hat{j}} &= \frac{1}{2}(L_{[\hat{i}\hat{j}]} + \overset{\hat{i}}{M}_{[\hat{j}\hat{0}]} - \overset{\hat{j}}{M}_{[\hat{0}\hat{i}]}) = -z_{\hat{i}|\hat{j}}. && (3.46)
\end{aligned}$$

3.2.3 $N_{\hat{a}\hat{b}}$ coefficients

As in the previous section, we will obtain $N_{\hat{a}\hat{b}}$ from the knowledge of antisymmetric parts of rotation coefficients and equation (2.18). Also some of the rotation coefficients will be found to be zero by constraints (2.34) and we will simplify calculations because we will be able to express some of $N_{\hat{a}\hat{b}}$ through $L_{\hat{a}\hat{b}}$ with the help of constraints (2.35)

$$\begin{aligned}
N_{\hat{1}\hat{0}} &= N_{\hat{1}\hat{1}} = N_{\hat{1}\hat{i}} = 0, && (3.47)\\
N_{\hat{0}\hat{0}} &= -L_{\hat{1}\hat{0}} = 0, && (3.48)\\
N_{\hat{0}\hat{1}} &= -L_{\hat{1}\hat{1}} = -\mathcal{H}_{|\hat{0}}, && (3.49)\\
N_{\hat{0}\hat{i}} &= -L_{\hat{1}\hat{i}} = 0, && (3.50)\\
N_{\hat{i}\hat{0}} &= \frac{1}{2}(N_{[\hat{i}\hat{0}]} + \overset{\hat{i}}{M}_{[\hat{0}\hat{1}]} - L_{[\hat{1}\hat{i}]}) = 0, && (3.51)\\
N_{\hat{i}\hat{1}} &= \frac{1}{2}(N_{[\hat{i}\hat{1}]} - N_{[\hat{1}\hat{i}]}) = -\mathcal{H}_{|\hat{i}} - \mathcal{H}z_{\hat{i}|\hat{1}}, && (3.52)\\
N_{\hat{i}\hat{j}} &= \frac{1}{2}(N_{[\hat{i}\hat{j}]} + \overset{\hat{i}}{M}_{[\hat{j}\hat{1}]} - \overset{\hat{j}}{M}_{[\hat{1}\hat{i}]}) = \mathcal{H}z_{\hat{j}|\hat{i}}. && (3.53)
\end{aligned}$$

3.2.4 $\overset{\hat{i}}{M}_{\hat{a}\hat{b}}$ coefficients

As in the previous two sections, we will obtain $\overset{\hat{i}}{M}_{\hat{a}\hat{b}}$ from the knowledge of antisymmetric parts of rotation coefficients and equation (2.18). Also some of the rotation coefficents will be found to be zero by constraints (2.34) and we will simplify calculations because we will be able to express some of $\overset{\hat{i}}{M}_{\hat{a}\hat{b}}$ through $L_{\hat{a}\hat{b}}$ and$N_{\hat{a}\hat{b}}$ with help of constraints (2.35). Indeed, these constraints will save us so much work, that we will have to calculate just coeffcients $\overset{\hat{i}}{M}_{\hat{i}\hat{j}}$ and $\overset{\hat{i}}{M}_{\hat{j}\hat{a}}$

$$\overset{\hat{i}}{M}_{\hat{i}\hat{0}} = \overset{\hat{i}}{M}_{\hat{i}\hat{1}} = \overset{\hat{i}}{M}_{\hat{i}\hat{j}} = 0, \tag{3.54}$$

$$\overset{\hat{i}}{M}_{\hat{0}\hat{0}} = -L_{\hat{i}\hat{0}} = z_{\hat{i}|\hat{0}}, \tag{3.55}$$

$$\overset{\hat{i}}{M}_{\hat{0}\hat{1}} = -L_{\hat{i}\hat{1}} = z_{\hat{i}|\hat{1}}, \tag{3.56}$$

$$\overset{\hat{i}}{M}_{\hat{0}\hat{j}} = -L_{\hat{i}\hat{j}} = z_{\hat{i}|\hat{j}}, \tag{3.57}$$

$$\overset{\hat{i}}{M}_{\hat{1}\hat{0}} = -N_{\hat{i}\hat{0}} = 0, \tag{3.58}$$

$$\overset{\hat{i}}{M}_{\hat{1}\hat{1}} = -N_{\hat{i}\hat{1}} = +\mathcal{H}_{|\hat{i}} + \mathcal{H}z_{\hat{i}|\hat{1}}, \tag{3.59}$$

$$\overset{\hat{i}}{M}_{\hat{1}\hat{j}} = -N_{\hat{i}\hat{j}} = -\mathcal{H}z_{\hat{j}|\hat{i}}, \tag{3.60}$$

$$\overset{\hat{i}}{M}_{\hat{j}\hat{0}} = \frac{1}{2}(\overset{\hat{i}}{M}_{[\hat{j}\hat{0}]} + \overset{\hat{j}}{M}_{[\hat{0}\hat{i}]} - L_{[\hat{i}\hat{j}]}) = 0, \tag{3.61}$$

$$\overset{\hat{i}}{M}_{\hat{j}\hat{1}} = \frac{1}{2}(\overset{\hat{i}}{M}_{[\hat{j}\hat{1}]} + \overset{\hat{j}}{M}_{[\hat{1}\hat{i}]} - N_{[\hat{i}\hat{j}]}) = 2\mathcal{H}z_{[\hat{i}|\hat{j}]}, \tag{3.62}$$

$$\overset{\hat{i}}{M}_{\hat{j}\hat{k}} = 0. \tag{3.63}$$

3.2.5 Condition of geodecity

Physically the most interesting case is when the null congruence is geodetic. As it was shown in the previous chapter in such a case matrix $L_{\hat{i}\hat{j}}$ has special geometric meaning and characterize null congruence in an invariant way. Thus, we have to prove that null congruence is geodetic in a case of Kerr-Schild spacetimes. In four dimensions this is done for example in the standard reference book [29] and its straightforward generalization to higher dimensions was given in [39].

Explicitly, Proposition 1 from [39] states that:

The null-vector ℓ in the Kerr-Schild metric (3.1) of an arbitrary dimension is geodetic, if and only if the energy-momentum tensor satisfies $T_{ab}k^a k^b = 0$

This condition is clearly satisfied in a case of vacuum space-times, which will be of our main interest here, but let us note that it is also satisfied in spacetimes with cosmological constant and in the presence of matter fields aligned with the KS vector ℓ, for example Maxwell field.

Thus, proposition 1 from [39] is satisfied in space-times, which are of our interest here and we can follow discussion in the previous chapter. We found that condition of geodecity is equivalent to the vanishing of coefficient $L_{\hat{i}\hat{0}}$. In the previous calculations we found that $\overset{\hat{i}}{M}_{\hat{0}\hat{0}}$ is given by this coefficient. Thus we can simplify our calculations by setting both of these coefficients to zero

$$L_{\hat{i}\hat{0}} = -\overset{\hat{i}}{M}_{\hat{0}\hat{0}} = 0. \tag{3.64}$$

As a consequence of this we are obtaining constraint

$$z_{\hat{i}|\hat{0}} = 0, \tag{3.65}$$

which will then simplify some other coefficients.
Note, that if ℓ is geodetic and

$$N_{\hat{i}\hat{0}} = \overset{\hat{i}}{M}_{\hat{j}\hat{0}} = 0 \tag{3.66}$$

holds, then frame (3.5)-(3.7) is parallely transported along ℓ.

Condition of geodecity will also simplify commutation relations of frame directional derivatives (3.15)-(3.18), which will be useful in the calculations of Riemann tensor.

$$\begin{aligned}
D\Delta - \Delta D &= -\mathcal{H}_{|\hat{0}}D + z_{\hat{i}|\hat{1}}\delta_{\hat{i}}, && (3.67)\\
D\delta_{\hat{i}} - \delta_{\hat{i}}D &= z_{\hat{j}|\hat{i}}\delta_{\hat{j}}, && (3.68)\\
\Delta\delta_{\hat{i}} - \delta_{\hat{i}}\Delta &= z_{\hat{i}|\hat{1}}\Delta + (\mathcal{H}_{|\hat{i}} + \mathcal{H}z_{\hat{i}|\hat{1}})D - \mathcal{H}z_{\hat{j}|\hat{i}}\delta_{\hat{j}}, && (3.69)\\
\delta_{\hat{i}}\delta_{\hat{j}} - \delta_{\hat{j}}\delta_{\hat{i}} &= 2z_{[\hat{j}|\hat{i}]}(\Delta + \mathcal{H}D). && (3.70)
\end{aligned}$$

Note, that if we use our results for rotation coefficients, we can easily check that these commutators are in agreement with the general results given in [35].

3.2.6 Summary

Here we can summarize all rotation coefficients found in previous calculations in a clear way. We will use representation by tables, where rows are running first indix and columns second one.

$L_{\hat{a}\hat{b}}$	$\hat{b}=\hat{0}$	$\hat{b}=\hat{1}$	$\hat{b}=\hat{j}$
$\hat{a}=\hat{0}$	0	0	0
$\hat{a}=\hat{1}$	0	$\mathcal{H}_{\vert\hat{0}}$	0
$\hat{a}=\hat{i}$	0	$-z_{\hat{i}\vert\hat{1}}$	$-z_{\hat{i}\vert\hat{j}}$

$N_{\hat{a}\hat{b}}$	$\hat{0}$	$\hat{1}$	$\hat{j}$
$\hat{0}$	0	$-\mathcal{H}_{\vert\hat{0}}$	0
$\hat{1}$	0	0	0
$\hat{i}$	0	$-\mathcal{H}_{\vert\hat{i}} - \mathcal{H}z_{\hat{i}\vert\hat{1}}$	$\mathcal{H}z_{\hat{j}\vert\hat{i}}$

$\overset{\hat{i}}{M}_{\hat{a}\hat{b}}$	$\hat{0}$	$\hat{1}$	$\hat{k}$
$\hat{0}$	0	$z_{\hat{i}\vert\hat{1}}$	$z_{\hat{i}\vert\hat{k}}$
$\hat{1}$	0	$\mathcal{H}z_{\hat{i}\vert\hat{1}} + \mathcal{H}_{\vert\hat{i}}$	$-\mathcal{H}z_{\hat{k}\vert\hat{i}}$
$\hat{i}$	0	0	0
$\hat{j}$	0	$2\mathcal{H}z_{[\hat{i}\vert\hat{j}]}$	0

Table 3.1: Rotation coefficients $L_{\hat{a}\hat{b}}$, $N_{\hat{a}\hat{b}}$, $\overset{\hat{i}}{M}_{\hat{a}\hat{b}}$ for geodetic congruence ℓ in the KS spacetime (3.1) in the frame (3.5)-(3.7).

All these coefficients hold if condition of geodecity is satisfied, which is the most interesting physical case, and with the help of which we were able to simplify our calculations. In terms of functions $z_{\hat{i}}$ this condition is

$$z_{\hat{i}|\hat{0}} = 0. \tag{3.71}$$

3.3 Connection and curvature forms

From the knowledge of rotation coefficients we can now easily find the connection forms which will be needed for finding curvature forms and consequently curvature tensor. We have just these independent connection forms $\Gamma_{\hat{0}\hat{1}}, \Gamma_{\hat{0}\hat{i}}, \Gamma_{\hat{1}\hat{i}}, \Gamma_{\hat{i}\hat{j}}$, which are given by

$$
\begin{aligned}
\Gamma_{\hat{0}\hat{1}} &= -L_{\hat{1}\hat{a}}m^{\hat{a}} = N_{\hat{0}\hat{a}}m^{\hat{a}} = -\mathcal{H}_{|\hat{0}}\ell, && (3.72)\\
\Gamma_{\hat{0}\hat{i}} &= -L_{\hat{i}\hat{a}}m^{\hat{a}} = \overset{\hat{i}}{M}_{\hat{0}\hat{a}}m^{\hat{a}} = z_{\hat{i}|\hat{a}}m^{\hat{a}} = dz_{\hat{i}}, && (3.73)\\
\Gamma_{\hat{1}\hat{i}} &= -N_{\hat{i}\hat{a}}m^{\hat{a}} = \overset{\hat{i}}{M}_{\hat{1}\hat{a}}m^{\hat{a}} = (\mathcal{H}_{|\hat{i}} + \mathcal{H}z_{\hat{i}|\hat{1}})\ell - \mathcal{H}z_{\hat{j}|\hat{i}}m^{\hat{j}} , && (3.74)\\
\Gamma_{\hat{i}\hat{j}} &= -\overset{\hat{i}}{M}_{\hat{j}\hat{a}}m^{\hat{a}} = \overset{\hat{j}}{M}_{\hat{i}\hat{a}}m^{\hat{a}} = -2\mathcal{H}z_{[\hat{i}|\hat{j}]}\ell. && (3.75)
\end{aligned}
$$

Exterior derivatives of these connection forms are found to be

$$
\begin{aligned}
d\Gamma_{\hat{0}\hat{1}} &= -\mathcal{H}_{|\hat{0}|\hat{a}}m^{\hat{a}} \wedge \ell + \mathcal{H}_{|\hat{0}}z_{\hat{i}|\hat{a}}m^{\hat{a}} \wedge m^{\hat{i}} , && (3.76)\\
d\Gamma_{\hat{0}\hat{i}} &= ddz_{\hat{i}} = 0, && (3.77)\\
d\Gamma_{\hat{1}\hat{i}} &= [(\mathcal{H}_{|\hat{i}} + \mathcal{H}z_{\hat{i}|\hat{1}})_{|\hat{a}} + \mathcal{H}^2 z_{\hat{k}|\hat{i}}z_{\hat{k}|\hat{a}}]m^{\hat{a}} \wedge \ell - \mathcal{H}z_{\hat{k}|\hat{i}}z_{\hat{k}|\hat{a}}m^{\hat{a}} \wedge n \\
&- [(\mathcal{H}_{|\hat{i}} + \mathcal{H}z_{\hat{i}|\hat{1}})z_{\hat{k}|\hat{a}} + (\mathcal{H}z_{\hat{k}|\hat{i}})_{|\hat{a}}]m^{\hat{a}} \wedge m^{\hat{k}} , && (3.78)\\
d\Gamma_{\hat{i}\hat{j}} &= -2(\mathcal{H}z_{[\hat{i}|\hat{j}]})_{|\hat{a}}m^{\hat{a}} \wedge \ell + 2\mathcal{H}z_{[\hat{i}|\hat{j}]}z_{\hat{k}|\hat{a}}m^{\hat{a}} \wedge m^{\hat{k}} , && (3.79)
\end{aligned}
$$

and corresponding curvature forms are given by equations (2.44)-(2.47).

$$
\begin{aligned}
\frac{1}{2}\mathcal{R}_{\hat{0}\hat{1}} &= [\mathcal{H}_{|\hat{0}|\hat{a}} + z_{\hat{i}|\hat{a}}(\mathcal{H}_{|\hat{i}} + \mathcal{H}z_{\hat{i}|\hat{1}})]\ell \wedge m^{\hat{a}} \\
&+ (\mathcal{H}z_{\hat{k}|\hat{a}}z_{\hat{j}|\hat{k}} + \mathcal{H}_{|\hat{0}}z_{\hat{j}|\hat{a}})m^{\hat{a}} \wedge m^{\hat{j}} , && (3.80)\\
\frac{1}{2}\mathcal{R}_{\hat{0}\hat{i}} &= (-\mathcal{H}_{|\hat{0}}z_{\hat{i}|\hat{a}} + 2\mathcal{H}z_{\hat{j}|\hat{a}}z_{[\hat{j}|\hat{i}]})\ell \wedge m^{\hat{a}} , && (3.81)\\
\frac{1}{2}\mathcal{R}_{\hat{1}\hat{i}} &= [(\mathcal{H}_{|\hat{i}} + \mathcal{H}z_{\hat{i}|\hat{1}})_{|\hat{a}} + \mathcal{H}^2 z_{\hat{k}|\hat{i}}z_{\hat{k}|\hat{a}}]m^{\hat{a}} \wedge \ell + \mathcal{H}z_{\hat{k}|\hat{i}}z_{\hat{k}|\hat{a}}n \wedge m^{\hat{a}} \\
&- [(\mathcal{H}_{|\hat{i}} + \mathcal{H}z_{\hat{i}|\hat{1}})z_{\hat{k}|\hat{a}} + (\mathcal{H}z_{\hat{k}|\hat{i}})_{|\hat{a}}]m^{\hat{a}} \wedge m^{\hat{k}} \\
&- (\mathcal{H}\mathcal{H}_{|\hat{0}}z_{\hat{k}|\hat{i}} - 2\mathcal{H}^2 z_{[\hat{i}|\hat{j}]}z_{\hat{k}|\hat{j}})\ell \wedge m^{\hat{k}} , && (3.82)\\
\frac{1}{2}\mathcal{R}_{\hat{i}\hat{j}} &= 2[(\mathcal{H}z_{[\hat{i}|\hat{j}]})_{|\hat{a}} + (\mathcal{H}_{[\hat{j}} + \mathcal{H}\Delta z_{[\hat{j}})z_{\hat{i}]|\hat{a}}]\ell \wedge m^{\hat{a}} \\
&+ 2\mathcal{H}(z_{\hat{k}|[\hat{i}}z_{\hat{j}]|\hat{a}} - z_{[\hat{i}|\hat{j}]}z_{\hat{k}|\hat{a}})m^{\hat{k}} \wedge m^{\hat{a}} . && (3.83)
\end{aligned}
$$

3.3.1 Curvature Tensor

Now, we can easily find components of the Riemann tensor using relation

$$\mathcal{R}_{\hat{a}\hat{b}} = R_{\hat{a}\hat{b}\hat{c}\hat{d}} m^{\hat{c}} \wedge m^{\hat{d}} \ . \tag{3.84}$$

Independent components of Riemann tensor can be summarized in the following table

$R_{\hat{0}\hat{1}\hat{0}\hat{1}}$	$-\mathcal{H}_{\vert\hat{0}\vert\hat{0}}$
$R_{\hat{0}\hat{1}\hat{0}\hat{i}}$	0
$R_{\hat{0}\hat{1}\hat{1}\hat{i}}$	$\mathcal{H}_{\vert\hat{0}\vert\hat{i}} + \mathcal{H}_{\vert\hat{0}} z_{\hat{i}\vert\hat{1}} + \mathcal{H}_{\vert\hat{j}} z_{\hat{j}\vert\hat{i}} + \mathcal{H} z_{\hat{j}\vert\hat{i}} z_{\hat{j}\vert\hat{1}} + \mathcal{H} z_{\hat{j}\vert\hat{1}} z_{\hat{i}\vert\hat{j}}$
$R_{\hat{0}\hat{1}\hat{i}\hat{j}}$	$2\mathcal{H} z_{\hat{k}\vert[\hat{i}} z_{\hat{j}]\vert\hat{k}} + 2\mathcal{H}_{\vert\hat{0}} z_{[\hat{j}\vert\hat{i}]}$
$R_{\hat{0}\hat{i}\hat{0}\hat{j}}$	0
$R_{\hat{0}\hat{i}\hat{1}\hat{j}}$	$-\mathcal{H}_{\vert\hat{0}} z_{\hat{i}\vert\hat{j}} - 2\mathcal{H} z_{\hat{k}\vert\hat{j}} z_{[\hat{i}\vert\hat{k}]}$
$R_{\hat{0}\hat{i}\hat{j}\hat{k}}$	0
$R_{\hat{1}\hat{i}\hat{1}\hat{j}}$	$-(\mathcal{H}_{\vert\hat{i}} + \mathcal{H} z_{\hat{i}\vert\hat{1}})_{\vert\hat{j}} - \mathcal{H}^2 z_{\hat{k}\vert\hat{i}} z_{\hat{k}\vert\hat{j}} - (\mathcal{H}_{\vert\hat{i}} + \mathcal{H} z_{\hat{i}\vert\hat{1}}) z_{\hat{j}\vert\hat{1}} - (\mathcal{H} z_{\hat{j}\vert\hat{i}})_{\vert\hat{1}} - \mathcal{H}\mathcal{H}_{\vert\hat{0}} z_{\hat{j}\vert\hat{i}} + 2\mathcal{H}^2 z_{[\hat{i}\vert\hat{k}]} z_{\hat{j}\vert\hat{k}}$
$R_{\hat{1}\hat{i}\hat{j}\hat{k}}$	$2(\mathcal{H}_{\vert\hat{i}} + \mathcal{H} z_{\hat{i}\vert\hat{1}}) z_{[\hat{j}\vert\hat{k}]} + 2(\mathcal{H}\delta_{\hat{i}} z_{[\hat{k}})_{\vert\hat{j}]}$
$R_{\hat{i}\hat{j}\hat{k}\hat{l}}$	$2\mathcal{H}(z_{\hat{k}\vert[\hat{i}} z_{\hat{j}]\vert[\hat{l}]} - z_{\hat{l}\vert[\hat{i}} z_{\hat{j}]\vert\hat{k}} - 2z_{[\hat{i}\vert\hat{j}]} z_{[\hat{k}\vert\hat{l}]})$

and other dependent components of curvature tensor can be summarized in the following table

$R_{\hat{0}\hat{i}\hat{0}\hat{1}}$	0
$R_{\hat{1}\hat{i}\hat{0}\hat{1}}$	$(\mathcal{H}_{\vert\hat{i}} + \mathcal{H} z_{\hat{i}\vert\hat{1}})_{\vert\hat{0}} + \mathcal{H} z_{\hat{k}\vert\hat{i}} z_{\hat{k}\vert\hat{1}}$
$R_{\hat{1}\hat{i}\hat{0}\hat{j}}$	$\mathcal{H} z_{\hat{k}\vert\hat{i}} z_{\hat{k}\vert\hat{j}} - (\mathcal{H} z_{\hat{j}\vert\hat{i}})_{\vert\hat{0}}$
$R_{\hat{i}\hat{j}\hat{0}\hat{1}}$	$-2(\mathcal{H} z_{[\hat{i}\vert\hat{j}]})_{\vert\hat{0}}$
$R_{\hat{i}\hat{j}\hat{0}\hat{k}}$	0
$R_{\hat{i}\hat{j}\hat{1}\hat{k}}$	$2(\mathcal{H} z_{[\hat{i}\vert\hat{j}]})_{\vert\hat{k}} + 2(\mathcal{H}_{\vert[\hat{j}} + \mathcal{H}\Delta z_{[\hat{j}}) z_{\hat{i}]\vert\hat{k}} - 2\mathcal{H}(z_{\hat{k}\vert[\hat{i}} z_{\hat{j}]\vert\hat{l}} - z_{[\hat{i}\vert\hat{j}]} z_{\hat{k}\vert\hat{l}})$

Components in these two tables are related each to other by symmetry (2.57), i.e. symmetry under exchange of pair of indices. For non-zero components these symmetries are

$$R_{\hat{1}\hat{i}\hat{0}\hat{1}} = R_{\hat{0}\hat{1}\hat{1}\hat{i}}, \tag{3.85}$$

$$R_{\hat{1}\hat{i}\hat{0}\hat{j}} = R_{\hat{0}\hat{j}\hat{1}\hat{i}}, \tag{3.86}$$

$$R_{\hat{i}\hat{j}\hat{0}\hat{1}} = R_{\hat{0}\hat{1}\hat{i}\hat{j}}, \tag{3.87}$$

$$R_{\hat{i}\hat{j}\hat{1}\hat{k}} = R_{\hat{1}\hat{k}\hat{i}\hat{j}}. \tag{3.88}$$

To check this explicitly we need just commutators (3.67)-(3.70). It is a straightforward but tedious work, which would take a lot of place and thus we will not write it in this thesis. To check the symmetries (2.55)-(2.56) (i.e. antisymmetry under exchange of indices inside the first or second pair of indices) is trivial and thus only symmetry which is left is as follows

$$R_{\hat{1}\hat{i}\hat{1}\hat{j}} = R_{\hat{1}\hat{j}\hat{1}\hat{i}}. \tag{3.89}$$

At the first sight this symmetry seems to not be satisfied, because $R_{\hat{1}\hat{i}\hat{1}\hat{j}}$ does not seem to be symetric under exchange of indices i and j, but if we use commutators (3.67)-(3.70) we will find that this symmetry holds. Thus Riemann tensor which we have just found, satisfies all symmetries of Riemann tensor.

We can use it as a check of our calculations and also, as we can see, to simplify our expressions for components for Riemann tensor, because some of the components of Riemann tensor in the second table are in more compact form than in the first one. Let us pick up each component in its most compact form from either of two previous tables and all non-zero independent components can then be summarized in the following table

$R_{\hat{0}\hat{1}\hat{0}\hat{1}}$	$-\mathcal{H}_{\|\hat{0}\|\hat{0}}$
$R_{\hat{0}\hat{1}\hat{1}\hat{i}}$	$(\mathcal{H}_{\|\hat{i}} + \mathcal{H} z_{\hat{i}\|\hat{1}})_{\|\hat{0}} + \mathcal{H} z_{\hat{k}\|\hat{i}} z_{\hat{k}\|\hat{1}}$
$R_{\hat{0}\hat{1}\hat{i}\hat{j}}$	$2(\mathcal{H} z_{[\hat{j}\|\hat{i}]})_{\|\hat{0}}$
$R_{\hat{0}\hat{i}\hat{1}\hat{j}}$	$-\mathcal{H}_{\|\hat{0}} z_{\hat{i}\|\hat{j}} - 2\mathcal{H} z_{\hat{k}\|\hat{j}} z_{[\hat{i}\|\hat{k}]}$
$R_{\hat{1}\hat{i}\hat{1}\hat{j}}$	$-(\delta_{\hat{j}} + z_{\hat{j}\|\hat{1}})(\mathcal{H}_{\|\hat{i}} + \mathcal{H} z_{\hat{i}\|\hat{1}}) - \mathcal{H}^2(z_{\hat{k}\|\hat{i}} z_{\hat{k}\|\hat{j}} - 2 z_{[\hat{i}\|\hat{k}]} z_{\hat{j}\|\hat{k}}) - (\mathcal{H} z_{\hat{j}\|\hat{i}})_{\|\hat{1}} - \mathcal{H}\mathcal{H}_{\|\hat{0}} z_{\hat{j}\|\hat{i}}$
$R_{\hat{1}\hat{i}\hat{j}\hat{k}}$	$2(\mathcal{H}_{\|\hat{i}} + \mathcal{H} z_{\hat{i}\|\hat{1}}) z_{[\hat{j}\|\hat{k}]} + 2(\mathcal{H}\delta_{\hat{i}} z_{[\hat{k}})_{\|\hat{j}]}$
$R_{\hat{i}\hat{j}\hat{k}\hat{l}}$	$2\mathcal{H}(z_{\hat{k}\|[\hat{i}} z_{\hat{j}]\|\hat{l}} - z_{[\hat{i}\|\hat{j}]} z_{\hat{k}\|\hat{l}})$

Table 3.2: Independent non-zero components of the curvature tensor

3.4 Einstein equations

In order to write Einstein equations we have to find components of Ricci tensor given by (2.49)-(2.54). Using the previous results can be found that they are given by

$$
\begin{aligned}
R_{\hat{0}\hat{0}} &= R_{\hat{0}\hat{i}} = 0, && (3.90)\\
R_{\hat{0}\hat{1}} &= \mathcal{H}_{|\hat{0}|\hat{0}} - \mathcal{H}_{|\hat{0}} z_{\hat{i}|\hat{i}} - 2\mathcal{H} z_{\hat{k}|\hat{i}} z_{[\hat{i}|\hat{k}]}, && (3.91)\\
R_{\hat{1}\hat{1}} &= -(\delta_{\hat{i}} + z_{\hat{i}|\hat{1}})(\mathcal{H}_{|\hat{i}} + \mathcal{H} z_{\hat{i}|\hat{1}}) - \mathcal{H}^2 (z_{\hat{k}|\hat{i}} z_{\hat{k}|\hat{i}} - 2 z_{[\hat{i}|\hat{k}]} z_{\hat{i}|\hat{k}}) && (3.92)\\
&- (\mathcal{H} z_{\hat{i}|\hat{i}})_{|\hat{1}} - \mathcal{H}\mathcal{H}_{|\hat{0}} z_{\hat{i}|\hat{i}}, && (3.93)\\
R_{\hat{1}\hat{i}} &= (\mathcal{H}_{|\hat{i}} + \mathcal{H} z_{\hat{i}|\hat{1}})_{|\hat{0}} + \mathcal{H} z_{\hat{k}|\hat{i}} z_{\hat{k}|\hat{1}} + 2(\mathcal{H}_{|\hat{j}} + \mathcal{H} z_{\hat{j}|\hat{1}}) z_{[\hat{i}|\hat{j}]} + 2(\mathcal{H}\delta_{\hat{j}} z_{[\hat{i}})_{|\hat{j}]}, && (3.94)\\
R_{\hat{i}\hat{j}} &= -2\mathcal{H} z_{\hat{i}|\hat{k}} z_{\hat{j}|\hat{k}} - 2(\mathcal{H}_{|\hat{0}} - \mathcal{H} z_{\hat{k}|\hat{k}}) z_{(\hat{i}|\hat{j})}. && (3.95)
\end{aligned}
$$

We can transform two of these components to more elegant and convenient form using commutators (3.68)-(3.70)

$$
\begin{aligned}
R_{\hat{1}\hat{1}} &= -\mathcal{H}_{|\hat{i}|\hat{i}} - 2(\mathcal{H} z_{\hat{i}|\hat{1}})_{|\hat{i}} - 2\mathcal{H} z_{\hat{i}|\hat{1}} z_{\hat{i}|\hat{1}} - (\mathcal{H}_{|\hat{1}} + \mathcal{H}\mathcal{H}_{|\hat{0}}) z_{\hat{i}|\hat{i}}, && (3.96)\\
R_{\hat{1}\hat{i}} &= (\mathcal{H}_{|\hat{i}} + \mathcal{H} z_{\hat{i}|\hat{1}})_{|\hat{0}} + 2(\mathcal{H} z_{[\hat{i}|\hat{j}]})_{|\hat{j}} + 2\mathcal{H}_{|[\hat{j}} z_{\hat{i}]|\hat{j}} + \mathcal{H} z_{\hat{i}|\hat{j}} z_{\hat{j}|\hat{1}}. && (3.97)
\end{aligned}
$$

Now, we can summarize all independent non-zero components of Ricci tensor in the following table If we use commutators (3.68)-(3.70) and expressions for rotation

$R_{\hat{0}\hat{1}}$	$\mathcal{H}_{\|\hat{0}\|\hat{0}} - \mathcal{H}_{\|\hat{0}} z_{\hat{i}\|\hat{i}} - 2\mathcal{H} z_{\hat{k}\|\hat{i}} z_{[\hat{i}\|\hat{k}]}$
$R_{\hat{1}\hat{1}}$	$-\mathcal{H}_{\|\hat{i}\|\hat{i}} - 2(\mathcal{H} z_{\hat{i}\|\hat{1}})_{\|\hat{i}} - 2\mathcal{H} z_{\hat{i}\|\hat{1}} z_{\hat{i}\|\hat{1}} - (\mathcal{H}_{\|\hat{1}} + \mathcal{H}\mathcal{H}_{\|\hat{0}}) z_{\hat{i}\|\hat{i}}$
$R_{\hat{1}\hat{i}}$	$(\mathcal{H}_{\|\hat{i}} + \mathcal{H} z_{\hat{i}\|\hat{1}})_{\|\hat{0}} + 2(\mathcal{H} z_{[\hat{i}\|\hat{j}]})_{\|\hat{j}} + 2\mathcal{H}_{\|[\hat{j}} z_{\hat{i}]\|\hat{j}} + \mathcal{H} z_{\hat{i}\|\hat{j}} z_{\hat{j}\|\hat{1}}$
$R_{\hat{i}\hat{j}}$	$-2\mathcal{H} z_{\hat{i}\|\hat{k}} z_{\hat{j}\|\hat{k}} - 2(\mathcal{H}_{\|\hat{0}} - \mathcal{H} z_{\hat{k}\|\hat{k}}) z_{(\hat{i}\|\hat{j})}$

Table 3.3: Non-zero components of Ricci tensor.

coefficients from the table 3.1, we can check whether our results are compatible with [39]. After some straight-forward but tedious algebra we can see that our results agree with the results of this paper if we make identification

$$\mathcal{H} \to -\mathcal{H},$$

which is correct, because in [39] Kerr-Schild metric (3.1) with a minus sign is used, while we are using a plus sign. Thus we can conclude that our results presented here are fully consistent with [39]. With the knowledge of all components of Ricci tensor we can write down vacuum Einstein's equations for KS spacetimes (3.1) in the frame (3.5)-(3.7) as follows

$$R_{\hat{a}\hat{b}} = 0, \tag{3.98}$$

where all $R_{\hat{a}\hat{b}}$ are given in the table 3.3. Solutions, or at least our attempts to solve these equations, will be presented in the following chapter.

Chapter 4

Vacuum Kerr-Schild solutions

In preceding chapter we found vacuum Einstein's equations for metric in a form of the Kerr-Schilld ansatz (3.1). In this chapter we will try to solve these equations, or at least to analyze them and show solutions in some special cases.

In a following calculations, values of *optical scalars* (expansion, shear and twist) will play important role. Let us recall definitions of optical scalars (2.42) and express them in terms of $z_{\hat{i}|\hat{j}}$

$$
\begin{aligned}
L_{\hat{i}\hat{j}} &= -z_{\hat{i}|\hat{j}}, &(4.1)\\
\theta &= -\frac{1}{n-2} z_{\hat{i}|\hat{i}}, &(4.2)\\
\sigma^2 &= \sigma_{\hat{i}\hat{j}}\sigma_{\hat{i}\hat{j}}, &(4.3)\\
\omega^2 &= z_{[\hat{i}|\hat{j}]} z_{[\hat{i}|\hat{j}]}, &(4.4)
\end{aligned}
$$

where shear matrix $\sigma_{\hat{i}\hat{j}}$ is

$$
\sigma_{\hat{i}\hat{j}} = -\theta\delta_{\hat{i}\hat{j}} - z_{(\hat{i}|\hat{j})}. \tag{4.5}
$$

4.1 Vacuum Einstein's equations

Einstein's equations in vacuum can be found from results of previous chapter. They are explicitly following equations, which we have rearranged according to increasing complexity

$$\mathcal{H} z_{\hat{\imath}|\hat{k}} z_{\hat{\jmath}|\hat{k}} + (\mathcal{H}_{|\hat{0}} - \mathcal{H} z_{\hat{k}|\hat{k}}) z_{(\hat{\imath}|\hat{\jmath})} = 0, \tag{4.6}$$

$$\mathcal{H}_{|\hat{0}|\hat{0}} - \mathcal{H}_{|\hat{0}} z_{\hat{\imath}|\hat{\imath}} - 2\mathcal{H} z_{\hat{k}|\hat{\imath}} z_{[\hat{\imath}|\hat{k}]} = 0, \tag{4.7}$$

$$\mathcal{H}_{|\hat{\imath}|\hat{\imath}} + 2(\mathcal{H} z_{\hat{\imath}|\hat{1}})_{|\hat{\imath}} + 2\mathcal{H} z_{\hat{\imath}|\hat{1}} z_{\hat{\imath}|\hat{1}} + (\mathcal{H}_{|\hat{1}} + \mathcal{H}\mathcal{H}_{|\hat{0}}) z_{\hat{\imath}|\hat{\imath}} = 0, \tag{4.8}$$

$$(\mathcal{H}_{|\hat{\imath}} + \mathcal{H} z_{\hat{\imath}|\hat{1}})_{|\hat{0}} + 2(\mathcal{H} z_{[\hat{\imath}|\hat{\jmath}]})_{|\hat{\jmath}} + 2\mathcal{H}_{|[\hat{\jmath}} z_{\hat{\imath}]|\hat{\jmath}} + \mathcal{H} z_{\hat{\imath}|\hat{\jmath}} z_{\hat{\jmath}|\hat{1}} = 0. \tag{4.9}$$

Let us start our analysis with the first one. We can contract this equation with $\delta_{\hat{\imath}\hat{\jmath}}$. Note, that then this is equivalent to Einstein's equation

$$R_{\hat{\imath}\hat{\imath}} = 0,$$

to obtain

$$\mathcal{H} z^2_{\hat{\imath}|\hat{k}} + \mathcal{H}_{|\hat{0}} z_{\hat{\imath}|\hat{\imath}} - \mathcal{H} z^2_{\hat{\imath}|\hat{\imath}} = 0, \tag{4.10}$$

where we have used notation

$$z^2_{\hat{\imath}|\hat{k}} = z_{\hat{\imath}|\hat{k}} z_{\hat{\imath}|\hat{k}}, \qquad z^2_{\hat{\imath}|\hat{\imath}} = (z_{\hat{\imath}|\hat{\imath}})^2. \tag{4.11}$$

From equation (4.10) we can find expression for $\mathcal{H}_{|\hat{0}}$, but just under assumption that the expansion, i.e. $z_{\hat{\imath}|\hat{\imath}}$ is non-zero. So, solutions with zero and nonzero expansion have to be treated separately.

4.2 Non-expanding solutions

The case with zero expansion θ, i.e. $z_{\hat{\imath}|\hat{\imath}} = 0$ is the simplest case, (4.10) implies that

$$z_{\hat{\imath}|\hat{k}} = 0. \tag{4.12}$$

Using this, we can find other Einstein's equations to be

$$\mathcal{H}_{|\hat{0}|\hat{0}} = 0, \tag{4.13}$$

$$\mathcal{H}_{|\hat{\imath}|\hat{\imath}} + 2(\mathcal{H} z_{\hat{\imath}|\hat{1}})_{|\hat{\imath}} + 2\mathcal{H} z_{\hat{\imath}|\hat{1}} z_{\hat{\imath}|\hat{1}} = 0, \tag{4.14}$$

$$(\mathcal{H}_{|\hat{\imath}} + \mathcal{H} z_{\hat{\imath}|\hat{1}})_{|\hat{0}} = 0. \tag{4.15}$$

With the help of commutators (3.68)-(3.70) we can find identities

$$
\begin{aligned}
z_{\hat{\imath}|\hat{1}|\hat{\imath}} &= -z_{\hat{\imath}|\hat{1}} z_{\hat{\imath}|\hat{1}}, && (4.16) \\
z_{\hat{\imath}|\hat{1}|\hat{0}} &= z_{\hat{\imath}|\hat{0}|\hat{1}} = 0, && (4.17) \\
\mathcal{H}_{|\hat{\imath}|\hat{0}} &= \mathcal{H}_{|\hat{0}|\hat{\imath}}, && (4.18)
\end{aligned}
$$

which simplify Einstein's equations to the form

$$
\begin{aligned}
\mathcal{H}_{|\hat{0}|\hat{0}} &= 0, && (4.19) \\
(\delta_{\hat{\imath}} + z_{\hat{\imath}|\hat{1}})\mathcal{H}_{|\hat{0}} &= 0, && (4.20) \\
(\delta_{\hat{\imath}} + 2z_{\hat{\imath}|\hat{1}})\mathcal{H}_{|\hat{\imath}} &= 0. && (4.21)
\end{aligned}
$$

Note, that this case was fully solved in [39], where it was found that non-expanding KS spacetimes are equivalent to the Kundt solutions [40] of Weyl type N, which belongs to the family of spacetimes with vanishing scalar invariants (VSI). Similiar as in $n = 4$, in higher dimensional spacetimes Kundt solutions of Weyl type N consist of two subfamilies: Kundt waves and pp-waves. For more detail discussion of these solutions see [39], [40], [41].

4.3 Expanding solutions

In a case of non-zero expansion θ, i.e. $z_{\hat{\imath}|\hat{\imath}} \neq 0$, we can express $\mathcal{H}_{|\hat{0}}$ out of equation (4.10) and obtain

$$
\mathcal{H}_{|\hat{0}} = \mathcal{H} \frac{z^2_{\hat{\imath}|\hat{\imath}} - z^2_{\hat{\imath}|\hat{k}}}{z_{\hat{\jmath}|\hat{\jmath}}}. \qquad (4.22)
$$

This can be inserted into the first Einstein's equation (4.6), which will then take form

$$
z_{\hat{\imath}|\hat{k}} z_{\hat{\jmath}|\hat{k}} = \frac{z_{\hat{l}|\hat{k}} z_{\hat{l}|\hat{k}}}{z_{\hat{m}|\hat{m}}} z_{(\hat{\imath}|\hat{\jmath})}, \qquad (4.23)
$$

which can be expressed in terms of rotation coefficients as

$$
L_{\hat{\imath}\hat{k}} L_{\hat{\jmath}\hat{k}} = \frac{L_{\hat{l}\hat{k}} L_{\hat{l}\hat{k}}}{(n-2)\theta} S_{\hat{\imath}\hat{\jmath}}, \qquad (4.24)
$$

what agrees with the result in [39].

It is important to observe that this condition on $z_{\hat{\imath}|\hat{\jmath}}$ does not contain $\mathcal{H}$ and so it is purely geometrical condition on the null congruence ℓ in the Minkowskian metric. We will refer to it, in agreement with [39], as the *optical constraint*.

Because we have expression for $\mathcal{H}_{|\hat{0}}$, we can try to simplify Einstein's equation (4.7) by substituting this expression there

$$\left(\mathcal{H}\frac{z^2_{\hat{\imath}|\hat{\imath}} - z^2_{\hat{\imath}|\hat{k}}}{z_{\hat{\jmath}|\hat{\jmath}}}\right)_{|\hat{0}} - \mathcal{H}_{|\hat{0}} z_{\hat{\imath}|\hat{\imath}} - 2\mathcal{H} z_{\hat{k}|\hat{\imath}} z_{[\hat{\imath}|\hat{k}]} = 0, \tag{4.25}$$

which we can simplify as

$$-\mathcal{H}_{|\hat{0}}\frac{z^2_{\hat{\imath}|\hat{k}}}{z_{\hat{\jmath}|\hat{\jmath}}} + \mathcal{H}\left(\frac{z^2_{\hat{\imath}|\hat{\imath}} - z^2_{\hat{\imath}|\hat{k}}}{z_{\hat{\jmath}|\hat{\jmath}}}\right)_{|\hat{0}} - 2\mathcal{H} z_{\hat{k}|\hat{\imath}} z_{[\hat{\imath}|\hat{k}]} = 0. \tag{4.26}$$

The term in brackets is

$$\left(\frac{z^2_{\hat{\imath}|\hat{\imath}} - z^2_{\hat{\imath}|\hat{k}}}{z_{\hat{\jmath}|\hat{\jmath}}}\right)_{|\hat{0}} = \frac{(z^2_{\hat{\imath}|\hat{\imath}} + z^2_{\hat{\imath}|\hat{k}}) z_{\hat{m}|\hat{l}} z_{\hat{l}|\hat{m}} - 2 z_{\hat{\imath}|\hat{k}} z_{\hat{m}|\hat{k}} z_{\hat{\imath}|\hat{m}} z_{\hat{\jmath}|\hat{\jmath}}}{z^2_{\hat{\jmath}|\hat{\jmath}}}, \tag{4.27}$$

and with the help of this and (4.22)we can simplify equation (4.25) to the following form

$$\frac{z_{\hat{\imath}|\hat{k}} z_{\hat{\imath}|\hat{k}} z_{\hat{\jmath}|\hat{l}} z_{(\hat{\jmath}|\hat{l})}}{z^2_{\hat{m}|\hat{m}}} = \frac{z_{\hat{\imath}|\hat{k}} z_{\hat{\jmath}|\hat{k}} z_{\hat{\imath}|\hat{\jmath}}}{z_{\hat{m}|\hat{m}}}. \tag{4.28}$$

With the help of optical constraint we find out that this *Einstein's equation* (4.7) *is in a case of expanding null congruence trivially satisfied* .Unfortunately, it seems that we are unable to simplify other Einstein's equations significantly in the most general case.

Before we continue our analyzes, let us summarize Einstein's equations for expanding null congruence ℓ

$$z_{\hat{\imath}|\hat{k}} z_{\hat{\jmath}|\hat{k}} = \frac{z_{\hat{l}|\hat{k}} z_{\hat{l}|\hat{k}}}{z_{\hat{m}|\hat{m}}} z_{(\hat{\imath}|\hat{\jmath})}, \tag{4.29}$$

$$\mathcal{H}_{|\hat{0}} = \mathcal{H}\frac{z^2_{\hat{\imath}|\hat{\imath}} - z^2_{\hat{\imath}|\hat{k}}}{z_{\hat{\jmath}|\hat{\jmath}}}, \tag{4.30}$$

$$\mathcal{H}_{|\hat{\imath}|\hat{\imath}} + 2(\mathcal{H} z_{\hat{\imath}|\hat{1}})_{|\hat{\imath}} + 2\mathcal{H} z_{\hat{\imath}|\hat{1}} z_{\hat{\imath}|\hat{1}} + (\mathcal{H}_{|\hat{1}} + \mathcal{H}\mathcal{H}_{|\hat{0}}) z_{\hat{\imath}|\hat{\imath}} = 0, \tag{4.31}$$

$$(\mathcal{H}_{|\hat{\imath}} + \mathcal{H} z_{\hat{\imath}|\hat{1}})_{|\hat{0}} + 2(\mathcal{H} z_{[\hat{\imath}|\hat{\jmath}]})_{|\hat{\jmath}} + 2\mathcal{H}_{|[\hat{\jmath}} z_{\hat{\imath}]|\hat{\jmath}} + \mathcal{H} z_{\hat{\imath}|\hat{\jmath}} z_{\hat{\jmath}|\hat{1}} = 0. \tag{4.32}$$

We have to keep in mind that beside these equations, our null congruence also has to satisfy the condition of geodecity

$$z_{\hat{\imath}|\hat{0}} = 0. \tag{4.33}$$

4.3.1 Comments on solving Einstein's equations

Our strategy in solving these equations can be divided into two steps:

1. At the beginning we should determine null congruence ℓ. We can see that the condition of geodecity (4.33) together with the optical constraint (4.29) contains no terms with function $\mathcal{H}$, i.e. these two equations determine the form of null congruence ℓ. Solution of these two equations should be the most general null congruence admitted in a case of geodetic Kerr-Schild spacetime.

2. With the knowledge of most general admitted null congruence we can solve rest of Einstein's equations (4.30)-(4.32) in order to determine function $\mathcal{H}$.

Let us just note that problem of finding most general null congruence admitted by optical constraint and condition of geodecity, i.e. step 1., can be considered on its own. It is analogue of Kerr theorem in a four dimensions, mentioned at the end of section 1.2, and results are interesting even without solving other Einstein's equations.

Problem is that resulting equations (4.30)-(4.33) are extremely complicated and idea to solve them analytically in general seems to be hopeless. If we will take a look just on simplest of them, i.e. condition of geodecity (4.33), we can see big difference from a four dimensional case. If we rewrite this equation in terms of partial derivatives using (3.9)

$$(\partial_v - z_j\partial_j - \frac{1}{2}z^2\partial_u)z_i = 0, \tag{4.34}$$

we can see that it is a system of quasilinear partial differential equations with non-linear coupling. To find a general analytic solution is not possible, or at least there exist no general procedure how to do it. In a four dimensions situations is much simpler, condition of geodecity is just a quasilinear partial differential equation $Y_{|\hat{0}} = 0$ together with its complex conjugate [27], i.e. quasilinear PDE without non-linear coupling. This is actually one of great advantages of using complex frame in four dimensions, because then we have just one variable Y and solution is easy.

The optical constraint (4.29) is a system of non-linear partial differential equations, which also we do not know how to solve in fully general case. Rest of Einstein's equations (4.30)-(4.32) seems also hopeless to solve in full generality.

This means that we are forced to give up hope to find general Kerr-Schild solution in a higher dimensions and we have to be satisfied with much less general and complete solutions. There are three different ways how to proceed now in order to find at least some Kerr-Schild solutions.

1. To try to analyze case when null congruence ℓ is non-twisting. We will see that this will simplify our equations very much and we will be able to find some restriction on solutions.

2. To try to guess some solutions which can be admitted by our equations. We will find such ansatzes, which will be based either on some symmetry or on Myers-Perry ansatz based on analogy with a four dimensional Kerr-Schild solutions.

3. It seems that one of most general results which one can obtain is by method started in [39]. This method is based on fixing some kind of r-dependency along null congruence ℓ and investigating just r-dependency and asymptotic behavior of Kerr-Schild solutions. This seems to be possible to do in fully general case.

4.4 Non-twisting solutions

In this case we have $\omega = 0$, i.e.

$$z_{\hat{\imath}|\hat{\jmath}} = z_{\hat{\jmath}|\hat{\imath}}. \tag{4.35}$$

From what follows that

$$z_{(\hat{\imath}|\hat{\jmath})} = z_{\hat{\imath}|\hat{\jmath}} \qquad z_{[\hat{\imath}|\hat{\jmath}]} = 0. \tag{4.36}$$

Thus the optical constraint will take form

$$z_{\hat{\imath}|\hat{k}} z_{\hat{\jmath}|\hat{k}} = \mathcal{F} z_{\hat{\imath}|\hat{\jmath}}. \tag{4.37}$$

where

$$\mathcal{F} = \frac{z_{\hat{l}|\hat{k}} z_{\hat{l}|\hat{k}}}{z_{\hat{m}|\hat{m}}}. \tag{4.38}$$

We can start our analysis of optical constraint with dividing our analysis into two main categories according to degeneracy of the matrix $z_{\hat{\imath}|\hat{\jmath}}$.

Non-degenerate $z_{\hat{\imath}|\hat{\jmath}}$

In a case of non-degenerate $z_{\hat{\imath}|\hat{\jmath}}$, i.e. $\det z_{\hat{\imath}|\hat{\jmath}} \neq 0$, we can always find its inverse $z^{-1}_{\hat{\imath}|\hat{\jmath}}$. Let us multiply with $z^{-1}_{\hat{\jmath}|\hat{n}}$ the optical constraint (4.37) to obtain

$$z_{\hat{\imath}|\hat{n}} = \mathcal{F}\delta_{\hat{\imath}\hat{n}}, \tag{4.39}$$

i.e. that matrix $z_{\hat{\imath}|\hat{\jmath}}$ is diagonal. Thus we proved that in a case of a non-degenerate $z_{\hat{\imath}|\hat{\jmath}}$ non-twisting solutions are also non-shearing. From [39] we know that only non-shearing non-twisting solution is Schwarzschild-Tangherlini solution.

Degenerate $z_{\hat{\imath}|\hat{\jmath}}$

Solutions with $\det z_{\hat{\imath}|\hat{\jmath}} = 0$ should represent black-string, according to [39]. For more details and analyzes in general case consult [39]. In the following we will find some ansatz, which will satisfy this condition and we will observe that it indeed represents black string.

4.5 Some explicit solutions

As was mentioned in the section 4.3.1, to solve system (4.29)-(4.33) analytically in a general case seems to be hopeless and as we could see that even in non-twisting case we were not able to find general solution. What we can do now, is to try to guess some solutions which would solve the system (4.29)-(4.33).

Solutions with dependency $z(x^i)$

Our guess, when we are attempting to solve condition of geodecity (4.33) is to try ansatz $z(x^i)$, i.e non-twisting solutions independent of coordinates u and v. Indeed, in a such case condition of geodecity (4.33) will take simple form

$$z_j\partial_j z_i = 0. \tag{4.40}$$

We may restrict ourselves to the case with zero twist, i.e.

$$\partial_i z_j = \partial_j z_i. \tag{4.41}$$

If we combine these two conditions, by inserting second one to first one, we obtain equation

$$\partial_i z_j z_j \equiv \partial_i z^2 = 0, \tag{4.42}$$

from where follows that

Null congruence given by ansatz $z_i = \frac{\sqrt{2}x^i}{r}$

We can guess a particular solution

$$z_i = \sqrt{2}\frac{x^i}{r}, \tag{4.43}$$

where r is given by expression

$$r = \sqrt{x^i x_i} = \sqrt{(x^2)^2 + (x^3)^2 + \cdots + (x^{n-1})^2}. \tag{4.44}$$

Note that

$$z^2 = 2. \tag{4.45}$$

We can derive important identities which we will use in following computation:

$$\partial_i r = \frac{z_i}{\sqrt{2}}, \tag{4.46}$$

and

$$r\partial_r = x^i \partial_i, \tag{4.47}$$

from which it follows

$$z_i \partial_i = \sqrt{2}\frac{x^i}{r}\partial_i = \sqrt{2}\partial_r. \tag{4.48}$$

We can easily find that directional derivatives of z_i are

$$\begin{aligned} z_{\hat{i}|\hat{0}} &= 0, & (4.49)\\ z_{\hat{i}|\hat{1}} &= 0, & (4.50)\\ z_{\hat{i}|\hat{j}} &= \partial_j z_i = \sqrt{2}\frac{\delta_{ij}r^2 - x_i x_j}{r^3} = z_{\hat{j}|\hat{i}}. & (4.51) \end{aligned}$$

and from last one we can obtain following useful relations

$$\begin{aligned} z_{\hat{i}|\hat{i}} &= \sqrt{2}\frac{n-3}{r}, & (4.52)\\ z_{\hat{i}|\hat{j}} z_{\hat{i}|\hat{j}} &= \frac{2}{r^6}(\delta_{ij}r^2 - x_i x_j)(\delta_{ij}r^2 - x_i x_j) = 2\frac{n-3}{r^2}. & (4.53) \end{aligned}$$

We can easily verify that optical constraint is satisfied. Determinant of $z_{\hat{i}|\hat{j}}$ is zero, i.e. this solution corresponds to degenerate case mentioned in previous section.

Einstein's equations for the above ansatz $z_i = \sqrt{2}\frac{x^i}{r}$

Because (4.43) satisfies optical constraint, we can continue calculation of the rest of Einstein's equations in order to determine function $\mathcal{H}$.
Einstein's equation (4.30) will take form

$$\mathcal{H}_{|\hat{0}} = \sqrt{2}\mathcal{H}\frac{(n-4)}{r}, \tag{4.54}$$

Second Einstein's equation (4.32) in a case of non-twisting null congruence ℓ is

$$\mathcal{H}_{|\hat{i}|\hat{0}} + 2\mathcal{H}_{|[\hat{j}}z_{\hat{i}]|\hat{j}} = 0, \tag{4.55}$$

which we can, with help of commutator (3.68), transform to the following form

$$\mathcal{H}_{|\hat{0}|\hat{i}} + 2\mathcal{H}_{|\hat{j}}z_{\hat{i}|\hat{j}} - \mathcal{H}_{|\hat{i}}z_{\hat{j}|\hat{j}} = 0. \tag{4.56}$$

Inserting previous result (4.54) we obtain

$$\mathcal{H}_{|\hat{i}} - \mathcal{H}\frac{(n-4)}{\sqrt{2}}\frac{z_{\hat{i}}}{r} - \mathcal{H}_{|\hat{j}}z_{\hat{i}}z_{\hat{j}} = 0 \tag{4.57}$$

and remaining Einstein's equation (4.31) become

$$\mathcal{H}_{|\hat{i}|\hat{i}} = -\sqrt{2}\partial_u\mathcal{H}\frac{n-3}{r} \tag{4.58}$$

Thus, equations (4.54)-(4.57) determine fully function $\mathcal{H}$ in a case of null congruence given by (3.5) with (4.43). These equations contain directional derivatives along frame vectors, so in order to solve (i.e. integrate) these equations we have to shift to standard partial derivatives. Thus, $\mathcal{H}_{|\hat{i}}$ expressed through partial derivative will be

$$\mathcal{H}_{|\hat{i}} = \partial_i\mathcal{H} + z_i\partial_u\mathcal{H}. \tag{4.59}$$

If we insert this to equation (4.57) we will obtain

$$\partial_i\mathcal{H} - z_i\partial_u H - \mathcal{H}\frac{(n-4)}{\sqrt{2}}\frac{z_i}{r} - \sqrt{2}z_i\partial_r\mathcal{H} = 0 \tag{4.60}$$

We can multiply this equation by z_i and we will get

$$\sqrt{2}\partial_u\mathcal{H} = -\partial_r\mathcal{H} - \mathcal{H}\frac{(n-4)}{r}, \tag{4.61}$$

which we can insert back to equation (4.60) to obtain

$$\partial_i \mathcal{H} = \frac{z_i}{\sqrt{2}} \partial_r \mathcal{H}. \tag{4.62}$$

Note, that, because of (4.46) , this equation tells us that dependence of function $\mathcal{H}$ on x^i is fully given by its dependence on r. Using previous two equations we can find frame directional derivative (4.59)

$$\mathcal{H}_{|\hat{i}} = -\mathcal{H}\frac{(n-4)}{\sqrt{2}}\frac{z_i}{r}. \tag{4.63}$$

Taking directional frame derivative of this expression we find

$$\mathcal{H}_{|\hat{i}|\hat{i}} = -\frac{(n-4)}{\sqrt{2}}(\partial_i + z_i\partial_u)\left[\mathcal{H}\frac{z_i}{r}\right] = -\mathcal{H}\frac{(n-4)^2}{r^2} - \partial_r\mathcal{H}\frac{(n-4)}{r} - \sqrt{2}\partial_u H\frac{(n-4)}{r}, \tag{4.64}$$

and inserting previous result (4.61) we find that

$$\mathcal{H}_{|\hat{i}|\hat{i}} = 0, \tag{4.65}$$

which allow us to simplify Einstein's equation (4.58) to

$$\partial_u \mathcal{H} = 0. \tag{4.66}$$

Using (4.61) we can find frame derivative $\mathcal{H}_{|\hat{0}}$ to be

$$\mathcal{H}_{|\hat{0}} = \partial_v \mathcal{H} - \sqrt{2}\partial_r\mathcal{H} - \frac{1}{2}\partial_u\mathcal{H} = \partial_v\mathcal{H} + \sqrt{2}\mathcal{H}\frac{(n-4)}{r}, \tag{4.67}$$

with this we can simplify Einstein's equation (4.54) to

$$\partial_v \mathcal{H} = 0. \tag{4.68}$$

Thus Einstein's equations (4.54)-(4.58) for congruence determined by (4.43) were found to be

$$\begin{aligned}
\partial_u \mathcal{H} &= 0, & (4.69)\\
\partial_v \mathcal{H} &= 0, & (4.70)\\
\partial_r \mathcal{H} &= -\mathcal{H}\frac{(n-4)}{r}. & (4.71)
\end{aligned}$$

First two equations simply tell us that function $\mathcal{H}$ is independent of coordinates u and v. According to (4.62) dependence of $\mathcal{H}$ on x^i is fully given by its dependency on

r. Thus $\mathcal{H}$ is function of only coordinate r and this dependence is fully determined by third equation (4.71). We can easily integrate this equation and find that

$$\mathcal{H} = \frac{M}{r^{(n-4)}}, \tag{4.72}$$

where M is an arbitrary constant.
With use of (3.5) and (4.43) we can find that

$$\ell = (du - dv - \sqrt{2}dr). \tag{4.73}$$

Thus metric for ansatz (4.43) is given by

$$ds^2_{BS} = 2dvdu + dx_i dx^i + 2\frac{M}{r^{(n-4)}}\left(du - dv + \sqrt{2}dr\right)^2. \tag{4.74}$$

To understand this solution it is more instructive to make transformation of coordinates. Using (3.4) null vector (4.73) in Cartesian coordinates will take form

$$\ell = -\sqrt{2}(dt + dr), \tag{4.75}$$

then metric will be

$$ds^2_{BS} = -dt^2 + (dx^1)^2 + dr^2 + r^2 d\Omega^2_{n-3} + 4\frac{M}{r^{(n-4)}}(dt + dr)^2, \tag{4.76}$$

where was used relation

$$dx_i dx^i = dr^2 + r^2 d\Omega^2_{n-3}. \tag{4.77}$$

We can make trasformation

$$dt + dr = dU, \tag{4.78}$$

where U is some new coordinate and we will obtain

$$ds^2_{BS} = -\left(1 - \frac{R_g}{r^{(n-4)}}\right) dU^2 + 2dUdr + r^2 d\Omega^2_{n-3} + (dx^1)^2, \tag{4.79}$$

where we redefined constant $R_g = 4M$. Using information from first chapter and [17] we can easily see that first three terms of this metric corresponds to $n-1$ dimensional Schwarzschild-Tangherlini solution in Eddington-Finkelstein coordinates. If we denote by $ds^2_{n-1}(\mathcal{S})$ metric of $n-1$ dimensional Schwarzschild-Tangherlini solution then (4.79) we can rewrite as

$$ds^2_{BS} = ds^2_{n-1}(\mathcal{S}) + (dx^1)^2, \tag{4.80}$$

from where we can see that it corresponds to **black string** [24], [13].

Generalization to black p-branes

In previous calculations we derived analytically black string solution by guessing ansatz for functions z_i in a form (4.43). We know that black string is topologically $\mathcal{S}^{(n-1)} \times \mathcal{R}$, where $\mathcal{S}^{(n-1)}$ represents $(n-1)$ dimensional Schwarzchschild-Tangherlini solution.

This solution is easy to generalize to *black p-branes* [13], which play important role in string theory. Black p-branes are topologically $\mathcal{S}^{(n-p)} \times \mathcal{R}^p$, i.e. black string being just special name for black 1-brane.

Such generalization can be easily done by omitting $(p-1)$ coordinates from ansatz (4.43) and definition of r-parameter (4.44). We can directly see that metric in such case would be

$$ds^2_{pB} = -\left(1 - \frac{\tilde{R}_g}{r_p^{(n-p-3)}}\right) dU^2 + 2dU dr_p + r_p^2 d\Omega^2_{n-p-2} + \sum_{\alpha=1}^{p} dx_\alpha dx^\alpha, \tag{4.81}$$

where dr_p represents r-parameter defined as in (4.44), but with $(p-1)$ coordinates omitted, i.e.

$$r_p = \sum_{\alpha=p+1}^{n-1} \sqrt{x^\alpha x_\alpha}. \tag{4.82}$$

We can rewrite this as

$$ds^2_{pB} = ds^2_{n-p}(\mathcal{S}) + \sum_{\alpha=1}^{p} dx_\alpha dx^\alpha, \tag{4.83}$$

from where we can directly see that black p-brane is topologically $\mathcal{S}^{(n-p)} \times \mathcal{R}^p$.

4.6 Fixing r-dependence

As we mentioned before, it is possible to choose some r dependence and analyze KS spacetimes then. This is very useful mainly in analyzes of asymptotical properties of the Riemann tensor. Natural such dependence is to choose r as a affine parameter along null vector ℓ, i.e. then

$$D = \frac{\partial}{\partial r}. \tag{4.84}$$

This means that we have to make transform of coordinates

$$(u, v, x^i) \rightarrow (r, x^A), \tag{4.85}$$

where index A runs from 1 to $n-1$.

In such coordinates other frame derivatives are given by

$$\Delta = U\frac{\partial}{\partial r} + Y^A \frac{\partial}{\partial x^A}, \tag{4.86}$$

and

$$\delta_{\hat{i}} = \omega_{\hat{i}}\frac{\partial}{\partial r} + \xi^A_{\hat{i}} \frac{\partial}{\partial x^A}. \tag{4.87}$$

In order to analyze asymptotical properties of Riemann tensor (table 3.2), we have to find asymptotical r-dependance of given directional derivatives, function $\mathcal{H}$ and directional derivatives of functions $z_{\hat{i}}$.

We may start with analyzing r-dependance of directional derivatives of functions $z_{\hat{i}}$. Condition of geodetecity (4.33) tells us that

$$z_{\hat{i}|\hat{0}} = \partial_r z_{\hat{i}} = 0. \tag{4.88}$$

To analyze r-dependance of $z_{\hat{i}|\hat{j}}$ we use commutator (3.68) and we obtain equation

$$\partial_r z_{\hat{i}|\hat{j}} = z_{\hat{i}|\hat{k}} z_{\hat{k}|\hat{j}}. \tag{4.89}$$

Note that this is equivalent to the Sachs equation used in [39].

Nondegenerate $z_{\hat{i}|\hat{j}}$

Now, we focus on case when $\det z_{\hat{i}|\hat{j}} \neq 0$ and we can obtain

$$z^{-1}_{\hat{m}|\hat{i}} \partial_r z_{\hat{i}|\hat{j}} = z_{\hat{m}|\hat{j}}. \tag{4.90}$$

Here we can use identity

$$\partial_r(\delta_{ij}) = \partial_r(z^{-1}_{\hat{i}|\hat{k}} z_{\hat{k}|\hat{j}}) = (\partial_r z^{-1}_{\hat{i}|\hat{k}}) z_{\hat{k}|\hat{j}} + z^{-1}_{\hat{i}|\hat{k}} \partial_r z_{\hat{k}|\hat{j}} = 0, \tag{4.91}$$

from where we can find that

$$z^{-1}_{\hat{i}|\hat{k}} \partial_r z_{\hat{k}|\hat{j}} = -(\partial_r z^{-1}_{\hat{i}|\hat{k}}) z_{\hat{k}|\hat{j}}. \tag{4.92}$$

If we insert this result back to (4.90) we obtain

$$(\partial_r z^{-1}_{\hat{m}|\hat{i}}) z_{\hat{i}|\hat{j}} = z_{\hat{m}|\hat{j}}, \tag{4.93}$$

from where it follows

$$\partial_r z^{-1}_{\hat{m}|\hat{i}} = -\delta_{\hat{m}\hat{i}}. \tag{4.94}$$

We can easily integrate this to obtain

$$z^{-1}_{\hat{m}|\hat{i}} = -r\delta_{\hat{m}\hat{i}} + a_{\hat{m}\hat{i}}, \tag{4.95}$$

where $a_{\hat{m}\hat{i}}$ is matrix of functions with the property $\partial_r a_{\hat{m}\hat{i}} = 0$. We can easily invert this equation under assumption that r goes to infinity

$$z_{\hat{i}|\hat{j}} = -\frac{\delta_{\hat{i}\hat{j}}}{r} + \frac{a_{\hat{i}\hat{j}}}{r^2} + O(r^{-3}). \tag{4.96}$$

Generalization of this result is the expansion [39], [42]

$$z_{\hat{i}|\hat{j}} = \sum_{p=1}^{k} \frac{(a^{p-1})_{\hat{i}\hat{j}}}{r^p} + O(r^{-p-1}), \tag{4.97}$$

where $O(r^{-n})$ represents all terms which goes to zero as r^{-n} and faster and $(a^{p-1})_{\hat{i}\hat{j}}$ is $p-1$ power of matrix $a_{\hat{i}\hat{j}}$.

With knowledge of (4.97), we can now easily find the r-dependance of $z_{\hat{i}|\hat{1}}$. We start with commutator (3.68), which if we apply on $z_{\hat{i}}$ we obtain

$$\partial_r z_{\hat{i}|\hat{1}} = z_{\hat{i}|\hat{j}} z_{\hat{j}|\hat{1}}. \tag{4.98}$$

We can multiply this equation by $z^{-1}_{\hat{i}|\hat{k}}$ and we obtain

$$z^{-1}_{\hat{i}|\hat{k}} \partial_r z_{\hat{i}|\hat{1}} = z_{\hat{i}|\hat{1}}, \tag{4.99}$$

and because of (4.95) we can use

$$\partial_r z^{-1}_{\hat{i}|\hat{k}} = -\delta_{\hat{i}\hat{k}}, \tag{4.100}$$

to simplify this equation to

$$\partial_r (z^{-1}_{\hat{i}|\hat{k}} z_{\hat{i}|\hat{1}}) = 0. \tag{4.101}$$

Solution of this equation is

$$z_{\hat{i}|\hat{1}} = b_{\hat{j}} z_{\hat{i}|\hat{j}}, \tag{4.102}$$

where $z_{\hat{i}|\hat{j}}$ is defined by (4.97) and $b_{\hat{i}}$ are some functions independent of r.

Using previous results we can find r-dependance of operators of directional derivatives (4.86) and (4.87). In order to do so we have to use commutators (3.68)-(3.70). Let us start with (4.87). Commutator (3.68) is given by

$$D\delta_{\hat{i}} - \delta_{\hat{i}}D = z_{\hat{j}|\hat{i}}\delta_{\hat{j}}, \tag{4.103}$$

we can apply this on function r to obtain

$$\partial_r \omega_{\hat{i}} = z_{\hat{j}|\hat{i}}\omega_{\hat{j}}. \tag{4.104}$$

If we multiply this equation by $z^{-1}_{\hat{i}|\hat{k}}$ we obtain

$$z^{-1}_{\hat{i}|\hat{k}}\partial_r \omega_{\hat{i}} = \omega_{\hat{k}}, \tag{4.105}$$

and using (4.100) we can simplify it to

$$\partial_r (z^{-1}_{\hat{i}|\hat{k}}\omega_{\hat{i}}) = 0. \tag{4.106}$$

This equation is solved by

$$\omega_{\hat{i}} = \omega^0_{\hat{j}} z_{\hat{i}|\hat{j}}, \tag{4.107}$$

where $\omega^0_{\hat{j}}$ is some integration constant of equation (4.106).
Analogously, if we apply commutator (4.103) on coordinates x^A we find that

$$\xi^A_{\hat{i}} = \xi^{A^0}_{\hat{j}} z_{\hat{i}|\hat{j}}, \tag{4.108}$$

where $\xi^{A^0}_{\hat{i}}$ are some integration constants. Thus we can write that r-dependance of operator $\delta_{\hat{i}}$ is given by

$$\delta_{\hat{i}} = z_{\hat{i}|\hat{j}}(\omega^0_{\hat{j}}\partial_r + \xi^{A^0}_{\hat{j}}\partial_A) \tag{4.109}$$

In similar way we can find r-dependance of Δ operator. Commutator (3.67) is given by

$$D\Delta - \Delta D = -\mathcal{H}_{|\hat{0}}D + z_{\hat{i}|\hat{1}}\delta_{\hat{i}}. \tag{4.110}$$

We can apply this commutator on r to obtain

$$\partial_r U = -\partial_r \mathcal{H} + z_{\hat{i}|\hat{1}}\omega_{\hat{i}}, \tag{4.111}$$

using result (4.107) we can rewrite this equation as

$$\partial_r (U + H) = \omega^0_{\hat{j}} z_{\hat{i}|\hat{j}} z_{\hat{i}|\hat{1}}, \tag{4.112}$$

which we can with help of (4.98) simplify to

$$\partial_r (U + H) = \omega^0_{\hat{i}} \partial_r z_{\hat{i}|\hat{1}}. \tag{4.113}$$

Solution to this equation is easily found to be

$$U = U^0 - \mathcal{H} + \omega^0_{\hat{i}} z_{\hat{i}|\hat{1}}, \tag{4.114}$$

where U^0 is some function independent of r. Similarly, if we apply commutator (4.110) on function x^A we can obtain

$$Y^A = Y^{A^0} + \xi^{A^0}_{\hat{i}} z_{\hat{i}|\hat{1}}. \tag{4.115}$$

Thus we can conclude that r dependance of operator Δ is given by

$$\Delta = (U^0 - \mathcal{H} + \omega^0_{\hat{i}} z_{\hat{i}|\hat{1}})\partial_r + (Y^{A^0} + \xi^{A^0}_{\hat{i}} z_{\hat{i}|\hat{1}})\partial_A. \tag{4.116}$$

Now, only remains to determine the r-dependance of KS function $\mathcal{H}$. If we consider just the leading term in expansion (4.97) then equation (4.30) has a simple form

$$\partial_r \mathcal{H} = -\mathcal{H}\frac{(n-3)}{r}, \tag{4.117}$$

which solution si given by

$$\mathcal{H} = \frac{\mathcal{H}_0}{r^{n-3}}, \tag{4.118}$$

where $\mathcal{H}_0$ is function independent of r. Generalization of this result to arbitrary term in r was already done in [39] and for non-degenerate case is given by

$$\mathcal{H} = \frac{\mathcal{H}_0}{r^{n-2q-3}} \prod_{\mu=1}^{q} \frac{1}{r^2 + (a^0_{(2\mu)})^2}, \tag{4.119}$$

where $\mathcal{H}_0$ is function independent of r, $0 \leq 2q \leq n-2$. For more details about meaning of functions $(a^0_{(2\mu)})$ see [39].

Let us **summarize** previous results:

$$z_{\hat{i}|\hat{j}} = \sum_{p=1}^{k} \frac{(a^{p-1})_{\hat{i}\hat{j}}}{r^p} + O(r^{-p-1}), \tag{4.120}$$

$$z_{\hat{i}|\hat{1}} = b_{\hat{j}} z_{\hat{i}|\hat{j}}, \tag{4.121}$$

$$\mathcal{H} = \frac{\mathcal{H}_0}{r^{n-2q-3}} \prod_{\mu=1}^{q} \frac{1}{r^2 + (a^0_{(2\mu)})^2}, \tag{4.122}$$

$$
\begin{aligned}
D &= \partial_r, && (4.123)\\
\Delta &= U\frac{\partial}{\partial r} + Y^A\frac{\partial}{\partial x^A}, && (4.124)\\
\delta_{\hat{i}} &= \omega_{\hat{i}}\frac{\partial}{\partial r} + \xi^A_{\hat{i}}\frac{\partial}{\partial x^A}, && (4.125)
\end{aligned}
$$

where $a_{\hat{i}\hat{j}}$ are matrices and $\mathcal{H}_0$ function independent of r, $(a^0_{(2\mu)})$ are some constant defined in [39], $0 \leq 2q \leq n-2$ and coefficients U, Y^A, ω_i and ξ^A_i are given by

$$
\begin{aligned}
\omega_{\hat{i}} &= \omega^0_{\hat{j}} z_{\hat{i}|\hat{j}}, && (4.126)\\
\xi^A_{\hat{i}} &= \xi^{A^0}_{\hat{j}} z_{\hat{i}|\hat{j}}, && (4.127)\\
U &= U^0 - \mathcal{H} + \omega^0_{\hat{i}} z_{\hat{i}|\hat{1}}, && (4.128)\\
Y^A &= Y^{A^0} + \xi^{A^0}_{\hat{i}} z_{\hat{i}|\hat{1}}, && (4.129)
\end{aligned}
$$

where $\omega^0_{\hat{j}}$, $\xi^{A^0}_{\hat{i}}$, $\omega^0_{\hat{i}}$ and $\xi^{A^0}_{\hat{i}}$ are functions independent of r.

Now, we can start to analyze the properties of Riemann tensor given in table 3.3. With the previous knowledge we can expand all obtained expressions just in terms of $z_{\hat{i}|\hat{j}}$ and $\mathcal{H}$. We will use shortcut $\partial_r\mathcal{H}$ whenever it will be convenient, because we know that

$$
\partial_r\mathcal{H} = \mathcal{H}\frac{z^2_{\hat{i}|\hat{i}} - z^2_{\hat{i}|\hat{k}}}{z_{\hat{j}|\hat{j}}}. \qquad (4.130)
$$

Then Riemann tensor will be

$$
\begin{aligned}
R_{\hat{0}\hat{1}\hat{0}\hat{1}} &= \partial_r\partial_r\mathcal{H}, && (4.131)\\
R_{\hat{0}\hat{1}\hat{1}\hat{i}} &= (\omega^0_{\hat{j}}\partial_r\mathcal{H} + \xi^{A^0}_{\hat{j}}\partial_A\mathcal{H}) z_{\hat{i}|\hat{k}} z_{\hat{k}|\hat{j}} + z_{\hat{i}|\hat{j}}(\omega^0_{\hat{j}}\partial_r\partial_r\mathcal{H} + \xi^{A^0}_{\hat{j}}\partial_r\partial_A\mathcal{H}) \\
&+ \partial_r\mathcal{H} z_{\hat{i}|\hat{j}} b_{\hat{j}} + 2\mathcal{H} z_{\hat{i}|\hat{j}} z_{\hat{j}|\hat{k}} b_{\hat{k}}, && (4.132)\\
R_{\hat{0}\hat{1}\hat{i}\hat{j}} &= 2z_{[\hat{j}|\hat{i}]}\partial_r\mathcal{H} - 2\mathcal{H}(z_{\hat{i}|\hat{k}} z_{\hat{k}|\hat{j}} - z_{\hat{j}|\hat{k}} z_{\hat{k}|\hat{i}}), && (4.133)\\
R_{\hat{0}\hat{i}\hat{1}\hat{j}} &= -\partial_r\mathcal{H} z_{\hat{i}|\hat{j}} - 2\mathcal{H} z_{\hat{k}|\hat{j}} z_{[\hat{i}|\hat{k}]}, && (4.134)\\
R_{\hat{1}\hat{i}\hat{1}\hat{j}} &= -z_{\hat{j}|\hat{k}}(\omega^0_{\hat{k}}\partial_r + \xi^{A^0}_{\hat{k}}\partial_A)\{z_{\hat{i}|\hat{l}}[(\omega^0_{\hat{l}}\partial_r + \xi^{A^0}_{\hat{l}}\partial_A)\mathcal{H}]\} - z_{\hat{j}|\hat{k}} z_{\hat{i}|\hat{l}}(\omega^0_{\hat{k}}\partial_r\mathcal{H} + \xi^{A^0}_{\hat{k}}\partial_A\mathcal{H}) b_{\hat{l}} \\
&- z_{\hat{i}|\hat{k}} z_{\hat{j}|\hat{l}}(\omega^0_{\hat{k}}\partial_r\mathcal{H} + \xi^{A^0}_{\hat{k}}\partial_A\mathcal{H}) b_{\hat{l}} - \mathcal{H} z_{\hat{i}|\hat{k}} z_{\hat{j}|\hat{l}} b_{\hat{l}} b_{\hat{k}} \\
&- \mathcal{H} z_{\hat{j}|\hat{k}}[(\omega^0_{\hat{k}}\partial_r + \xi^{A^0}_{\hat{k}}\partial_A) z_{\hat{i}|\hat{l}} b_{\hat{l}}] - \mathcal{H}^2(z_{\hat{k}|\hat{i}} z_{\hat{k}|\hat{j}} - 2z_{[\hat{i}|\hat{k}]} z_{\hat{j}|\hat{k}}) - \mathcal{H}(\partial_r\mathcal{H}) z_{\hat{j}|\hat{i}} \\
&- (U^0 - \mathcal{H} - \omega^0_{\hat{k}} z_{\hat{k}|\hat{l}} b_{\hat{l}}) z_{\hat{j}|\hat{i}}\partial_r\mathcal{H} - (U^0 - \mathcal{H} - \omega^0_{\hat{k}} z_{\hat{k}|\hat{l}} b_{\hat{l}})\mathcal{H} z_{\hat{j}|\hat{m}} z_{\hat{m}|\hat{i}} \\
&- \xi^{A^0}_{\hat{k}} z_{\hat{k}|\hat{l}} b_{\hat{l}}\partial_A(z_{\hat{j}|\hat{i}}\mathcal{H}) - Y^{A^0}\partial_A(z_{\hat{j}|\hat{i}}\mathcal{H}), && (4.135)
\end{aligned}
$$

$$
\begin{aligned}
R_{\hat{1}\hat{i}\hat{j}\hat{k}} &= 2(\omega_{\hat{l}}^{0}\partial_r\mathcal{H} + \xi_{\hat{l}}^{A^0}\partial_A\mathcal{H} + \mathcal{H}b_{\hat{l}})z_{\hat{i}|\hat{l}}z_{[\hat{j}|\hat{k}]} \\
&+ (\omega_{\hat{l}}^{0}\partial_r\mathcal{H} + \xi_{\hat{l}}^{A^0}\partial_A\mathcal{H})(z_{\hat{j}|\hat{l}}z_{\hat{k}|\hat{i}} - z_{\hat{k}|\hat{l}}z_{\hat{j}|\hat{i}}) + \mathcal{H}[z_{\hat{j}|\hat{l}}(\omega_{\hat{l}}^{0}z_{\hat{k}|\hat{m}}z_{\hat{m}|\hat{i}} + \xi_{\hat{l}}^{A^0}\partial_A z_{\hat{k}|\hat{i}}) \\
&- z_{\hat{k}|\hat{l}}(\omega_{\hat{l}}^{0}z_{\hat{j}|\hat{m}}z_{\hat{m}|\hat{i}} + \xi_{\hat{l}}^{A^0}\partial_A z_{\hat{j}|\hat{i}})], \qquad (4.136) \\
R_{\hat{i}\hat{j}\hat{k}\hat{l}} &= 2\mathcal{H}(z_{\hat{k}|[\hat{i}}z_{\hat{j}]|\hat{l}} - z_{[\hat{i}|\hat{j}]}z_{\hat{k}|\hat{l}}). \qquad (4.137)
\end{aligned}
$$

Here we found r-dependance of all components of Riemann tensor in general case. This may be usefull in further calculations. For now, let us analyze only properties of Riemann tensor at infinity. Because of previous results we know that to the leading term holds

$$
\begin{aligned}
z_{\hat{i}|\hat{j}} &= O(r^{-1}), \qquad (4.138) \\
\mathcal{H} &= O(r^{n-3}). \qquad (4.139)
\end{aligned}
$$

If we use these two relations we can find that all components of Riemann tensor seems to be to the leading term in r

$$R_{\hat{a}\hat{b}\hat{c}\hat{d}} = O(r^{1-n}), \qquad (4.140)$$

except component $R_{\hat{1}\hat{i}\hat{1}\hat{j}}$ where last term seems to be of order $O(r^{2-n})$. However, if we take trace of this component we obtain

$$R_{\hat{1}\hat{1}} = R_{\hat{1}\hat{i}\hat{1}\hat{i}}. \qquad (4.141)$$

If we use Einstein's equation $R_{\hat{1}\hat{1}} = 0$ we can easily find that last

$$\partial_A(z_{\hat{i}|\hat{i}}\mathcal{H}) = O(r^{1-n}). \qquad (4.142)$$

With this we proved that all components of Riemann tensor to the leading term are

$$R_{\hat{a}\hat{b}\hat{c}\hat{d}} = O\left(\frac{1}{r^{n-1}}\right). \qquad (4.143)$$

We can see that this result is in a agreement with general analyzes [42]. This means that Kerr-Schild spacetimes do not "peel off" and do not contain radiation what is also in agreement with our knowldege about Kerr-Schild spacetimes in four dimensions.

Conclusion

In this thesis we investigated Kerr-Schild spacetimes in higher ($n > 4$) dimensions. For this we used a generalization of the Newman-Penrose formalism to higher dimensions. This has been developed only very recently and our work is thus one of the first investigations that use it in its full strength.

We used special ansatz, so-called *Kerr-Schild ansatz*, where spacetime is fully determined by null congruence ℓ and by a function $\mathcal{H}$. For such ansatz we found explicitly all Ricci rotation coefficients, connection forms and curvature forms. We could observe that condition of ℓ to be geodetic is very important from both physical viewpoint and viewpoint of calculations (it allow us to simplify many calculations). Then using this knowledge we were able to find frame components of Riemann and Ricci tensor. From there we found general form of gravitational part of Einstein's equations in higher dimensions for Kerr-Schild ansatz. These results were derived in full generality and thus they may be useful in future studies of Kerr-Schild spacetimes in higher dimensions. For example, in analyzes of solutions with electromagnetic field or cosmological constant.

We restricted ourselves to the vacuum case and we proceeded to analyze Einstein's equations. We found that one of Einstein's equations does not contain function $\mathcal{H}$ and together with condition of geodecity it determines fully null-congruence ℓ. Because of this we named this equation, in agreement with references, *optical constraint*. Rest of Einstein's equations determines unknown functions $\mathcal{H}$.

We found that we have to distinguish two main classes of solutions, according to value of expansion. Because non-expanding case is already well-studied and understood, we focused on expanding solutions. In this case we found that one of Einstein's equations is trivially satisfied as a consequence of rest of equations.

We summarized remaining Einstein's equations and we discussed possibility to solve them. We found that situation in $n > 4$ dimensions is much more complicated than 4 dimensional case, where general solutions exists. We could see that these equations are systems of non-linear PDEs with nonlinear couplings. Because we are unable to solve such systems we had to give up our hope for finding analytical solution in general case.

Therefore, we argued that best what we can hope is to find some partial results. We gave three possible ways how to proceed. First of them is to restrict ourselves to the case of non-twisting ℓ. We found that in such case we have to divide our analyzes to two classes, according to degeneracy of matrix $z_{\hat{\imath}|\hat{\jmath}}$. We proved that, if $\det(z_{\hat{\imath}|\hat{\jmath}}) \neq 0$, then non-twisting solution is automatically non-shearing. We know that the most general solution of this kind is Schwarzschild-Tangherlini. Solutions with $\det(z_{\hat{\imath}|\hat{\jmath}}) = 0$ were little discussed in general, according to earlier works this case should include black strings.

Other way how to deal with impossibility to solve Einstein's equations in general case, was idea to guess some ansatz null-congruence ℓ, which would satisfy these equations and condition of geodecity. We restricted ourselves to the non-twisting case and we gave some instructions how to guess such ansatz. We were successful and we found one, which would satisfy condition of geodecity and optical constraint. We solved rest of Einstein's equations and determined function $\mathcal{H}$ in this special case. We could see that this corresponds to black strings and with some modifications to more general black p-branes. This is in agreement with previous analyzes, because our ansatz for ℓ was satisfying condition $\det(z_{\hat{\imath}|\hat{\jmath}}) = 0$ and thus black strings are solution which we would expect.

Then we decided to fix r-dependence and analyze asymptotical properties of Riemann tensor. We choosed affine parameter r along null vector ℓ. We had to make transformation of coordinates and we were able to find r-dependance of matrix $z_{\hat{\imath}|\hat{\jmath}}$ and $z_{\hat{\imath}|\hat{1}}$ expressed as expansions. With knowledge of this and using properties of commutators of operators of directional derivatives we could find r-dependance of operators of directional derivatives.

With knowledge of this we could find expansions of all components of Riemann tensor to arbitrary order in r. This result we obtained in generality and so it can be usefull in future calculations. Because of complexity of Riemann tensor we decided to ananalyze properties of Riemann tensor at infinity. We kept just leading terms in all components and we were able to find that all components of Riemann tensor are $O(r^{1-n})$ to the leading term in r. This result is in an agreement with four dimensions and its implication is that KS spacetimes does not contain gravitational radiation.

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Appendix A

Notation

Notation used here and in main references [27], [23], [39] are different from each other. Here we adopted notation which is mostly combination of all of them and is trying to be most instructive and clear. In following we give review of this notation and also notation used in [27], [23], [39]. We hope that it will help the reader to orientate faster here and in main references.

Metric, indices

Metric in a null frame herein is given by

$$g_{\hat{a}\hat{b}} = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & \ddots \end{pmatrix},$$

(where $\ldots$ represents diagonal matrix) while in [23] is difference in signs

$$g_{\hat{a}\hat{b}} = \begin{pmatrix} 0 & -1 & \\ -1 & 0 & \\ & & \ddots \end{pmatrix}.$$

Space-time indices herein are written in Latin lower case, same as in [39], while [27] and [23] are using Greek lower-case.

Frame or **abstract indices** are herein Latin lower case with hats. This notation is based on [43] and is very handful, because it makes clear difference between frame and spacetime indices. In [39] frame indices are written usually in same way as spacetime indices, just in case where authors want to stress difference are frame indices written in round brackets. Other references [27] and [23] are using Latin lower case for frame indices.

Beside this, reader should keep in mind that **ranges of indices**, according to place of index in the alphabet, are as follows

$a, b, \cdots$ - Latin lower case from beginning of alphabet runs from 0 to $n-1$.

$i, j, \cdots$ - Latin lower case from middle of alphabet runs from 2 to $n-1$.

$\alpha, \beta, \cdots$ - Greek lower case runs from 1 to $\lfloor \frac{n-2}{2} \rfloor$, where $\lfloor \cdots \rfloor$ represents integer part of given number.

Note, that these conventions about ranges of indices holds equally for frame and space-time indices.

Symbols used

In following we give list of symbols used in this thesis and in main references. Note, that some symbols used here do not have to have analogues in other references, this denoted as "$-$" in the table. Note, that we give list of corresponding symbols in main references, but they do not necessarily describe same quantity, but *corresponding quantity*. For example, functions z_i used herein are not same functions as $Y, \bar{Y}$ used in [27] in 4 dimensions, but they do play same role. Symbols are divided to categories for easier orientation.

Metric:

Here	[27]	[23]	[39]	Description
n	4	$N+1$	n	number of dimensions
η_{ab}	$\eta_{\mu\nu}$	$\eta_{\mu\nu}$	η_{ab}	Minkowski metric
g_{ab}	$g_{\mu\nu}$	$g_{\mu\nu}$	g_{ab}	spacetime metric
$g_{\hat{a}\hat{b}}$	g_{ab}	g_{ab}	g_{ab}	frame metric

Frame, frame derivatives:

$m_{\hat{a}}$	e_a	E_a	$m_{(a)}$	covariant frame vectors
$m^{\hat{a}}$	e^a	E^a	$m^{(a)}$	contra-variant frame vectors
n	e^3	–	n	special symbol for $m^{\hat{0}}$
ℓ	e^4	–	ℓ	special symbol for $m^{\hat{1}}$
k^a	k^μ	k^μ	k^a	Kerr-Schild null-vector, $\ell = k$
$z_i, z_{\hat{i}}$	$Y, \bar{Y}$	A^k	–	functions determining Kerr-Schild vector
$\mathcal{H}$	h	H	$\mathcal{H}$	Kerr-Schild function
$\delta_{\hat{a}}$	–	δ_a	δ_a	directional derivative along a-th frame vector
$_{\|\hat{a}}$	$_{,a}$	–	–	short notation of directional derivative
D	–	D	D	directional derivative along 0-th frame vector
Δ	–	Δ	Δ	directional derivative along 1-th frame vector

Connection:

$\Gamma^{\hat{a}}{}_{\hat{b}\hat{c}}$	$\Gamma^a{}_{bc}$	–	–	Ricci rotation coefficients
$N_{\hat{i}\hat{j}}$	–	–	N_{ij}	$\Gamma^{\hat{0}}{}_{\hat{i}\hat{j}}$ coefficient
$L_{\hat{i}\hat{j}}$	–	–	L_{ij}	$\Gamma^{\hat{1}}{}_{\hat{i}\hat{j}}$ coefficient
$\overset{\hat{i}}{M}{}_{\hat{j}\hat{k}}$	–	–	$\overset{i}{M}{}_{jk}$	$\Gamma^{\hat{i}}{}_{\hat{j}\hat{k}}$ coefficient
$S_{\hat{i}\hat{j}}$	–	–	S_{ij}	symmetric part of $L_{\hat{i}\hat{j}}$
$A_{\hat{i}\hat{j}}$	–	–	A_{ij}	anti-symmetric part of $L_{\hat{i}\hat{j}}$, twist matrix
$\sigma_{\hat{i}\hat{j}}$	–		σ_{ij}	shear matrix
θ	θ	–	θ	expansion scalar
σ	0	–	σ	shear scalar
ω	ω	–	ω	twist scalar
$\Gamma_{\hat{a}\hat{b}}$	Γ_{ab}	–	–	connection form

Curvature:

$\mathcal{R}_{\hat{a}\hat{b}}$	$\mathcal{R}_{\hat{a}\hat{b}}$			curvature form
$R_{\hat{a}\hat{b}\hat{c}\hat{d}}$	R_{abcd}	R_{abcd}	R_{abcd}	frame components of curvature tensor
$C_{\hat{a}\hat{b}\hat{c}\hat{d}}$	–	C_{abcd}	C_{abcd}	frame components of Weyl tensor
$R_{\hat{a}\hat{b}}$	R_{ab}	R_{ab}	R_{ab}	frame components of Ricci tensor

Other:

r	r	r	r	affine parameter, "radius"
r_p	$-$	$-$	$-$	affine parameter in $n-p$ space dimensions. See sec. 4.5.
$d\Omega_n$	$-$		$-$	angular part of spherical metric in n dimenions
λ	$-$	$-$	$-$	boost weight, see sec. 2.2.
X_i^j	$-$	$-$	$-$	Spin matrix, $SO(n-2)$ see sec. 2.2.