Evoluční rovnice
Evolution equations

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1 Introduction

1.1 What is an evolution equation?

The theory of evolution equations can be considered as a part of the vast area called *applied mathematics*. The most fundamental idea behind the theory of evolution equations is the *principle of determinism*:

The time evolution of a given (physical, biological, chemical,...) system is completely determined by its initial state.

What we understand by *state of a system* is to be determined in accordance with our best knowledge of the part of "real life" we want to examine. In most of the cases we shall deal with, the state $U = U(t)$ of a given system will belong to an abstract space X, and the corresponding evolution equation takes the form

$$\frac{\mathrm{d}}{\mathrm{d}t} U(t) = \mathcal{A}(t, U(t)), \tag{1.1}$$

where $\mathcal{A}(t, \cdot) : X \to X$ is a mapping of the *state space* X into itself.

In order to define the time derivative of U, we need both an *algebraic* and *topological* structure to be given on X. Typically, X will be a Banach or even more general topological vector space. In the simplest case when the dimension of X is finite, the abstract equation (1.1) reduces to a system of ordinary differential equations. Here, we focus on infinite dimensional state spaces X, where $\mathcal{A}$ may be a general differential operator and, accordingly, (1.1) represents one or several partial differential equations.

1.2 Basic principles

In accordance with the principle of determinism stated above, any "reasonable" evolution equation should comply with the following three principles:

Existence of solutions. Given a time interval $[0, T)$ (the initial time $t = 0$ is only for convenience), the equation (1.1) should possesses a solution for any given admissible *initial state* U_0 defined for all $t \in [0, T)$. More precisely, U as a function of the *time* t should be continuous in some sense, and $U(0) = U_0$. The set of admissible states U_0 is typically a proper subspace (or only subset) of X. Obviously, the model cannot be good if the solution does not exist.

Uniqueness. For any given initial state U_0, there exists at most one solution U of (1.1) on $[0,T)$ satisfying $U(0) = U_0$. This is, in fact, a mathematical formulation of the principle of determinism. Although it seems to be the real corner stone of any "sensible" mathematical theory, there are many equations which have been studied for many years and for which uniqueness is still an open problem.

Stability. Stability or *correctness in the sense of Hadamard* means that for given initial data U_0^1, U_0^2 which are close in some sense, the corresponding solutions stay close at least on a given compact time interval. If a given system does not satisfy the principle of stability, it is of no use for applications even if the solutions were uniquely determined by their initial state. In practice, the initial state is never known with an absolute accuracy as the physical (or other) measurements are always influenced be errors given the device we have at hand.

The above principles are of theoretical nature, and we shall examine evolution equations from their point of view. As we shall see, these quations are largely open for many equations arising in rather different areas of applications.

1.3 Classical classification

In many text books on partial differential equations, the following three examples are presented as model problems:

the Poisson equation:

$$\Delta u = f, \text{ where } \Delta \equiv \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} \text{ is the Laplace operator;} \tag{1.2}$$

the heat equation:

$$\frac{\partial}{\partial t} u - \Delta u = 0; \tag{1.3}$$

the wave equation:

$$\frac{\partial^2}{\partial t^2} u - \Delta u = 0. \tag{1.4}$$

The Poisson equation is taken as a prototype of *elliptic* equations, the heat equation is *parabolic*, and the wave equation *hyperbolic*. The terminology originates from the so-called symbolic calculus. Taking the Fourier transform of (1.2) - (1.4) respectively we get

$$\sum_{i=1}^{N} \xi_i^2 \mathcal{F}_{x\to\xi}[u] + \mathcal{F}_{x\to\xi}[f] = 0,$$

$$\Big(\mathrm{i}\xi_0 + \sum_{i=1}^{N} \xi_i^2\Big)\mathcal{F}_{(t,x)\to\xi}[u] = 0,$$

$$\Big(\xi_0^2 - \sum_{i=1}^{N} \xi_i^2\Big) \mathcal{F}_{(t,x)\to\xi}[u] = 0.$$

From the mathematical point of view, the properties of these equations are rather different and very often studied separately. Only (1.3), (1.4) represent evolution equations in the spirit of the philosophy delineated above.

1.4 Linear parabolic equations

It can be shown by means of the Fourier transform that any function of the form

$$u(t,x) = \frac{1}{(4\pi t)^{\frac{N}{2}}} \int_{R^N} \exp\Big(\frac{-|x-y|^2}{4t}\Big) u_0(y)\ \mathrm{d}y \tag{1.5}$$

is a solution of the equation (1.3) satisfying the initial condition $u(0,x) = u_0(x)$. There several important consequences of formula (1.5) which are, in fact, characteristic for parabolic like equations.

- **Smoothing effect.** The solution is smooth (differentiable) for any $t > 0$ even if u_0 is only a bounded measurable function. Regularity of solutions improves with time.

- **Maximum principle.** Assuming that u_0 is a bounded continuous function we have
$$\sup_{t>0, x\in R^N} u(t,x) \leq \sup_{x \in R^N} u_0(x).$$

- **Decay to zero for integrable data.** Suppose $u_0 \in L^1(R)$, then
$$\sup_{x\in R^N} u(t,x) \to 0 \text{ as } t \to \infty.$$

- **Uniqueness.** For any initial datum u_0 which is bounded and continuous, there exists a unique solution of (1.3) in the *same class.*

As for the *maximum principle*, there are several ways how to deduce it. Another possibility is to consider a function

$$v = e^{-\lambda t} u \text{ where } u \text{ solves (1.3).}$$

It is easy to check that v is a solution of the problem

$$\frac{\partial}{\partial t} v - \Delta v + \lambda v = 0,\ \ v(0,x) = u_0(x). \tag{1.6}$$

Now, assume there are $t_0 > 0$ and $x_0 \in R$ such that

$$v(t_0, x_0) = \sup_{t\in[0,t_0], x\in R^N} v(t,x) > \sup_{x\in R^N} v_0(x) = \sup_{x\in R^N} u_0(x) \geq 0.$$

Thus we have $u_t(t_0, x_0) \geq 0$, $\Delta v(t_0, x_0) \leq 0$, and $v(t_0, x_0) > 0$ which *is not compatible* with (1.6).

Yet another way to see the maximum principle is to use *test functions* of the form $b'(u)$ where b is a non-negative, convex function, $b(0) = 0$. Multiplying (1.3) by $b'(u)$ and integrating by parts, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{R^N} b(u)(t) \, \mathrm{d}x \leq 0.$$

Taking $b = [u - M]^+$ for a suitable M yields the desired result. The method of test functions is very useful in applications. In particular, it works for more general equations which may be even non-linear in u.

1.5 Hyperbolic equations

Consider the wave equation (1.4) for $N = 1$. One checks easily that

$$u = w(t + x), \;\; u = z(t - x)$$

are both solutions of (1.4) provided the functions w and z are smooth enough.

Consequently, it is easy to construct a counterexample to see that the maximum principle *does not hold* for hyperbolic equations.

Moreover, it is also clear that there is no smoothing effect, i.e., regularity of solutions is the same as that of the initial data.

1.6 Non-linear equations

In this course, we shall mostly discuss non-linear equations. Like (1.2) - (1.4) are considered as archetypes of linear equations, there are three model examples of *non-linear equations*:

the Burgers (transport) equation:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0; \tag{1.7}$$

the Korteweg-de Vries equation:

$$\frac{\partial u}{\partial t} - 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0; \tag{1.8}$$

the Navier-Stokes system of equations:

$$\frac{\partial u^i}{\partial t} - \nu \Delta u^i + \sum_{j=1}^{N} \frac{\partial u^i}{\partial x_j} u^j = \frac{\partial p}{\partial x_i} + f^i(x), \;\; i = 1, ..., N, \tag{1.9}$$

$$\sum_{j=1}^{N} \frac{\partial u^j}{\partial x_j} = 0. \tag{1.10}$$

The Burgers equation (1.7) is the simples example of a single non-linear conservation law whose solutions develop *shock waves*, i.e., discontinuities in solutions after a finite time no matter how smooth the initial data are. This phenomenon prevents us from using *classical concepts* of differentiability as appropriate when dealing with hyperbolic problems. Well-posedness of hyperbolic equations represents a very interesting issue which led finally to new concepts of so-called *entropy solutions* and many other interesting ideas in the theory of hyperbolic partial differential equations.

The Korteweg-de Vries equation (1.8) admits *soliton solutions* - a concept which some people consider to one of the major achievements of non-linear sience of the last (20th) century. Solitons are spatially localized stable objects which, after mutual interaction, remain basically of the same form. The soliton theory led finally to the discovery of the so-called *completely integrable* non-linear equations studied by the method of *inverse scattering* where an analogue of the (linear) Fourier transorm is introduced to obtain analytic solutions of a given (non-linear) equation.

Finally, the Navier-Stokes equations are considered as a prototype of a system of partial differential equations with *chaotic behaviour* though exact analytic results seem to be still in a short supply. Chaos, in general, means exponential in time separation of two different solutions resulting in non-stable long-time behaviour very often related to the phenomenon of turbulence. In particular, the future state of a "chaotic system" is very hard to be determined from the knowledge of the initial data. This popularly well-known as the "butterfly effect".

1.7 Shock waves

Consider the equation (1.7). After a short inspection, one observes that any (smooth) solutions emanating from the initial point $u_0(x_0)$ is constant along the so-called *characteristic line*

$$x = x_0 + u_0(x_0)t, \ t \geq 0.$$

Consequently, a solution u satisfying the initial condition

$$u(0, x) = u_0(x), \ x \in R$$

is given through

$$u(t, x) = u_0(x - tu(t, x)), \ x \in R, \ t \geq 0.$$

or

$$u(t, x + tu_0(x)) = u_0(x). \tag{1.11}$$

Differentiating (1.11) with respect to x, we get

$$u_x(t, x + tu_0(x)) = \frac{(u_0)_x(x)}{1 + t(u_0)_x(x)}.$$

If $(u_0)_x$ is negative for some $x \in R$ then the solution *cannot be differentiable* for *any* $t > 0$.

1.8 Solitons

A simple form of a solitary wave solution for the Korteweg-deVries equation (1.9) can be obtained in the form

$$u(t,x) = U(x - ct).$$

Substituting into (1.9) we get a non-linear ordinary differential equation which yields

$$U = U(z) = -\frac{1}{2}c\ \mathrm{sech}^2\Big(\frac{1}{2}\sqrt{c}(z - z_0)\Big).$$

The solution of this form is located in space and moves with a speed c.

1.9 Chaos

Chaotic behaviour, very often attributed to solutions of the Navier-Stokes system (1.9), (1.10) has never been verified rigorously. However, there are other partial differential equations, for instance, the sin-Gordon or beam equation, where exact results are available. To understand what is going on, the best example is furnished by finite-dimensional dynamical systems.

Let z be a stationary point, i.e., a solution of the evolution problem which is *independent of* time. Let $\omega > 0$ be given. We define the *stable manifold*

$$W_s(z) = \{U \mid U(n\omega) \to z \text{ as } n \to \infty\},$$

and the *unstable manifold*,

$$W_u(z) = \{U \mid U(n\omega) \to z \text{ as } n \to -\infty\}.$$

The simplest example of a complex behaviour of a dynamical system is a transversal intersection of the stable and unstable manifold of a single equilibrium point. If this is the case, using the continuity and uniqueness arguments, we obtain a dense set of homoclinic orbits allowing a dense system of homoclinic orbits. If there are two equilibria, the phenomena are even more complex, there are trajectories spending an arbitrarily long time in a neighbourhood of each equilibria etc.

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2 Semigroups theories - advantages and limits

The concept of *semigroup* is a rigorous statement of what has been vaguely said above concerning correctness of a problem related to an evolution equation. A *semigroup* on a Banach space X is a family of (non-linear) mappings

$$S[t] : X \to X \text{ for any } t \geq 0$$

having the following properties:

- $$S[0](x) = x \text{ for all } x \in X;$$

- $$S[t+s](x) = S[t]\Big(S[s](x)\Big) \text{ for all } x \in X,\ t,s \geq 0;$$

- the vector valued function $t \mapsto S[t](x)$ is continuous for any fixed $x \in X$;

- the mapping $x \mapsto S[t](x)$ is continuous on X for any fixed $t \geq 0$.

As an example, consider a semilinear parabolic equation

$$\frac{\partial}{\partial t}u - \Delta u = f(u) \text{ on } (0,T) \times \Omega \tag{2.1}$$

supplemented by the Dirichlet boundary conditions

$$u|_{\partial\Omega} = 0. \tag{2.2}$$

We shall assume that $\Omega \subset R^N$ is a bounded regular domain and $f : R \to R$ a globally Lipschitz consitnuous function.

Let u_1, u_2 be two solutions of (2.1), (2.2). Multiplying the difference of the corresponding equations by $u_1 - u_2$ and integrating by parts, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u_1 - u_2|^2 \, \mathrm{d}x + \int_\Omega |\nabla(u_1 - u_2)|^2 \, \mathrm{d}x = \int_\Omega \Big(f(u_1) - f(u_2)\Big)(u_1 - u_2) \, \mathrm{d}x.$$

Since f is Lipschitz continuous, we get

$$\int_\Omega \Big(f(u_1) - f(u_2)\Big)(u_1 - u_2) \, \mathrm{d}x \leq L \int_\Omega |u_1 - u_2|^2 \, \mathrm{d}x,$$

and, by virtue of the Gronwall lemma,

$$\int_\Omega |(u_1 - u_2)(t)|^2 \, \mathrm{d}x \leq \exp(2Lt) \int_\Omega |(u_1 - u_2)(0)|^2 \, \mathrm{d}x, \ t \geq 0. \tag{2.3}$$

In other words, the problem (2.1), (2.2), *provided it is solvable*, generates a Lipschitz semigroup on the Hilbert space $L^2(\Omega)$.

Similar examples may be obtained even for quasilinear equation in the so-called divergence form

$$\frac{\partial}{\partial t}u - \sum_{i=1}^N \frac{\partial}{\partial x_i} F_i(\nabla u) = f(u)$$

provided $F : R^N \to R^N$ is *monotone*, i.e.,

$$\sum_{i=1}^N \Big(F_i(w_1, ..., w_N) - F_i(v_1, ..., v_N)\Big)(w_i - v_i) \geq 0. \tag{2.4}$$

A very popular example of such an operator is the p-Laplacean

$$F_i(\nabla u) = |\nabla u|^{p-2} \frac{\partial}{\partial x_i} u, \ i = 1, ..., N.$$

The next example is a scalar conservation law of the form

$$\frac{\partial}{\partial t}u + \frac{\partial}{\partial x} f(u) = 0, \tag{2.5}$$

where f is an increasing function.

Now, let u_1, u_2 be two solutions of (2.5). We multiply the difference of the corresponding equations by the expression

$$\mathrm{sgn}(u_1 - u_2) = \mathrm{sgn}(f(u_1) - f(u_2))$$

to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_R |u_1 - u_2| \, \mathrm{d}x + \int_R \frac{\partial}{\partial x}|f(u_1) - f(u_2)| \, \mathrm{d}x = 0.$$

Assuming that both u_1 and u_2 vanish for large arguments, we get

$$\int_R |(u_1 - u_2)(t)| \, \mathrm{d}x \leq \int_R |(u_1 - u_2)(0)| \, \mathrm{d}x \text{ for any } t \geq 0. \tag{2.6}$$

Similarly as above, we can conclude that the problem (2.5), *provided it is solvable*, generates a Lipschitz semigroup on the space $L^1(R)$.

Remark: The equation (2.5) generates a Lipschitz continuous semigroup on a non-reflexive space $L^1(R)$. In fact, if the space was reflexive, we would get the existence of the spatial derivative in the same space which is impossible as the solutions can be discontinuous.

2.1 Accretive operators

Consider and abstract problem

$$\frac{\mathrm{d}}{\mathrm{d}t}U(t) + A[U(t)] = 0,\ U(0) = U_0 \in X, \tag{2.7}$$

where X is a Banach space. Trying to solve (2.7) by an *implicit scheme* we get

$$U(t+h) + hA[U(t+h)] = U(t),\ U(0) = U_0,\ h > 0.$$

To make this work, we need

- the equation $v + \lambda A[v] = f$ to be solvable for any $f \in X$ and $\lambda > 0$;
- the inverse $(\mathrm{Id} + \lambda A)^{-1}$ to be a contraction.

This motivates the following definition.

Let $A : \mathcal{D}(A) \subset X \to X$ be a densely defined (nonlinear) mapping. We shall say that A is accretive if

$$\|z_1 + \lambda A[z_1] - \Big(z_2 + \lambda A[z_2]\Big)\|_X \geq \|z_1 - z_2\|_X \textit{ for any } z_1, z_2 \in \mathcal{D}(A),$$

and any $\lambda > 0$.
A is m-accretive if it is accretive and

$$\mathcal{R}(\mathrm{Id} + \lambda A) = X.$$

The solutions of the problem (2.7) will be constructed by means of the implicit time-discretization. This will be done in several steps.

(i) Consider first the equidistant partition and define

$$U_\lambda^n = (\mathrm{Id} + \lambda A)^{-1} U_\lambda^{n-1},\ \lambda = \frac{T}{N}, n = 1, ..., N.$$

Equivalently, we get

$$U_\lambda^n + \lambda A[U_\lambda^n] = U_\lambda^{n-1},$$

i.e.,

$$U_\lambda^n + \lambda A[U_\lambda^n] - U_\lambda^{n-1} - \lambda A[U_\lambda^{n-1}] = -\lambda A[U_\lambda^{n-1}] = U_\lambda^{n-1} - U_\lambda^{n-2}$$

By accretivity

$$\|U_\lambda^n - U_\lambda^{n-1}\|_X \leq \|U_\lambda^{n-1} - U_\lambda^{n-2}\|_X \leq ... \leq \lambda \|A[U_0]\|_X, \tag{2.8}$$

and

$$\|U_\lambda^n - U_0\|_X \leq n\lambda \|A[U_0]\|_X. \tag{2.9}$$

(ii) Now, we compare two sequences: U_λ^n defined as above with

$$U_\tau^m = (\mathrm{Id} + \tau A)^{-m}[U_0],\ 1 \leq m \leq M,\ \tau = \frac{T}{M},$$

where $M > N$.

We have

$$U_n^\lambda - U_m^\tau = U_{n-1}^\lambda - U_{m-1}^\tau - \lambda A[U_n^\lambda] + \tau A[U_m^\tau] =$$
$$U_{n-1}^\lambda - U_{m-1}^\tau - \tau(A[U_n^\lambda] - A[U_m^\tau]) - (\lambda - \tau)A[U_n^\lambda],$$

where

$$(\lambda - \tau)A[U_n^\lambda] = \frac{\lambda - \tau}{\lambda}(U_{n-1}^\lambda - U_n^\lambda).$$

Thus

$$U_n^\lambda - U_m^\tau + \tau(A[U_n^\lambda] - A[U_m^\tau]) = \theta(U_{n-1}^\lambda - U_{m-1}^\tau) + (1-\theta)(U_n^\lambda - U_{m-1}^\tau,$$

where $\theta = \frac{\tau}{\lambda} \in (0,1)$; whence

$$\|U_n^\lambda - U_m^\tau\|_X \leq \theta \|U_{n-1}^\lambda - U_{m-1}^\tau\|_X + (1-\theta)\|U_n^\lambda - U_{m-1}^\tau\|_X, \tag{2.10}$$
$$\theta = \frac{\tau}{\lambda},\ 1 \leq n \leq N,\ 1 \leq m \leq M.$$

(iii) Denoting

$$a_{n,m} = \|U_b^\lambda - U_m^\tau\|_X$$

we deduce from (2.9), (2.10) that

$$a_{n,0} \leq n\lambda \|A[U_0]\|_X,\ a_{0,m} \leq m\tau \|A[U_0]\|_X,$$

and

$$a_{n,m} \leq \theta a_{n-1,m-1} + (1-\theta)a_{n,m-1}$$

from which we deduce

$$\|U_n^\lambda - U_m^\tau\|_X \leq \Big(m\tau(\lambda - \tau) + (n\lambda - m\tau)^2\Big)^{\frac{1}{2}} \|A[U_0]\|_X. \tag{2.11}$$

(iv) Now, let

$$t = n\lambda + \xi = m\tau + \eta,\ 0 \le \xi < \lambda,\ 0 \le \eta < \tau.$$

We set

$$U_N(t) = U_n^\lambda + (U_{n+1}^\lambda - U_n^\lambda)\frac{\xi}{\lambda},\ U_M(t) = U_m^\tau + (U_{m+1}^\tau - U_m^\tau)\frac{\eta}{\tau}. \tag{2.12}$$

Consequently,

$$\|U_N(t) - U_M(t)\|_X \le \|U_n^\lambda - U_m^\tau\|_X + \|U_{n+1}^\lambda - U_n^\lambda\|_X + \|U_{m+1}^\tau - U_m^\tau\|_X \le$$

$$\left(T(\lambda-\tau) + |\xi-\eta|^2\right)^{\frac{1}{2}}\|A[U_0]\|_X + (\lambda+\tau)\|A[U_0]\|_X.$$

Thus we have

$$U_N(t) \to U(t) \text{ in } X \text{ uniformly with respect to } t \in [0,T].$$

The function U is a continuous function on $[0,T]$ satisfying $U(0) = U_0$.

We have proved the following theorem:

Theorem 2.1 [Crandall-Liggett] *Let X be a Banach space and $A : \mathcal{D}(A) \subset X \to X$ a densely defined m-accretive operator. Let $U_0 \in \mathcal{D}(A)$, and let $U_N = U_N[U_0] \in C([0,T];X)$ be the sequence of functions defined through (2.12).*
Then there is a unique function $U = U[U_0] \in C([0,T];X)$ such that

$$U_N(t) \to U(t) \text{ in } X \text{ uniformly with respect to } t \in [0,T].$$

2.2 A solution semigroup

Motivated by Theorem 2.1, we set

$$S[t](U_0) = U(t), \text{ where } U \text{ is the function obtained in Theorem 2.1.} \tag{2.13}$$

We are going to verify that $\{S\}_t \ge 0$ defines a semigroup on X. First of all consider the difference

$$S[t](U^1) - S[t](U^2) = \lim_{N\to\infty} U_N[U^1](t) - U_N[U^2](t)$$

where

$$U_N[U^1](t) - U_N[U^2](t) = (\mathrm{Id} + \lambda A)^{-n}[U_1] - (\mathrm{Id} + \lambda A)^{-n}[U_2] \text{ provided } t = n\lambda,\ \lambda = \frac{T}{N}$$

for certain N and $n \in \{0, ..., N\}$.

Since A is accretive, we get in this case

$$\|S[t](U^1) - S[t](U^2)\|_X \leq \|U^1 - U^2\|_X \text{ in } t = \frac{Tn}{N}.$$

As the times t in the above form are dense in $[0, T]$ and U is continuous we get

$$\|S[t](U^1) - S[t](U_2)\|_X \leq \|U^1 - U^2\|_X \text{ for any } U^1, U^2 \in \mathcal{D}(A) \text{ and any } t \in [0, T].$$

Thus $S[t]$ can be extended in a unique way to be a Lipschitz mapping defined on the whole X for any fixed $t \in [0, T]$. The rest of the semigroup properties is verified analogously.

The function

$$U(t) = S[t](U_0),\ t \geq 0,\ U_0 \in X,$$

with

$$S[t] = \lim_{N \to \infty} U_N[U_0](t) \textit{ for } U_0 \in \mathcal{D}(A),$$

where the functions U_N are uniquely determined through Theorem 2.1, is termed a **mild solution** *of the problem (2.7).*

2.3 Differentiability

It follows from (2.7) that the functions U_N defined in Theorem 2.1 are uniformly Lipschitz continuous with a Lipschitz constant not exceeding $\|A[U_0]\|_X$ provided $U_0 \in \mathcal{D}(A)$. Consequently, the mild solution $U(t)$ is Lipschitz continuous with the same constant. The following theorem holds:

Proposition 2.1 (Brezis [Appendice]) *Let X be a reflexive Banach space and let $U \in C([0, T]; X)$ be Lipschitz continuous.*

Then U is differentiable a.a. on $[0,T]$, the derivative $\mathrm{d}U/\mathrm{dt}$ *is a bounded measurable function, and*

$$U(t) = \int_0^t \frac{\mathrm{d}U}{\mathrm{d}t}(\tau)\ \mathrm{d}\tau \textit{ for any } t \in [0, T].$$

2.4 Strong solutions

Our goal is to show that the mild solutions constructed above represent strong solutions of the problem (2.7) provided $U_0 \in \mathcal{D}(A)$. As we have already seen in the example of the Burgers equation, this cannot be true in general. Thus we shall make the following additional hypotheses:

- The space X is a separable Hilbert space, $\mathcal{D}(A) = V$ is another Hilbert space continuously embedded into X.
- A set B is bounded in V if and only if $\sup_{u \in B}(\|u\|_X + \|A[u]\|_X)$ is bounded.

- The mapping $t \mapsto A[u + t\phi]$ is continuous for any $u,\ \phi \in \mathcal{D}(A)$.

Together with U_N, let us define a sequence V_N of piecewise constant functions,

$$V_N(t) = U_\lambda^{n+1} \text{ for } t = n\lambda + \xi,\ \xi \in [0, \lambda),\ \lambda = \frac{T}{N}.$$

We have

$$\frac{\mathrm{d}U_N(t)}{\mathrm{d}t} = \frac{U_\lambda^{n+1} - U_\lambda^n}{\lambda} = -A[U_\lambda^{n+1}] = -A[V_N(t)] \text{ for any } t = n\lambda + \xi,\ \xi \in (0, \lambda).$$

As

$$U_N(t) - V_N(t) = (U_\lambda^{n+1} - U_\lambda^n)\frac{\xi}{\tau},\ t = n\lambda + \xi$$

and (2.8) holds, we have

$$V_N(t) \to U(t) \text{ in } X \text{ uniformly in } t \in [0, T].$$

Since the time derivative

$$\frac{\mathrm{d}U_N}{\mathrm{d}t} \text{ is bounded in the space } L^\infty(0, T; X)$$

we can pass to the limit for $N \to \infty$ to obtain

$$\frac{\mathrm{d}U_N}{\mathrm{d}t} \to \frac{\mathrm{d}U}{\mathrm{d}t} \text{ weakly star in } L^\infty(0, T; X). \tag{2.14}$$

Consequently the sequence of functions $A[V_N]$ is bounded in $L^\infty(0, T; X)$; whence, by virtue of our hypotheses in V_N is bounded in $L^\infty(0, T; V)$. Thus we have

$$V_N \to U \text{ in } L^\infty(0, T; V), \tag{2.15}$$

in particular, $U(t) \in \mathcal{D}(A)$ for a.a. $t \in [0, T]$ and

$$A[U] \in L^\infty(0, T; X). \tag{2.16}$$

As $U \in C([0, T]; X)$ and the norm in V is weakly lower semicontinuous, we get, in fact

$$U(t) \in \mathcal{D}(A) \text{ for all } t \in [0, T]. \tag{2.17}$$

Finally, we get

$$A[V_N] \to \overline{A[U]} \text{ weakly star in } L^\infty(0, T; X),$$

and our ultimate task is to show

$$A[U](t) = \overline{A[U]}(t) \text{ for a.a. } t \in [0, T].$$

To this end, we use the so-called Minty's trick. Since A is m-accretive on X, i.e., monotone, we have

$$\int_0^T < A[V_N] - A[\psi], V_N - \psi >_X \ \mathrm{d}t \geq 0 \text{ for any } \psi \in L^2(0,T;V).$$

We compute the limit

$$\lim_{N\to\infty} \int_0^T < A[V_N], V_N >_X \ \mathrm{d}t = \lim_{N\to\infty} \int_0^T < A[V_N], U_N >_X \ \mathrm{d}t =$$

$$- \lim_{N\to\infty} < \frac{\mathrm{d}U_N}{\mathrm{d}t}, U_N > \ \mathrm{d}t = \|U_0\|_X^2 - \lim_{N\to\infty} \|U_N(T)\|_X^2 =$$

$$= \|U_0\|_X^2 - \|U(T)\|_X^2 = \int_0^T < \overline{A[U]}, U >_X \ \mathrm{d}t.$$

Thus we conclude,

$$\int_0^T < \overline{A[U]} - A[\psi], U - \psi >_X \ \mathrm{d}t \geq 0 \tag{2.18}$$

from which we deduce, taking $\psi = U + \epsilon\phi$,

$$\overline{A[U]}(t) = A[U](t) \text{ for a.a. } t \in [0,T]. \tag{2.19}$$

We have proved the following theorem.

Theorem 2.2 *Let $V \subset X$ be a separable Hilbert spaces, and $A : V \to X$ an m-accretive (monotone) mapping such that bounded sets in V are characterized by boundedness of the "graph norm"*

$$\|v\|_X + \|A[v]\|_X.$$

Let, moreover, the function $t \in [0,T] \mapsto A[U + t\phi] \in X$ is continuous for any U, $\phi \in \mathcal{D}(A)$.

Then for any $U_0 \in \mathcal{D}(A)$, the problem

$$\frac{\mathrm{d}U}{\mathrm{d}t} + A[U] = 0, \ U(0) = U_0 \tag{2.20}$$

admits a unique strong solution, specifically, the function U is Lipschitz continuous on $[0,T]$, $U(t) \in \mathcal{D}(A)$ for any $t \in [0,T]$, $U \in C([0,T];X)$, and (2.20) holds for a.a. $t \in [0,T]$.

2.5 A non-linear hypebolic equation

As another application od the abstract theory developed above, we consider a scalar conservation law of the form:

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0 \tag{2.21}$$

where the unknown function $u = u(t,x)$, $t \in [0,T]$, $x \in R$ is spatially periodic, i.e., there is $\omega > 0$ such that

$$u(t, x+\omega) = u(t, x) \text{ for all } t, x. \tag{2.22}$$

We impose the initial conditions

$$u(0, x) = u_0(x), \ \ x \in R. \tag{2.23}$$

The associated function space will be a Banach space

$$X = L^1_\omega(R) \approx L^1(0, \omega),$$

and we set

$$\mathcal{D}(A) = \{v \mid v \text{ continuous and } \omega\text{−periodic on } R, \ f(v) \text{ absolutely continuous}\},$$

$$A[v] = \frac{\mathrm{d}}{\mathrm{d}x} f(v).$$

Lemma 2.1 *Let $f : R \to R$ be a continuously differentiable, strictly increasing function;*

$$f(0) = 0, \ \ f(R) = R.$$

Then the mapping A defined above is m-accretive on $X = L^1_\omega(R)$.

Proof:

Consider the difference

$$v + \lambda A[v] - w - \lambda A[w].$$

We divide the interval $[0, \omega]$ into three parts:

$$\left\{ \begin{array}{l} I_- = \{x \in [0, \omega] \mid v(x) < w(x)\}, \\ \\ I_0 = \{x \in [0, \omega] \mid v(x) = w(x)\}, \\ \\ I_+ = \{x \in [0, \omega] \mid v(x) > w(x)\}. \end{array} \right\}$$

Obviously,

$$\|v - w\|_{L^1_\omega(R)} = \int_{I_-} w(x) - v(x) \ \mathrm{d}x + \int_{I_+} v(x) - w(x) \ \mathrm{d}x.$$

On the other hand, we have

$$\int_{I_-} |v(x) + \lambda A[v](x) - w(x) - \lambda A[w](x)| \ \mathrm{d}x \geq \int_{I_-} w(x) - v(x) + \lambda (A[v](x) - A[w](x)) \ \mathrm{d}x =$$

$$\int_{I_-} |v(x) - w(x)| \ \mathrm{d}x + \lambda \sum_{j \in J} \int_{J^j_-} A[v](x) - A[w](x) \ \mathrm{d}x,$$

where $\{J^j_-\}_{j \in J}$ is at most countable system of open interval on which $w(x) > v(x)$.

The are two posibilities: either $\overline{J_-^j} \subset (0,\omega)$ in which case, necessarily,

$$\int_{J_-^j} A[w](x) - A[v](x)\ \mathrm{d}x = \int_{J_-^j} \frac{\mathrm{d}}{\mathrm{d}x}(f(w) - f(v))\ \mathrm{d}x = 0$$

as $v = w$ at the end points of J_-^j; or there two intervals, say, J_-^1, J_2^- such that

$$0 \in \overline{J_-^1},\ \ \omega \in \overline{J_-^2},$$

and, since the functions are $\omega-$periodic

$$\int_{J_-^1 \cup J_-^2} A[w] - A[v]\ \mathrm{d}x = \int_{J_-^1 \cup (J_-^2 \oplus \omega)} \frac{\mathrm{d}}{\mathrm{d}x}(f(w) - f(v))\ \mathrm{d}x = 0.$$

Thus we have proved that A is accretive on X.

Finally, we have to show that the image of $\mathcal{D}(A)$ under the mapping $\mathrm{Id} + \lambda A$ is the whole space X. Let $g \in L^1_\omega$. We have to solve the problem

$$\frac{\mathrm{d}}{\mathrm{d}x} f(v) + \mu v = G,\ \ v(0) = v(\omega), \tag{2.24}$$

where we have set $\mu = \frac{1}{\lambda}$, $G = \frac{1}{\lambda} g$.

We consider an approximate problem

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big(f(v) + \epsilon v\Big) + \mu v = G_\delta,$$

where G_δ is continuous and $\epsilon, \delta > 0$.

As f is continuous, strictly increasing, and $f(R) = R$, corresponding initial value problem

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big(f(v) + \epsilon v\Big) + \mu v = G_\delta,\ \ v(0) = v_0 \in R$$

possesses a unique solution defined on $[0, T]$.

Moreover, we have

$$\frac{\mathrm{d}v(x)}{\mathrm{d}x} < 0 \text{ whenever } v(x) > \frac{1}{m} \sup_{y \in [0,\omega]} G_\delta(y)$$

and, similarly,

$$\frac{\mathrm{d}v(x)}{\mathrm{d}x} > 0 \text{ whenever } v(x) < \frac{1}{\mu} \inf_{y \in [0,\omega]} G_\delta(y).$$

Consequently, there exists an $M > 0$ such that the period map

$$\Pi : v_0 \mapsto v(\omega) \text{ maps } [-M, M] \text{ into itself.}$$

Thus this map possesses a fixed point - a solution of the periodic problem.

Letting $\epsilon \to 0$ we get that the corresponding periodic solutions v_ϵ are uniformly bounded, and, consequently,

$$f(v_\epsilon) + \epsilon v_\epsilon \text{ uniformly Lipschitz continuous.}$$

By virtue of the Arzela-Ascoli theorem, we can pass to a subsequence such that

$$v_\epsilon \to v,\ f(v_\epsilon) + \epsilon v_\epsilon \to f(v) \text{ uniformly on } [0, \omega],$$

where the limit v is a continuous, $\omega-$periodic function solving the problem (2.24) for $G = G_\delta$.

Finally, we take a sequence

$$G_\delta \to G \text{ in } L^1_\omega(R) \text{ as } \delta \to 0.$$

Since we have already shown that A is m-accretive, we get for the corresponding solutions

$$v_\delta \to v \text{ in } L^1_\Omega.$$

We deduce from the equation that $f(v_\delta)$ are uniformly absolutely continuous, in paricular, they are uniformly bounded. Thus, by virtue of the Lebesgue theorem,

$$f(v_\delta) \to f(v) \text{ in } L^1_\omega(R),$$

and $f(v)$ solves (2.24) in the sense of distributions.

Thus $f(v)$ is absolutely continuous and v solves (2.24) a.a. on R.

$\square$

The problem (2.21) - (2.23) satisfies all hypotheses of the Crandall-Liggett theorem (Theorem 2.1). Consequently, it possesses mild solution which is a continuous function of $t \in [0, T]$ with values in the Banach space $L^1_\omega(R)$. However, as we have seen in Section 1, the solutions can be dicontimuous in x no matter how smooth the initial data are. That means that in general one has

$$u(t) \notin \mathcal{D}(A) \text{ for all } t \text{ large enough even if } u(0) \in \mathcal{D}(A).$$

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3 Compensated compactness

We shall investigate once more the scalar hyperbolic conservation law of the form

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0 \tag{3.1}$$

suppelemented by the initial condition

$$u(0, x) = u_0(x), \tag{3.2}$$

where we look for solutions periodic in space, i.e.,

$$u(t, x + \omega) = u(t, x),\ \omega > 0.$$

3.1 The vanishing viscosity method

We consider an approximate problem

$$\frac{\partial}{\partial t}u_\varepsilon + \frac{\partial}{\partial x}f(u) = \varepsilon\frac{\partial^2}{\partial x^2}u_\varepsilon,\ \varepsilon > 0 \tag{3.3}$$

complemented with the same set of initial and boundary conditions.

We shall assume that (3.3), (3.2) possesses a regular solution u_ε on a time interval $(0,T)$ for any $\varepsilon > 0$.

Our aim will be to pass to the limit for $\varepsilon \to 0$ to obtain a solution of the original problem (3.1), (3.2).

3.2 Estimates

Take $b : R^1 \to R^1$ a differentiable function. Multiplying (3.3) by $b'(u_\varepsilon)$ we obtain

$$\partial_t b(u_\varepsilon) + \partial_x F_b(u) = \varepsilon\partial^2_{xx}u_\varepsilon b'(u_\varepsilon)$$

from which we deduce

$$\partial_t b(u_\varepsilon) + \partial_x F_b(u_\varepsilon) = \varepsilon\partial_x\Big(\partial_x u_\varepsilon b'(u_\varepsilon)\Big) - \varepsilon b''(u_\varepsilon)(\partial_x u_\varepsilon)^2, \tag{3.4}$$

where

$$F_b(u) \equiv \int_0^u f'(z)b'(z)\ \mathrm{d}z.$$

In particular, if b is convex, we can integrate by parts to deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\omega b(u_\varepsilon)\ \mathrm{d}x \le 0 \text{ for any convex } b. \tag{3.5}$$

Let

$$\underline{u} \le u_0(x) \le \overline{u} \text{ for all } x \in R.$$

Then we can take $b(u) = [u - \overline{u}]^+$ to deduce

$$u_\varepsilon(t,x) \le \overline{u} \text{ for any } t,x \text{ independently of } \varepsilon; \tag{3.6}$$

and, similarly,

$$u_\varepsilon(t,x) \ge \underline{u} \text{ for any } t,x \text{ independently of } \varepsilon. \tag{3.7}$$

In other words, we have verified the *maximum principle* for (3.3).

Moreover, if $b(u) = u$ we get

$$\varepsilon\int_0^T\int_0^\omega |\partial_x u|^2\ \mathrm{d}x\ \mathrm{d}t < c \text{ independently of } \varepsilon > 0. \tag{3.8}$$

3.3 Compactness

The uniform estimates above are not enough to prevent oscillations in the sequence of approximate solutions $\{u_\varepsilon\}_{\varepsilon>0}$. However, the relation (3.4) can be used to show that oscillations are not present. This is an interesting phenomenon of cancellations of oscillations.

Let us start with a simple situation when the right-hand side of (3.4) vanishes. We prove the following "div-curl" lemma.

Lemma 3.1 *Let $\vec{U}_n$, $\boldsymbol{V}_n$ be two sequences of vector functions such that*

$$\vec{U}_n \to \vec{U} \textit{ weakly star in } L^\infty(R^2; R^2),$$

$$\vec{V}_n \to \vec{V} \textit{ weakly star in } L^\infty(R^2; R^2)$$

such that

$$\mathrm{div}(\vec{U}_n) = \mathrm{div}(\vec{V}_n) = 0.$$

Then

$$U_n^1 V_n^2 - V_n^1 U_n^2 \to U^1 V^2 - V^1 U^2 \textit{ weakly star in } L^\infty(R).$$

Proof: We have that

$$[V_n^2, -V_n^1] \text{ is a potential vector field.}$$

In other words, there exists $I_n : R^1 \to R^1$ such that

$$\partial_t I_n = V_n^2,\ \partial_x I_n = -V_n^1$$

In accordance with our hypotheses, I_n are uniformly Lipschitz continuous and, consequently, we can assume

$$I_n \to I \text{ in } C(B) \text{ for any compact } B \subset R^2,$$

where

$$\partial_t I = V^2,\ \partial_x I = -V^1.$$

Thus taking a smooth test function ϕ, we get

$$\int_{R^2} \phi\Big(U_n^1 V_n^2 - U_n^2 V_n^1\Big)\ \mathrm{d}x\ \mathrm{d}t = \int_{R^2} \phi\Big(U_n^1 \partial_t I_n + U_n^2 \partial_x I_n\Big)\ \mathrm{d}x\ \mathrm{d}t =$$

$$\int_{R^2} \partial_t \phi U_n^1 I_n + \partial_x \phi U_n^2 I_n\ \mathrm{d}x\ \mathrm{d}t + \int_{R^2} \phi I_n \Big(\partial_t U_n^1 + \partial_x U_n^2\Big)\ \mathrm{d}x\ \mathrm{d}t \to$$

$$\int_{R^2} \partial_t \phi U^1 I + \partial_x \phi U^2 I\ \mathrm{d}x\ \mathrm{d}t =$$

$$\int_{R^2} \phi\Big(U^1 V^2 - U^2 V^1\Big)\ \mathrm{d}x\ \mathrm{d}t$$

□

Thus assuming

$$b^1(u_\varepsilon) \to \overline{b^1(u)},\ F_b^1(u_\varepsilon) \to \overline{F_b^1(u)} \text{ weakly star in } L^\infty((0,T)\times R),$$
$$b^2(u_\varepsilon) \to \overline{b^2(u)},\ F_b^2(u_\varepsilon) \to \overline{F_b^2(u)} \text{ weakly star in } L^\infty((0,T)\times R)$$

we get

$$b^1(u_\varepsilon)F_b^2(u_\varepsilon) - b^2(u_\varepsilon)F_b^1(u_\varepsilon) \to \overline{b^1(u)}\ \overline{F_b^2(u)} - \overline{b^2(u)}\ \overline{F_b^1(u)} \tag{3.9}$$
$$\text{weakly star in } L^\infty((0,T)\times R).$$

We shall use (3.9) to deduce more information on convergence of the sequence $\{u_\varepsilon\}_{\varepsilon>0}$. To this end, we take $b^1(u) = u$, $b^2(u) = b(u)$ and write

$$(u - u_\varepsilon)F_b(u_\varepsilon) + (f(u_\varepsilon) - \overline{f(u)})b(u_\varepsilon) \to 0 \text{ weakly star in } L^\infty((0,T)\times R).$$

Now, take
$$b(u) = |u-c|,\ F_b(u) = \mathrm{sgn}(u-c)\big(f(u) - f(c)\big)$$
which yields, for the choice $c = u(t,x)$

$$\begin{gathered}(u - u(t,x))(f(u(t,x)) - f(u_\varepsilon))\mathrm{sgn}(u(t,x) - u_\varepsilon)+ \\ |u(t,x) - u_\varepsilon|\Big(f(u(t,x)) - f(u_\varepsilon)\Big) + |u(t,x) - u_\varepsilon|\Big(f(u_\varepsilon) - \overline{f(u)}\Big) = \\ (u - u(t,x))(f(u(t,x)) - f(u_\varepsilon))\mathrm{sgn}(u(t,x) - u_\varepsilon)+ \\ |u(t,x) - u_\varepsilon|\Big(f(u(t,x)) - \overline{f(u)}\Big) \to 0 \text{ weakly star in } L^\infty((0,T)\times R)\end{gathered} \tag{3.10}$$

for a.a. $t \in (0,T)$, $x \in R$.

Assume that (t,x) is a Lebesgue point of both u and $\overline{f(u)}$. We deduce from (3.10) that

$$\Big(f(u(t,x) - \overline{f(u)}(t,x)\Big) \lim_{r\to 0}\Big(\lim_{\varepsilon\to 0}\frac{1}{|B_r(t,x)|}\int_{B_r(t,x)} |u_\varepsilon - u|\ \mathrm{d}t\ \mathrm{d}x\Big) = 0,$$

where $B_r(t,x)$ is a ball of radius r centered at (t,x).

This implies that u_ε converges strongly to u in $L^1(M)$, where

$$M = \{(t,x)\mid \overline{f(u)}(t,x) \neq f(u(t,x))\}.$$

Thus necessarily $|M| = 0$ and $\overline{f(u)} = f(u)$ a.a. on $(0,T)\times R$ which implies that the limit function u solve the problem (3.1), (3.2).

Remark: Of course, the previous argument must be slightly modified to accommodate the vanishing viscosity. It can be shown that the conclusion of Lemma 3.1 still holds, if

$$\mathrm{divU_n} = \partial_x(g_n) + \mu_n,\ \mathrm{divV_n} = \partial_x(h_n) + \nu_n$$

where

$$g_n \to 0,\ h_n \to 0 \text{ in } L^2(R^2),$$

and

$$\mu_n, \nu_n \text{ are measures bounded independently of } n = 1, 2, ...$$

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